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Asymptotic Chow Semistability Implies Ding Polystability for Gorenstein Toric Fano Varieties

Naoto Yotsutani

Faculty of Education, Mathematics, Kagawa University, Saiwaicho 1-1, Takamatsu 760-8522, Japan; yotsutani.naoto@kagawa-u.ac.jp

Abstract: In this paper, we prove that if a *Gorenstein* toric Fano variety $(X, -K_X)$ is asymptotically Chow semistable, then it is Ding polystable with respect to toric test configurations (Theorem 3). This extends the known result obtained by others (Theorem 2) to the case where X admits *Gorenstein singularity*. We also show the additivity of the Mabuchi constant for the product toric Fano varieties in Proposition 2 based on the author's recent work (Ono, Sano and Yotsutani in arxiv:2305.05924). Applying this formula to certain toric Fano varieties, we construct *infinitely many* examples that clarify the difference between relative K-stability and relative Ding stability in a systematic way (Proposition 1). Finally, we verify the relative Chow stability for Gorenstein toric del Pezzo surfaces using the combinatorial criterion developed in (Yotsutani and Zhou in *Tohoku Math. J.* **71** (2019), 495–524.) and specifying the symmetry of the associated polytopes as well.

Keywords: Chow stability; relative stability; Fano varieties

MSC: 14L24; 14M25; 53C55

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1. Introduction

Let (X, L) be a polarized projective variety of complex dimension n. One of the outstanding problems in Kähler geometry is to distinguish whether the first Chern class $c_1(L)$ contains a Kähler metric ω with constant scalar curvature (cscK metric). A parallel reasoning question in algebraic geometry is to study an appropriate notion of stability of (X, L) in the sense of Geometric Invariant Theory (GIT). This leads us to investigate various notions of GIT stability and study the relations among them. For example, Ross-Thomas clarified the following implications among GIT stability in their paper [1]:

$$\begin{array}{rcl} \text{Asymptotic Chow stability} & \Rightarrow & \text{Asymptotic Hilbert stability} \\ \Rightarrow \ \text{Asymptotic Hilbert semistability} & \Rightarrow & \text{Asymptotic Chow semistability} \\ & \Rightarrow & \text{K-semistability.} \end{array}$$

In [2], Mabuchi proved that Chow stability and Hilbert stability asymptotically coincide. We remark that, for a fixed positive integer $i \in \mathbb{Z}_+$, Chow stability for $(X, L^{\otimes i})$ implies Hilbert stability for $(X, L^{\otimes i})$ (i.e., not necessarily an asymptotic stability case) by the classical result due to Forgaty [3]. See also ([4] Corollary 3.4) for more combinatorial description of this result in terms of GIT weight polytopes.

In order to describe our issue more precisely, we first recall that a complex normal variety X is said to be *Fano* if its anticanonical divisor $-K_X$ is ample. It is called *Gorenstein* if $-K_X$ is Cartier. Suppose that X is a *smooth* Fano variety (i.e., a Fano manifold) with a Kähler metric

$$\omega = \sqrt{-1} g_{i\bar{j}} dz_i \wedge d\bar{z}_j \in 2\pi c_1(X).$$

We recall that $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ is Kähler–Einstein if and only if φ is a critical point of either the K-energy ν_ω or the Ding functional $\mathcal{D}_\omega$, where both functionals are defined on the space of Kähler potentials $\mathcal{H}_\omega = \{\varphi \in C^\infty(X) | \omega_\varphi > 0\}$. It is known that these functionals satisfy the inequality $\mathcal{D}_\omega \leqslant \nu_\omega$, which shows that the Ding invariant is less than or equal to the Donaldson–Futaki (DF) invariant [5]. In the case where X is toric, Yao gave an explicit description of the inequality between the Ding invariant and the DF invariant in terms of the associated polytope ([6] Proposition 4.6). In particular, Ding polystability implies K-polystability for a toric Fano manifold. Moreover, the converse direction has been proven in [7] for (not necessarily toric) Fano manifolds.

Theorem 1 (Fujita)**.** *Let X be a Fano manifold. Then, Ding semistability is equivalent to K-semistability. Furthermore, Ding polystability (resp. Ding stability) is also equivalent to K-polystability (resp. K-stability).*

On the one hand, from a differential-geometrical point of view, Theorem 1 corresponds to the fact that cscK metrics in the anticanonical classes of Fano manifolds are Kähler–Einstein metrics. Recall that for a compact Kähler manifold X with a fixed Kähler class $[\omega]$, φ is a critical point of ν_ω if and only if ω_φ is a cscK metric. On the other hand, we conclude that if a Fano manifold X is asymptotically Chow semistable, then it is Ding *semistable* according to the previous argument. In the case where X is a toric Fano manifold, it is known that X is K-semistable if and only if it is K-polystable [5,8,9]. Summing up these arguments, we have the following.

Theorem 2 (Berman, Ono, Yao)**.** *Let $(X, -K_X)$ be a smooth toric Fano variety. If $(X, -K_X)$ is asymptotically Chow semistable with respect to toric test configurations, then it is Ding polystable with respect to toric test configurations.*

In this article, we show a more general result via a combinatorial proof.

Theorem 3. *Let $(X, -K_X)$ be a* Gorenstein *toric Fano variety. If $(X, -K_X)$ is asymptotically Chow semistable with respect to toric test configurations, then it is Ding polystable with respect to toric test configurations.*

Essentially, the proof of Theorem 3 is based on the *Ehrhart reciprocity law* and the fact that any toric Fano variety is K-polystable if and only if the barycenter of the associated reflexive polytope $\Delta \subseteq M_{\mathbb{R}}$ is the origin. As mentioned above, another advantage of our combinatorial approach is that X may admit Gorenstein singularity (i.e., not necessarily smooth) in our main theorem. However, it does not work for a $\mathbb{Q}$-Gorenstein toric variety since the corresponding polytope Δ contains not only the origin, but also other lattice points. It also should be noted that we only assume $(X, -K_X)$ to be asymptotically Chow *semistable* and *do not assume* $(X, -K_X)$ to be asymptotically Chow polystable in Theorem 3.

In the following Section 4, we discuss the relative stability of the toric Fano variety. Recently, we found that there are at least four examples of smooth toric Fano varieties that clarify the difference between relative K-stability and relative Ding stability in [10]. In order to discover these four examples of a relatively K-polystable toric Fano variety, but which is relatively Ding unstable, we focused on the geometrical description such that they are all $\mathbb{P}^1$-bundles over $\mathbb{P}^m$. In particular, we consider the case of Picard number one projective toric varieties. Based on a recent argument discussed in [11], we systematically construct such examples in arbitrary dimension.

Proposition 1 (See Corollary 4)**.** *Fixing a positive integer r, we consider an extremal smooth toric Fano variety X_k with the associated polytope Δ_k, for $1 \leqslant k \leqslant r$. Suppose $\theta_{\Delta_k}(\boldsymbol{x}_k)$ to be the potential function of Δ_k defined in (10) with $\frac{1}{r} \leqslant \theta_{\Delta_k} < 1$. For the product polytope $\Delta = \prod_{k=1}^{r} \Delta_k$, the associated smooth toric Fano variety $(X_\Delta, -K_{X_\Delta})$ is relatively K-polystable, but it is relatively Ding unstable.*

In order to prove Proposition 1, we shall use the following additive property of the Mabuchi constant M_{X_Δ} for the products of toric Fano varieties.

Proposition 2 (See Corollary 3)**.** *For the product polytope Δ of reflexive polytopes Δ_k for $k = 1, \ldots, r$, let M_{X_Δ} and $M_{X_{\Delta_k}}$ be the Mabuchi constant defined in* (18)*. Then, we have the equality*

$$M_{X_\Delta} = M_{X_{\Delta_1}} + \cdots + M_{X_{\Delta_r}}.$$

We give a purely combinatorial proof of Proposition 2 in Section 4.3. In the following Section 4.4, we classify Gorenstein toric del Pezzo surfaces in terms of (asymptotic) relative Chow polystability. We use the criteria (12) to verify the asymptotic relative Chow stability of the polarized toric variety. However, it is very difficult to verify the asymptotic relative Chow stability of a given polarized toric variety because we have to prove that there exists $t_i \in \mathbb{R}$ satisfying the equality in (12) for *any* positive integer i. In order to solve this difficulty, we consider the *symmetry of the associated polytopes* $\Delta \subset M_{\mathbb{R}}$, which works very well for two-dimensional reflexive polygons (16 types). Adapting the symmetry of reflexive polygons and a combinatorial criterion (12) investigated by Zhou and the author in [12], we verify the relative Chow stability of each Gorenstein toric del Pezzo surface.

Proposition 3 (See Proposition 6)**.** *Among all* 16 *isomorphism classes of Gorenstein toric del Pezzo surfaces, there are* 5 *isomorphism classes of asymptotically relatively Chow polystable surfaces and* 4 *isomorphism classes of asymptotically relatively Chow unstable surfaces. The remaining* 7 *classes are relatively Chow polystable with respect to the anticanonical polarization.*

All the results are listed in Table 1. We also refer the reader to Table 2, specifying the symmetry of each reflexive polygon $\Delta \subset M_{\mathbb{R}}$.

This paper is organized as follows. Section 2 is a brief review of Gorenstein toric Fano varieties, Ding stability and asymptotic Chow stability. The proof of Theorem 2 is given in Section 3. Section 4 collects the results of relative algebro-geometric stability. In Sections 4.1 and 4.2, we recall the criteria of relative Chow stability of polarized toric varieties investigated by the author and B. Zhou in [12]. We prove Proposition 2 in Section 4.3 by applying the product formulas regarding convex polytopes, which were also used in [11]. See Lemma 3 and the proof of Proposition 5 for further details. Section 4.4 is devoted to verifying the asymptotic relative Chow stability of Gorenstein toric del Pezzo surfaces. All the results and practical values of invariants are summarized in Proposition 6 and Table 1.

Table 1. Relative Chow stability of Gorenstein toric del Pezzo surfaces.

Label in [13]	**Stability**	t_1 **in** (12)
3 ($\mathbb{C}P^2$)	Asymptotically relatively Chow polystable	No need
$4A$ ($\mathbb{C}P^1 \times \mathbb{C}P^1$)	Asymptotically relatively Chow polystable	No need
$4B$ (dP_8)	Relatively Chow polystable with respect to $\mathcal{O}_X(-K_X)$	$-65/828$
$4C$	Relatively Chow polystable with respect to $\mathcal{O}_X(-K_X)$	$-5/72$
$5A$ (dP_7)	Relatively Chow polystable with respect to $\mathcal{O}_X(-K_X)$	$69/665$
$5B$	Asymptotically relatively Chow unstable	–
$6A$ (dP_6)	Asymptotically relatively Chow polystable	No need
$6B$	Relatively Chow polystable with respect to $\mathcal{O}_X(-K_X)$	$-259/1944$
$6C$	Asymptotically relatively Chow unstable	–
$6D$	Asymptotically relatively Chow unstable	–
$7A$	Relatively Chow polystable with respect to $\mathcal{O}_X(-K_X)$	$-409/2646$
$7B$	Asymptotically relatively Chow unstable	–
$8A$	Asymptotically relatively Chow polystable	No need

Table 1. *Cont.*

Label in [13]	Stability	t_1 in (12)
$8B$	Relatively Chow polystable with respect to $\mathcal{O}_X(-K_X)$	$-33/200$
$8C$	Relatively Chow polystable with respect to $\mathcal{O}_X(-K_X)$	$-3/19$
9	Asymptotically relatively Chow polystable	No need

Table 2. Combinatorial data and the delta invariant of Gorenstein toric del Pezzo surfaces.

Label	$\Delta \subseteq M_{\mathbb{R}}$ in [13]	Symmetry of Δ
3	$\mathrm{conv}\{\begin{pmatrix}-1\\1\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}-1\\-2\end{pmatrix}\}$	No need
$4A$	$\mathrm{conv}\{\begin{pmatrix}-1\\1\end{pmatrix},\begin{pmatrix}-1\\-1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\}$	No need
$4B$	$\mathrm{conv}\{\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}-1\\-1\end{pmatrix}\}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$4C$	$\mathrm{conv}\{\begin{pmatrix}-1\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}-1\\-1\end{pmatrix}\}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$
$5A$	$\mathrm{conv}\{\begin{pmatrix}-1\\-1\end{pmatrix},\begin{pmatrix}0\\-1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}-1\\0\end{pmatrix}\}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$5B$	$\mathrm{conv}\{\begin{pmatrix}-1\\0\end{pmatrix},\begin{pmatrix}1\\-2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix}\}$	–
$6A$	$\mathrm{conv}\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix},\begin{pmatrix}-1\\0\end{pmatrix},\begin{pmatrix}0\\-1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix}\}$	No need
$6B$	$\mathrm{conv}\{\begin{pmatrix}-1\\1\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\end{pmatrix}\}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$
$6C$	$\mathrm{conv}\{\begin{pmatrix}-1\\-1\end{pmatrix},\begin{pmatrix}0\\-1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix}\}$	–
$6D$	$\mathrm{conv}\{\begin{pmatrix}-1\\1\end{pmatrix},\begin{pmatrix}-1\\-2\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\}$	–
$7A$	$\mathrm{conv}\{\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix}\}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$7B$	$\mathrm{conv}\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}-3\\-1\end{pmatrix},\begin{pmatrix}0\\-1\end{pmatrix}\}$	–
$8A$	$\mathrm{conv}\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}-1\\0\end{pmatrix},\begin{pmatrix}0\\-1\end{pmatrix}\}$	No need
$8B$	$\mathrm{conv}\{\begin{pmatrix}-1\\2\end{pmatrix},\begin{pmatrix}2\\-1\end{pmatrix},\begin{pmatrix}0\\-1\end{pmatrix},\begin{pmatrix}-1\\0\end{pmatrix}\}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$8C$	$\mathrm{conv}\{\begin{pmatrix}-1\\2\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}-1\\-2\end{pmatrix}\}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$
9	$\mathrm{conv}\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}-1\\-1\end{pmatrix}\}$	No need

2. Preliminaries

2.1. Gorenstein Toric Fano Varieties

We first recall the standard notation and basic definitions of Gorenstein toric Fano varieties, as can be found in [14].

Let $N \cong \mathbb{Z}^n$ be a lattice of rank n, while $M = \mathrm{Hom}(N, \mathbb{Z})$ is the $\mathbb{Z}$-dual of M. Let $P \subseteq N_{\mathbb{R}} \cong \mathbb{R}^n$ be a lattice polytope with $\mathbf{0} \in \mathrm{Int}(P)$. We assume that all vertices of P are primitive elements in N. For a subset S of $N_{\mathbb{R}}$, we denote the positive hull of S by $\mathrm{pos}(S)$, i.e., $\mathrm{pos}(S) = \sum_{v \in S} \mathbb{R}_{\geqslant 0} v$. Then,

$$\Sigma_P := \{\mathrm{pos}(F) | F \text{ is a face of } P\}$$

forms the fan, which is often called the *normal fan* of P. It is well known that the fan $\Sigma = \Sigma_P$ associates a toric variety X_Σ with the complex torus $T_N := \mathrm{Spec}\, \mathbb{C}[M]$ action. Here and hereafter, we denote the associated toric variety by X for simplicity. Recall that the anticanonical divisor of X is given by $-K_X = \sum_\rho D_\rho$, where D_ρ is the torus invariant Weil divisor corresponding to a ray $\rho \in \Sigma(1)$. Then, the dual polytope of P (with respect to $-K_X$) is defined by

$$\Delta = \{y \in M_{\mathbb{R}} | \prec x, y \succ \geq -1 \text{ for all } x \in P\}$$

which is also an n-dimensional (rational) polytope in $M_{\mathbb{R}}$ with $\mathbf{0} \in \mathrm{Int}(\Delta)$. Then, Δ is called *reflexive* if it is a lattice polytope. There is a bijective correspondence between isomorphism classes of reflexive polytopes and isomorphism classes of Gorenstein toric Fano varieties. For a fixed dimension n, there are only finitely many isomorphism classes of n-dimensional reflexive polytopes [15,16]. They found $1, 16, 4319$ and 473800776 isomorphism classes for $n = 1, 2, 3$ and 4. Throughout the paper, we assume that a (toric) Fano variety X admits at worst Gorenstein singularities.

2.2. Ding Stability for Fano Varieties

In this section, we briefly review the notion of Ding stability; see [5–7] for more details.

Let (X, ω) be an n-dimensional Fano manifold with a Kähler metric $\omega \in 2\pi c_1(X)$. We set V to be the volume $V := \int_X \omega^n$ of the given Fano manifold X. Recall that the *Ding functional* $\mathcal{D}_\omega : \mathcal{H}_\omega \to \mathbb{R}$ is given by

$$\mathcal{D}_\omega := -\frac{1}{V} \int_0^1 \int_X \dot{\varphi}_t (1 - e^{\rho_{\omega_t}}) \omega_t^n \, dt,$$

where φ_t is a smooth path in $\mathcal{H}_\omega$ joining 0 with φ and ρ_ω is the function that satisfies

$$\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\bar{\partial}\rho_\omega \quad \text{and} \quad \int_X (e^{\rho_\omega} - 1)\omega^n = 0. \tag{1}$$

Then, we readily see that φ is a critical point of $\mathcal{D}_\omega$ if and only if ω_φ is a Kähler–Einstein metric.

Next, we recall the notion of a test configuration. A *test configuration* for a Fano variety $(X, -K_X)$ is a polarized scheme $(\mathcal{X}, \mathcal{L})$ with

- a $\mathbb{C}^\times$-action and a $\mathbb{C}^\times$-equivariant proper flat morphism $\pi : \mathcal{X} \to \mathbb{C}$, where $\mathbb{C}^\times$ acts on the base by multiplication;
- a $\mathbb{C}^\times$-equivariant line bundle $\mathcal{L} \to \mathcal{X}$, which is ample over all fiber $\mathcal{X}_z := \pi^{-1}(z)$ for $z \neq 0$, and $(X, -K_X)$ is isomorphic to $(\mathcal{X}_z, \mathcal{L}_z)$ with $\mathcal{L}_z = \mathcal{L}|_{\mathcal{X}_z}$.

Taking a Hermitian metric h_0 on $\mathcal{O}_X(-K_X)$ with positive curvature, we can construct the Phong–Sturm geodesic ray h_t that emanates from h_0 in $\mathcal{H}_\omega$ [17]. In [5], Berman defined the Ding invariant as the asymptotic slope of the Ding functional along the geodesic rays. Moreover, he showed that

$$DF(\mathcal{X}, \mathcal{L}) = \lim_{t \to \infty} \frac{1}{V} \frac{d\mathcal{D}_\omega(h_t)}{dt} + q$$

where the error term q is non-negative and $DF(\mathcal{X}, \mathcal{L})$ is the *Donaldson–Futaki invariant*. Then, the *Ding invariant* $\mathrm{Ding}(\mathcal{X}, \mathcal{L})$ is given by

$$\mathrm{Ding}(\mathcal{X}, \mathcal{L}) = \lim_{t \to \infty} \frac{1}{V} \frac{d\mathcal{D}_\omega(h_t)}{dt}.$$

A Gorenstein Fano variety X is said to be *Ding semistable* if, for any test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, -K_X)$, we have $\mathrm{Ding}(\mathcal{X}, \mathcal{L}) \geqslant 0$. Moreover, X is said to be *Ding polystable* if X is Ding semistable and $\mathrm{Ding}(\mathcal{X}, \mathcal{L}) = 0$ if and only if $(\mathcal{X}, \mathcal{L})$ is equivariantly isomorphic to $(X \times \mathbb{C}, p_1^*(-K_X))$, where $p_1 : X \times \mathbb{C} \to X$ is the projection.

Now, we consider the toric case. Let X be an n-dimensional toric Fano variety and $\Delta \subseteq M_\mathbb{R}$ the corresponding reflexive polytope with the coordinates $\boldsymbol{x} = (x_1, \ldots, x_n)$. Recall that a piecewise linear convex function $u = \max\{f_1, \ldots, f_\ell\}$ on Δ is called *rational* if $f_k = \sum a_{k,i} x_i + c_k$ with $(a_{k,1}, \ldots, a_{k,n}) \in \mathbb{Q}^n$ and $c_k \in \mathbb{Q}$ for $k = 1, \ldots, \ell$. A *toric test configuration* for $(X, -iK_X)$, introduced by Donaldson [18], is a test configuration associated with a rational piecewise linear convex function u on Δ, so that iQ is a lattice polytope in $M_\mathbb{R} \times \mathbb{R} \cong \mathbb{R}^{n+1}$. Here, Q is given by

$$Q = \{(\boldsymbol{x}, t) | \boldsymbol{x} \in \Delta,\ 0 \leqslant t \leqslant R - u(\boldsymbol{x})\}$$

and R is an integer such that $u \leqslant R$. Then, iQ defines the $n+1$-dimensional polarized toric variety $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$ and a flat morphism $\overline{\mathcal{X}} \to \mathbb{C}P^1$. Hence, the family restricted to $\mathbb{C}$ gives a torus equivariant test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, -iK_X)$.

The toric geodesic ray h_t associated with a toric test configuration was described by Song-Zeldich [19]. In [6], Yao detected an explicit description of the Ding invariants of toric Fano varieties.

Theorem 4 (Yao). *Let $(X, -K_X)$ be a Gorenstein toric Fano variety with the associated reflexive polytope Δ. Let u be a piecewise linear convex function. The Ding invariant of the toric test configuration associated with u is given by*

$$\begin{aligned}\mathrm{Ding}(\mathcal{X}, \mathcal{L}) &= \lim_{t \to \infty} \frac{1}{\mathrm{vol}(\Delta)} \frac{d\mathcal{D}_\omega(h_t)}{dt} \\ &= -u(0) + \frac{1}{\mathrm{vol}(\Delta)} \int_\Delta u(\boldsymbol{x})\, dv =: \mathcal{I}_\Delta(u).\end{aligned} \tag{2}$$

Then, a reflexive polytope $\Delta \subseteq M_\mathbb{R}$ is said to be *Ding polystable* if $\mathcal{I}_\Delta(u) \geqslant 0$ for all convex piecewise linear functions u and the equality holds if and only if u is affine linear. One can observe that $\mathcal{I}_\Delta(u)$ is invariant when we add affine linear functions to convex piecewise linear functions. Hence, it suffices to consider *normalized* convex piecewise linear functions u on Δ for our purpose, i.e., $u(\boldsymbol{x}) \geqslant u(0) = 0$. The following observation was given by Yao [6], and we provide the details for the reader's convenience.

Proposition 4 (Yao). *If Δ is a reflexive polytope, then the associated Gorenstein toric Fano variety $(X, -K_X)$ is Ding polystable if and only if the barycenter of Δ is $\mathbf{0}$.*

Proof. Suppose that Δ is Ding polystable. Hence,

$$\frac{1}{\mathrm{vol}(\Delta)} \int_\Delta u(\boldsymbol{x})\, dv \geqslant 0 \tag{3}$$

for any normalized convex piecewise linear function u. Applying (3) to linear functions, i.e., $u = \pm x_i$ for $i = 1, \ldots, n$, we conclude $\int_\Delta \boldsymbol{x}\, dv = \mathbf{0}$.

Conversely, we assume that $\int_\Delta \boldsymbol{x}\, dv = \mathbf{0}$. Then, for any normalized convex piecewise linear function u, Jensen's inequality implies that

$$\int_\Delta u(\boldsymbol{x})\, dv \geqslant u\Big(\int_\Delta \boldsymbol{x}\, dv\Big) = u(\mathbf{0}) = 0.$$

Hence, Δ is Ding polystable. $\square$

2.3. Asymptotic Chow Stability of Toric Varieties

In this section, let us briefly recall the notion of Chow stability; see [20,21] for more details.

Let $X \subset \mathbb{C}P^N$ be an n-dimensional irreducible complex projective variety of degree $d \geqslant 2$. Recall that for a projectively embedded n-dimensional complex subvariety $X \subset \mathbb{C}P^N$, the *degree* d of X is a number of intersection of X with a linear subspace L in a general position, such that $n + \dim L = N$. Let us denote the Grassmann variety by $\mathbb{G}(k, \mathbb{C}P^N)$. We define the *associated hypersurface* of $X \subset \mathbb{C}P^N$ by

$$Z_X := \{L \in \mathbb{G}(N-n-1, \mathbb{C}P^N) | L \cap X \neq \emptyset\}.$$

Remark that the construction of Z_X can be regarded as an analog of the projective dual varieties as in ([22] Chapter1) . In fact, it is well known that Z_X is an irreducible divisor in $\mathbb{G}(N-n-1, \mathbb{C}P^N)$ with $\deg Z_X = d$ in the Plücker coordinates. Therefore, there exists $R_X \in H^0(\mathbb{G}(N-n-1, \mathbb{C}P^N), \mathcal{O}_{\mathbb{G}}(d))$ such that $Z_X = \{R_X = 0\}$. We call R_X the *X-resultant*. Since there is a natural action of $\mathrm{SL}(N+1, \mathbb{C})$ on $H^0(\mathbb{G}(N-n-1, \mathbb{C}P^N), \mathcal{O}_{\mathbb{G}}(d))$, we define GIT stability for the X-resultant R_X as follows.

Definition 1. Let $X \subset \mathbb{C}P^N$ be an n-dimensional irreducible complex projective variety. X is said to be *Chow semistable* if the closure of the $\mathrm{SL}(N+1, \mathbb{C})$-orbit of the X-resultant R_X does not contain the origin. X is said to be *Chow polystable* if the orbit $\mathrm{SL}(N+1, \mathbb{C}) \cdot R_X$ is closed. We call X *Chow unstable* if it is not Chow semistable.

Definition 2. Let (X, L) be a polarized projective variety. For $i \gg 0$, we denote the Kodaira embedding by $\Psi_i : X \to \mathbb{P}(H^0(X, L^{\otimes i})^*)$. (X, L) is said to be *asymptotically Chow semistable* (resp. *polystable*) if there is an i_0 such that $\Psi_i(X)$ is Chow semistable (resp. polystable) for each $i \geqslant i_0$. (X, L) is called *asymptotically Chow unstable* if it is not asymptotically Chow semistable.

Next, we will give a brief review of Ono's necessary condition for the Chow semistability of polarized toric varieties. Let Δ be an n-dimensional lattice polytope in $M_{\mathbb{R}} \cong \mathbb{R}^n$. The *Euler–Maclaurin summation formula* for polytopes provides a powerful connection between the integral over a polytope Δ and the summation of lattice points in Δ. More specifically, for any polynomial function ϕ on $\mathbb{R}^n$, we would like to see how the summation

$$\sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} \phi(\mathbf{a}) =: I(\phi, \Delta)(i)$$

will behave for a positive integer i. If we take ϕ to be 1, $I(\phi, \Delta)(i)$ is the so-called Ehrhart polynomial, which counts the number of lattice points in the i-th dilation of a polytope Δ:

$$I(1, \Delta)(i) = \#(\Delta \cap (\mathbb{Z}/i)^n).$$

Recall that the Ehrhart polynomial has the expression

$$E_\Delta(t) := I(1, \Delta)(t) = \mathrm{vol}(\Delta)t^n + \frac{\mathrm{vol}(\partial\Delta)}{2} t^{n-1} + \cdots + 1$$

where $\partial\Delta$ is the boundary of a lattice polytope Δ. Similarly, if we take ϕ to be the coordinate functions $\boldsymbol{x} = (x_1, \ldots, x_n)$, then $I(\phi, \Delta)(i)$ counts the weight of lattice points in the i-th dilation of a polytope Δ:

$$I(\boldsymbol{x}, \Delta)(i) = \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} \mathbf{a}. \tag{4}$$

Similar to the Ehrhart polynomial, it is also known that (5) gives the $\mathbb{R}^n$-valued polynomial satisfying

$$\begin{aligned} s_\Delta(t) &:= I(\boldsymbol{x}, \Delta)(t) \\ &= t^n \int_\Delta \boldsymbol{x}\, dv + \frac{t^{n-1}}{2} \int_{\partial\Delta} \boldsymbol{x}\, d\sigma + \cdots + c, \qquad s_\Delta(i) = \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} \mathbf{a} \end{aligned} \tag{5}$$

for any positive integer i. We call $s_\Delta(t)$ the *lattice points sum polynomial*. The following necessary condition of the Chow semistability of projective toric varieties was obtained in [20].

Theorem 5 (Ono)**.** *Let Δ be a lattice polytope, $E_\Delta(t)$ the Ehrhart polynomial and $s_\Delta(t)$ the lattice points sum polynomial. We fix a positive integer $i \in \mathbb{Z}_+$. If the associated toric variety X with respect to $L^{\otimes i}$ is Chow semistable, then the following equality holds:*

$$s_\Delta(i) = \frac{E_\Delta(i)}{\mathrm{vol}(\Delta)} \int_\Delta \boldsymbol{x}\, dv. \tag{6}$$

Suppose that a projective polarized toric variety (X, L) associated with a lattice polytope Δ is asymptotically Chow semistable. Then, there is an $i_0 \in \mathbb{Z}_+$ such that (6) holds for any positive integer $i \geqslant i_0$. On the other hand, we observe that $E_\Delta(t)$ and $s_\Delta(t)$ are ($\mathbb{R}^n$-valued) polynomials. Hence, the polynomial identity theorem gives the following (see also ([20] Theorem 1.4).

Lemma 1. *Let Δ be a lattice polytope. If the associated projective polarized toric variety (X, L) is asymptotically Chow semistable, then* (6) *holds for* any *(not necessarily positive) integer $i \in \mathbb{Z}$.*

3. Proof of Theorem 3

3.1. Ehrhart Reciprocity Law for Polynomial Functions

Let Δ be an n-dimensional lattice polytope in $M_\mathbb{R} \cong \mathbb{R}^n$ and ϕ a polynomial function on $\mathbb{R}^n$. As in Section 2.3, we consider

$$I(\phi, \Delta)(i) = \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} \phi(\mathbf{a})$$

and

$$I(\phi, \mathrm{Int}(\Delta))(i) = \sum_{\mathbf{a} \in \mathrm{Int}(\Delta) \cap (\mathbb{Z}/i)^n} \phi(\mathbf{a})$$

for a positive integer i. Remark that $I(1, \mathrm{Int}(\Delta))(i) = \#(\mathrm{Int}(\Delta) \cap (\mathbb{Z}/i)^n)$. The classical result of the *Ehrhart reciprocity law* says that the following equality holds for any positive integer $i \in \mathbb{Z}_+$:

$$I(1, \mathrm{Int}(\Delta))(i) = (-1)^n I(1, \Delta)(-i).$$

Brion and Vergne gave the following beautiful generalization of this reciprocity law [23].

Theorem 6 (Brion–Vergne)**.** *Let Δ be an n-dimensional lattice polytope. If ϕ is a homogeneous polynomial function of degree d on Δ, then the following reciprocity law*

$$I(\phi, \mathrm{Int}(\Delta))(i) = (-1)^{n+d} I(\phi, \Delta)(-i) \tag{7}$$

holds for any positive integer $i \in \mathbb{Z}_+$.

We use this result to prove the following.

Lemma 2. *Let Δ be an n-dimensional reflexive polytope in $M_{\mathbb{R}}$. Let $E_\Delta(t)$ be the Ehrhart polynomial and $s_\Delta(t)$ the lattice point sum polynomial. Then, we have*

$$E_\Delta(-1) = (-1)^n \qquad \text{and} \qquad s_\Delta(-1) = \mathbf{0}.$$

Proof. We note that $\mathrm{Int}(\Delta) \cap \mathbb{Z}^n = \{\mathbf{0}\}$ since Δ is a reflexive polytope. Taking $\phi = 1$ and $i = 1$ in (7), we have

$$E_\Delta(-1) = (-1)^n \cdot \#(\mathrm{Int}(\Delta) \cap \mathbb{Z}^n) = (-1)^n.$$

Similarly, if we take $\phi = \boldsymbol{x}$ and $i = 1$, then (7) becomes

$$s_\Delta(-1) = (-1)^{n+1} \cdot \sum_{\mathbf{a} \in \mathrm{Int}(\Delta) \cap \mathbb{Z}^n} \mathbf{a} = \mathbf{0}.$$

□

3.2. A Combinatorial Proof

Now, we prove Theorem 3.

Proof of Theorem 3. If a Gorenstein toric Fano variety $(X, -K_X)$ is asymptotically Chow semistable, then (6) holds for any integer $i \in \mathbb{Z}$, by Lemma 1. Taking $i = -1$ in (6), we have

$$\int_\Delta \boldsymbol{x}\, dv = \mathbf{0}$$

by Lemma 2. Thus, Proposition 4 implies that $(X, -K_X)$ is Ding polystable. This completes the proof. □

3.3. Conclusion of the Proof of Theorem 3

If Δ is a simple reflexive polytope, then the corresponding toric Fano variety $(X, -K_X)$ may admit only orbifold singularities. Combining Theorem 3 and the result of [24,25], we conclude the following.

Corollary 1. *Let X be a toric Fano orbifold. If $(X, -K_X)$ is asymptotic Chow semistable, then X admits a Kähler–Einstein metric in $c_1(-K_X)$.*

We finish this section with the following example, which illustrates the combinatorial proof of Theorem 3 by using a Gorenstein toric del Pezzo surface.

Example 1. Let Δ be the polygon in $M_{\mathbb{R}} \cong \mathbb{R}^2$ whose vertices are given by

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\},$$

which is the polytope labeled with 9 in Table 2. Then, the associated polarized toric variety (X, L) is the cubic surface

$$X = \{[x : y : z : w] \in \mathbb{P}^3 | xyz = w^3\}$$

with the anticanonical line bundle $L = \mathcal{O}_X(-K_X)$. It is known that $(X, L^{\otimes i})$ is Chow polystable for any integer $i > 0$ by Theorem 1.2 (3) in [26]. Thus, $(X, -K_X)$ is asymptotically Chow semistable.

Let us compute the $\mathbb{R}^2$-valued polynomial function $s_\Delta(t)$. Firstly, straightforward computation shows that

$$\int_\Delta \boldsymbol{x}\, dv = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for the standard volume form $dv = dx \wedge dy$ of $M_\mathbb{R}$. Secondarily, we shall compute $\int_{\partial\Delta} \boldsymbol{x} d\sigma$ (which is equal to the second leading coefficient of $s_\Delta(t)$) as follows: the polygon Δ has three facets $F_i = \{\boldsymbol{x} \in \Delta | \ell_i(\boldsymbol{x}) = 0\}$ for $i = 1, 2, 3$ whose defining equations are given by

$$\ell_1(\boldsymbol{x}) = 1 - x - y, \quad \ell_2(\boldsymbol{x}) = 1 + 2x - y, \quad \text{and} \quad \ell_3(\boldsymbol{x}) = 1 - x + 2y,$$

respectively. Then, the boundary measure $d\sigma_i$ on each facet F_i is determined by

$$dv = \pm d\sigma_i \wedge d\ell_i. \tag{8}$$

Thus, (8) shows that we can take

$$d\sigma_1 = -dx, \quad d\sigma_2 = -dx, \quad \text{and} \quad d\sigma_3 = \frac{1}{2}dx$$

as the boundary measures on $\partial\Delta$. Consequently, the x-coordinate of the barycenter of each facet F_i is given by

$$\int_{F_1} x\, d\sigma_1 = \int_1^0 x(-dx) = \frac{1}{2}, \qquad \int_{F_2} x\, d\sigma_2 = \int_{-1}^0 x\, dx = -\frac{1}{2}$$
$$\text{and} \qquad \int_{F_3} x\, d\sigma_3 = \frac{1}{2}\int_{-1}^1 x\, dx = 0,$$

respectively. By the symmetry of Δ, we find that

$$\int_{\partial\Delta} \boldsymbol{x}\, d\sigma = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, $s_\Delta(t)$ has the form of

$$s_\Delta(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{9}$$

for some constants c_1 and c_2. See (5). In order to determine c_1 and c_2, we plug the value of $s_\Delta(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ into (9), which yields that $c_1 = c_2 = 0$. Thus, we see that $s_\Delta(t) \equiv \mathbf{0}$ and this is consistent with the Ehrhart reciprocity law

$$s_\Delta(-1) = (-1)^3 \cdot \sum_{\boldsymbol{a} \in \mathrm{Int}(\Delta) \cap \mathbb{Z}^2} \boldsymbol{a} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Moreover, we already see that $\int_\Delta \boldsymbol{x}\, dv = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in the above computation. Consequently, $(X, -K_X)$ is Ding polystable by Proposition 4.

4. Relative Algebro-Geometric Stability

In order to deal with the existence problem of extremal Kähler metrics, the definition of K-stability was extended by Székelyhidi in [27] to Kähler classes with a non-vanishing Futaki invariant, which was called *relative K-stability*. Analogously, we can extend the notion of Chow stability to *relative Chow stability*, which has been also investigated by many researchers [28,29].

In this section, we study the relative Chow/K-stability of toric Fano varieties, which were dealt with in [10,12]. The product formulas for potential functions θ_Δ and the additivity of the constant M_{X_Δ} defined in (18) are discussed in Section 4.3. Then, in Section 4.4, we verify the (asymptotic) relative Chow stability of Gorenstein toric del Pezzo surfaces, by applying our combinatorial criterion of relative Chow stability (see Corollary 2) in the toric setting, and we list the results in Table 1. In Section 4.5, we systematically construct examples of relatively K-polystable toric Fano manifolds, but which are relatively Ding unstable, building upon the works of [10,11]. See Corollary 4 and Example 2 for more details.

4.1. Fundamental Results on Relative Chow stability

Firstly, we quickly review the notion of relative Chow stability and related results. See [12] for more details.

Let us consider a reductive complex algebraic group G with Lie algebra $\mathfrak{g}$. Suppose that G acts linearly on a finite-dimensional complex vector space $\mathbf{V}$. This induces a natural G-action on $\mathbb{P}(\mathbf{V})$. We will abbreviate $v \in \mathbb{P}(\mathbf{V})$ and its representatives in $\mathbf{V}$. Let T be a torus in G with Lie algebra $\mathfrak{t}$. We assume that T fixes the point v. Using an inner product $\langle\ ,\ \rangle$ and the Lie bracket $[\ ,\ ]$, we define the subalgebras of $\mathfrak{g}$ by

$$\mathfrak{g}_T = \{\alpha \in \mathfrak{g} | [\alpha, \beta] = 0 \ \text{ for all } \ \beta \in \mathfrak{t}\},$$
$$\mathfrak{g}_{T^\perp} = \{\alpha \in \mathfrak{g}_T | \langle \alpha, \beta \rangle = 0 \ \text{ for all } \ \beta \in \mathfrak{t}\}.$$

Then, the corresponding Lie group of $\mathfrak{g}_T$ (resp. $\mathfrak{g}_{T^\perp}$) is denoted by G_T (resp. $G_{T^\perp}$). Following classical GIT (see Section 2.3), we call $v \in \mathbb{P}(\mathbf{V})$ *semistable relative to* T if the closure of the $G_{T^\perp}$ orbit $\mathcal{O}_{G_{T^\perp}}(v)$ does not contain the origin. v is *polystable relative to* T if $\mathcal{O}_{G_{T^\perp}}(v)$ is a closed orbit. v is said to be *unstable relative to* T if it is not semistable relative to T.

Let us consider the relative stability of the Chow form. For an irreducible complex projective variety $X \subset \mathbb{C}P^N$, we choose $G = \mathrm{SL}(N+1, \mathbb{C})$ and T to be the $\mathbb{C}^\times$-action induced by the extremal vector field.

Definition 3. A complex irreducible projective variety $X \subset \mathbb{C}P^N$ is said to be *relatively Chow polystable (resp. semistable, unstable)* if the X-resultant R_X of X is $\mathrm{SL}(N+1, \mathbb{C})$-polystable (resp. semistable, unstable) relative to T.

The definition of asymptotic relative Chow stability is analogous to Definition 2; hence, we do not repeat the definition in this paper (see ([12] Definition 3.6)).

4.2. Toric Reduction of Relative Chow stability

We consider the toric case. In particular, we are interested in the case where X is an n-dimensional Gorenstein toric Fano variety with the associated reflexive polytope $\Delta \subseteq M_\mathbb{R} \cong \mathbb{R}^n$. As in [6], *the Ricci affine function* ℓ_Δ associated with Δ is the unique function determined by $\int_\Delta \ell_\Delta u \, dv = u(0)$ for any affine linear function u—namely, one can solve the linear system

$$\int_\Delta \ell_\Delta(\boldsymbol{x}) \, dv = 1, \qquad \int_\Delta \ell_\Delta(\boldsymbol{x}) \cdot x_i \, dv = 0 \qquad \text{for} \qquad i = 1, \ldots, n$$

in order to find $\ell_\Delta(\boldsymbol{x}) = \sum a_i x_i + c$ with a_i and c. Let us define the *potential function* of Δ by

$$\theta_\Delta := 1 - \mathrm{vol}(\Delta)\ell_\Delta. \tag{10}$$

Then, we consider its average

$$\bar{\theta}_\Delta = \frac{1}{N+1} \sum_{j=1}^{N+1} \theta_\Delta(\mathbf{a}_j),$$

where $\{\mathbf{a}_1, \ldots, \mathbf{a}_{N+1}\}$ are lattice points in Δ. Denoting

$$d_\Delta = (1, \ldots, 1), \qquad \tilde{\theta}_\Delta = ((\theta_\Delta(\mathbf{a}_1) - \bar{\theta}_\Delta), \ldots, (\theta_\Delta(\mathbf{a}_{N+1}) - \bar{\theta}_\Delta))$$

in $\mathbb{R}^{N+1}$, we can show the following.

Theorem 7 (Theorem 3.8 in [12])**.** *Let* $\mathrm{Ch}(\Delta)$ *be the Chow polytope of an n-dimensional Gorenstein toric Fano variety* $X_\Delta \subset \mathbb{C}P^N$*. Then,* X_Δ *is relatively Chow polystable in the toric sense if and only if there exists* $t \in \mathbb{R}$ *such that*

$$\frac{(n+1)!\mathrm{vol}(\Delta)}{N+1}(d_\Delta + t\tilde{\theta}_\Delta) \in \mathrm{Int}(\mathrm{Ch}(\Delta)). \tag{11}$$

Let $\bar{\theta}_{i\Delta} = \frac{1}{E_\Delta(i)} \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} \theta_\Delta(\frac{\mathbf{a}}{i})$. Defining $d_{i\Delta}$ and $\tilde{\theta}_{i\Delta}$ by

$$d_{i\Delta}(\mathbf{a}) = 1, \quad \tilde{\theta}_{i\Delta}(\mathbf{a}) = \frac{\theta_\Delta(\mathbf{a}) - \bar{\theta}_{i\Delta}}{i}, \quad \text{for} \quad \mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n,$$

we obtain a necessary condition for the associated polarized toric variety to be asymptotically relatively Chow semistable.

Corollary 2 (Corollary 3.11 in [12])**.** *If* $(X_\Delta, -K_{X_\Delta})$ *is asymptotically relatively Chow semistable, then, for any* $i \in \mathbb{Z}_+$*, there exists* $t_i \in \mathbb{R}$ *satisfying*

$$\sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} i\mathbf{a} + t_i \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} \tilde{\theta}_{i\Delta}(\mathbf{a})\mathbf{a} = \frac{iE_\Delta(i)}{\mathrm{vol}(\Delta)} \int_\Delta \boldsymbol{x}\, dv. \tag{12}$$

4.3. Product Formulas for Potential Functions

Recently, Ono, Sano and the author proved that the only Bott manifolds such that the Futaki invariant vanishes for any Kähler class are isomorphic to the products of the projective lines [11]. The key to proving the main theorem in [11] is the analysis of the product of two polytopes. By applying this technique to the potential functions in (10), we derive the product formula in this section.

Now, let us discuss the *product* of two (or more) convex polytopes. For this, we consider the full-dimensional polytopes $\Delta_1 \subseteq \mathbb{R}^{n_1}$ and $\Delta_2 \subseteq \mathbb{R}^{n_2}$ and define

$$\Delta_1 \times \Delta_2 := \{ \begin{pmatrix} x \\ \boldsymbol{y} \end{pmatrix} \in \mathbb{R}^{n_1+n_2} | \boldsymbol{x} = (x_1, \ldots, x_{n_1}) \in \Delta_1, \;\; \boldsymbol{y} = (y_1, \ldots, y_{n_1}) \in \Delta_2 \}.$$

Setting $\Delta = \Delta_1 \times \Delta_2$, we see that Δ is a polytope of dimension $n_1 + n_2 (= n)$, whose any nonempty face is given by the product of a nonempty face F of Δ_1, and a nonempty face G of Δ_2. For $i = 1, 2$, let dv_i be the standard volume form of $\mathbb{R}^{n_i}$. Then, $dv = dv_1 \wedge dv_2$ defines the volume form of Δ.

For a given arbitrary (not necessarily product) convex polytope P with $\dim P = n$, we consider the functional $\mathscr{L}_P(u)$ defined by

$$\mathscr{L}_P(u) = \int_{\partial P} u\, d\sigma - \int_P \left(\frac{\mathrm{vol}(\partial P)}{\mathrm{vol}(P)} + \theta_P \right) u\, dv. \tag{13}$$

Here, u is a convex function, θ_P is the potential function defined in (10) and $d\sigma$ is the $(n-1)$-dimensional Lebesgue measure of ∂P, defined as follows: let $\ell_j(\boldsymbol{x}) = \langle \boldsymbol{x}, v_j \rangle + c_j$ be the defining equation of a facet F_j of P, where $c_j \in \mathbb{Z}$ and v_j is a primitive vector. Recall that $dv = dx_1 \wedge \cdots \wedge dx_n$ is the standard volume form of $\mathbb{R}^n$. On each facet $F_j = \{\boldsymbol{x} \in P | \ell_j(\boldsymbol{x}) = 0\} \subset \partial P$, we define the $(n-1)$-dimensional Lebesgue measure $d\sigma_j$ of ∂P by

$$dv = \pm d\sigma_j \wedge d\ell_j. \tag{14}$$

Then, $d\sigma$ is uniquely determined as the $(n-1)$-dimensional Lebesgue measure of ∂P so that $d\sigma_j = d\sigma|_{F_j}$, up to the sign.

Let us go back to the product polytope $\Delta = \Delta_1 \times \Delta_2$. Let $d\sigma_1$ (resp. $d\sigma_2$) be the (n_1-1)-dimensional (resp. (n_2-1)-dimensional) Lebesgue measure of $\partial\Delta_1$ (resp. $\partial\Delta_2$) defined in (14). Since any nonempty face of Δ is obtained by the product of a nonempty face $F \preceq \Delta_1$ and a nonempty face $G \preceq \Delta_2$, we see that the boundary of Δ is written as

$$\partial\Delta = \partial\Delta_1 \times \Delta_2 \cup \Delta_1 \times \partial\Delta_2. \tag{15}$$

Moreover, see (4.18) in [11]. In particular, we find the following equalities by direct computation.

Lemma 3. *Let $\Delta = \Delta_1 \times \Delta_2$ be the product of two polytopes Δ_k with $\dim \Delta_k = n_k$ for $k = 1,2$. Let $\boldsymbol{x} = (x_1, \dots, x_{n_1})$ and $\boldsymbol{y} = (y_1, \dots, y_{n_2})$ be the coordinates of Δ_1 and Δ_2, respectively. We denote the volume form of Δ (resp. Δ_k) by dv (resp. dv_k), and the volume form of $\partial\Delta$ (resp. $\partial\Delta_k$) by $d\sigma$ (resp. $d\sigma_k$). For $i = 1, \dots, n_1$ and $j = 1, \dots, n_2$, we have*

$$\begin{aligned}
&\mathrm{vol}(\Delta) = \mathrm{vol}(\Delta_1)\mathrm{vol}(\Delta_2),\\
&\int_\Delta x_i\, dv = \mathrm{vol}(\Delta_2)\int_{\Delta_1} x_i\, dv_1, \qquad \int_\Delta y_j\, dv = \mathrm{vol}(\Delta_1)\int_{\Delta_2} y_j\, dv_2,\\
&\mathrm{vol}(\partial\Delta) = \mathrm{vol}(\partial\Delta_1)\mathrm{vol}(\Delta_2) + \mathrm{vol}(\Delta_1)\mathrm{vol}(\partial\Delta_2),
\end{aligned}$$

$$\begin{aligned}
&\int_{\partial\Delta} x_i\, d\sigma = \mathrm{vol}(\Delta_2)\int_{\partial\Delta_1} x_i\, d\sigma_1 + \mathrm{vol}(\partial\Delta_2)\int_{\Delta_1} x_i\, dv_1, \qquad \textit{and}\\
&\int_{\partial\Delta} y_j\, d\sigma = \mathrm{vol}(\Delta_1)\int_{\partial\Delta_2} y_j\, d\sigma_2 + \mathrm{vol}(\partial\Delta_1)\int_{\Delta_2} y_j\, dv_2.
\end{aligned}$$

We finish this subsection with the following additive property of the potential functions θ_Δ and the Mabuchi constants M_{X_Δ} for the product polytopes.

Proposition 5. *Let $\Delta = \Delta_1 \times \Delta_2$ be the product of two polytopes as in Lemma 3. Then, the potential function θ_Δ defined in (10) satisfies the equality*

$$\theta_\Delta(\boldsymbol{x},\boldsymbol{y}) = \theta_{\Delta_1}(\boldsymbol{x}) + \theta_{\Delta_2}(\boldsymbol{y}).$$

Moreover, for the product $\Delta = \prod_{k=1}^r \Delta_k$, we see that $\theta_\Delta(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_r) = \sum_{k=1}^r \theta_{\Delta_k}(\boldsymbol{x}_k)$.

Proof. As was described in [12] (p. 496), the potential function θ_Δ is uniquely determined by solving the $n+1$-linear system

$$\mathscr{L}_\Delta(1) = 0, \quad \mathscr{L}_\Delta(x_i) = 0, \quad \mathscr{L}_\Delta(y_j) = 0 \qquad \text{for} \quad i = 1, \dots, n_1, \;\; j = 1, \dots, n_2, \tag{16}$$

where $\mathscr{L}_\Delta(u)$ is the function defined in (13). Since θ_{Δ_k} is the potential function of Δ_k for each $k = 1,2$, we have

$$\mathscr{L}_{\Delta_1}(1) = \mathscr{L}_{\Delta_2}(1) = 0, \quad \mathscr{L}_{\Delta_1}(x_i) = 0, \quad \text{and} \quad \mathscr{L}_{\Delta_2}(y_j) = 0. \tag{17}$$

In order to prove our assertion, it suffices to show that $\theta_\Delta(\boldsymbol{x},\boldsymbol{y}) := \theta_{\Delta_1}(\boldsymbol{x}) + \theta_{\Delta_2}(\boldsymbol{y})$ satisfies the $(n+1)$-equalities in (16) using our assumption (17).

Firstly, we find that

$$\int_\Delta \theta_\Delta(\boldsymbol{x},\boldsymbol{y})dv = \int_{\Delta_1} \theta_{\Delta_1}(\boldsymbol{x})dv_1 + \int_{\Delta_2} \theta_{\Delta_2}(\boldsymbol{y})dv_2,$$

which equals 0, by our assumption $\mathscr{L}_{\Delta_1}(1) = \mathscr{L}_{\Delta_2}(1) = 0$.

Secondly, for $i = 1, \dots, n_1$, we prove that $\mathscr{L}_\Delta(x_i) = 0$. To see this, we compute that

$$\int_\Delta \left(\frac{\mathrm{vol}(\partial\Delta)}{\mathrm{vol}(\Delta)} + \theta_\Delta(\boldsymbol{x}, \boldsymbol{y}) \right) x_i \, dv = \frac{\mathrm{vol}(\partial\Delta)}{\mathrm{vol}(\Delta)} \int_\Delta x_i \, dv + \int_\Delta (\theta_{\Delta_1}(\boldsymbol{x}) + \theta_{\Delta_2}(\boldsymbol{y})) x_i \, dv$$
$$= \frac{\mathrm{vol}(\partial\Delta_1)\mathrm{vol}(\Delta_2) + \mathrm{vol}(\Delta_1)\mathrm{vol}(\partial\Delta_2)}{\mathrm{vol}(\Delta_1)} \int_{\Delta_1} x_i \, dv_1 + \mathrm{vol}(\Delta_2) \int_{\Delta_1} \theta_{\Delta_1}(\boldsymbol{x}) x_i \, dv_1.$$

By applying Lemma 3 into $\int_{\partial\Delta} x_i \, d\sigma$, we find that

$$\mathscr{L}_\Delta(x_i) = \mathrm{vol}(\Delta_2)\mathscr{L}_{\Delta_1}(x_i) = 0,$$

where we use (17) for the last equality.

Finally, for $j = 1, \dots, n_2$, we have $\mathscr{L}_\Delta(y_j) = \mathrm{vol}(\Delta_1)\mathscr{L}_{\Delta_2}(y_j) = 0$ in the same manner as the above computation. This completes the proof of $\theta_\Delta(\boldsymbol{x}, \boldsymbol{y}) = \theta_{\Delta_1}(\boldsymbol{x}) + \theta_{\Delta_2}(\boldsymbol{y})$.

In order to see the second assertion

$$\theta_\Delta(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_r) = \sum_{k=1}^{r} \theta_{\Delta_k}(\boldsymbol{x}_k),$$

for the product polytope $\Delta = \prod_{k=1}^r \Delta_k$, we use the inductive argument. Hence, the assertion is verified. $\square$

For later use, we consider the value of constant

$$M_{X_\Delta} = \max_{x \in \Delta} \{\theta_\Delta(\boldsymbol{x})\}, \tag{18}$$

which verifies the relative Ding stability of the corresponding toric (Fano) variety. See Section 4.5 for further discussion. After posting this version of the paper on arXiv (version 5, arXiv:1711.10113v5), the author found that the following additivity of the constant M_{X_Δ} is mentioned by Mabuchi in ([30] Theorem 9.9) for general (not necessarily toric) Fano manifolds. However, it is worth mentioning that we derive a direct combinatorial proof for the case of toric Fano manifolds from Proposition 5 and (18).

Corollary 3. *Let $\Delta = \prod_{k=1}^r \Delta_k$ be the product of (reflexive) polytopes. Then, the constant of M_{X_Δ} has the additive property such that*

$$M_{X_\Delta} = M_{X_{\Delta_1}} + \cdots + M_{X_{\Delta_r}}. \tag{19}$$

4.4. Asymptotic Relative Chow Stability of Gorenstein Toric Del Pezzo surfaces

As mentioned in Section 2.1, there are 16 isomorphism classes of Gorenstein toric del Pezzo surfaces. See [13] for more details. On the one hand, the relative Ding stability of Gorenstein toric del Pezzo surfaces has been verified in ([6] Example 5.14). On the other hand, it is difficult to verify the asymptotic relative Chow stability of a polarized toric variety because we have to show that there exists $t_i \in \mathbb{R}$ satisfying (12) for *any* positive integer i (cf. [26] for (not relative) Chow stability case). However, we can solve this difficulty in the case of two dimensions by using the symmetry of the associated reflexive polytopes. See Case 3 in the proof of Proposition 6 below. As a consequence, we verify the relative Chow stability of each Gorenstein toric del Pezzo surface. We list all the results in Table 1.

Proposition 6. *Among all 16 isomorphism classes of Gorenstein toric del Pezzo surfaces, there are 5 isomorphism classes of asymptotically relatively Chow polystable surfaces and 4 isomorphism classes of asymptotically relatively Chow unstable surfaces. The remaining 7 classes are relatively Chow polystable with respect to the anticanonical polarization ($i = 1$).*

Proof. Case 1. Note that any toric surface has at worst orbifold singularities. There are 5 isomorphism classes of Kähler–Einstein Gorenstein toric del Pezzo surfaces with the vanishing Futaki character, i.e., $\mathbb{C}P^2, \mathbb{C}P^1 \times \mathbb{C}P^1, S_6, \mathbb{C}P^1 \times \mathbb{C}P^1/\mathbb{Z}_2$ and $\mathbb{C}P^2/\mathbb{Z}_3$. Hence, the relative Chow stability coincides with Chow stability for these 5 classes of del Pezzo surfaces. In particular, the vanishing Futaki character, i.e., $\int_\Delta \boldsymbol{x}\, dv = \mathbf{0}$, implies $\theta_\Delta \equiv 0$. This means $\tilde{\theta}_{i\Delta}(\mathbf{a}) = \mathbf{0}$ for any $i \in \mathbb{Z}_+$ and a necessary condition of the asymptotic relative Chow semistability of a polarized toric variety (12) becomes

$$\sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} i\mathbf{a} = \frac{iE_\Delta(i)}{\mathrm{vol}(\Delta)} \int_\Delta \boldsymbol{x}\, dv$$

for all $i \in \mathbb{Z}_+$. Hence, we obtain the same equality in (6). Moreover, $\int_\Delta \boldsymbol{x}\, dv = \mathbf{0}$ implies that $\sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} i\mathbf{a} = \mathbf{0}$ for any $i \in \mathbb{Z}_+$. Remark that this is equivalent to the vanishing of the obstruction for asymptotic Chow semistability defined in [31] (see [20] (p. 1385)). Since X admits a Kähler–Einstein metric, it must be asymptotically Chow polystable for $X = \mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$ and S_6 due to the result in ([32] Main Theorem). Hence, we have verified the assertion for these 3 classes.

For the remaining two orbifold cases $X = \mathbb{C}P^2/\mathbb{Z}_3$ (labeled 9 in Table 1) and $\mathbb{C}P^1 \times \mathbb{C}P^1/\mathbb{Z}_2$ (labeled $8A$ in Table 1), asymptotic Chow polystability of $(X, -K_X)$ has been verified in Theorem 1.2 (3) in [26]. We remark that the minimal embeddings of these del Pezzo surfaces are given by

$$\mathbb{C}P^2/\mathbb{Z}_3 = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 | z_0^3 - z_1 z_2 z_3 = 0\}$$

with three A_2 singularities, and

$$\mathbb{C}P^1 \times \mathbb{C}P^1/\mathbb{Z}_2 = \{[z_0 : z_1 : z_2 : z_3 : z_4] \in \mathbb{C}P^4 | z_1 z_3 - z_0^2 = 0,\ \ z_2 z_4 - z_0^2 = 0\}$$

with four A_1 singularities, respectively. See [33] for further details.

Case 2. Let X be a Goresntein toric del Pezzo surface labeled with $5B$ in Table 1. Then, the associated reflexive polytope $\Delta \subseteq M_\mathbb{R}$ is given by

$$\Delta = \mathrm{conv}\{(-1,0), (1,-2), (0,1), (-1,1)\}.$$

We claim that X is asymptotically relatively Chow unstable by using Corollary 2. Hence, it suffices to show that there is no $t_1 \in \mathbb{R}$ satisfying (12) for $i = 1$. See Remark 3.12 and Proposition 5.4 in [12]. We readily see that

$$E_\Delta(i) = \frac{5}{2}i^2 + \frac{5}{2}i + 1,\ \int_\Delta \boldsymbol{x}\, dv = \left(-\frac{1}{3}, -\frac{1}{3}\right),\ \sum_{\mathbf{a} \in \Delta \cap \mathbb{Z}^2} \mathbf{a} = (-1,-1),$$
$$\theta_\Delta(\boldsymbol{x}) = -\frac{1}{529}(1032x_1 + 648x_2 + 224) \quad \text{and} \quad \bar{\theta}_\Delta = \frac{56}{529}.$$

Therefore,

$$t_1 \sum_{\mathbf{a} \in \Delta \cap \mathbb{Z}^2} \tilde{\theta}_\Delta(\mathbf{a})\mathbf{a} = -t_1 \left(\frac{872}{529}, \frac{1160}{529}\right).$$

This yields that there is no $t_1 \in \mathbb{R}$ satisfying (12).

Case 3. Let X be a weighted projective space $\mathbb{C}P(1,1,2)$. This is a Gorenstein toric del Pezzo surface labeled with $8C$ in Table 1 and the corresponding reflexive polytope Δ is

$$\Delta = \mathrm{conv}\{(-1,2), (1,0), (-1,-2)\}.$$

We prove that $(X, -K_X)$ is relatively Chow polystable. Straightforward computation shows that

$$E_\Delta(i) = 4i^2 + 4i + 1, \int_\Delta \boldsymbol{x}\, dv = \left(-\frac{4}{3}, 0\right), \sum_{\mathbf{a} \in \Delta \cap \mathbb{Z}^2} \mathbf{a} = (-4, 0),$$
$$\theta_\Delta(\boldsymbol{x}) = -\frac{3}{2}x_1 - \frac{1}{2} \quad \text{and} \quad \bar{\theta}_\Delta = \frac{1}{6}.$$

Taking $i = 1$ in (12), we find that $t_1 = -3/19$ satisfies the equation

$$\sum_{\mathbf{a} \in \Delta \cap \mathbb{Z}^2} \mathbf{a} + t_1 \sum_{\mathbf{a} \in \Delta \cap \mathbb{Z}^2} \tilde{\theta}_\Delta(\mathbf{a})\mathbf{a} = \frac{E_\Delta(1)}{\mathrm{vol}(\Delta)} \int_\Delta \boldsymbol{x}\, dv.$$

Moreover, Δ is invariant under unimodular transformation $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which gives the coordinate interchange $x_2 \mapsto -x_2$. By this symmetry, we conclude that there exists t_i for *any* $i \in \mathbb{Z}_+$ such that (12) holds.

Next, we verify (11). For $t = -3/19$, we readily see that the left-hand side of (11) is given by $p := \frac{4}{19}(11, 14, 17, 14, 11, 11, 11, 11, 14)$. On the other hand, the Chow polytope $\mathrm{Ch}(\Delta)$ is the 6-dimensional polytope in $\mathbb{R}^9$ with 296 vertices. In particular, $\mathrm{Ch}(\Delta)$ (we used package `TOPCOM` for the computation) is determined by three defining equations $f_i(\boldsymbol{x}) = 0$ (i=1,2,3) and 26 defining inequalities $h_j(\boldsymbol{x}) \geqslant 0$ $(j = 1, \ldots, 26)$ in $\mathbb{R}^9$. By direct computation, one can see that $f_i(p) = 0$ and $h_j(p) > 0$ hold for all i, j. This implies $p \in \mathrm{Int}(\mathrm{Ch}(\Delta))$ and the assertion is verified. Other cases are similar and further details are left to the reader. $\square$

Remark 1.

1. *Using the symmetry of polytopes, one can verify the existence of t_i for $i \gg 0$ satisfying* (12) *for each case* ($4B, 4C, 5A, 6B, 7A, 8B$ *and* $8C$ *in Table* 1)*. We mention that this is only a necessary condition for (X, L) to be asymptotically relatively Chow semistable (Corollary* 2*).*

2. *On the other hand,* $\mathrm{Ch}(i\Delta)$ *will be a huge number of vertices in a multidimensional Euclidean space if $i > 0$ is a sufficiently large positive integer. Hence, it is generally impossible to verify the condition*
$$\frac{i^n(n+1)!\,\mathrm{vol}(\Delta)}{E_\Delta(i)}(d_{i\Delta} + t_i\tilde{\theta}_{i\Delta}) \in \mathrm{Int}(\mathrm{Ch}(i\Delta))$$
for arbitrary positive integer i. See [4,22] for more combinatorial descriptions of $\mathrm{Ch}(\Delta)$.

4.5. Relative Ding/K-Stability

In [10], we found that there are several examples of toric Fano manifolds that clarify the difference between relative K-stability and relative Ding stability. More specifically, we verified that if X is either

- a toric Fano 3-fold $\mathcal{B}_1 = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(2))$, or
- toric Fano 4-folds (which are all $\mathbb{P}^1$-bundles over $\mathbb{P}^3$) $B_1 = \mathbb{P}_{\mathbb{P}^3}(\mathcal{O} \oplus \mathcal{O}(3))$, $B_2 = \mathbb{P}_{\mathbb{P}^3}(\mathcal{O} \oplus \mathcal{O}(2))$, $L_1 = \mathbb{P}_{\mathbb{P}^3}(\mathcal{O} \oplus \mathcal{O}(1,1,1))$,

then $(X, -K_X)$ is relatively K-polystable, but it is relatively Ding unstable. In order to prove that these four examples ($\mathcal{B}_1$, B_1, B_2 and L_1) admit extremal Kähler metrics in their first Chern classes, which in turn are relatively K-polystable, we focused on their geometric structures, such as projective bundles, Bott structures, etc. [34–37]. On the one hand, the relative Ding stability of toric Fano manifolds is determined by the value of constant M_{X_Δ} defined in (18) being larger than 1 or not, due to the work of Yao [6]. On the other hand, Proposition 5 implies that the products of (higher-dimensional) toric extremal manifolds are more likely to be relatively Ding unstable, by the additive property of M_{X_Δ} (see (19) and Corollary 4). In this section, we systematically construct examples of a relatively K-polystable toric Fano manifold, but it is relatively Ding unstable.

Let us quickly review the notions of relative K-stability and relative Ding stability for a (smooth) toric Fano variety. Remark that we only consider a *toric* (or *T-equivariant*) *test configuration* for the definitions of relative Ding/K-stability. This is because, for polarized toric varieties, it suffices to check only toric test configurations of relative Ding/K-stability as in [38] and [39]. We refer the reader to Section 2 in [10], for more details.

Let $\Delta \subseteq M_{\mathbb{R}}$ be an n-dimensional reflexive Delzant polytope. In this case, the average of the scalar curvature, i.e., $\overline{S} = \mathrm{vol}(\partial\Delta)/\mathrm{vol}(\Delta)$, is equal to n, and hence the functional defined in (13) will be

$$\mathscr{L}_\Delta(u) = \int_{\partial\Delta} u\, d\sigma - \int_\Delta (n + \theta_\Delta) u\, dv,$$

where u is a convex function of Δ. A convex function $u : \Delta \to \mathbb{R}$ is called *rational PL convex* if u has the form of

$$u(\boldsymbol{x}) = \max\{f_1(\boldsymbol{x}), \dots, f_m(\boldsymbol{x})\}$$

with each f_k a rational affine function. The associated anticanonically polarized smooth toric Fano variety $(X_\Delta, -K_{X_\Delta})$ is *relatively K-polystable* if $\mathscr{L}_\Delta(u) \geqslant 0$ for any rational PL convex function u, and the equality holds if and only if u is affine linear. Let M_{X_Δ} be the *Mabuchi constant* defined in (18). $(X_\Delta, -K_{X_\Delta})$ is *relatively Ding polystable* if $M_{X_\Delta} \leqslant 1$. Conversely, it is called *relatively Ding unstable* if $M_{X_\Delta} > 1$. See [6] and ([10] Proposition 1.2) for further details.

On the other hand, Corollary 3 implies that X_Δ is more likely to be relatively Ding unstable if the dimension of X_Δ becomes higher and higher. Meanwhile, for given extremal Kähler manifolds (X_k, g_k) with $1 \leqslant k \leqslant r$, the product manifold $X = \prod_{k=1}^r X_k$ admits the product extremal Kähler metric $\prod_{k=1}^r g_k$. Thus, $(X, -K_X)$ must be relatively K-polystable. In particular, X is Fano. From this observation, one can expect that there are more examples of toric Fano manifolds that clarify the difference between relative K-stability and relative Ding stability. As a consequence of (19), we systematically construct infinitely many examples of relatively K-polystable extremal toric Fano manifolds that are relatively Ding unstable.

Corollary 4. *For $1 \leqslant k \leqslant r$, let X_k be an extremal toric Fano manifold with the associated polytope Δ_k and let $\theta_{\Delta_k}(\boldsymbol{x}_k)$ be the potential function of Δ_k satisfying $\frac{1}{r} \leqslant \theta_{\Delta_k} < 1$. Let Δ be the product of polytopes Δ_k for $1 \leqslant k \leqslant r$. Then, the associated anticanonically polarized toric Fano manifold $(X_\Delta, -K_{X_\Delta})$ is relatively K-polystable, but it is relatively Ding unstable.*

Using Table 3 in [10], we obtain the following examples.

Example 2. Let dP_{9-i} denote a smooth del Pezzo surface with degree $(9-i)$, which is obtained by the blow-up of $\mathbb{P}^2$ at i points. Fixing a positive integer r, we denote a copy of dP_8 by X_k for $1 \leqslant k \leqslant r$. It is known that X_k admits an extremal Kähler metric in every Kähler class [40], and this yields that $X = \prod_{k=1}^r X_k$ also admits the extremal Kähler metric in its first Chern class. Hence, $(X, -K_X)$ is relatively K-polystable for any positive integer r.

On the other hand, direct computation shows that $M_{X_k} = 5/11$. See ([10] Table 1 , No.3). Thus, we conclude that $M_X = 5r/11$ by (19). Consequently, $(X, -K_X)$ is relatively (uniform) Ding polystable if $r = 2$, whereas it is relatively Ding unstable if $r \geqslant 3$. We note that the toric Fano 4-fold $dP_8 \times dP_8$ is denoted by L_7 (No. 55) in ([10] Table 3). In particular, there are other examples, such as $Q_{10} = dP_7 \times dP_8$ (No. 93) and $dP_7 \times dP_7$ (No. 119) in the four-dimensional case.

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