

Article

Approximate Controllability for a Class of Semi-Linear Fractional Integro-Differential Impulsive Evolution Equations of Order $1 < \alpha < 2$ with Delay

Daliang Zhao

School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, China; dlzhao928@sdu.edu.cn

Abstract: This article is mainly concerned with the approximate controllability for some semi-linear fractional integro-differential impulsive evolution equations of order $1 < \alpha < 2$ with delay in Banach spaces. Firstly, we study the existence of the PC-mild solution for our objective system via some characteristic solution operators related to the Mainardi's Wright function. Secondly, by using the spatial decomposition techniques and the range condition of control operator B , some new results of approximate controllability for the fractional delay system with impulsive effects are obtained. The results cover and extend some relevant outcomes in many related papers. The main tools utilized in this paper are the theory of cosine families, fixed-point strategy, and the Grönwall-Bellman inequality. At last, an example is given to demonstrate the effectiveness of our research results.

Keywords: approximate controllability; fractional integro-differential impulsive evolution equation; delay; Schaefer's fixed point theorem; range condition of control operator

MSC: 26A33; 34K37; 93B05; 46T99



Citation: Zhao, D. Approximate Controllability for a Class of Semi-Linear Fractional Integro-Differential Impulsive Evolution Equations of Order $1 < \alpha < 2$ with Delay. *Mathematics* **2023**, *11*, 4069. <https://doi.org/10.3390/math11194069>

Academic Editors: Pengyu Chen, Haiyong Qin and Haoyu Niu

Received: 8 September 2023

Revised: 23 September 2023

Accepted: 24 September 2023

Published: 25 September 2023



Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

It is an undeniable actuality at the present stage that the fractional derivative has stronger expressiveness than the integral order derivative. Since fractional calculus is more suitable for describing objective reality, there is an increasing amount of valuable outcomes about various fractional systems, including theoretical aspects and application fields in recent years. Currently, the theory of fractional calculus has been extensively employed in disciplines such as viscoelasticity and rheology, physics, signal processing, control engineering, etc. For further information regarding these studies, please go through [1–8].

Controllability of control systems is an important component and research direction of control theory, as well as the foundation of optimal control and optimal estimation. In recent years, the controllability of various types of fractional dynamic systems, including fractional impulsive systems [9,10], delay systems [11], stochastic systems [12,13], neutral systems [14], nonlocal systems [15], damped systems [16], integro-differential systems [17], measure evolution systems [18], etc., has been studied extensively and deeply. For example, in [10], the authors derived some new results of the total controllability (a type of exact controllability) for a fractional control system with non-instantaneous impulse by means of Krasnoselskii's fixed-point theorem. H. Gou et al. [19] proved the exact controllability for fractional integro-differential system with impulsive effects via the theory of resolvent operators and measures of noncompactness. The exact controllability of a neutral fractional evolution system was also investigated by fixed point theory and the measures of noncompactness in [20]. Y. Yi et al. [21] addressed the exact controllability for fractional integro-differential equations with input delay via Mittag-Leffler functions and nonlinear functional analysis theory in finite spaces.

We should note that exact controllability can manipulate the target system to any specified endpoint state, but the requirement for the control operator is that it must be

reversible. Moreover, an important fact that needs to be clarified is that when the semigroup and sine family are compact, the evolution systems of the first and second orders are never exactly controllable [22]. Undoubtedly, this means that exact controllability has significant limitations in practice. Therefore, as an extension of its concept, which indicates that target system can be manipulated to the neighborhood of a specified endpoint state, approximate controllability is provided with more widespread practical application prospects. For instance, the authors in [23] formulated and demonstrated several sufficient conditions for approximate controllability of a class of stochastic fractional system by using solution operator theory and fixed-point strategy. In [24,25], the authors established some sufficient conditions for approximate controllability of fractional differential equations of order $\alpha \in (1, 2)$ with finite delay and infinite delay via the theory of strongly continuous cosine and sine family, respectively. C. Dineshkumar et al. [26] discussed the approximate controllability of fractional neutral integro-differential systems by applying Schauder's fixed-point theorem. N. I. Mahmudov [27] proved the partial-approximate controllability of some fractional nonlocal dynamic equations by using an approximating technique due to the non-compactness of associated C_0 -semigroup at $t = 0$.

What we need to emphasize here is that such outcomes mentioned above regarding approximate controllability are achieved under some special resolvent conditions related to the resolvent operators associated with the studied systems (see [28–30] for further details). However, there are only a few articles on investigating approximate controllability by decomposing the given Banach space into the direct sum of its orthogonal subspaces (we call them spatial decomposition techniques). For example, K. Naito [31] studied the approximate controllability of a class of integer order system

$$\begin{cases} \frac{dx}{dt} = -Ax(t) + F(x(t)) + Bu(t), & t \in [0, T], \\ x(0) = 0, \end{cases}$$

where F is Lipschitz continuous, $-A$ generates a C_0 -semigroup, and B is a bounded linear operator. The results were firstly derived through some range conditions of the operator B , and a proper decomposition of certain space related with the C_0 -semigroup. S. Kumar et al. [32] studied the approximate controllability of the following fractional differential equation with finite delay

$$\begin{cases} D^\alpha x(t) = Ax(t) + f(t, x(t-h)) + Bu(t), & t \in [0, b], \\ x(t) = \varphi(t), & t \in [-h, 0], \end{cases}$$

where $\alpha \in (\frac{1}{2}, 1)$, f is nonlinear, φ is continuous, and A generates a C_0 -semigroup. By using Banach contraction principle and spatial decomposition methods, the authors obtained the existence of mild solutions and the approximate controllability of the fractional system.

Thus, it can be observed that although results on the approximate controllability of integer order and fractional order evolution systems have been presented one after another in recent years, and also only a few papers have considered the approximate controllability for such systems by using spatial decomposition techniques (see [31–33]), there is currently no report on the approximate controllability of fractional integro-differential impulsive evolution equations of order $1 < \alpha < 2$ with delay under a new definition of the PC -mild solution by using spatial decomposition techniques.

Motivated by these considerations, in this work, we take advantage of spatial decomposition techniques and the theory of the strongly continuous cosine family to study the approximate controllability of the following fractional delay system with impulsive effects

$$\begin{cases} {}^C D^\alpha x(t) = Ax(t) + \tilde{A}x(t-b) + Bu(t) + f\left(t, x(t-b), \int_0^t k(t,s, x(s-b))ds\right), \text{ a.e. } t \in I, \\ x(t) = \phi(t), \quad t \in [-b, 0], \\ x'(0) = x_0, \\ \Delta x(t_k) = z_k, \quad k = 1, 2, \dots, m, \\ \Delta x'(t_k) = \tilde{z}_k, \quad k = 1, 2, \dots, m, \end{cases} \quad (1)$$

where ${}^C D^\alpha$ is the Caputo derivative with order $1 < \alpha < 2$. The state x takes value in a Banach space X with norm $\|\cdot\|$. The linear closed operator $A : D(A) \subset X \rightarrow X$ generates a strongly continuous cosine family $\{C(t)\}_{t \geq 0}$ in X . Let $Z = L^2(I; X)$, $Z_b = L^2(J; X)$, where $I = [0, a]$ and $J = [-b, a]$. Define $Y = L^2(I; U)$, and U is also a Banach space with norm $\|\cdot\|_U$. The control u is introduced in Y , and B is a bounded linear operator from Y to Z . The impulse point set $\{t_k\}$ satisfies $0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = a < +\infty$, $m \in \mathbb{N}$. $\tilde{A}$ is a bounded linear operator on X . ϕ is continuous on $[-b, 0]$. $x_0, z_k, \tilde{z}_k \in X$. The functions f, k are some given functions that will be mentioned in detail later. If $f \equiv 0$, the system (1) degenerates into a linear system denoted by (2); that is,

$$\begin{cases} {}^C D^\alpha x(t) = Ax(t) + \tilde{A}x(t-b) + Bu(t), \text{ a.e. } t \in I, \\ x(t) = \phi(t), \quad t \in [-b, 0], \\ x'(0) = x_0, \\ \Delta x(t_k) = z_k, \quad k = 1, 2, \dots, m, \\ \Delta x'(t_k) = \tilde{z}_k, \quad k = 1, 2, \dots, m. \end{cases} \quad (2)$$

The major features of this article are as follows. (i) The *PC*-mild solution of fractional integro-differential impulsive evolution equations of order $1 < \alpha < 2$ with delay (1) is firstly presented by some characteristic solution operators based on the Mainardi's Wright function. (ii) Compared with the way of spatial decomposition in some of the previous relevant literature, approximate controllability of fractional system (1) is investigated by utilizing some more extensive spatial decomposition techniques, which covers and extends some related conclusions obtained in many papers [24,31–34].

The structure of the present paper is arranged as follows. Section 2 provides some known fundamental theories and some necessary preparations for the new definition of the *PC*-mild solution. Section 3 gives and proves the existence of the new *PC*-mild solution for the considered system by utilizing fixed-point strategy. Approximate controllability for fractional integro-differential impulsive system of order $1 < \alpha < 2$ with delay (1) is proved by means of spatial decomposition techniques in Section 4. In the last section, an example is proposed to demonstrate the obtained controllability results.

2. Preliminaries

Denote $PC(J; X) = \{x : J \rightarrow X \mid x \in C((t_k, t_{k+1}]; X), x(t_k^-) \text{ and } x(t_k^+) \text{ exist with } x(t_k^-) = x(t_k), k = 1, 2, \dots, m\}$. It is easy to see that $PC(J; X)$ is a Banach space provided with the norm $\|x\|_{PC(J; X)} = \sup_{t \in J} \|x(t)\|$. Let $\mathfrak{L}(X)$ be a space of all bounded linear operators from

X into itself endowed with the norm $\|\cdot\|_{\mathfrak{L}}$. Also, $C([-b, 0]; X)$ represents a Banach space of all continuous function from $[-b, 0]$ to X with the norm $\|\cdot\|_C$.

Definition 1 ([35]). The Caputo fractional derivative with order $\alpha \in (1, 2)$ is denoted by

$$D_{0+}^\alpha x(\tau) = \frac{1}{\Gamma(2-\alpha)} \int_0^\tau \frac{x^{(2)}(s)}{(\tau-s)^{\alpha-1}} ds,$$

where $x \in C^1([0, a]; X)$.

Definition 2 ([36]). The family $\{C(\tau)\}_{\tau \in \mathbb{R}} \subseteq \mathfrak{L}(X)$ is said to be a cosine family if it satisfies

$$(i) \quad C(\zeta + \tau) + C(\zeta - \tau) = 2C(\zeta)C(\tau), \quad \forall \zeta, \tau \in \mathbb{R};$$

- (ii) $C(0) = I$;
- (iii) $C(\tau)\xi$ is strongly continuous in τ on $\mathbb{R}$ for every fixed $\xi \in X$.

Definition 3 ([36]). The Mainardi's Wright function M_β is defined as

$$M_\beta(\omega) = \sum_{n=0}^{\infty} \frac{(-\omega)^n}{n! \Gamma(-\beta n + 1 - \beta)}, \quad \beta \in (0, 1), \quad \omega \in \mathbb{C},$$

$$\text{and it satisfies } \int_0^\infty \omega^\varsigma M_\beta(\omega) d\omega = \frac{\Gamma(1 + \varsigma)}{\Gamma(1 + \beta\varsigma)}, \quad -1 < \varsigma < \infty.$$

The family $\{S(\tau)\}_{\tau \in \mathbb{R}} \subseteq \mathfrak{L}(X)$ is said to be a sine family associated with $\{C(\tau)\}_{\tau \in \mathbb{R}}$ if it satisfies

$$S(\tau)\xi = \int_0^\tau C(s)\xi ds, \quad \xi \in X, \quad \tau \in \mathbb{R}.$$

The operator A denoted by

$$A\xi = \frac{d^2}{dt^2} C(\tau)\xi|_{\tau=0}, \quad \forall \xi \in D(A),$$

is said to be an infinitesimal generator of $\{C(\tau)\}_{\tau \in \mathbb{R}}$ where

$$D(A) = \{\xi \in X : C(\tau)\xi \text{ is a twice continuously differentiable function of } \tau\}.$$

In addition, from [36,37], we know that there exists a constant $M \geq 1$ such that $\|C(\tau)\|_{\mathfrak{L}} \leq M$ for $\tau \geq 0$.

Lemma 1 ([36]). Assume that $\{C(\tau)\}_{\tau \in \mathbb{R}}$ is a strongly continuous cosine family in X and satisfies $\|C(\tau)\|_{\mathfrak{L}} \leq Me^{\sigma|\tau|}$, $\tau \in \mathbb{R}$. Then, for λ with $\operatorname{Re} \lambda > \sigma$, $\lambda^2 \in \rho(A)$ and

$$\lambda R(\lambda^2; A)\xi = \int_0^\infty e^{-\lambda\tau} C(\tau)\xi d\tau, \quad R(\lambda^2; A)\xi = \int_0^\infty e^{-\lambda\tau} S(\tau)\xi d\tau, \quad \forall \xi \in X.$$

Before deducing the mild solution of system (1), let us consider the following linear fractional impulsive system with delay

$$\begin{cases} {}^C D^\alpha x(t) = Ax(t) + h(t), \text{ a.e. } t \in I, \\ x(0) = \phi(0), \\ x'(0) = x_0, \\ \Delta x(t_k) = z_k, \quad k = 1, 2, \dots, m, \\ \Delta x'(t_k) = \tilde{z}_k, \quad i = 1, 2, \dots, m. \end{cases} \quad (3)$$

By using a similar derivation to the mild solution of fractional impulsive systems in [9,38], we can transform system (3) into an equivalent integral expression

$$x(t) = \begin{cases} \phi(0) + x_0 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (Ax(s) + h(s)) ds, & t \in [0, t_1], \\ \phi(0) + x_0 t + z_1 + \tilde{z}_1(t-t_1) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (Ax(s) + h(s)) ds, & t \in (t_1, t_2], \\ \phi(0) + x_0 t + z_1 + \tilde{z}_1(t-t_1) + z_2 + \tilde{z}_2(t-t_2) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (Ax(s) + h(s)) ds, & t \in (t_2, t_3], \\ \vdots \\ \phi(0) + x_0 t + \sum_{k=1}^m z_k + \sum_{k=1}^m \tilde{z}_k(t-t_k) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (Ax(s) + h(s)) ds, & t \in (t_m, a]. \end{cases} \quad (4)$$

Obviously, (4) can be expressed in the following form

$$x(t) = \phi(0) + x_0 t + \sum_{k=1}^m \chi_k(t) z_k + \sum_{k=1}^m \chi_k(t) \tilde{z}_k(t - t_k) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (Ax(s) + h(s)) ds, \quad t \in I, \quad (5)$$

where

$$\chi_k(t) = \begin{cases} 0, & t \leq t_k, \\ 1, & t > t_k. \end{cases} \quad (6)$$

For simplicity, denote $\beta = \frac{\alpha}{2}$ for $\alpha \in (1, 2)$. Then, one presents the next important lemma.

Lemma 2. Suppose that (5) holds. Then, one has

$$x(t) = C_\beta(t)\phi(0) + K_\beta(t)x_0 + \sum_{k=1}^m \chi_k(t)[C_\beta(t - t_k)z_k + K_\beta(t - t_k)\tilde{z}_k] + \int_0^t (t-s)^{\beta-1} P_\beta(t-s)h(s)ds, \quad t \in I, \quad (7)$$

where

$$C_\beta(t) = \int_0^\infty M_\beta(s)C(t^\beta s)ds, \quad K_\beta(t) = \int_0^t C_\beta(s)ds, \quad P_\beta(t) = \int_0^\infty \beta s M_\beta(s)S(t^\beta s)ds.$$

Proof. First, it is not difficult to check the following Laplace transform:

$$\mathcal{L}[\chi_k(t)](\lambda) = \frac{e^{-\lambda t_k}}{\lambda}, \quad \mathcal{L}[\chi_k(t)(t - t_k)](\lambda) = \frac{e^{-\lambda t_k}}{\lambda^2}. \quad (8)$$

Furthermore, using the similar way in [37] implies

$$\lambda^{\beta-1} \int_0^\infty e^{-\lambda^\beta t} C(t)\phi(0)dt = \mathcal{L}[C_\beta(t)\phi(0)](\lambda), \quad (9)$$

$$\int_0^\infty e^{-\lambda^\beta t} S(t)\mathcal{L}[h(t)](\lambda)dt = \mathcal{L}\left[\int_0^t (t-s)^{\beta-1} P_\beta(t-s)h(s)ds\right](\lambda). \quad (10)$$

Let $\lambda^\alpha \in \rho(A)$. From Lemma 1 and (8)–(10) and taking the Laplace transform to (5) on both sides, it follows that

$$\begin{aligned} \mathcal{L}[x(t)](\lambda) &= \frac{1}{\lambda}\phi(0) + \frac{1}{\lambda^2}x_0 + \sum_{k=1}^m \frac{e^{-\lambda t_k}}{\lambda}z_k + \sum_{k=1}^m \frac{e^{-\lambda t_k}}{\lambda^2}\tilde{z}_k + \frac{1}{\lambda^\alpha}A\mathcal{L}[x(t)](\lambda) + \frac{1}{\lambda^\alpha}\mathcal{L}[h(t)](\lambda) \\ &= \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}\phi(0) + \lambda^{\alpha-2}(\lambda^\alpha I - A)^{-1}x_0 + \sum_{k=1}^m e^{-\lambda t_k}\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}z_k \\ &\quad + \sum_{k=1}^m e^{-\lambda t_k}\lambda^{\alpha-2}(\lambda^\alpha I - A)^{-1}\tilde{z}_k + (\lambda^\alpha I - A)^{-1}\mathcal{L}[h(t)](\lambda) \\ &= \lambda^{\beta-1} \int_0^\infty e^{-\lambda^\beta t} C(t)\phi(0)dt + \lambda^{-1}\lambda^{\beta-1} \int_0^\infty e^{-\lambda^\beta t} C(t)x_0dt + \sum_{k=1}^m e^{-\lambda t_k}\lambda^{\beta-1} \int_0^\infty e^{-\lambda^\beta t} C(t)z_kdt \\ &\quad + \sum_{k=1}^m e^{-\lambda t_k}\lambda^{-1}\lambda^{\beta-1} \int_0^\infty e^{-\lambda^\beta t} C(t)\tilde{z}_kdt + \int_0^\infty e^{-\lambda^\beta t} S(t)\mathcal{L}[h(t)](\lambda)dt \\ &= \int_0^\infty e^{-\lambda t} C_\beta(t)\phi(0)dt + \int_0^\infty e^{-\lambda t} \left(\int_0^t C_\beta(s)x_0ds \right)dt + \sum_{k=1}^m e^{-\lambda t_k} \int_0^\infty e^{-\lambda t} C_\beta(t)z_kdt \\ &\quad + \sum_{k=1}^m e^{-\lambda t_k} \int_0^\infty e^{-\lambda t} \left(\int_0^t C_\beta(s)\tilde{z}_kds \right)dt + \int_0^\infty e^{-\lambda t} \left(\int_0^t (t-s)^{\beta-1} P_\beta(t-s)h(s)ds \right)dt \\ &= \mathcal{L}[C_\beta(t)\phi(0)](\lambda) + \mathcal{L}[K_\beta(t)x_0](\lambda) + \sum_{k=1}^m \mathcal{L}[\chi_k(t)C_\beta(t - t_k)z_k](\lambda) \\ &\quad + \sum_{k=1}^m \mathcal{L}[\chi_k(t)K_\beta(t - t_k)\tilde{z}_k](\lambda) + \mathcal{L}\left[\int_0^t (t-s)^{\beta-1} P_\beta(t-s)h(s)ds\right](\lambda). \end{aligned}$$

Then, the uniqueness theorem of Laplace transform guarantees that

$$x(t) = C_\beta(t)\phi(0) + K_\beta(t)x_0 + \sum_{k=1}^m \chi_k(t)[C_\beta(t-t_k)z_k + K_\beta(t-t_k)\tilde{z}_k] + \int_0^t (t-s)^{\beta-1} P_\beta(t-s)h(s)ds, \quad t \in I.$$

This ends the proof. $\square$

Lemma 3 ([37]). For any $\tau \in [0, +\infty)$ and $\xi \in X$, the following inequalities hold:

$$\|C_\beta(\tau)\xi\| \leq M\|\xi\|, \quad \|K_\beta(\tau)\xi\| \leq M\|\xi\|\tau, \quad \|P_\beta(\tau)\xi\| \leq \frac{M}{\Gamma(2\beta)}\|\xi\|\tau^\beta.$$

Lemma 4 ([37]). For any $\tau_1, \tau_2 \in [0, +\infty)$ and for any $\xi \in X$, the following estimations hold:

- (i) $\|C_\beta(\tau_2)\xi - C_\beta(\tau_1)\xi\| \rightarrow 0$, as $|\tau_1 - \tau_2| \rightarrow 0$;
- (ii) $\|K_\beta(\tau_2) - K_\beta(\tau_1)\|_{\mathcal{L}} \rightarrow 0$, as $|\tau_1 - \tau_2| \rightarrow 0$;
- (iii) $\|P_\beta(\tau_2) - P_\beta(\tau_1)\|_{\mathcal{L}} \leq \frac{M}{\Gamma(2\beta)}|\tau_2^\beta - \tau_1^\beta| \rightarrow 0$, as $|\tau_1 - \tau_2| \rightarrow 0$.

Lemma 5 ([38] Schaefer's fixed-point theorem). Suppose X to be a Banach space and the operator $\Psi : X \rightarrow X$ to be completely continuous. If the set

$$U(\Psi) = \{x \in X : x = \lambda\Psi x \text{ for certain } \lambda \in (0, 1)\}$$

is bounded, then the operator Ψ has at least a fixed point.

3. Existence of the Mild Solution

On the basis of Lemma 2, the new PC-mild solution of system (1) can be introduced as below.

Definition 4. A function $x \in PC(J; X)$ is called a PC-mild solution of system (1) for each control $u \in Y$ if it satisfies the piecewise equation

$$x(t) = \begin{cases} C_\beta(t)\phi(0) + K_\beta(t)x_0 + \sum_{k=1}^m \chi_k(t)[C_\beta(t-t_k)z_k + K_\beta(t-t_k)\tilde{z}_k] \\ + \int_0^t (t-s)^{\beta-1} P_\beta(t-s) \left[\tilde{A}x(s-b) + Bu(s) + f\left(s, x(s-b), \int_0^s k(s, \eta, x(\eta-b))d\eta\right) \right] ds, \quad t \in I, \\ \phi(t), \quad t \in [-b, 0]. \end{cases}$$

Now, we impose some necessary hypotheses on the nonlinear terms of the studied fractional system.

Hypothesis 1 (H1). The function $f : I \times X \times X \rightarrow X$ is continuous, and there exists a function $p(t) \in C(I; \mathbb{R}^+)$ such that

$$\|f(t, x, y)\| \leq p(t)(1 + \|x\| + \|y\|),$$

for all $x, y \in X$ and $t \in I$.

Hypothesis 2 (H2). The function $k : \Delta \times X \rightarrow X$ is continuous where $\Delta = \{(t, s) \in I \times I : s \leq t\}$, and there exists a function $m \in L^1(I; \mathbb{R}^+)$ such that $\|k(t, s, x)\| \leq m(s)$ for all $(t, s) \in \Delta$ and $x \in X$.

Lemma 6. Assume that (H1) and (H2) hold. Then, the fractional system (1) has at least one PC-mild solution.

Proof. Define an operator $\Psi : PC(J; X) \rightarrow PC(J; X)$ as

$$(\Psi x)(t) = \begin{cases} C_\beta(t)\phi(0) + K_\beta(t)x_0 + \sum_{k=1}^m \chi_k(t) [C_\beta(t-t_k)z_k + K_\beta(t-t_k)\tilde{z}_k] \\ + \int_0^t (t-s)^{\beta-1} P_\beta(t-s) \left[\tilde{A}x(s-b) + Bu(s) + f\left(s, x(s-b), \int_0^s k(s, \eta, x(\eta-b))d\eta\right) \right] ds, \quad t \in I, \\ \phi(t), \quad t \in [-b, 0]. \end{cases}$$

To make the proof more concise, we shall now divide it into four steps.

Step I. Ψ maps bounded set into bounded set in $PC(J; X)$.

For any $t \in I$, $r_1 > 0$ and any $x \in B_{r_1} = \{x \in PC(J; X) : \|x\|_{PC} \leq r_1\}$, one can obtain

$$\begin{aligned} \|(\Psi x)(t)\| &\leq \|C_\beta(t)\phi(0)\| + \|K_\beta(t)x_0\| + \sum_{k=1}^m [\|C_\beta(t-t_k)z_k\| + \|K_\beta(t-t_k)\tilde{z}_k\|] \\ &\quad + \int_0^t (t-s)^{\beta-1} \left\| P_\beta(t-s) \left[\tilde{A}x(s-b) + Bu(s) + f\left(s, x(s-b), \int_0^s k(s, \eta, x(\eta-b))d\eta\right) \right] \right\| ds \\ &\leq M\|\phi(0)\| + Ma\|x_0\| + \sum_{k=1}^m [M\|z_k\| + Ma\|\tilde{z}_k\|] \\ &\quad + \frac{Ma^{2\beta-1}}{\Gamma(2\beta)} \int_0^t [\|\tilde{A}x(s-b)\| + \|Bu(s)\| + p(s)(1 + \|x(s-b)\| + \|m\|_{L^1})] ds \\ &\leq M\|\phi(0)\| + Ma\|x_0\| + \sum_{k=1}^m [M\|z_k\| + Ma\|\tilde{z}_k\|] \\ &\quad + \frac{Ma^{2\beta-1}}{\Gamma(2\beta)} \left[ar_1\|\tilde{A}\| + \sqrt{a}\|Bu\|_Z + \max_{t \in I} \int_0^t p(s)ds \cdot (1 + r_1 + \|m\|_{L^1}) \right] := r_0, \end{aligned}$$

which implies that $\|\Psi x\|_{PC} \leq \max\{\|\phi\|_C, r_0\} := r_2$. Thereby, we have $\Psi(B_{r_1}) \subseteq B_{r_2}$.

Step II. Ψ is continuous.

Suppose that $\{x_n\}$ satisfies $x_n \rightarrow x$ in $PC(J; X)$. Since f, k are all continuous, then for every $t \in I$, we have from Lebesgue's domination convergence theorem that

$$\begin{aligned} &\|(\Psi x_n)(t) - (\Psi x)(t)\| \\ &\leq \int_0^t (t-s)^{\beta-1} \left\| P_\beta(t-s) [\tilde{A}x_n(s-b) - \tilde{A}x(s-b)] \right\| ds \\ &\quad + \int_0^t (t-s)^{\beta-1} \left\| P_\beta(t-s) \left[f\left(s, x_n(s-b), \int_0^s k(s, \eta, x_n(\eta-b))d\eta\right) - f\left(s, x(s-b), \int_0^s k(s, \eta, x(\eta-b))d\eta\right) \right] \right\| ds \\ &\leq \frac{M\|\tilde{A}\|}{\Gamma(2\beta)} \int_0^t (t-s)^{2\beta-1} \|x_n(s-b) - x(s-b)\| ds \\ &\quad + \frac{M}{\Gamma(2\beta)} \int_0^t (t-s)^{2\beta-1} \left\| f\left(s, x_n(s-b), \int_0^s k(s, \eta, x_n(\eta-b))d\eta\right) - f\left(s, x(s-b), \int_0^s k(s, \eta, x(\eta-b))d\eta\right) \right\| ds \\ &\leq \frac{a^{2\beta}M\|\tilde{A}\|}{\Gamma(2\beta)} \|x_n - x\|_{PC} \\ &\quad + \frac{a^{2\beta}M}{\Gamma(2\beta)} \left\| f\left(s, x_n(s-b), \int_0^s k(s, \eta, x_n(\eta-b))d\eta\right) - f\left(s, x(s-b), \int_0^s k(s, \eta, x(\eta-b))d\eta\right) \right\| \\ &\rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

In addition, it is obvious that $\|(\Psi x_n)(t) - (\Psi x)(t)\| = 0$ for each $t \in [-b, 0]$. Therefore, by using the Ascoli–Arzela theorem, it is not difficult to verify that $\|\Psi x_n - \Psi x\|_{PC} \rightarrow 0$, as $n \rightarrow \infty$.

Step III. Ψ maps bounded set into equicontinuous set in $PC(J; X)$.

For simplicity, let

$$F(t) = f\left(t, x(t-b), \int_0^t k(t, s, x(s-b))ds\right), \quad t \in I.$$

Then, for any $0 \leq \sigma_1 < \sigma_2 \leq t_1$ and $x \in B_{r_1}$, we have

$$\begin{aligned} & \|(\Psi x)(\sigma_2) - (\Psi x)(\sigma_1)\| \\ \leq & \|C_\beta(\sigma_2)\phi(0) - C_\beta(\sigma_1)\phi(0)\| + \|K_\beta(\sigma_2)x_0 - K_\beta(\sigma_1)x_0\| \\ & + \int_0^{\sigma_1} \left\| \left[(\sigma_2 - s)^{\beta-1} P_\beta(\sigma_2 - s) - (\sigma_1 - s)^{\beta-1} P_\beta(\sigma_1 - s) \right] \left[\tilde{A}x(s - b) + Bu(s) + F(s) \right] \right\| ds \\ & + \int_{\sigma_1}^{\sigma_2} \left\| (\sigma_2 - s)^{\beta-1} P_\beta(\sigma_2 - s) \left[\tilde{A}x(s - b) + Bu(s) + F(s) \right] \right\| ds \\ = & \sum_{i=1}^3 \Theta_i, \end{aligned}$$

where

$$\begin{aligned} \Theta_1 &= \|C_\beta(\sigma_2)\phi(0) - C_\beta(\sigma_1)\phi(0)\| + \|K_\beta(\sigma_2)x_0 - K_\beta(\sigma_1)x_0\|, \\ \Theta_2 &= \int_0^{\sigma_1} \left\| \left[(\sigma_2 - s)^{\beta-1} P_\beta(\sigma_2 - s) - (\sigma_1 - s)^{\beta-1} P_\beta(\sigma_1 - s) \right] \left[\tilde{A}x(s - b) + Bu(s) + F(s) \right] \right\| ds, \\ \Theta_3 &= \int_{\sigma_1}^{\sigma_2} \left\| (\sigma_2 - s)^{\beta-1} P_\beta(\sigma_2 - s) \left[\tilde{A}x(s - b) + Bu(s) + F(s) \right] \right\| ds. \end{aligned}$$

It can be seen from Lemma 4 that $C_\beta(t)$ is strongly continuous and $K_\beta(t)$ is uniformly continuous, and thus we have $\Theta_1 \rightarrow 0$ as $\sigma_2 \rightarrow \sigma_1$. In addition, the uniform continuity of $P_\beta(t)$ can imply the operator $\tilde{P}_\beta(t) := t^{\beta-1}P_\beta(t)$ is also uniformly continuous, which can guarantee that

$$\begin{aligned} \|\Theta_2\| &\leq \int_0^{\sigma_1-\varepsilon} \left\| \left[\tilde{P}_\beta(\sigma_2 - s) - \tilde{P}_\beta(\sigma_1 - s) \right] \left[\tilde{A}x(s - b) + Bu(s) + F(s) \right] \right\| ds \\ &\quad + \int_{\sigma_1-\varepsilon}^{\sigma_1} \left\| \left[\tilde{P}_\beta(\sigma_2 - s) - \tilde{P}_\beta(\sigma_1 - s) \right] \left[\tilde{A}x(s - b) + Bu(s) + F(s) \right] \right\| ds \\ &\leq \int_0^{\sigma_1-\varepsilon} \left[\|\tilde{A}x(s - b)\| + \|Bu(s)\| + p(s)(1 + \|x(s - b)\| + \|m\|_{L^1}) \right] ds \cdot \sup_{s \in [0, \sigma_1-\varepsilon]} \left\| \tilde{P}_\beta(\sigma_2 - s) - \tilde{P}_\beta(\sigma_1 - s) \right\|_{\mathfrak{L}} \\ &\quad + \frac{M}{\Gamma(2\beta)} \int_{\sigma_1-\varepsilon}^{\sigma_1} \left[\|\tilde{A}x(s - b)\| + \|Bu(s)\| + p(s)(1 + \|x(s - b)\| + \|m\|_{L^1}) \right] \cdot \left[(\sigma_2 - \sigma_1 + \varepsilon)^{2\beta-1} + \varepsilon^{2\beta-1} \right] ds \\ &\leq \left[ar_1 \|\tilde{A}\| + \sqrt{a} \|Bu\|_Z + (1 + r_1 + \|m\|_{L^1}) \int_0^{\sigma_1-\varepsilon} p(s) ds \right] \cdot \sup_{s \in [0, \sigma_1-\varepsilon]} \left\| \tilde{P}_\beta(\sigma_2 - s) - \tilde{P}_\beta(\sigma_1 - s) \right\|_{\mathfrak{L}} \\ &\quad + \frac{M}{\Gamma(2\beta)} \left[\varepsilon r_1 \|\tilde{A}\| + \sqrt{\varepsilon} \|Bu\|_Z + (1 + r_1 + \|m\|_{L^1}) \int_{\sigma_1-\varepsilon}^{\sigma_1} p(s) ds \right] \cdot \left[(\sigma_2 - \sigma_1 + \varepsilon)^{2\beta-1} + \varepsilon^{2\beta-1} \right] \\ &\rightarrow 0, \text{ as } \sigma_2 \rightarrow \sigma_1, \varepsilon \rightarrow 0. \end{aligned}$$

As for Θ_3 , we have

$$\begin{aligned} \|\Theta_3\| &\leq \frac{M}{\Gamma(2\beta)} \int_{\sigma_1}^{\sigma_2} (\sigma_2 - s)^{2\beta-1} \left[\|\tilde{A}x(s - b)\| + \|Bu(s)\| + p(s)(1 + \|x(s - b)\| + \|m\|_{L^1}) \right] ds \\ &\leq \frac{M}{\Gamma(2\beta)} (\sigma_2 - \sigma_1)^{2\beta-1} \left[(\sigma_2 - \sigma_1) r_1 \|\tilde{A}\| + \sqrt{\sigma_2 - \sigma_1} \|Bu\|_Z + (1 + r_1 + \|m\|_{L^1}) \int_{\sigma_1}^{\sigma_2} p(s) ds \right] \\ &\leq \frac{M}{\Gamma(2\beta)} \left[(\sigma_2 - \sigma_1)^{2\beta} r_1 \|\tilde{A}\| + (\sigma_2 - \sigma_1)^{2\beta-\frac{1}{2}} \|Bu\|_Z + (\sigma_2 - \sigma_1)^{2\beta-1} (1 + r_1 + \|m\|_{L^1}) \int_{\sigma_1}^{\sigma_2} p(s) ds \right] \\ &\rightarrow 0, \text{ as } \sigma_2 \rightarrow \sigma_1. \end{aligned}$$

Hence, we obtain that Ψ is equicontinuous on $[0, t_1]$. For the case of general interval $(t_k, t_{k+1}]$, it can be similarly proved that

$$\begin{aligned}
& \|(\Psi x)(\sigma_2) - (\Psi x)(\sigma_1)\| \\
& \leq \|C_\beta(\sigma_2)\phi(0) - C_\beta(\sigma_1)\phi(0)\| + \|K_\beta(\sigma_2)x_0 - K_\beta(\sigma_1)x_0\| \\
& \quad + \sum_{k=1}^m [\|C_\beta(\sigma_2 - t_k)z_k - C_\beta(\sigma_1 - t_k)z_k\| + \|K_\beta(\sigma_2 - t_k)\tilde{z}_k - K_\beta(\sigma_1 - t_k)\tilde{z}_k\|] \\
& \quad + \int_0^{\sigma_1} \left\| \left[(\sigma_2 - s)^{\beta-1} P_\beta(\sigma_2 - s) - (\sigma_1 - s)^{\beta-1} P_\beta(\sigma_1 - s) \right] \left[\tilde{A}x(s - b) + Bu(s) + F(s) \right] \right\| ds \\
& \quad + \int_{\sigma_1}^{\sigma_2} (\sigma_2 - s)^{\beta-1} \left\| P_\beta(\sigma_2 - s) \left[\tilde{A}x(s - b) + Bu(s) + F(s) \right] \right\| ds \\
& = \sum_{k=1}^m [\|C_\beta(\sigma_2 - t_k)z_k - C_\beta(\sigma_1 - t_k)z_k\| + \|K_\beta(\sigma_2 - t_k)\tilde{z}_k - K_\beta(\sigma_1 - t_k)\tilde{z}_k\|] + \sum_{i=1}^3 \Theta_i
\end{aligned}$$

$\rightarrow 0$, as $\sigma_2 \rightarrow \sigma_1$,

which means that Ψ is equicontinuous on $(t_k, t_{k+1}]$.

Consequently, we can obtain by Step I–III and the Arzela–Ascoli theorem that $\Psi(B_{r_1})$ is compact.

Step IV. A priori bound.

We shall prove that the set

$$U(\Psi) = \{x \in PC(J; X) : x = \lambda \Psi x \text{ for certain } \lambda \in (0, 1)\}$$

is bounded.

Assume that $x \in U(\Psi)$, then $x = \lambda \Psi x$ for certain $\lambda \in (0, 1)$. In general, we only consider the case $t \in (t_k, t_{k+1}]$. Hence,

$$\begin{aligned}
\|x(t)\| & \leq \|C_\beta(t)\phi(0)\| + \|K_\beta(t)x_0\| + \sum_{i=1}^k [\|C_\beta(t - t_k)z_i\| + \|K_\beta(t - t_i)\tilde{z}_i\|] \\
& \quad + \int_0^t (t - s)^{\beta-1} \left\| P_\beta(t - s) \left[\tilde{A}x(s - b) + Bu(s) + f\left(s, x(s - b), \int_0^s k(s, \eta, x(\eta - b))d\eta\right) \right] \right\| ds \\
& \leq M\|\phi(0)\| + Ma\|x_0\| + \sum_{i=1}^k [M\|z_k\| + Ma\|\tilde{z}_k\|] \\
& \quad + \frac{Ma^{2\beta-1}}{\Gamma(2\beta)} \int_0^t [\|\tilde{A}x(s - b)\| + \|Bu(s)\| + p(s)(1 + \|x(s - b)\| + \|m\|_{L^1})] ds \\
& \leq M\|\phi(0)\| + Ma\|x_0\| + \sum_{i=1}^k [M\|z_k\| + Ma\|\tilde{z}_k\|] + \frac{Ma^{2\beta-1}}{\Gamma(2\beta)} \|Bu\|_Z \\
& \quad + \frac{Ma^{2\beta}}{\Gamma(2\beta)} \max_{t \in I} \int_0^t p(s)ds \cdot (1 + \|m\|_{L^1}) + \frac{Ma^{2\beta-1}}{\Gamma(2\beta)} \int_0^t (p(s) + \|\tilde{A}\|) \|x(s - b)\| ds.
\end{aligned}$$

By the Grönwall–Bellman inequality, we can choose a constant $M_k > 0$ such that

$$\|x(t)\| \leq M_k, \quad \forall t \in (t_k, t_{k+1}].$$

Denote

$$\tilde{M} = \max \left\{ \|\phi\|_C, \max_{1 \leq k \leq m} \{M_k\} \right\}.$$

Then, for any $t \in J$, it has $\|x(t)\| \leq \tilde{M}$, which implies $\|x\|_{PC} \leq \tilde{M}$. The boundedness of $U(\Psi)$ has been proven.

From Lemma 5, the operator Ψ has a fixed point which is a PC-mild solution of fractional system (1). This ends the proof. $\square$

4. Approximate Controllability

In this section, we study the approximate controllability of the fractional system (1).

Denote the reachable set of system (1) by $\Lambda_a(f)$, where

$$\Lambda_a(f) = \{x(a) \in X : x \text{ is the mild solution of (1)}\}.$$

Definition 5. The fractional system (1) is called approximately controllable on I provided that $\overline{\Lambda_a(f)} = X$, where $\overline{\Lambda_a(f)}$ represents the closure of $\Lambda_a(f)$. Evidently, linear system (2) is approximately controllable provided that $\overline{\Lambda_a(0)} = X$.

We can define operators $F_b^{(i)} : Z_b \rightarrow Z$ ($i = 1, 2$) presented by

$$(F_b^{(1)}x)(t) = \tilde{A}x(t-b),$$

and

$$(F_b^{(2)}x)(t) = f\left(t, x(t-b), \int_0^t k(t,s,x(s-b))ds\right),$$

and denote

$$(F_b x)(t) = \tilde{A}x(t-b) + f\left(t, x(t-b), \int_0^t k(t,s,x(s-b))ds\right).$$

Clearly,

$$(F_b x)(t) = (F_b^{(1)}x)(t) + (F_b^{(2)}x)(t).$$

Define an operator $Q : Z \rightarrow X$ as

$$Qx = \int_0^a (a-s)^{\beta-1} P_\beta(t-s)x(s)ds,$$

and $W : Z \rightarrow Z$ is defined by

$$(Wx)(t) = \int_0^t (t-s)^{\beta-1} P_\beta(t-s)x(s)ds, \quad t \in I.$$

In the sequel, $\mathfrak{R}(B)$ and $\overline{\mathfrak{R}(B)}$ stand for the range of the operator B and its closure; $\mathfrak{N}_0(Q)$ and $\mathfrak{N}_0^\perp(Q)$ represent the null space of Q and its orthogonal space, respectively. Then, we have a unique decomposition $Z = \mathfrak{N}_0(Q) \oplus \mathfrak{N}_0^\perp(Q)$.

To demonstrate our controllability result, we also need the following hypotheses.

Hypothesis 3 (H3). The cosine operator family $\{C(t)\}_{t \in \mathbb{R}}$ is compact.

Hypothesis 4 (H4). For every $x \in Z$, there is a $y \in \overline{\mathfrak{R}(B)}$ satisfying $Qx = Qy$.

Hypothesis 5 (H5). (i) There exists a function $l \in L^1(I; \mathbb{R}^+)$ satisfying

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq l(t)(\|x_1 - x_2\| + \|y_1 - y_2\|), \quad \forall t \in I, x_i, y_i \in X, i = 1, 2.$$

(ii) There exists a function $q \in L^1(I; \mathbb{R}^+)$ satisfying

$$\|k(t, s, x) - k(t, s, y)\| \leq q(s)\|x - y\|, \quad \forall (t, s) \in \Delta, x, y \in X.$$

Hypothesis (H4) indicates that any $x \in Z$ can be decomposed as

$$x = \theta + y : \theta \in \mathfrak{N}_0(Q), y \in \overline{\mathfrak{R}(B)}.$$

Then, define an operator $G : \mathfrak{N}_0^\perp(Q) \rightarrow \overline{\mathfrak{R}(B)}$ as $G\tilde{x} = \tilde{y}$, where $\tilde{y}$ is the unique element with minimum norm in $\{x + \mathfrak{N}_0(Q)\} \cap \overline{\mathfrak{R}(B)}$ satisfying

$$\|G\tilde{x}\| = \|\tilde{y}\| = \min\{\|z\| : z \in \{\tilde{x} + \mathfrak{N}_0(Q)\} \cap \overline{\mathfrak{R}(B)}\}.$$

Note that G is a linear and continuous operator [32]. Hence, there exists a positive constant δ such that $\|G\| \leq \delta$.

Lemma 7 ([32]). *For each $x \in Z$ and corresponding $\theta \in \mathfrak{N}_0(Q)$, the following inequality holds $\|\theta\|_Z \leq (1 + \delta)\|x\|_Z$.*

Consider the following set

$$\Xi = \left\{ \begin{array}{l} \omega \in Z_b : \omega(t) = (W\theta)(t), \quad \theta \in \mathfrak{N}_0(Q), \quad t \in I; \\ \omega(t) = 0, \quad t \in [-b, 0]. \end{array} \right\}. \quad (11)$$

Obviously, Ξ is a subspace of Z_b and satisfies $\omega(a) = 0, \forall \omega \in \Xi$.

For each mild solution x of linear system (2) with control u , define operators $\gamma_x^{(1)}$ and $\gamma_x^{(2)}$ from Ξ to Ξ as

$$\gamma_x^{(1)}(\omega) = \begin{cases} W\theta_1, & t \in I; \\ 0, & t \in [-b, 0], \end{cases}$$

and

$$\gamma_x^{(2)}(\omega) = \begin{cases} W\theta_2, & t \in I; \\ 0, & t \in [-b, 0], \end{cases}$$

where θ_1 and θ_2 are given by the unique decompositions

$$F_b^{(1)}(x + \omega) = \theta_1 + \iota_1, \quad \theta_1 \in \mathfrak{N}_0(Q), \quad \iota_1 \in \overline{\mathfrak{R}(B)}, \quad (12)$$

and

$$F_b^{(2)}(x + \omega) = \theta_2 + \iota_2, \quad \theta_2 \in \mathfrak{N}_0(Q), \quad \iota_2 \in \overline{\mathfrak{R}(B)}, \quad (13)$$

respectively.

Lemma 8. *Assume that (H1)–(H3) hold. Then, the operator $\gamma_x^{(i)}$ ($i = 1, 2$) has a fixed point in Ξ , provided that $\frac{Ma^{2\beta}(1 + \delta)}{\Gamma(2\beta)} \max\{p^*, \|\tilde{A}\|\} < 1$, where $p^* = \max_{t \in I}\{p(t)\}$.*

Proof. We are about to take advantage of Schauder's fixed-point theorem. Consider the set $\Omega_r = \{\omega \in \Xi : \|\omega\|_{Z_b} \leq r\}$ where r is a certain positive constant. Let us prove that $\gamma_x^{(i)}(\Omega_r) \subseteq \Omega_r$ ($i = 1, 2$) first. If not, then there exists an element $\omega \in \Omega_r$ satisfying $\gamma_x^{(i)}(\omega) \notin \Omega_r$, i.e., $\|\gamma_x^{(i)}(\omega)\| > r$ ($i = 1, 2$). $\square$

From Hölder's inequality and Lemma 7, it follows that

$$\begin{aligned} \left(\int_0^t \|\theta_1(s)\| ds\right)^2 &\leq \left(\int_0^t 1^2 ds\right) \left(\int_0^t \|\theta_1(s)\|^2 ds\right) \\ &\leq t \|\theta_1\|_Z^2 \leq t(1 + \delta)^2 \|F_b^{(1)}(x + \omega)\|_Z^2 \\ &\leq t(1 + \delta)^2 \|\tilde{A}\|^2 \int_0^a \|(x + \omega)(t - b)\|^2 dt, \end{aligned}$$

and

$$\begin{aligned}
 \left(\int_0^t \|\theta_2(s)\| ds \right)^2 &\leq \left(\int_0^t 1^2 ds \right) \left(\int_0^t \|\theta_2(s)\|^2 ds \right) \\
 &\leq t \|\theta_2\|_Z^2 \leq t(1+\delta)^2 \left\| F_b^{(2)}(x+\omega) \right\|_Z^2 \\
 &\leq t(1+\delta)^2 \int_0^a [p(t)(1+\|(x+\omega)(t-b)\| + \|m\|_{L^1})]^2 dt \\
 &\leq t(1+\delta)^2 (p^*)^2 \int_0^a (1+\|m\|_{L^1} + \|(x+\omega)(t-b)\|)^2 dt.
 \end{aligned}$$

Then, we obtain

$$\begin{aligned}
 \|\gamma_x^{(1)}(\omega)\|_{Z_b}^2 &\leq \int_0^a \left\| \int_0^t (t-s)^{\beta-1} P_\beta(t-s) \theta_1(s) ds \right\|^2 dt \\
 &\leq \int_0^a \left(\frac{Ma^{2\beta-1}}{\Gamma(2\beta)} \right)^2 \left(\int_0^t \|\theta_1(s)\| ds \right)^2 dt \\
 &\leq \left(\frac{Ma^{2\beta-1}}{\Gamma(2\beta)} \right)^2 (1+\delta)^2 \|\tilde{A}\|^2 a^2 \int_0^a \|(x+\omega)(s-b)\|^2 ds \\
 &= \left(\frac{Ma^{2\beta}}{\Gamma(2\beta)} \right)^2 (1+\delta)^2 \|\tilde{A}\|^2 \int_0^a \|(x+\omega)(s-b)\|^2 ds,
 \end{aligned} \tag{14}$$

and

$$\begin{aligned}
 \|\gamma_x^{(2)}(\omega)\|_{Z_b}^2 &\leq \int_0^a \left\| \int_0^t (t-s)^{\beta-1} P_\beta(t-s) \theta_2(s) ds \right\|^2 dt \\
 &\leq \int_0^a \left(\frac{Ma^{2\beta-1}}{\Gamma(2\beta)} \right)^2 \left(\int_0^t \|\theta_2(s)\| ds \right)^2 dt \\
 &\leq \left(\frac{Ma^{2\beta-1}}{\Gamma(2\beta)} \right)^2 (1+\delta)^2 (p^*)^2 a^2 \int_0^a (1+\|(x+\omega)(t-b)\| + \|m\|_{L^1})^2 ds \\
 &= \left(\frac{Ma^{2\beta}}{\Gamma(2\beta)} \right)^2 (1+\delta)^2 (p^*)^2 \int_0^a (1+\|m\|_{L^1} + \|(x+\omega)(t-b)\|)^2 ds.
 \end{aligned} \tag{15}$$

For (14), it is easy to see that

$$\begin{aligned}
 r < \|\gamma_x^{(1)}(\omega)\|_{Z_b} &\leq \frac{Ma^{2\beta}}{\Gamma(2\beta)} (1+\delta) \|\tilde{A}\| \cdot \|x+\omega\|_{Z_b} \\
 &\leq \frac{Ma^{2\beta}}{\Gamma(2\beta)} (1+\delta) \|\tilde{A}\| \cdot (\|x\|_{Z_b} + \|\omega\|_{Z_b}) \\
 &\leq \frac{Ma^{2\beta}}{\Gamma(2\beta)} (1+\delta) \|\tilde{A}\| \cdot (\|x\|_{Z_b} + r).
 \end{aligned}$$

Divide both sides of the above inequality by r , and then make r tend towards $+\infty$ to take the limit, we imply

$$\frac{Ma^{2\beta}}{\Gamma(2\beta)} (1+\delta) \|\tilde{A}\| \geq 1.$$

This is a contradiction.

For (15), it can be derived from Minkowski's inequality that

$$\begin{aligned}
r < \|\gamma_x^{(2)}(\omega)\|_{Z_b} &\leq \frac{Ma^{2\beta}}{\Gamma(2\beta)}(1+\delta)p^* \left[\left(\int_0^a (1+\|m\|_{L^1})^2 ds \right)^{\frac{1}{2}} + \left(\int_0^a \|(x+\omega)(t-b)\|^2 ds \right)^{\frac{1}{2}} \right] \\
&\leq \frac{Ma^{2\beta}}{\Gamma(2\beta)}(1+\delta)p^* \left[(1+\|m\|_{L^1})a^{\frac{1}{2}} + \|x+\omega\|_{Z_b} \right] \\
&\leq \frac{Ma^{2\beta}}{\Gamma(2\beta)}(1+\delta)p^* \left[(1+\|m\|_{L^1})a^{\frac{1}{2}} + \|x\|_{Z_b} + \|\omega\|_{Z_b} \right] \\
&\leq \frac{Ma^{2\beta}}{\Gamma(2\beta)}(1+\delta)p^* \left[(1+\|m\|_{L^1})a^{\frac{1}{2}} + \|x\|_{Z_b} + r \right].
\end{aligned}$$

Also, divide by r and take the limit as $r \rightarrow +\infty$ to obtain

$$\frac{Ma^{2\beta}}{\Gamma(2\beta)}(1+\delta)p^* \geq 1,$$

which is also a contradiction. Therefore, we claim that $\gamma_x^{(i)}(\Omega_r) \subseteq \Omega_r$ ($i = 1, 2$).

From the compactness of cosine operator $C(t)$ presented by (H3), it follows that $P_\beta(t)$ is compact (see [37]). Hence, the integral operator W is compact and then $\gamma_x^{(i)}$ ($i = 1, 2$) is also compact. By using Schauder's fixed-point theorem, we obtain that $\gamma_x^{(i)}$ has a fixed point ω_i in the set Ξ ($i = 1, 2$).

Theorem 1. Assume that (H1)–(H5) hold. Then, the fractional system (1) is approximately controllable provided that $\hat{M} := \frac{Ma^{2\beta-1}\|l\|_{L^1}}{\Gamma(2\beta)}(1+\|q\|_{L^1}) < 1$.

Proof. Let x be a mild solution of linear system (2) which is defined by

$$x(t) = \begin{cases} C_\beta(t)\phi(0) + K_\beta(t)x_0 + \sum_{k=1}^m \chi_k(t)[C_\beta(t-t_k)z_k + K_\beta(t-t_k)\tilde{z}_k] \\ + \int_0^t (t-s)^{\beta-1} P_\beta(t-s) [\tilde{A}x(s-b) + Bu(s)] ds, & t \in I, \\ \phi(t), & t \in [-b, 0]. \end{cases} \quad (16)$$

Next, we shall show that $y = x + \omega_2$ is the mild solution of the following system:

$$\begin{cases} {}^C D^\alpha y(t) = Ay(t) + \tilde{A}y(t-b) + (Bu - F_b^{(1)}\omega_2 - \iota_2)(t) + f\left(t, y(t-b), \int_0^t k(t,s,y(s-b))ds\right), & a.e. t \in I, \\ y(t) = \phi(t), & t \in [-b, 0], \\ y'(0) = x_0, \\ \Delta y(t_k) = z_k, & k = 1, 2, \dots, m, \\ \Delta y'(t_k) = \tilde{z}_k, & i = 1, 2, \dots, m. \end{cases} \quad (17)$$

It is easy to see from (12) and (13) that

$$F_b^{(1)}(x+\omega)(t) = \theta_1(t) + \iota_1(t),$$

and

$$F_b^{(2)}(x+\omega)(t) = \theta_2(t) + \iota_2(t).$$

Note that ω_i is a fixed point of $\gamma_x^{(i)}$ ($i = 1, 2$). This together with (11) implies that

$$WF_b^{(1)}(x+\omega_1)(t) = W\theta_1(t) + W\iota_1(t) = \omega_1(t) + W\iota_1(t), \quad (18)$$

and

$$WF_b^{(2)}(x + \omega_2)(t) = W\theta_2(t) + Wl_2(t) = \omega_2(t) + Wl_2(t). \quad (19)$$

From (18), we have

$$W(F_b^{(1)}x)(t) - \omega_1(t) = Wl_1(t) - W(F_b^{(1)}\omega_1)(t). \quad (20)$$

Add the two sides of (18) and (19) separately, and then add x on both sides to obtain

$$x(t) + WF_b^{(1)}(x + \omega_1)(t) + WF_b^{(2)}(x + \omega_2)(t) = x(t) + \omega_1(t) + \omega_2(t) + Wl_1(t) + Wl_2(t).$$

Denote $y(t) = x(t) + \omega_2(t)$; we obtain

$$y(t) = x(t) - \omega_1(t) + WF_b^{(1)}(x + \omega_1)(t) + W(F_b^{(2)}y)(t) - Wl_1(t) - Wl_2(t),$$

and then by utilizing (16) and (20), it can be concluded that

$$\begin{aligned} y(t) &= C_\beta(t)\phi(0) + K_\beta(t)x_0 + \sum_{k=1}^m \chi_k(t)[C_\beta(t-t_k)z_k + K_\beta(t-t_k)\tilde{z}_k] + W(Bu)(t) \\ &\quad + W(F_b^{(1)}x)(t) - \omega_1(t) + WF_b^{(1)}(x + \omega_1)(t) + W(F_b^{(2)}y)(t) - Wl_1(t) - Wl_2(t) \\ &= C_\beta(t)\phi(0) + K_\beta(t)x_0 + \sum_{k=1}^m \chi_k(t)[C_\beta(t-t_k)z_k + K_\beta(t-t_k)\tilde{z}_k] + W(Bu)(t) \\ &\quad + Wl_1(t) - W(F_b^{(1)}\omega_1)(t) + WF_b^{(1)}(x + \omega_1)(t) + W(F_b^{(2)}y)(t) - Wl_1(t) - Wl_2(t) \\ &= C_\beta(t)\phi(0) + K_\beta(t)x_0 + \sum_{k=1}^m \chi_k(t)[C_\beta(t-t_k)z_k + K_\beta(t-t_k)\tilde{z}_k] + W(Bu)(t) \\ &\quad + W(F_b^{(2)}y)(t) + WF_b^{(1)}(x + \omega_1)(t) - W(F_b^{(1)}\omega_1)(t) - Wl_2(t) \\ &= C_\beta(t)\phi(0) + K_\beta(t)x_0 + \sum_{k=1}^m \chi_k(t)[C_\beta(t-t_k)z_k + K_\beta(t-t_k)\tilde{z}_k] + W(Bu)(t) \\ &\quad + [W(F_b^{(2)}y)(t) + W(F_b^{(1)}y)(t)] - W(F_b^{(1)}y)(t) + WF_b^{(1)}(x + \omega_1)(t) - W(F_b^{(1)}\omega_1)(t) - Wl_2(t) \\ &= C_\beta(t)\phi(0) + K_\beta(t)x_0 + \sum_{k=1}^m \chi_k(t)[C_\beta(t-t_k)z_k + K_\beta(t-t_k)\tilde{z}_k] + W(Bu)(t) \\ &\quad + W(F_b y)(t) - W(F_b^{(1)}y)(t) + WF_b^{(1)}(x + \omega_1)(t) - W(F_b^{(1)}\omega_1)(t) - Wl_2(t) \\ &= C_\beta(t)\phi(0) + K_\beta(t)x_0 + \sum_{k=1}^m \chi_k(t)[C_\beta(t-t_k)z_k + K_\beta(t-t_k)\tilde{z}_k] + W(Bu)(t) \\ &\quad + W(F_b y)(t) - W(F_b^{(1)}\omega_2)(t) - Wl_2(t) \\ &= C_\beta(t)\phi(0) + K_\beta(t)x_0 + \sum_{k=1}^m \chi_k(t)[C_\beta(t-t_k)z_k + K_\beta(t-t_k)\tilde{z}_k] \\ &\quad + W(Bu - F_b^{(1)}\omega_2 - l_2)(t) + W(F_b y)(t). \end{aligned}$$

This is exactly the mild solution of the system (17) under the control $(Bu - F_b^{(1)}\omega_2 - l_2)$.

From the fact $l_2 \in \overline{\mathfrak{R}(B)}$, for any $\varepsilon > 0$, we can find an element $v \in Y$ that satisfies

$$\|Bv - l_2\|_Z < \varepsilon_0,$$

where $\varepsilon_0 = \left\{ \left(1 - \widehat{M}\right)^{-1} \frac{Ma^{2\beta-1}}{\Gamma(2\beta)} \exp \left[\left(1 - \widehat{M}\right)^{-1} \frac{Ma^{2\beta}\|\tilde{A}\|}{\Gamma(2\beta)} \right] \right\}^{-1} \varepsilon$. Suppose that z is the mild solution of fractional system (1) under the control $(B(u - v) - F_b^{(1)}\omega_2)$, that is,

$$z(t) = C_\beta(t)\phi(0) + K_\beta(t)x_0 + \sum_{k=1}^m \chi_k(t)[C_\beta(t-t_k)z_k + K_\beta(t-t_k)\tilde{z}_k] + W(B(u - v) - F_b^{(1)}\omega_2)(t) + W(F_b z)(t).$$

Hence, from hypothesis (H5) and Hölder's inequality, it follows that

$$\begin{aligned}
& \|y(t) - z(t)\| \\
= & \|W(Bv - \iota_2)(t)\| + \|W(F_b y)(t) - W(F_b z)(t)\| \\
\leq & \left\| \int_0^t (t-s)^{\beta-1} P_\beta(t-s)(Bv - \iota_2)(s) ds \right\| + \left\| \int_0^t (t-s)^{\beta-1} P_\beta(t-s)(F_b y - F_b z)(s) ds \right\| \\
\leq & \frac{Ma^{2\beta-1}}{\Gamma(2\beta)} \int_0^t \|(Bv - \iota_2)(s)\| ds \\
& + \frac{Ma^{2\beta-1}}{\Gamma(2\beta)} \int_0^t \left[\|\tilde{A}\| \cdot \|y(s-b) - z(s-b)\| + l(s) \left(\|y(s-b) - z(s-b)\| + \int_0^s q(\eta) \|y(\eta-b) - z(\eta-b)\| d\eta \right) \right] ds \\
\leq & \frac{Ma^{2\beta-1}}{\Gamma(2\beta)} \sqrt{a} \|Bv - \iota_2\|_Z + \frac{Ma^{2\beta-1}}{\Gamma(2\beta)} \left(\|\tilde{A}\| \int_0^t \|y(s) - z(s)\| ds + \|l\|_{L^1} (1 + \|q\|_{L^1}) \|y - z\|_{PC} \right) \\
\leq & \frac{Ma^{2\beta-\frac{1}{2}}}{\Gamma(2\beta)} \varepsilon_0 + \frac{Ma^{2\beta-1}}{\Gamma(2\beta)} \|\tilde{A}\| \int_0^t \|y(s) - z(s)\| ds + \frac{Ma^{2\beta-1}}{\Gamma(2\beta)} \|l\|_{L^1} (1 + \|q\|_{L^1}) \|y - z\|_{PC},
\end{aligned}$$

which implies that

$$\begin{aligned}
\|y(a) - z(a)\| & \leq \left[1 - \frac{Ma^{2\beta-1}}{\Gamma(2\beta)} \|l\|_{L^1} (1 + \|q\|_{L^1}) \right]^{-1} \frac{Ma^{2\beta-\frac{1}{2}}}{\Gamma(2\beta)} \varepsilon_0 \\
& + \left[1 - \frac{Ma^{2\beta-1}}{\Gamma(2\beta)} \|l\|_{L^1} (1 + \|q\|_{L^1}) \right]^{-1} \frac{Ma^{2\beta-1} \|\tilde{A}\|}{\Gamma(2\beta)} \int_0^a \|y(s) - z(s)\| ds \\
= & \left(1 - \hat{M} \right)^{-1} \frac{Ma^{2\beta-\frac{1}{2}}}{\Gamma(2\beta)} \varepsilon_0 + \left(1 - \hat{M} \right)^{-1} \frac{Ma^{2\beta-1} \|\tilde{A}\|}{\Gamma(2\beta)} \int_0^a \|y(s) - z(s)\| ds.
\end{aligned}$$

By means of the Grönwall–Bellman inequality, we have

$$\|y(a) - z(a)\| \leq \left(1 - \hat{M} \right)^{-1} \frac{Ma^{2\beta-\frac{1}{2}}}{\Gamma(2\beta)} \varepsilon_0 \cdot \exp \left[\left(1 - \hat{M} \right)^{-1} \frac{Ma^{2\beta} \|\tilde{A}\|}{\Gamma(2\beta)} \right] < \varepsilon.$$

In addition, it is easy to see that

$$\omega_2(0) = 0 = \omega_2(a),$$

and

$$\begin{aligned}
y(0) &= x(0) + \omega_2(0) = x(0) = \phi(0), \\
y(a) &= x(a) + \omega_2(a) = x(a) \in \Lambda_a(0).
\end{aligned}$$

Therefore, for any $\varepsilon > 0$, we obtain

$$\|x(a) - z(a)\| = \|y(a) - z(a)\| < \varepsilon,$$

which indicates that $\Lambda_a(0) \subseteq \overline{\Lambda_a(f)}$. Since $\Lambda_a(0)$ is dense in X (hypothesis (H4) ensures that the system (2) is approximately controllable [32,33]), then we derive that $\overline{\Lambda_a(f)}$ is also dense in X . Consequently, the fractional system (1) is approximately controllable. This ends the proof. $\square$

5. An Example

Consider the following fractional control system with impulse and delay effects

$$\left\{ \begin{array}{l} \partial_t^{\frac{6}{5}} x(t, \zeta) = \partial_{\zeta}^2 x(t, \zeta) + \varrho x(t - \frac{1}{2}, \zeta) + f\left(t, x(t - \frac{1}{2}, \zeta), \int_0^t k(t, s, x(s - \frac{1}{2}, \zeta)) ds\right) + Bu(t, \zeta), \\ t \in (0, 1], t \neq t_1 = \frac{1}{3}; \zeta \in [0, \pi], \\ x(t, 0) = x(t, \pi) = 0, t \in [0, 1], \\ \partial_t x(0, \zeta) = x_0(\zeta), \zeta \in [0, \pi], \\ \Delta x(\frac{1}{3}, \zeta) = z_1(\zeta), \Delta \partial_t x(\frac{1}{3}, \zeta) = \tilde{z}_1(\zeta), \zeta \in [0, \pi], \\ x(t, \zeta) = \phi(t, \zeta), (t, \zeta) \in [-\frac{1}{2}, 0] \times [0, \pi], \end{array} \right. \quad (21)$$

where $\partial_t^{\frac{6}{5}}$ is the Caputo fractional partial derivative with order $\alpha = \frac{6}{5}$, ϱ is a positive constant, and ϕ is a continuous function endowed with some smoothness hypotheses.

Consider $X = L^2[0, \pi]$, and $A := \frac{\partial^2}{\partial x^2}$ with

$$D(A) = \{y \in X : y, \frac{\partial y}{\partial x} \in AC, \frac{\partial^2 y}{\partial x^2} \in X, y(0) = y(\pi) = 0\}.$$

Then, we know that A is the infinitesimal generator of strongly continuous cosine family $\{C(t), t \geq 0\}$, and

$$Ay = - \sum_{n=1}^{\infty} n^2 \langle y, \mu_n \rangle \mu_n, \quad y \in D(A),$$

where $\mu_n(\tau) = \sqrt{\frac{2}{\pi}} \sin(n\tau)$, $\tau \in [0, \pi]$, $n \in \mathbb{N}$. Thus, $\{\mu_n(\tau)\}$ stands for the orthonormal basis of X , and A possesses an eigenvalue denoted as $\lambda_n = -n^2$, and the eigenfunction is μ_n , $n \in \mathbb{N}$. Now, we let

$$U = \left\{ u : u = \sum_{n=2}^{\infty} u_n \mu_n \text{ with } \sum_{n=2}^{\infty} u_n^2 < \infty \right\}$$

provided with norm

$$\|u\|_U = \left(\sum_{n=2}^{\infty} u_n^2 \right)^{\frac{1}{2}}.$$

In addition, define the operator B from U into X as below

$$Bu = 2u_2 \mu_1 + \sum_{n=2}^{\infty} u_n \mu_n, \quad u = \sum_{n=2}^{\infty} u_n \mu_n \in U.$$

Take

$$\alpha = \frac{6}{5}, \quad \beta = \frac{3}{5}, \quad a = 1, \quad b = \frac{1}{2},$$

and

$$x(t)(\zeta) = x(t, \zeta), \quad u(t)(\zeta) = u(t, \zeta), \quad \phi(t)(\zeta) = \phi(t, \zeta), \quad t \in [0, 1], \quad \zeta \in [0, \pi],$$

and the impulsive point $t_1 = \frac{1}{3}$. Now, we can rewrite problem (21) into the following abstract system

$$\begin{cases} {}^C D^\alpha x(t) = Ax(t) + \tilde{A}x(t-b) + Bu(t) + f\left(t, x(t-b), \int_0^t k(t,s,x(s-b))ds\right), & t \in (0,a], t \neq t_1, \\ x(t) = \phi(t), & t \in [-b,0], \\ x'(0) = x_0, \\ \Delta x(t_1) = z_1, \\ \Delta x'(t_1) = \tilde{z}_1. \end{cases}$$

Additionally, impose certain appropriate assumptions on components of the considered system such that the hypotheses (H1)–(H5) hold, and then the fractional control system (21) is approximately controllable from Theorem 1.

Remark 1. (i) In fact, the nonlinear terms f and k are not difficult to verify in Theorem 1. For instance, take $f(t, x, y) = c_1 t(\sin x + \sin y)$ and $k(t, s, x) = \frac{c_2 s}{1+t} \sin x$, where c_i ($i = 1, 2$) is a constant. It is very easy to check that f and k satisfy the required nonlinear growth conditions and Lipschitz conditions. (ii) If $\tilde{A} = 0$, the main techniques of investigating approximate controllability in this paper (Lemma 8 and Theorem 1) degenerate into the cases of the previous literature [31–33,39]. Therefore, the present results can generalize and cover as special cases the method in [31,39] ($\tilde{A} = 0$, $b = 0$) and the method in [32,33] ($\tilde{A} = 0$).

6. Conclusions

Some new results of approximate controllability for the semi-linear fractional impulsive integro-differential evolution equations of order $1 < \alpha < 2$ with delay are derived by using the spatial decomposition techniques and the range condition of control operator B . We improve and generalize the decomposition techniques utilized in some related references [31–33,39]. A new representation of the PC-mild solution for the considered fractional evolution equations of order $\alpha \in (1, 2)$ is also deduced via some characteristic solution operators related to the fractional order $\beta \in (0, 1)$. The main tools used in this work are the theory of cosine families, fixed-point theorems, and the Grönwall–Bellman inequality. An example is also included to explain the validity of the new results.

In future work, we are about to continue our research and extend it to study the approximate controllability of the following fractional delay systems with nonlinear impulsive effects and nonlocal conditions:

$$\begin{cases} {}^C D^\alpha x(t) = Ax(t) + \tilde{A}x(t-b) + Bu(t) + f\left(t, x(t-b), \int_0^t k(t,s,x(s-b))ds\right), & a.e. t \in [0,a], \\ x(t) + g(t, x) = \phi(t), & t \in [-b,0], \\ x'(0) = x_0, \\ \Delta x(t_k) = Z_k(x(t_k)), & k = 1, 2, \dots, m, \\ \Delta x'(t_k) = \tilde{Z}_k(x(t_k)), & i = 1, 2, \dots, m, \end{cases} \quad (22)$$

and its corresponding fractional system without delay

$$\begin{cases} {}^C D^\alpha x(t) = Ax(t) + \tilde{A}x(t) + Bu(t) + f\left(t, x(t), \int_0^t k(t,s,x(s))ds\right), & a.e. t \in [0,a], \\ x(0) + g(0, x) = \phi(0), \\ x'(0) = x_0, \\ \Delta x(t_k) = Z_k(x(t_k)), & k = 1, 2, \dots, m, \\ \Delta x'(t_k) = \tilde{Z}_k(x(t_k)), & i = 1, 2, \dots, m, \end{cases} \quad (23)$$

where the nonlocal term $g(t, x)$ is continuous, and $Z_k, \tilde{Z}_k : X \rightarrow X$ are nonlinear impulsive functions. Some new efforts will be devoted to derive the relationship of approximate controllability between the fractional impulsive system with delay effects (22) and that of fractional impulsive system without delay effects (23) under certain range conditions of control operator B .

Funding: This research was funded by the National Natural Science Foundation of China (61873150) and the Natural Science Foundation of Shandong Province (ZR2020MA007).

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Mao, J.; Zhao, D. Multiple positive solutions for nonlinear fractional differential equations with integral boundary value conditions and a parameter. *J. Funct. Spaces* **2019**, *2019*, 2787569. [\[CrossRef\]](#)
2. Saravanakumar, S.; Balasubramaniam, P. Non-instantaneous impulsive Hilfer fractional stochastic differential equations driven by fractional Brownian motion. *Stoch. Anal. Appl.* **2021**, *39*, 549–566. [\[CrossRef\]](#)
3. Salem, A.; Alharbi, K.N.; Alshehri, H.M. Fractional evolution equations with infinite time delay in abstract phase space. *Mathematics* **2022**, *10*, 1332. [\[CrossRef\]](#)
4. Zhao, D. A study on controllability of a class of impulsive fractional nonlinear evolution equations with delay in Banach spaces. *Fractal Fract.* **2021**, *5*, 279. [\[CrossRef\]](#)
5. Zhao, D.; Liu, Y.; Li, X. Controllability for a class of semilinear fractional evolution systems via resolvent operators. *Commun. Pure Appl. Anal.* **2019**, *18*, 455–478. [\[CrossRef\]](#)
6. Zhao, D.; Liu, Y. Controllability of nonlinear fractional evolution systems in Banach spaces: A survey. *Electron. Res. Arch.* **2021**, *29*, 3551–3580. [\[CrossRef\]](#)
7. Zhou, Y. Infinite interval problems for fractional evolution equations. *Mathematics* **2022**, *10*, 900. [\[CrossRef\]](#)
8. Zhu, B.; Zhu, M. Existence theorems for fractional semilinear integrodifferential equations with noninstantaneous impulses and delay. *J. Funct. Spaces* **2020**, *2020*, 2914269. [\[CrossRef\]](#)
9. Gou, H.; Li, Y. A study on impulsive fractional hybrid evolution equations using sequence method. *Comput. Appl. Math.* **2020**, *39*, 225. [\[CrossRef\]](#)
10. Kumar, V.; Malik, M.; Debbouche, A. Total controllability of neutral fractional differential equation with non-instantaneous impulsive effects. *J. Comput. Appl. Math.* **2021**, *383*, 113158. [\[CrossRef\]](#)
11. You, Z.; Fečkan, M.; Wang, J. Relative controllability of fractional delay differential equations via delayed perturbation of Mittag-Leffler functions. *J. Comput. Appl. Math.* **2020**, *378*, 112939. [\[CrossRef\]](#)
12. Lu, L.; Liu, Z. Existence and controllability results for stochastic fractional evolution hemivariational inequalities. *Appl. Math. Comput.* **2015**, *268*, 1164–1176. [\[CrossRef\]](#)
13. Mahmudov, N.I. Finite-approximate controllability of semilinear fractional stochastic integro-differential equations. *Chaos Solitons Fractals* **2020**, *139*, 110277. [\[CrossRef\]](#)
14. Ravichandran, C.; Valliammal, N.; Nieto, J.J. New results on exact controllability of a class of fractional neutral integro-differential systems with state-dependent delay in Banach spaces. *J. Frankl. Inst.* **2019**, *356*, 1535–1565. [\[CrossRef\]](#)
15. Zhao, D.; Liu, Y. New discussion on approximate controllability for semilinear fractional evolution systems with finite delay effects in Banach spaces via differentiable resolvent operators. *Fractal Fract.* **2022**, *6*, 424. [\[CrossRef\]](#)
16. Kumar, V.; Malik, M.; Debbouche, A. Stability and controllability analysis of fractional damped differential system with non-instantaneous impulses. *Appl. Math. Comput.* **2021**, *391*, 125633. [\[CrossRef\]](#)
17. Zhao, D.; Liu, Y.; Li, H. Fast-time complete controllability of nonlinear fractional delay integrodifferential evolution equations with nonlocal conditions and a parameter. *Math. Methods Appl. Sci.* **2022**, *45*, 5649–5669. [\[CrossRef\]](#)
18. Gu, H.; Sun, Y. Nonlocal controllability of fractional measure evolution equation. *J. Inequalities Appl.* **2020**, *2020*, 60. [\[CrossRef\]](#)
19. Gou, H.; Li, Y. A study on controllability of impulsive fractional evolution equations via resolvent operators. *Bound. Value Probl.* **2021**, *2021*, 25. [\[CrossRef\]](#)
20. Kavitha, K.; Vijayakumar, V.; Udhayakumar, R. Results on controllability of Hilfer fractional neutral differential equations with infinite delay via measures of noncompactness. *Chaos Solitons Fractals* **2020**, *139*, 110035. [\[CrossRef\]](#)
21. Yi, Y.; Chen, D.; Xie, Q. Controllability of nonlinear fractional order integrodifferential system with input delay. *Math. Methods Appl. Sci.* **2019**, *42*, 3799–3817. [\[CrossRef\]](#)
22. Hernandez, M.E.; O'Regan, D. Controllability of Volterra-Fredholm type systems in Banach spaces. *J. Frankl. Inst.* **2009**, *346*, 95–101. [\[CrossRef\]](#)
23. Guendouzi, T.; Farahi, S. Approximate controllability of semilinear fractional stochastic dynamic systems with nonlocal conditions in Hilbert spaces. *Mediterr. J. Math.* **2016**, *13*, 637–656. [\[CrossRef\]](#)
24. Shukla, A.; Patel, R. Controllability results for fractional semilinear delay control systems. *J. Appl. Math. Comput.* **2021**, *65*, 861–875. [\[CrossRef\]](#)
25. Shukla, A.; Sukavanam, N.; Pandey, D.N. Approximate controllability of semilinear fractional control systems of order $\alpha \in (1, 2]$ with infinite delay. *Mediterr. J. Math.* **2016**, *13*, 2539–2550. [\[CrossRef\]](#)
26. Dineshkumar, C.; Udhayakumar, R.; Vijayakumar, V.; Nisar, K.S. A discussion on the approximate controllability of Hilfer fractional neutral stochastic integro-differential systems. *Chaos Solitons Fractals* **2021**, *142*, 110472. [\[CrossRef\]](#)
27. Mahmudov, N.I. Partial-approximate controllability of nonlocal fractional evolution equations via approximating method. *Appl. Math. Comput.* **2018**, *334*, 227–238. [\[CrossRef\]](#)

28. Raja, M.M.; Vijayakumar, V.; Udhayakumar, R.; Zhou, Y. A new approach on the approximate controllability of fractional differential evolution equations of order $1 < r < 2$ in Hilbert spaces. *Chaos Solitons Fractals* **2020**, *141*, 110310.
29. Raja, M.M.; Vijayakumar, V.; Udhayakumar, R. A new approach on approximate controllability of fractional evolution inclusions of order $1 < r < 2$ with infinite delay. *Chaos Solitons Fractals* **2020**, *141*, 110343.
30. Sakthivel, R.; Ganesh, R.; Ren, Y.; Anthoni, S.M. Approximate controllability of nonlinear fractional dynamical systems. *Commun. Nonlinear Sci. Numer. Simul.* **2013**, *18*, 3498–3508. [[CrossRef](#)]
31. Naito, K. Controllability of semilinear control systems dominated by the linear part. *SIAM J. Control. Optim.* **1987**, *25*, 715–722. [[CrossRef](#)]
32. Kumar, S.; Sukavanam, N. Approximate controllability of fractional order semilinear systems with bounded delay. *J. Differ. Equ.* **2012**, *252*, 6163–6174. [[CrossRef](#)]
33. Sukavanam, N.; Tafesse, S. Approximate controllability of a delayed semilinear control system with growing nonlinear term. *Nonlinear Anal.* **2011**, *74*, 6868–6875. [[CrossRef](#)]
34. Shukla, A.; Vijayakumar, V.; Nisar, K.S. A new exploration on the existence and approximate controllability for fractional semilinear impulsive control systems of order $r \in (1, 2)$. *Chaos Solitons Fractals* **2022**, *154*, 111615. [[CrossRef](#)]
35. Podlubny, I. *Fractional Differential Equations, Mathematics in Science and Engineering*; Academic Press: New York, NY, USA, 1999.
36. Travis, C.C.; Webb, G.F. Cosine families and abstract nonlinear second order differential equations. *Acta Math. Hung.* **1978**, *32*, 75–96. [[CrossRef](#)]
37. Zhou, Y.; He, J. New results on controllability of fractional evolution systems with order $\alpha \in (1, 2)$. *Evol. Equ. Control. Theory* **2021**, *10*, 491–509. [[CrossRef](#)]
38. Wang, J.; Li, X.; Wei, W. On the natural solution of an impulsive fractional differential equation of order $q \in (1, 2)$. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 4384–4394. [[CrossRef](#)]
39. Sukavanam, N. Approximate controllability of semilinear control system with growing nonlinearity. In *Mathematical Theory of Control, Proceedings of the International Conference (Bombay, 1990)*; Marcel Dekker: New York, NY, USA, 1993; pp. 353–357.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.