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Filamentation of a Hollow Gaussian Beam in a Nonlinear Optical Medium

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Abstract: This paper reports the filamentation of hollow Gaussian beams of the first, second, third, and fourth orders during propagation in a cubic and quintic nonlinear medium. Due to spatial modulation instability, the hollow Gaussian beams split to form either co-centric circular filaments or ultrashort pulses. It is found that the properties of the nonlinear medium used for propagation have a strong influence on certain characteristics of the formed filaments, such as peak intensity and pulse width. This correlation between the system parameters of the medium and filament characteristics represents a method for calculating the system parameters of the medium. This investigation can be helpful in the development of a hollow Gaussian beam-based artificial intelligence system that can be used to measure the system parameters of the studied nonlinear medium.

Keywords: filamentation; modulation instability; cubic and quintic nonlinear medium

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1. Introduction

Filamentation is a process by which a spatiotemporal or spatial optical pulse splits to form ultrashort pulses called filaments. This process occurs due to modulation instability in a nonlinear medium. A delicate balance among the effects of linear dispersion or between diffraction and nonlinear effects is necessary for the generation of filaments. Ultrashort laser pulses can undergo filamentation to form optical filaments in transparent media with different states, examples of which include gases, solids, and liquids [1]. Investigations determining the influence of air turbulence on femtosecond laser filamentation have reported that the formed optical filaments are robust in terms of beam-pointing accuracy and survival when moving through turbulent air [2]. The ellipticity of the input beam can induce multiple filamentation processes [3]. The ability to observe filamentation and the generation of a supercontinuum during the propagation of optical vortices in a nonlinear medium with self-focusing nonlinearity has been investigated previously [4]. Analyses of the impact of nonlinear Landau damping on the temporal growth rate of modulation instability and filament formation show that the amplification of stationary-state filaments is highly influenced by the amplitude of ultra-relativistic electromagnetic waves [5]. Moreover, the peak value of the growth rate of filamentation instability can be further enhanced in the presence of a nonlinear Landau damping term. The number of annular solitons formed as a result of the modulation instability of a vortex beam in a nonlinear metamaterial can be enhanced by increasing the topological charge or azimuthal index of the vortex [6]. Laser filamentation may be useful for applications related to atmospheric remote sensing [7].

On the other hand, the recent literature on optical beam dynamics indicates that the propagation of hollow Gaussian beams is worthy of study. Hollow Gaussian beams can be generated using various methods. The most popular methods include the nonlinear interaction of photons with orbital angular momentum [8], phase-only filtering [9], and Fresnel diffraction of the Gaussian beam using a spiral zone plate [10]. By adopting polynomial expansion, hollow Gaussian beams can be mathematically stated as a vector sum of a series of Laguerre–Gaussian modes [11]. The tight-focused beam formed during the self-trapping of hollow Gaussian beams in a material with a negative index may be useful for trapping nano-sized particles [12]. A mathematical expression to describe the intensity distribution of hollow Gaussian beams propagating in a spherically aberrated lens has been derived by adopting the Collins formula [13]. Optically tunable hollow Gaussian beams with controllable profiles have been generated using the reflection of a TEM_{00} Gaussian beam in a metal thin film [14].

In this paper, we report on the filamentation that occurs due to modulation instability in a non-Kerr nonlinear medium with cubic and quintic nonlinearity. We consider hollow Gaussian beams of the first, second, third, and fourth orders. A medium with both cubic and quintic nonlinearities supports the stable dynamics of the laser beam and the existence of stable stationary radially symmetric modes [15]. Numerical studies on the propagation characteristics of cosh-Gaussian laser beams show that in a Kerr medium the beams collapse, whereas in a defocusing quintic nonlinear medium the beams transform into sech, Gaussian, or flat-top beams depending upon the optical power of the input [16]. We found that filamentation occurs and forms a stable soliton-like structure as hollow Gaussian beams propagate in a medium. Tailored filaments or intense lattice solitons with controllable and regulated parameters can be formed by tuning the lattice parameters of the periodic waveguide array [17]. Similarly, we found that the properties of a propagating nonlinear medium have a strong influence on the parameters of formed filaments, such as the peak intensity and pulse width. This correlation between the system parameters of the medium and the filament characteristics provides a way to estimate the system parameters of the medium. We believe that this investigation can be helpful for the development of a hollow Gaussian beam-based artificial intelligence system that can be used to measure the system parameters of the studied nonlinear medium.

The remaining sections of this article are divided as follows: in Section 2, the theoretical model of the problem is presented; a numerical analysis and discussion is provided in Section 3; and our conclusions are described in Section 4.

2. Theoretical Formulation

In this study, we consider high-intensity electromagnetic pulse propagation in a nonlinear metamaterial with a negative index . We assume that free charges and free current flow are absent in the material. Moreover, the nonlinear waveguide possesses third-order and fifth-order nonlinear susceptibilities. Hence, the nonlinear electric polarization of the material can be represented as

$$P_{NL} = \varepsilon_0 \chi_E^{(3)} |E|^2 E + \varepsilon_0 \chi_E^{(5)} |E|^4 E, \tag{1}$$

where $\chi_E^{(3)}$ and $\chi_E^{(5)}$ are third-order and fifth-order susceptibilities, respectively, and ε_0 is the permittivity of the free space. Taking the influence of third-order and fifth-order susceptibilities into consideration, the nonlinear Schrödinger equation, which describes the propagation of electromagnetic waves in the cubic and quintic nonlinear medium, is derived, as expressed by the following nonlinear partial differential equation [18–20]:

$$\frac{\partial Q}{\partial z} = i\,\frac{1}{2}\nabla_\perp^2 Q + i\rho_1 |Q|^2 Q + i\rho_2 |Q|^4 Q, \tag{2}$$

where $Q(x, y, z)$ is the normalized complex amplitude of the wave, $\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the transverse Laplace operator, z is the propagation distance of the beam, and ρ_1 and ρ_2 are the self-phase modulation and quintic nonlinear coefficients, respectively. Now, we consider the propagation of a hollow Gaussian beam with the following form in the metamaterial modeled using Equation (2):

$$Q(z, x, y) = \Theta(z)((\frac{x}{\tau\eta_1(z)})^2 + (\frac{y}{\tau\eta_2(z)})^2)^m \\ e^{i\theta(z)} e^{-((\frac{x}{\tau\eta_1(z)})^2 + (\frac{y}{\tau\eta_2(z)}))^2}, \tag{3}$$

where τ is a constant, $m = 1, 2, 3, \ldots$ is the order of the hollow Gaussian beams , $\Theta(z)$ is the peak amplitude of the propagating beam, $\eta_1(z)$ and $\eta_2(z)$ are beam width parameters, and τ is a constant that determines the width of the beam. Equation (3) represents light beams with a dark hollow spatial intensity distribution, which are depicted in Figure 1 for $m = 1, 2, 3$ and 4. Now, we numerically solve the nonlinear Schrödinger model in Equation (2). We consider the initial beam profile as a hollow Gaussian beam with the form denoted by Equation (3) . We adopt the Crank–Nicholson method for the linear portion and the fourth-order Runge–Kutta method for the nonlinear portion. Compared to the forward–backward difference Euler method , the Crank–Nicholson method is unconditionally stable and has a higher order of accuracy. Additionally, the fourth-order Runge–Kutta method provides improved accuracy. After considering the numerical stability and accuracy of these methods, we have chosen to use them in our numerical simulation experiments.

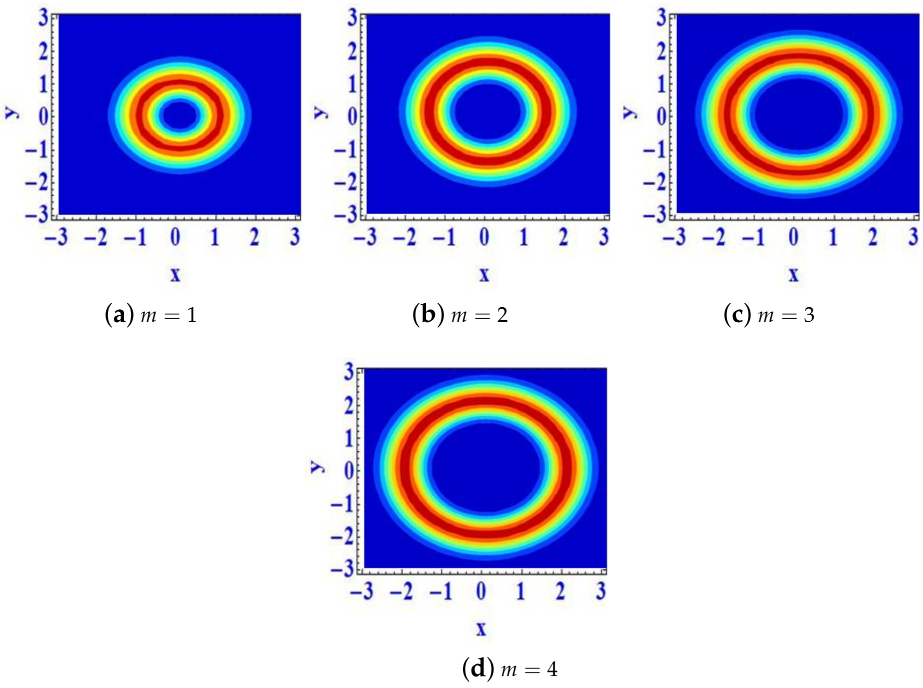


Figure 1. Hollow Gaussian beams with order m = 1, 2, 3, and 4.

To solve the linear portion of Equation (2), we sample the resulting partial differential equation at a point (i, j $n + \frac{1}{2}$) corresponding to the propagation distance (z) and two traverse coordinates (x and y), then apply the difference approximations to obtain

$$[D_z Q]^{n+\frac{1}{2}} = \theta[\alpha(D_x D_x Q + D_y D_y Q)]^{n+1} + (1-\theta)[\alpha(D_x D_x Q + D_y D_y Q)]^n, \tag{4}$$

which leads to

$$\frac{(Q_{i,j}^{n+1}-Q_{i,j}^{n})}{\Delta z}=\alpha\theta\frac{(Q_{i-1,j}^{n+1}-2Q_{i,j}^{n+1}+Q_{i+1,j}^{n+1})}{\Delta x^2}+\alpha\theta\frac{(Q_{i,j-1}^{n+1}-2Q_{i,j}^{n+1}+Q_{i,j+1}^{n+1})}{\Delta y^2}+\alpha(1-\theta)$$
$$(\frac{(Q_{i-1,j}^{n}-2Q_{i,j}^{n}+Q_{i+1,j}^{n})}{\Delta x^2})+\alpha(1-\theta)(\frac{(Q_{i,j-1}^{n}-2Q_{i,j}^{n}+Q_{i,j+1}^{n})}{\Delta y^2}), \tag{5}$$

where $\alpha = \sqrt{-1}$, $\theta = 1/2$, $x_i = i\ \Delta x$ with $i = 1,2,3\ldots\ldots N_x$, $y_j = i\ \Delta y$, with $j = 1,2,3\ldots\ldots N_y$ being equally spaced mesh points, and $Q_{i,j}^n$ represents the mesh function. Grouping the unknowns of Equation (5) on the left-hand side, we have

$$Q_{i,j}^{n+1}-\theta(F_x(Q_{i-1,j}^{n+1}-2Q_{i,j}^{n+1}+Q_{i+1,j}^{n+1}))-\theta(F_y(Q_{i,j-1}^{n+1}-2Q_{i,j}^{n+1}+Q_{i,j+1}^{n+1}))=$$
$$(1-\theta)(F_x(Q_{i-1,j}^{n}-2Q_{i,j}^{n}+Q_{i+1,j}^{n}))+(1-\theta)(F_y(Q_{i,j-1}^{n}-2Q_{i,j}^{n}+Q_{i,j+1}^{n}))+Q_{i,j}^{n}, \tag{6}$$

where $F_x = \frac{\alpha\Delta z}{\Delta x^2}$ and $F_y = \frac{\alpha\Delta z}{\Delta y^2}$ are the Fourier numbers in the x and y directions, respectively. Equation (6) represents a system of algebraic equations, which can be written as

$$Ac = b, \tag{7}$$

where A is the coefficient matrix, c represents the vector of unknowns, and b represents right-hand side of Equation (6). Now, we can solve Equation (7) for the vector of unknowns by adopting the Thomas algorithm.

The nonlinear part of Equation (2) can be represented as

$$\frac{\partial Q}{\partial z} = i\rho_1|Q|^2 Q + i\rho_2|Q|^4 Q = f(z, Q), \tag{8}$$

which can be solved by adopting a fourth-order Runge–Kutta integration scheme. At the propagation distance $z = 0$, the corresponding Q value is Q_0, which can be found using Equation (3). Now, we pick an appropriate step size $h > 0$ and define the parameters

$$Q_{n+1} = Q_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{9}$$

where

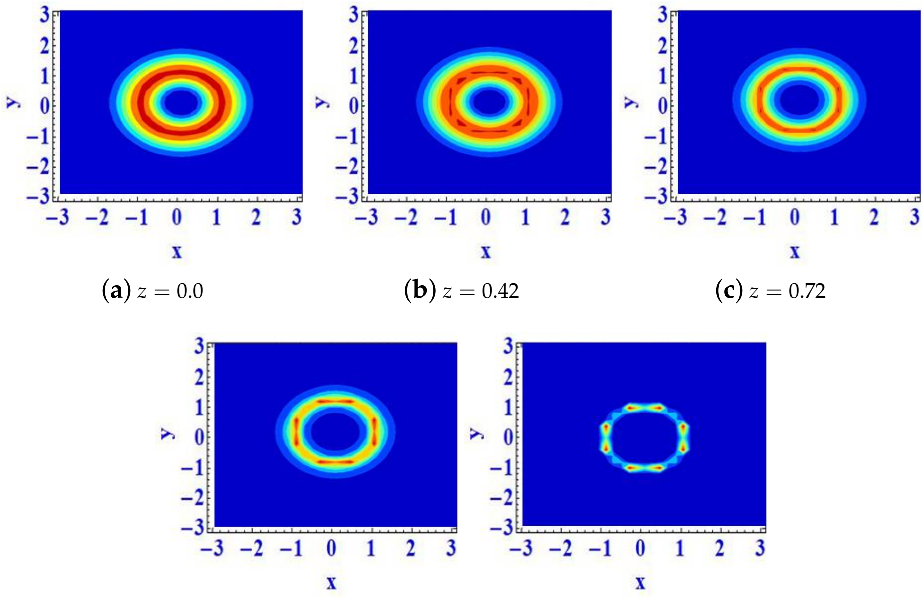
$$k_1 = f(z_n, Q_n), \tag{10}$$
$$k_2 = f(z_n + \frac{h}{2}, Q_n + h\frac{k_1}{2}), \tag{11}$$
$$k_3 = f(z_n + \frac{h}{2}, Q_n + h\frac{k_2}{2}), \tag{12}$$
$$k_4 = f(z_n + h, Q_n + hk_3). \tag{13}$$

Now, we choose $\rho_1 = 1$ and $\rho_2 = 0.3$ as the cubic and quintic nonlinear coefficients. In addition, we consider the profile of the input beam to be a form of hollow Gaussian beam, as shown in Equation (3). We consider hollow Gaussian beams of the first, second, third, and fourth orders in our numerical experiment. The numerical experiment was implemented using Python, and the obtained results are discussed in the following section.

3. Numerical Results

We now discuss the results obtained through our numerical experiments by considering the dynamic model in Equation (2). We consider a nonlinear medium with cooperating cubic and quintic nonlinearities. A nonlinear medium with cooperating cubic and quintic nonlinearities shows the phenomenon of vortex beam filamentation, whereas the competing cubic and quintic nonlinearities in the medium may not support splitting, and instead have the effect of stabilizing the dynamics [21]. As an illustrative example, we choose $\rho_1 = 1$ and $\rho_2 = 0.3$ to be the cubic and quintic nonlinear coefficients. Here, all of the parameters are in normalized units, justifying the propagation model in Equation (2). Similar works using normalized units have been reported, such as those using temporal splitting in nonlinear metamaterials [22], breather generation of vortex light bullets [23], and gap soliton formation in an optical metamaterial coupler [24], to mention only a few. Here, we consider the profile of the input beam to be a form of hollow Gaussian beam, as shown in Equation (3). Additionally, we consider hollow Gaussian beams of the first, second, third, and fourth orders for our numerical experiments. In particular, we focus on the dependence of the system parameters on the attributes of filaments formed via the filamentation of hollow Gaussian beams in a cooperating cubic and quintic nonlinear medium.

In a medium with cooperating cubic and quintic nonlinearities, the propagation of high-intensity laser beams undergoes intrinsic collapse in two and three dimensions, resulting in unstable dynamics. In a nonlinear medium with cooperating cubic and quintic nonlinearities, a hollow Gaussian beam propagating with adequate optical power may experience instability and become split into several uncorrelated light structures due to the phenomenon of modulation instability, which results in the generation of optical filaments. Figure 2 depicts filamentation resulting from the modulation instability of a first-order hollow Gaussian beam in a nonlinear medium with cubic and quintic nonlinearities. The parameters used for the simulation are $\rho_1 = 1$ and $\rho_2 = 0.3$, and the input for the optical power is $\Theta = 2$. It is clear from Figure 2 that when the light beam passes through a nonlinear medium with cooperating cubic and quintic nonlinearities it breaks into filaments after a certain distance. This transverse filamentation arises due to spatial modulation instability, and results in the formation of soliton clusters. When the diffraction is balanced with nonlinearities, spatial modulation instability occurs and the hollow Gaussian beam splits to form solitons. During further propagation, the formed soliton clusters maintain their shape and evolve tangentially relative to the input ring-shaped hollow Gaussian beam profile. When the propagation distance is z = 1.17, a stable filament is generated; we found that this propagates further without any additional changes in shape up to z = 2.25. When it does propagate further, the filaments become unstable due to nonlinear focusing.

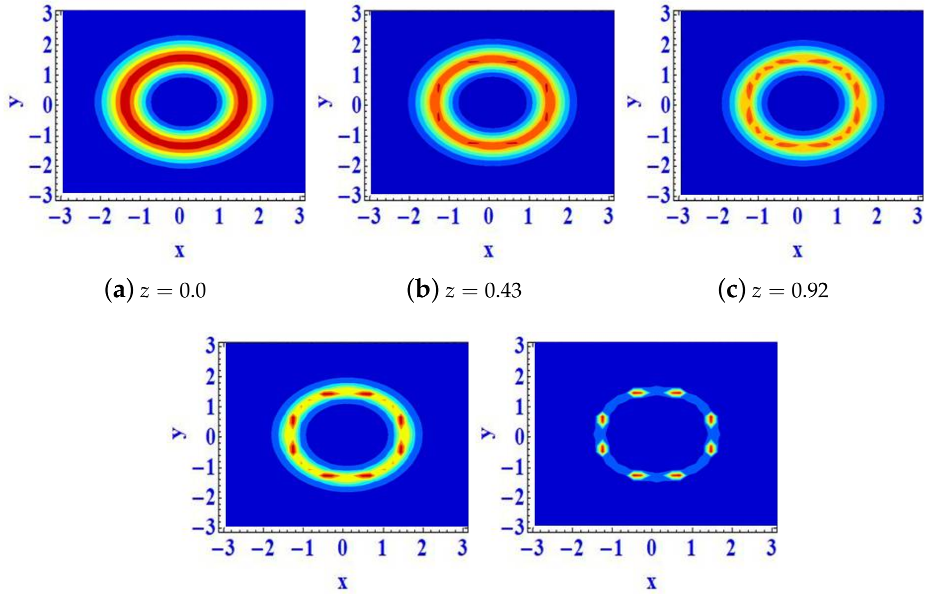


(a) $z = 0.0$ **(b)** $z = 0.42$ **(c)** $z = 0.72$

(d) $z = 0.94$ **(e)** $z = 1.17$

Figure 2. Filamentation of first-order hollow Gaussian beam in a nonlinear medium with cubic and quintic nonlinearities when $\rho_1 = 1$, $\rho_2 = 0.3$, and $\Theta = 2$.

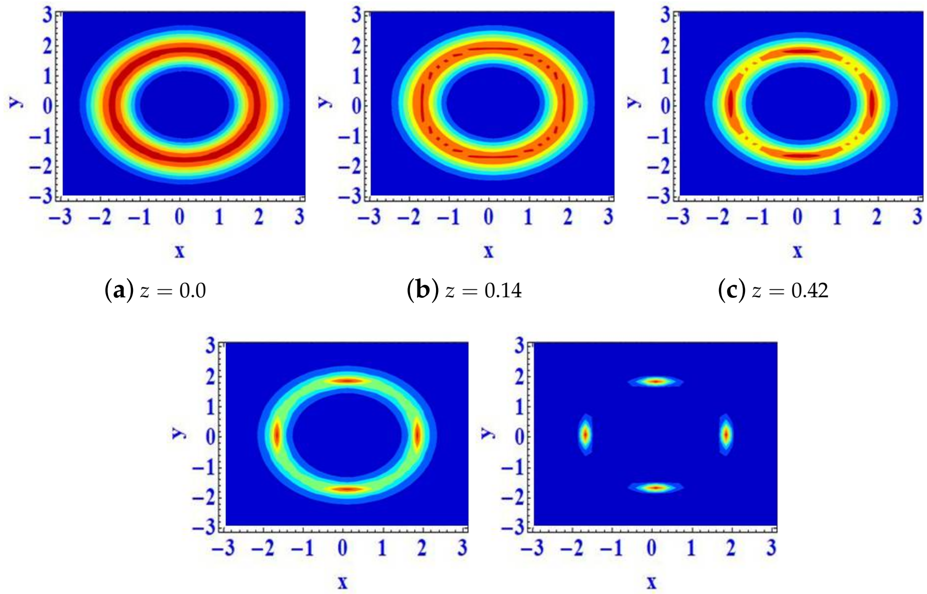
The filamentation of higher-order hollow Gaussian beams is depicted in Figures 3–5. Figures 3–5 correspond to the filamentation of higher-order hollow Gaussian beams of the second, third, and fourth orders, respectively. It can be observed that when the input power is above a certain threshold the hollow Gaussian beam splits to form soliton clusters, which is due to the fundamental phenomenon of modulation instability . In each case, the number of solitons in the cluster and their attributes, such as maximum intensity and pulse width, may differ.



(**a**) $z = 0.0$ (**b**) $z = 0.43$ (**c**) $z = 0.92$

(**d**) $z = 1.18$ (**e**) $z = 1.42$

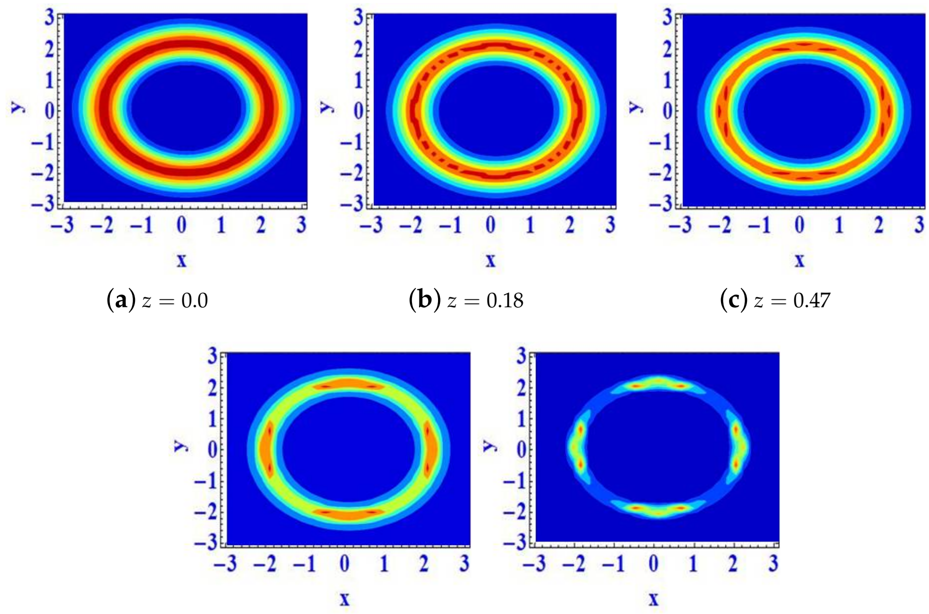
Figure 3. Filamentation of second-order hollow Gaussian beam in a nonlinear medium with cubic and quintic nonlinearities. The system parameters are $\rho_1 = 1$, $\rho_2 = 0.3$, and $\Theta = 2$.



(**a**) $z = 0.0$ (**b**) $z = 0.14$ (**c**) $z = 0.42$

(**d**) $z = 0.79$ (**e**) $z = 1.02$

Figure 4. Filamentation of third-order hollow Gaussian beam in a nonlinear medium with cubic and quintic nonlinearities. The system parameters are $\rho_1 = 1$, $\rho_2 = 0.3$, and $\Theta = 2$.



(**a**) $z = 0.0$ (**b**) $z = 0.18$ (**c**) $z = 0.47$

(**d**) $z = 0.73$ (**e**) $z = 1.14$

Figure 5. Filamentation of fourth-order hollow Gaussian beam in a nonlinear medium with cubic and quintic nonlinearities. The system parameters are $\rho_1 = 1$, $\vartheta = \rho_2$, and $\Theta = 2$.

We found that the properties of a propagating nonlinear medium have a strong influence on the characteristics of formed filaments, such as the peak intensity and pulse width. When changing the system parameters that characterize the nonlinear medium, we found them to have considerable influence on the attributes of the filaments. This correlation between the system parameters of the medium and filament characteristics provides a way of estimating the system parameters of the medium by measuring the filament attributes. In general, our findings may be useful to estimate the system parameters of any nonlinear medium using the known values of filament attributes. If this relationship between the system parameters of the medium and filament characteristics is programmed properly to detect and display the system parameters, then the findings of our study can provide a means of developing a hollow Gaussian beam-based artificial intelligence system. Hence, this investigation should be helpful in the development of a hollow Gaussian beam-based artificial intelligence system for measuring the system parameters of the studied nonlinear medium .

4. Conclusions

In this paper, we have investigated the filamentation and formation of soliton rings from hollow Gaussian beams. We numerically studied the propagation of hollow Gaussian beams of the first, second, third, and fourth orders in a nonlinear medium with cooperating cubic and quintic nonlinearities. Due to the phenomenon of modulation instability, the hollow Gaussian beam splits to form either co-centric stable circular filaments or ultrashort pulses. The stability of the formed filament was found to be a function of the system parameters characterizing the medium. We found that the properties of the propagating nonlinear medium highly influence the characteristics of the formed filaments, such as the peak intensity and pulse width. This correlation between the system parameters of the medium and the filament characteristics provides a way to estimate the system parameters of the medium. Hence, this study reports a method for the development of a hollow Gaussian beam-based artificial intelligence system to measure the system parameters of the studied nonlinear medium.

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Conflicts of Interest: The authors declare that they have no personal relationships or known competing financial interest that could have appeared to impact the work described in this manuscript.

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