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Some Results on Third-Order Differential Subordination and Differential Superordination for Analytic Functions Using a Fractional Differential Operator

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Abstract: In this study, we explore the implications of a third-order differential subordination in the context of analytic functions associated with fractional differential operators. Our investigation involves the consideration of specific admissible classes of third-order differential functions. We also extend this exploration to establish a dual principle, resulting in a sandwich-type outcome. We introduce these admissible function classes by employing the fractional derivative operator $\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z)$ and derive conditions on the normalized analytic function f that lead to sandwich-type subordination in combination with an appropriate fractional differential operator.

Keywords: analytic functions; differential subordination; differential superordination; best dominant; best subordinate; fractional derivative; 30C45

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1. Introduction

Complex numbers have been provided in the past for solving real cubic equations, which paved the way for the emergence of an interesting theory called the theory of functions of a complex variable (complex analysis). The history of this discipline goes back to the 17th century. Pivotal scientists include Riemann, Gauss, Euler, Cauchy, Mittag–Leffler, and others. In the 19th century, Riemann proposed the delightful Riemann mapping theorem in 1851, which led to the birth of a significant and attractive theory, the Geometric Function Theory (GFT) [1]. It has developed remarkably and has implementations in operator theory, differential inequality theory, and other scientific fields. In 1907, Koebe [1] employed a univalent (one-to-one) function defined on an open unit disk to refine the Riemann mapping theorem. Later, Lindeöf [1] presented the subordinate concept in 1909. This idea is utilized to analyze two complex functions in terms of the Schwarz function. In this regard, differential subordination theory on a complex domain is the generalization of differential inequality theory on a real domain, which is a substantial theme that Miller and Mocanu excitingly discussed in 1978 [2], 1981 [3], and 2000 [4]. In 2003, Miller and Mocanu [5] developed the dual idea of differential subordination theory, namely differential superordination. Actually, differential subordination and superordination are essential tools in GFT utilized in investigations to acquire sandwich outcomes. In 2011, Antonino and Miller [6] extended the differential subordination formula of the second order to the third order. In 2014, Tang et al. [7] presented the dual idea of third-order differential superordination by utilizing admissible functions. They also investigated third-order differential subordination and superordination outcomes for meromorphically p-valent functions involving a complex operator. In addition, sandwich outcomes were also yielded.

This theory is very significant, and several complex analysts have contributed outstandingly to investigating the connected problems, such as Ibrahim et al. [8], Srivastava et al. [9], Tang et al. [10], Ghanim et al. [11,12], Al-Janaby et al. [13], Lupas and Oros [14], Morais and Zayed [15], Attiya et al. [16], Ghanim et al. [17], Mishra and Gochhayat [18], and others.

In 2015, Ibrahim et al. [8] defined a new operator by using a convolution tool between a fractional integral operator and the Carlson Shaffer operator to study the properties of the subordination and superordination. In 2021, Morais and Zayed [15] extended a fractional derivative operator for higher-order derivatives for certain analytic multivalent functions. In 2021, Lupas and Oros [14] studied subordination and subordination properties using the fractional integral of the confluent Hypergeometric function. In 2022, some other researchers worked on some subordination and subordination properties [19–21].

The fractional integral operator is a mathematical concept used in various fields of science and engineering. It has applications in several domains. The integration of a fractional calculus into physical models has been accomplished in recent decades. The use of the generalized Mittag–Leffler function has also been employed in both mathematical and physical contexts since it naturally represents solutions to fractional integral and differential equations. A fractional order calculus is often used in several practical applications, for example [22–26]. This work makes a contribution to the field of mathematical applications by using fractional operators in the resolution of differential equations. Additionally, it highlights the significance of these operators in the domains of physics and engineering, especially for the development of geometric function theory, a specialized area within complex analysis.

The approach of subordination is applied to relevant types of admissible functions. Antonino and Miller [6] define the admissible functions as follows:

Let $\mathcal{H}(\mathcal{U})$ be the class of functions which are regular in the open unit disk, $\mathcal{U} = \{z : z \in C$ and $|z| < 1\}$, whenever $a \in C$ and $n \in N$ (for C is the complex plane and N is a positive integer), and let

$$\mathcal{H}[a,\ n] = \{\vartheta \in \mathcal{H}(\mathcal{U}) : \vartheta\ (z) = \mu + \mu_\kappa\ z^\kappa + \mu_{\kappa+1}\ z^{\kappa+1} + \ldots\},$$

which is called the class of the regular functions in the unit disk $\mathcal{U}$ and suppose that $\mathcal{H}_0 = \mathcal{H}[0,\ 1]$ and $\mathcal{H}_1 = \mathcal{H}[1,\ 1]$.

Also, let $\mathcal{A}$ denote the class of analytic functions in $\mathcal{U}$ and have the normalized form

$$\vartheta\ (z) = z + \sum_{\kappa=2}^{\infty} \mu_\kappa\ z^\kappa,\ \ (z \in\ \mathcal{U}\ ) \tag{1}$$

For two functions ϑ_1 and ϑ_2 analytic in $\mathcal{U}$, the function ϑ_1 is subordinate to ϑ_2, written as

$$\vartheta_1 <\ \vartheta_2 \text{ in } \mathcal{U} \text{ or } \vartheta_1\ (z)\ <\ \vartheta_2(z)\ (z \in\ \mathcal{U}\ )$$

if there exists a function w analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying

$$\vartheta_1(z) = \vartheta_2(w(z)), \text{ for } (z \in \mathcal{U}\ ).$$

In particular, if the function ϑ_2 is univalent in $\mathcal{U}$, then ϑ_1 (z) < ϑ_2(z) (z ∈ $\mathcal{U}$) is equivalent to ϑ_1 (0) = ϑ_2(0) and ϑ_1 ($\mathcal{U}$) ⊂ ϑ_2($\mathcal{U}$).

Definition 1 ([6]). *Let $\mathcal{T}$: $C^4 \times \mathcal{U}\ \rightarrow\ C$ and suppose that the function $\hbar(z)$ is univalent in $\mathcal{U}$. If the function $p\ (z)$ is analytic in $\mathcal{U}$ and satisfies the following third-order differential subordination:*

$$\mathcal{T}\Big(p(z),\ zp'(z),\ z^2\omega p''(z),\ z^3 p'''(z); z\Big) <\ \hbar(z), \tag{2}$$

then $p(z)$ is called a solution of the differential subordination (2).

In addition, a given univalent function $q(z)$ is referred to as a dominant of the solutions of the differential subordination (2); or, to put it another way, it is a dominant if $p(z) < q(z)$

for every $p(z)$ satisfying (2). A dominant $\widetilde{q}(z)$ that satisfies $\widetilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (2) is said to be the best dominant.

Definition 2 ([7]). *Let $\mathcal{T}$: $C^4 \times \mathcal{U} \to C$ and let the function $\mathfrak{k}(z)$ be univalent in $\mathcal{U}$. If the functions $p(z)$ and $\mathcal{T}\left(p(z),\ zp'(z),\ z^2p''(z),\ z^3p'''(z); z\right)$, are univalent in $\mathcal{U}$ and satisfy the following third-order differential superordination*

$$q(z) \prec \mathcal{T}\Big(p(z),\ zp'(z),\ z^2p''(z),\ z^3p'''(z); z\Big), \tag{3}$$

then $p(z)$ is referred to as a differential superordination solution according to (3). An analytic function $q(z)$ is defined as a subordinant of the solutions of the differential superordination provided by (3) (or, in simpler terms, a subordinant) if $q(z) \prec p(z)$ for all $p(z)$ satisfies (3).

The best subordinate of the differential superordination provided by (3) is a univalent subordinate $\widetilde{q}(z)$ that fulfills q $(z) \prec \widetilde{q}(z)$ for all subordinates $q(z)$ of (3).

Definition 3 ([27]). *The fractional derivative of order α is defined for function f by*

$$\mathcal{D}_z^{\alpha}\vartheta(z) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_0^z \frac{\vartheta(\zeta)}{(z-\zeta)^{\alpha}}d\zeta,\ 0 \le \alpha < 1. \tag{4}$$

Let ϑ be a regular function in a simply connected region of the complex z-plane C involving the origin and the multiplicity of $(z-\tau)^{-\alpha}$ is extracted by demanding $\log(z-\tau)$ to be real when $(z-\zeta) > 0,\ \alpha > 0$.

$$\mathcal{D}_z^{\alpha} z^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma\gamma-\alpha+1}\, z^{\gamma-\alpha},\ \gamma > -1,\ 0 \le \alpha < 1$$

$$\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z) = \frac{z^{\alpha+1}}{\Gamma 2+\alpha}\mathcal{N}_{\rho,\,\sigma}(z) = \frac{1}{\Gamma 2+\alpha}\sum_{\mathfrak{n}=0}^{\infty}\frac{\Gamma(\rho+\sigma\mathfrak{n})}{\Gamma(\rho)\,\Gamma(\mathfrak{n}+1)}\, z^{\mathfrak{n}+\alpha+1}$$

$$\mathcal{N}_{p,\,\sigma}(z) = \sum_{\mathfrak{n}=0}^{\infty}\frac{\Gamma(\rho+\sigma\,\kappa)}{\Gamma(\rho)\,\Gamma(\kappa+1)}\, z^{\kappa},\ (z,\rho\ \in C;\ \ 0 < \mathcal{R}(\sigma)\ ). \tag{5}$$

Then, by using $\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z)$, we consider a new complex linear operator $\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z)$: $\mathcal{A} \to \mathcal{A}$ which has the following convolution definition:

$$\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}} = \mathcal{D}_z^{\alpha} z^{\gamma} * \mathfrak{S}_{\mathcal{N},\mathcal{S}} = \mathrm{z} + \sum\nolimits_{\kappa=2}^{\infty}\frac{\Gamma(\rho+\sigma(\ \kappa-1)\Gamma(\kappa+\alpha+1)}{\Gamma(2+\alpha)\Gamma(\rho)\Gamma n\ \Gamma(\kappa+1)}\, z^{\kappa} \tag{6}$$

$$\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z) = \mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}} * \vartheta(z) = \mathrm{z} + \sum\nolimits_{\kappa=2}^{\infty}\frac{\Gamma(\rho+\sigma(\ \kappa-1)\Gamma(\kappa+\alpha+1)}{\Gamma(2+\alpha)\Gamma(\rho)\Gamma\kappa\ \Gamma(\kappa+1)} a_{\kappa}\, z^{\kappa}$$

$$z\big(\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z)\big)' = \frac{\mathcal{N}}{\partial}\,\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z) - \left(\frac{\mathcal{N}}{\partial} - 1\right)\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z). \tag{7}$$

The following specific cases related to the operator $\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\partial}\vartheta(z)$ are also introduced by assumption values of the parameters. We obtain

1. $\mathcal{D}_z^{1}\mathcal{M}_{1,1}\vartheta(z) =\ \mathrm{z} + \sum_{\kappa=2}^{\infty}\frac{(\kappa+1)}{2}\mu_{\kappa}\, z^{\kappa}$;
2. $\mathcal{D}_z^{1}\mathcal{M}_{2,1}\vartheta(z) =\ \mathrm{z} + \sum_{\kappa=2}^{\infty}\frac{\kappa(\ \kappa+1)}{2}\mu_{\kappa}\, z^{\kappa}$.

Definition 4 ([6]). *Denote by Q the set of all functions q that are analytic and injective on $\overline{\mathcal{U}} \setminus E(q)$, where $\overline{\mathcal{U}} = \mathcal{U}\ \bigcup\{z\ \in\ \partial\mathcal{U}\}$ and*

$$E(q) = \left\{\zeta \in \partial\mathcal{U} : \lim_{n\to\infty} q(z) = \infty\right\} \tag{8}$$

are such that $f'(\zeta) \neq 0$ *for* $\zeta \in \frac{\partial \mathcal{U}}{E(q)}$. *Further, let the subclass of Q for which* $q(0) = a$ *be denoted by* (a),

$$Q(0) = Q_0 \text{ and } Q(1) = Q_1 \tag{9}$$

Definition 5 ([6]). *Let* $\mho$ *be a set in* C, $q \in \mathcal{H}\ [a,\ n]$ *with* $q\prime(z) \neq 0$ *and* $n \in N\backslash\{1\}$. *The class of admissible functions* $\mathcal{T}_n[\mho, q]$ *consists of those functions* $\mathcal{T} : C^4 \times \mathcal{U} \to C$ *that satisfy the following admissibility condition:*

$$\mathcal{T}(\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4; z) \notin \mho,$$

whenever

$$\lambda_1 = q(\zeta), \lambda_2 = kzq'(\zeta),$$

$$\mathcal{R}e\left\{\frac{\lambda_3}{\lambda_2} + 1\right\} \geq k\mathcal{R}e\left\{1 + \frac{zq''(\zeta)}{q'(\zeta)}\right\},$$

and

$$\mathcal{R}e\left\{\frac{\lambda_4}{\lambda_2}\right\} \leq k^2\mathcal{R}e\left\{\frac{z^2 q'''(\zeta)}{q'(\zeta)}\right\}$$

where $z \in \mathcal{U}$, $\zeta \in \partial\, \mathcal{U}$*-E(q) and* $k \geq n$.

The next lemma is the foundation result in the theory of third-order differential subordination.

Lemma 1 ([6]). $p \in \mathcal{H}[a,\ n]$ *with* $n \geq 2$. *Also, let* $q \in Q(a)$ *and satisfy the following conditions:*

$$\mathcal{R}e\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}\right\} \geq 0, \quad \left|\frac{zq'(z)}{q'(\zeta)}\right| \leq k,$$

where $z \in \mathcal{U}, \zeta \in \partial U \backslash E(q)$ *and* $k \geq n$. *If* $\mho$ *is a set in* C, $\mathcal{T} \in \mathcal{T}_n[\mho, q]$ *and*

$$\mathcal{T}\Big(p(z),\ zp'(z),\ z^2p''(z),\ z^3p'''(z); z\Big) \in \mho;$$

then

$$p(z) < q(z)\ (z \in \mathcal{U}).$$

Definition 6 ([7]). *Let* $\mho$ *be a set in* C, $q \in \mathcal{H}\ [a,\ n]$ *with* $q\prime(z) \neq 0$ *and* $n \in N\backslash\{1\}$. *The class of admissible functions* $\mathcal{T}_n[\Omega, q]$ *consists of those functions* $\mathcal{T} : C^4 \times \overline{\overline{\mathcal{U}}} \to C$ *that satisfy the following admissibility condition,*

$$\mathcal{T}(\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4; z) \in \mho,$$

whenever

$$\lambda_1 = q(z), \lambda_2 = \frac{zq'(z)}{m}, \mathcal{R}e\left(\frac{\lambda_3}{\lambda_2} + 1\right) \leq \frac{1}{m}\mathcal{R}e\left\{1 + \frac{zq''(z)}{q'(z)}\right\}$$

and

$$\mathcal{R}e\left(\frac{\lambda_4}{\lambda_2}\right) \leq \frac{1}{m^2}\mathcal{R}e\left\{\frac{z^2q'''(z)}{q'(z)}\right\},$$

and when $z \in \mathcal{U}$, $\zeta \in \partial\mathcal{U}$ *and* $m \geq n$.

Lemma 2 ([7]). *Let* $\mathcal{T} \in \mathcal{T}_n'[\mho, q]$. *If the function*

$$\mathcal{T}\Big(p(z),\ zp'(z),\ z^2p''(z),\ z^3p'''(z); z\Big)$$

is univalent in $\mathcal{U}$, $p \in Q(a)$ and $p \in \mathcal{H}[a, n]$, satisfy the following conditions:

$$\mathcal{R}e\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}\right\} \geq 0, \quad \left|\frac{zp'(z)}{q'(\zeta)}\right| \leq m,$$

where $z \in \mathcal{U}$, $\zeta \in \partial\mathcal{U}$ and $m \geq n \geq 2$; then

$$\mho \subset \left\{\mathcal{T}\left(p(z),\ zp'(z),\ z^2p''(z),\ z^3p'''(z); z\right) : \mathrm{z} \in \mathcal{U}\right\},$$

implies that

$$q(z) < p(z),\ (z \in \mathcal{U}).$$

In this investigation, the fractional calculus concept for complex numbers is utilized, and a new complex fractional operator of normalized analytic function is stated. Then, differential subordination theory is employed to admissible functions in order to examine the condition of the sandwich-type complex of the following form holds:

$$\hbar 1(z) < \mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z) < q_2(z)(z \in \mathcal{U}),$$

where q_1, q_2 are univalent in $\mathcal{U}$ and $\varnothing$ is a suitable operator.

2. Results Related to the Third-Order Subordination

In this part, we start by considering a given set $\mho$ and a given function q and proceed to ascertain a set of acceptable operators $\mathcal{T}$ so that the condition expressed in Equation (2) is satisfied. In order to establish the fundamental third-order differential subordination theorems for the operator $\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z)$ described using Equation (5), we develop a new class of admissible functions. These functions will play an important role in the proof.

Definition 7. *Let $\mho$ be a set in C, $q \in \mathcal{H}\ [a,\ n]$ with $q\prime(z) \neq 0$ and $n \in Q_0 \cap \mathcal{H}_0$. The class of admissible functions $\phi_j[\mho, q]$ consists of those functions $\varphi : C^4 \times \mathcal{U} \to C$ that satisfy the following admissibility condition:*

$$\varphi(y_1, ,\ y_2,\ y_3,\ y_4; z) \notin \mho;$$

whenever $y_1 = q(\zeta)$, $y_2 = z\frac{k\zeta q'(\zeta) + \left(\frac{\mathcal{N}}{\mathcal{S}} - 1\right)q(z)}{\frac{\mathcal{N}}{\mathcal{S}}}$,

$$Re\left(\frac{y_3\left(\frac{\mathcal{N}(\mathcal{N}+1)}{\mathcal{S}}\right) - \frac{(\mathcal{N}-\mathcal{S})\ (\mathcal{N}+1-\mathcal{S})}{\mathcal{S}^2}y_1}{y_2\left(\frac{\mathcal{N}}{\mathcal{S}}\right) - \left(\frac{\mathcal{N}}{\mathcal{S}} - 1\right)y_1} - \left(\frac{2\mathcal{N}+1}{\mathcal{S}} - 2\right)\right) \geq k\mathcal{R}e\left\{1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right\},$$

$$Re\left\{\begin{array}{c}\dfrac{\frac{\mathcal{N}(\mathcal{N}+1)\ (\mathcal{N}+2)}{\mathcal{S}^2} - 3y_3\ \left(\frac{(\mathcal{N}+1)}{\mathcal{S}}\right)\left(\frac{\mathcal{N}(\mathcal{N}+1)}{\mathcal{S}}\right) + \left[\frac{(3\mathcal{N}+3\ )(\mathcal{N}-\mathcal{S})(\mathcal{N}+1-\mathcal{S}) - (\mathcal{N}-\mathcal{S})(\mathcal{N}+1-\mathcal{S})(\mathcal{N}+2-\mathcal{S})}{\mathcal{S}^3}\right]y_1}{y_2\left(\frac{\mathcal{N}}{\mathcal{S}}\right) - \left(\frac{\mathcal{N}}{\mathcal{S}} - 1\right)y_1} \\ + \left[\frac{(3\mathcal{N}+3\ )(2\mathcal{N}+1-\mathcal{S}) - (\mathcal{N}-\mathcal{S})(\mathcal{N}+1-\mathcal{S})(\mathcal{N}+2-\mathcal{S})}{\mathcal{S}^3}\right]\end{array}\right\}$$

$$\leq k^2\mathcal{R}e\left\{\frac{z^2q'''(\zeta)}{q'(\zeta)}\right\}$$

whenever $\mathrm{z} \in \mathcal{U}$, $\zeta \in \mathcal{U}\partial$-E$(q)$ and $k \geq 2$.

Theorem 1. *Let $\varphi \in \phi_\iota[\mho, q]$. If $\vartheta \in \mathcal{A}$ and $q \in Q_0$ attain the following situations:*

$$\mathcal{R}e\left(\frac{\varsigma q''(\varsigma)}{q'(\varsigma)}\right) \geq 0,\ \left|\frac{\mathcal{D}_z^{\alpha}\mathcal{M}_{\mathcal{N},\mathcal{S}}\vartheta(z)}{q'(\varsigma)}\right| \leq\ k\ , \tag{10}$$

$$\left\{\varphi\left(\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z),\ \mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z),\ \mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+2,\mathcal{S}}\vartheta(z),\ \mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+3,\mathcal{S}}\vartheta(z); z \in \mathcal{U}\right)\right\} \subset \mho; \tag{11}$$

then

$$\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z) \prec \hbar(z).$$

Proof. Consider an analytic function $p(z)$ in $\mathcal{U}$ as:

$$p(z) = \mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z). \tag{12}$$

It follows from (7) and (12) that

$$\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z) = \frac{zp'(z) + \left(\frac{\mathcal{N}}{\mathcal{S}} - 1\right) p(z)}{\frac{\mathcal{N}}{\mathcal{S}}}. \tag{13}$$

A comparable argument is obtained as follows:

$$\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+2,\mathcal{S}}\vartheta(z) = \frac{z^2p''(z) + \left(\frac{2\mathcal{N}+1}{\mathcal{S}} - 1\right)zp'(z) + \frac{(\mathcal{N}-\mathcal{S})\ (\mathcal{N}+1-\mathcal{S})}{\mathcal{S}^2}p(z)}{\frac{\mathcal{N}\ (\mathcal{N}+1)}{\mathcal{S}^2}}, \tag{14}$$

and

$$\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+3,\mathcal{S}}\vartheta(z) =$$

$$\frac{z^3p'''(z) + 3\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)z^2p''(z) + \left(\frac{(\mathcal{N}-\mathcal{S})(\mathcal{N}+1-\mathcal{S})((\mathcal{N}+2)(2\mathcal{N}+1-\mathcal{S}))}{\mathcal{S}^2}\right)zp'(z) + \frac{(\mathcal{N}-\mathcal{S})(\mathcal{N}+1-\mathcal{S})(\mathcal{N}+2-\mathcal{S})}{\mathcal{S}^3}p(z)}{\frac{\mathcal{N}(\mathcal{N}+1)(\mathcal{N}+2)}{\mathcal{S}^3}}. \tag{15}$$

Now, we will define a transformation from $\mathbb{C}^4$ to $\mathbb{C}$ using

$$y_1 = \lambda_1, y_2 = \frac{\lambda_2 + \left(\frac{\mathcal{N}}{\mathcal{S}} - 1\right)\lambda_1}{\frac{\mathcal{N}}{\mathcal{S}}}, y_3 = \frac{\lambda_3 + \left(\frac{2\mathcal{N}+1}{\mathcal{S}} - 1\right)\lambda_2 + \frac{(\mathcal{N}-\mathcal{S})\ (\mathcal{N}+1-\mathcal{S})}{\mathcal{S}^2}\lambda_1}{\frac{\mathcal{N}(\mathcal{N}+1)}{\mathcal{S}^2}}, \tag{16}$$

and

$$y_4 = \frac{\lambda_4 + 3\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)\lambda_3 + \left(\frac{(\mathcal{N}-\mathcal{S})(\mathcal{N}+1-\mathcal{S})((\mathcal{N}+2)(2\mathcal{N}+1-\mathcal{S}))}{\mathcal{S}^2}\right)\lambda_2 + \frac{(\mathcal{N}-\mathcal{S})(\mathcal{N}+1-\mathcal{S})(\mathcal{N}+2-\mathcal{S})}{\mathcal{S}^3}\lambda_1}{\frac{\mathcal{N}(\mathcal{N}+1)(\mathcal{N}+2)}{\mathcal{S}^3}}. \tag{17}$$

Let $\mathcal{T}(\lambda_1, \lambda_2, \lambda_3, \lambda_4; z) = \varphi(y_1, y_2, y_3, y_4; z)$

$$= \varphi\left(\begin{array}{c} \lambda_1,\ \frac{\lambda_2 + \left(\frac{\mathcal{N}}{\mathcal{S}} - 1\right)\lambda_1}{\frac{\mathcal{N}}{\mathcal{S}}},\ \frac{\lambda_3 + \left(\frac{2\mathcal{N}+1}{\mathcal{S}} - 1\right)\lambda_2 + \frac{(\mathcal{N}-\mathcal{S})\ (\mathcal{N}+1-\mathcal{S})}{\mathcal{S}^2}\lambda_1}{\frac{\mathcal{N}(\mathcal{N}+1)}{\mathcal{S}^2}}, \\ \frac{\lambda_4 + 3\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)\lambda_3 + \left(\frac{(\mathcal{N}-\mathcal{S})(\mathcal{N}+1-\mathcal{S})((\mathcal{N}+2)(2\mathcal{N}+1-\mathcal{S}))}{\mathcal{S}^2}\right)\lambda_2 + \frac{(\mathcal{N}-\mathcal{S})(\mathcal{N}+1-\mathcal{S})(\mathcal{N}+2-\mathcal{S})}{\mathcal{S}^3}\lambda_1}{\frac{\mathcal{N}(\mathcal{N}+1)(\mathcal{N}+2)}{\mathcal{S}^3}}; z \end{array}\right) \tag{18}$$

Employing (12) to (15), we yield

$$\mathcal{T}\left(p(z),\ zp'(z),\ z^2p''(z),\ z^3p'''(z); z\right) =$$

$$\varphi\left(\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z),\ \mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z),\ \mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+2,\mathcal{S}}\vartheta(z),\ \mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+3,\mathcal{S}}\vartheta(z); z\right). \tag{19}$$

Therefore, evidently, (11) is given as follows:

$$\mathcal{T}\left(p(z),\ zp'(z),\ z^2p''(z),\ z^3p'''(z); z\right) \in \mho$$

$$\frac{\lambda_3}{\lambda_2} + 1 = \left(\frac{y_3 \left(\frac{\mathcal{N}(\mathcal{N}+1)}{\mathcal{S}} \right) - \frac{(\mathcal{N}-\partial)\ (\mathcal{N}+1-\partial)}{\mathcal{S}^2} y_1}{y_2 \left(\frac{\mathcal{N}}{\mathcal{S}} \right) - \left(\frac{\mathcal{N}}{\mathcal{S}} - 1 \right) y_1} - \left(\frac{2\mathcal{N}+1}{\mathcal{S}} - 2 \right) \right),$$

and

$$\frac{\lambda_4}{\lambda_2} = \frac{\frac{\mathcal{N}(\mathcal{N}+1)\ (\mathcal{N}+2)}{\mathcal{S}^2} - 3y_3 \left(\frac{(\mathcal{N}+1)}{\mathcal{S}} \right) \left(\frac{\mathcal{N}(\mathcal{N}+1)}{\mathcal{S}} \right) + \left[\frac{(3\mathcal{N}+3\ )(\mathcal{N}-\mathcal{S})(\mathcal{N}+1-\mathcal{S}) - (\mathcal{N}-\mathcal{S})(\mathcal{N}+1-\mathcal{S})(\mathcal{N}+2-\mathcal{S})}{\mathcal{S}^3} \right] y_1}{y_2 \left(\frac{\mathcal{N}}{\mathcal{S}} \right) - \left(\frac{\mathcal{N}}{\mathcal{S}} - 1 \right) y_1} + \left[\frac{(3\mathcal{N}+3\ )(2\mathcal{N}+1-\mathcal{S}) - (\mathcal{N}-\mathcal{S})(\mathcal{N}+1-\mathcal{S})(\mathcal{N}+2-\mathcal{S})}{\mathcal{S}^2} \right]$$

Hence, evidently, the admissibility situation for $\varphi \in \phi_\iota[\mho, q]$ in Definition 6 is equivalent to an admissibility situation for $\mathcal{T} \in \mathcal{T}_2[\mathcal{D}, \hbar]$. Thus, by means of Definition 4 and Lemma 1 with n = 2, and by using (10), we acquire

$$\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N},\mathcal{S}} \vartheta(z) < q(z).$$

This completes the theorem.

The next outcome is an expansion of Theorem 1 to the state when the behavior of $q(z)$ is on $\partial\mathcal{U}$. □

Corollary 1. *Let $\mho \subset C$ and $q(z)$ be univalent in $\mathcal{U}$, $q(0) = 0$. Let $\varphi \in \phi_\iota[\mho,\ q_\sigma]$ for some $\sigma \in (0,\ 1)$ where $q_\sigma(z) = q(\sigma z)$. If $\varphi \in \mathcal{A}$ attains*

$$\mathcal{R}e\left(\frac{\varsigma q_\sigma''(\varsigma)}{q_\sigma'(\varsigma)} \right) \geq 0,\ \left| \frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+1,\mathcal{S}} \vartheta(z)}{q_\sigma'(\varsigma)} \right| \leq k$$

and

$$\varphi\left(\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N},\mathcal{S}} \vartheta(z),\ \mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+1,\mathcal{S}} \vartheta(z),\ \mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+2,\mathcal{S}} \vartheta(z),\ \mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+3,\mathcal{S}} \vartheta(z); z \right) \in \mho,$$

then

$$\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N},\mathcal{S}} \vartheta(z) < q(z), \text{ where z} \in \mathcal{U}.$$

Proof. Theorem 1 leads to

$$\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N},\mathcal{S}} \vartheta(z) < q_\sigma(z).$$

Therefore

$$q_\sigma(z) < q(z)\ (\text{z} \in \mathcal{U}).$$

This completes the proof of Corollary 1.

If $\mho \neq C$ is a simply connected domain, then $\mho = \hbar(\mathcal{U})$ for some conformal mapping (z) of $\mathcal{U}$ onto $\mho$. In this case, the class $\phi_j[\hbar(\mathcal{U}), q]$ is written as $\phi_j[\hbar, q]$. The next result is an immediate implication of Theorem 1. □

Theorem 2. *Let $\varphi \in \phi_j[\mho, q]$. If $\vartheta \in \mathcal{A}$ and $q \in Q_0$ attain the situations, then*

$$\mathcal{R}e\left(\frac{\varsigma q_\sigma''(\varsigma)}{q'(\varsigma)} \right) \geq 0,\ \left| \frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+1,\mathcal{S}} \vartheta(z)}{q'(\varsigma)} \right| \leq k. \tag{20}$$

Also, if

$$\varphi\left(\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N},\mathcal{S}} \vartheta(z),\ \mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+1,\mathcal{S}} \vartheta(z),\ \mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+2,\mathcal{S}} \vartheta(z),\ \mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+3,\mathcal{S}} \vartheta(z); z \right) < (z), \tag{21}$$

then $\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N},\mathcal{S}} \vartheta(z) < q(z)\ (z \in \mathcal{U})$.

The conclusion presented is a direct outcome of Corollary 1.

Corollary 2. *Let $\mho \subset C$ and $q(z)$ be univalent in $\mathcal{U}$, $q(0) = 0$, and let $\varphi \in \phi_j[\hbar, q_\sigma]$ for some $\sigma \in (0,\, 1)$ where $q_\sigma(z) = q(\sigma z)$. If $\vartheta \in \mathcal{A}$ attains*

$$\mathcal{R}e\left(\frac{\varsigma q_\sigma{}''(\varsigma)}{q_\sigma{}'(\varsigma)}\right) \geq 0,\ \left|\frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z)}{q_\sigma{}'(\varsigma)}\right| \leq k,\ \text{where } z \in \mathcal{U} \text{ and } \varsigma \in \partial\mathcal{U} \setminus E(q_\sigma)$$

and

$$\varphi\left(\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z),\ \mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z),\ \mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+2,\mathcal{S}}\vartheta(z),\ \mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+3,\mathcal{S}}\vartheta(z); z\right) \prec \hbar(z),$$

then $\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z) \prec q(z)$ $(z \in \mathcal{U})$.

The next outcome produces the best dominant of the differential subordination Equation (21).

Theorem 3. *Let $\hbar$ be univalent in $\mathcal{U}$. Further, let $\varphi : C^4 \times \mathcal{U} \longrightarrow C$ and $\mathcal{T}$ be given by (18). Consider the differential equation*

$$\mathcal{T}\Big(p(z),\ zp'(z),\ z^2p''(z),\ z^3p'''(z); z\Big) = \hbar(z), \tag{22}$$

Which has a solution $q(z)$ with $q(0) = 0$, which attains (10). If $\vartheta \in \mathcal{A}$ attains (21) and

$$\varphi\left(\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z),\ \mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z),\ \mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+2,\mathcal{S}}\vartheta(z),\ \mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+3,\mathcal{S}}\vartheta(z); z\right)$$

is analytic in $\mathcal{U}$, then

$$\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z) \prec q(z)(z \in \mathcal{U})$$

and $q(z)$ is the best dominant.

Proof. Theorem 1 leads to q is dominant of (21). Since q attains (22), it is also a resolution of (21). Thus, q will be dominated by all dominants. Therefore, q is the best dominant. □

In light of Definition 6 and the specific case $q(z) = Mz$ $(M > 0)$, the admissible class of functions $\phi_j[\mho, q]$ indicated by $\phi_j[\mho, M\,]$] is stated as:

Definition 8 *Let $\mho$be a set in C and $M > 0$; the admissible class of function $\phi_j[\mho,\ M]$ includies $\varphi : C^4 \times \mathcal{U} \to C$ which attain the following:*

$$\varphi\left(\begin{array}{c} Me^{i\theta},\ \left(\frac{\left(K+\left(\frac{\mathcal{N}}{\mathcal{S}}-1\right)\right)Me^{i\theta}}{\frac{\mathcal{N}}{\mathcal{S}}}\right),\ 1+\frac{L+\left(\frac{2\mathcal{N}+1}{\mathcal{S}}-1\right)K+\left(\frac{\mathcal{N}+1}{\mathcal{S}}-1\right)\left(\frac{\mathcal{N}}{\mathcal{S}}-1\right)Me^{i\theta}}{\left(\frac{\mathcal{N}}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)}, \\ \frac{\mathrm{N}+3\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)L+\left[\frac{(\mathcal{N}-\mathcal{S})(\mathcal{N}+1-\mathcal{S})+(\mathcal{N}+2)(2\mathcal{N}+1-\mathcal{S})}{\mathcal{S}^2}\right]K+\left[\frac{(\mathcal{N}-\mathcal{S})(\mathcal{N}+1-\mathcal{S})(\mathcal{N}+2-\mathcal{S})}{\partial^3}\right]Me^{i\theta}}{\frac{\mathcal{N}(\mathcal{N}+1)(\mathcal{N}+2)}{\mathcal{S}^3}},\ z \end{array}\right) \notin \mho, \tag{23}$$

where $z \in \mathcal{U}$,

$$\mathcal{R}\mathrm{e}\Big(Le^{-i\theta}\Big) \geq (k-1)k\,M,$$

and

$$\mathcal{R}e\,\Big(\mathrm{N}e^{-i\theta}\Big) \geq 0 \text{ for all real } \theta \in R; k \geq 2.$$

Corollary 3. *Let $\varphi \in \phi_j[\mho,\ M]$. If $\vartheta \in \mathcal{A}$ attains the following situation:*

$$\left|\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z)\right| \leq kM, z \in \mathcal{U}; k \geq 2; M > 0,$$

and

$$\varphi\left(\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z), \mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z), \mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+2,\mathcal{S}}\vartheta(z), \mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+3,\mathcal{S}}\vartheta(z); z\right) \in \mho,$$

then

$$\left|\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z)\right| < M.$$

In the specific case $\mho = q(\mathcal{U}) = \{\tau : |\tau| < M\}$, $(M > 0)$, *the class* $\phi_j[\mho,\ M]$ *is simply symbolized by* $\phi_j[M]$.

Corollary 4. *Let* $\varphi \in \phi_j[M]$. *If* $\vartheta \in \mathcal{A}$ *attains the following situation:*

$$\left|\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z)\ \right| \leq k\mathrm{M}.$$

Also if

$$\left|\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z),\ \mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z),\ \mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+2,\mathcal{S}}\vartheta(z),\ \mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+3,\mathcal{S}}\vartheta(z);z\right| < \mathrm{M},$$

we obtain

$$\left|\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z)\ \right| < \mathrm{M}.$$

Corollary 5. *Let* $k \geq 2,\ 0 \neq \mathcal{N} \in C$ *and* $M > 0$. *If* $\vartheta \in \mathcal{A}$ *attains the following situation:*

$$\left|\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z)\ \right| \leq k\mathrm{M}.$$

Also, if

$$\left|\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z) - \mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z)\right| \leq \frac{\mathrm{M}}{\frac{\mathcal{N}}{\mathcal{S}}}$$

we obtain

$$\left|\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z)\ \right| < \mathrm{M}.$$

Proof. Let $\varphi(y_1,\ y_2,\ y_3,\ y_4;z) = y_2 - y_1$ and $\mho \in \hbar(\mathcal{U})$, whenever

$$\hbar(\mathrm{z}) = \frac{\mathrm{M}_z}{\frac{\mathcal{N}}{\mathcal{S}}}\ (M > 0).$$

Apply Corollary 3 to demonstrate that $\varphi \in \phi_\iota[\mho,\ M]$; the admissibility condition (23) achieves the next condition:

$$\varphi(y_1,\ y_2,\ y_3,\ y_4;z) = \left|\frac{(k-1)Me^{i\theta}}{\frac{\mathcal{N}}{\mathcal{S}}}\right| \geq \frac{M}{\left|\frac{\mathcal{N}}{\mathcal{S}}\right|}.$$

When $z \in \mathcal{U}, k \geq 2$ and $\theta \in R$, the required result follows from Corollary 3. □

Definition 9. *Let* $\mho$ *be a set in* C, $q \in Q_1 \cap \mathcal{H}_1$. *The class of admissible functions* $\phi_{j,1}[\mho, q]$ *consists of those functions* $\varphi : C^4 \times \mathcal{U} \to C$ *that satisfy the following admissibility condition:*

$$\varphi(y_1,\ y_2,\ y_3,\ y_4;z) \notin \mho,$$

where

$$y_1 = q(\zeta), y_2 = \frac{\beta\zeta q'(\zeta) + (\frac{\mathcal{N}}{\mathcal{S}})q(\zeta)}{\frac{\mathcal{N}}{\mathcal{S}}},$$

$$Re\left\{\left(\frac{\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)(y_3 - y_1)}{y_2 - y_1} - \left(\frac{2\mathcal{N}+1}{\mathcal{S}}\right)\right)\right\} \geq kRe\left\{1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right\},$$

and

$$Re\left\{\frac{\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+2}{\mathcal{S}}\right)(y_4-y_1)+\left[\left(1+\frac{(\mathcal{N}+1)}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)\right]3(y_3-y_1)}{y_2-y_1}+\left(1+\frac{(2\mathcal{N}+1)}{\mathcal{S}}\right)\left[3\left(1+\frac{(\mathcal{N}+1)}{\mathcal{S}}\right)-\left(1+\frac{(\mathcal{N}+1)}{\mathcal{S}}\right)\right]+\left(\frac{(\mathcal{N}+1)}{\mathcal{S}}\right)\left(\frac{(\mathcal{N})}{\mathcal{S}}\right)\right\}\leq k^2 Re\left\{\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right\},$$

whenever $z \in \mathcal{U}$, $\zeta \in \partial\mathcal{U}$-$E(q)$ *and* $k \geq 2$.

Theorem 4. *Let* $\varphi \in \phi_{j,1}[\mho, q]$. *If* $\vartheta \in \mathcal{A}$ *and* $q \in Q_1$ *attain the following situations:*

$$\mathcal{R}e\left(\frac{\varsigma q''(\varsigma)}{q'(\varsigma)}\right) \geq 0,\ \left|\frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z)}{q'(\varsigma)}\right| \leq k \tag{24}$$

and

$$\left\{\varphi\left(\frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z)}{z}, \frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z)}{z}, \frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+2,\mathcal{S}}\vartheta(z)}{z}, \frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+3,\mathcal{S}}\vartheta(z)}{z}; z \in \mathcal{U}\right)\right\} \subset \mho. \tag{25}$$

Then,

$$\frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z)}{z} \prec q(z).$$

Proof. Consider an analytic function $p(z)$ as:

$$p(z) = \frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z)}{z}. \tag{26}$$

It follows from (7) and (26) that

$$\frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z)}{z} = \frac{zp'(z) + (\frac{\mathcal{N}}{\mathcal{S}})p(z)}{\frac{\mathcal{N}}{\mathcal{S}}}. \tag{27}$$

A comparable argument is obtained as follows:

$$\frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+2,\mathcal{S}}\vartheta(z)}{z} = \frac{z^2 p''(z) + \left(1 + \frac{2\mathcal{N}+1}{\mathcal{S}}\right)zp'(z) + \left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)\left(\frac{\mathcal{N}}{\mathcal{S}}\right)p(z)}{\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)\left(\frac{\mathcal{N}}{\mathcal{S}}\right)} \tag{28}$$

and

$$\frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+3,\mathcal{S}}\vartheta(z)}{z} = \frac{z^3 p'''(z)+3\left(1+\frac{\mathcal{N}+1}{\mathcal{S}}\right)z^2 p''(z)+\left(\left(\frac{\mathcal{N}}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)+\left(\frac{\mathcal{N}+2}{\mathcal{S}}+1\right)\left(\frac{2\mathcal{N}+1}{\mathcal{S}}+1\right)\right)zp'(z)+\left(\frac{\mathcal{N}}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+2}{\mathcal{S}}\right)p(z)}{\left(\frac{\mathcal{N}}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+2}{\mathcal{S}}\right)}. \tag{29}$$

Then, consider the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ using

$$y_1 = \lambda_1, y_2 = \frac{\lambda_2 + (\frac{\mathcal{N}}{\mathcal{S}})\,\lambda_1}{\frac{\mathcal{N}}{\mathcal{S}}},$$

$$y_3 = \frac{\lambda_3 + \left(\frac{2\mathcal{N}+1}{\mathcal{S}} + 1\right)\lambda_2 + \left(\frac{\mathcal{N}}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)\lambda_1}{\left(\frac{\mathcal{N}}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)}, \tag{30}$$

and

$$y_4 = \frac{\lambda_4 + 3\left(1 + \frac{\mathcal{N}+1}{\mathcal{S}}\right)\lambda_3 + \left(\left(\frac{\mathcal{N}}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right) + \left(\frac{\mathcal{N}+2}{\mathcal{S}} + 1\right)\left(\frac{2\mathcal{N}+1}{\mathcal{S}} + 1\right)\right)\lambda_2 + \left(\frac{\mathcal{N}}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+2}{\mathcal{S}}\right)\lambda_1}{\left(\frac{\mathcal{N}}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+2}{\mathcal{S}}\right)}. \tag{31}$$

Let $\mathcal{T}(\lambda_1, \lambda_2, \lambda_3, \lambda_4; z) = \varphi(y_1, y_2, y_3, y_4; z)$

$$= \varphi\left(\begin{array}{c} \lambda_1,\ \frac{\lambda_2 + \left(\frac{\mathcal{N}}{\mathcal{S}}\right)\lambda_1}{\frac{\mathcal{N}}{\mathcal{S}}},\ \frac{\lambda_3 + \left(\frac{2\mathcal{N}+1}{\mathcal{S}}+1\right)\lambda_2 + \left(\frac{\mathcal{N}}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)\lambda_1}{\left(\frac{\mathcal{N}}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)}, \\ \frac{\lambda_4 + 3\left(1 + \frac{\mathcal{N}+1}{\mathcal{S}}\right)\lambda_3 + \left(\left(\frac{\mathcal{N}}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right) + \left(\frac{\mathcal{N}+2}{\mathcal{S}}+1\right)\left(\frac{2\mathcal{N}+1}{\mathcal{S}}+1\right)\right)\lambda_2 + \left(\frac{\mathcal{N}}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+2}{\mathcal{S}}\right)\lambda_1}{\left(\frac{\mathcal{N}}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+2}{\mathcal{S}}\right)}; z \end{array}\right) \tag{32}$$

By means of (26) to (29) and (32), we yield

$$\mathcal{T}\Big(p(z),\ zp'(z),\ z^2p''(z),\ z^3p'''(z); z\Big) =$$

$$\varphi\left(\frac{\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z)}{z},\ \frac{\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z)}{z},\ \frac{\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+2,\mathcal{S}}\vartheta(z)}{z},\ \frac{\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+3,\mathcal{S}}\vartheta(z)}{z}; z \in \mathcal{U}\right). \tag{33}$$

Therefore, evidently, (25) is given as follows:

$$\mathcal{T}\Big(p(z),\ zp'(z),\ z^2p''(z),\ z^3p'''(z); z\Big) \in \mho$$

$$\frac{\lambda_3}{\lambda_2} + 1 = \left(\frac{\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)(y_3 - y_1)}{y_2 - y_1} - \left(\frac{2\mathcal{N}+1}{\mathcal{S}}\right)\right),$$

and

$$\frac{\lambda_4}{\lambda_2} = \frac{\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+2}{\mathcal{S}}\right)(y_4 - y_1) + \left[\left(1 + \frac{(\mathcal{N}+1)}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)\right]3(y_3 - y_1)}{y_2 - y_1}$$

$$+\left(1 + \frac{(2\mathcal{N}+1)}{\mathcal{S}}\right)\left[3\left(1 + \frac{(\mathcal{N}+1)}{\mathcal{S}}\right) - \left(1 + \frac{(\mathcal{N}+1)}{\mathcal{S}}\right)\right] + \left(\frac{(\mathcal{N}+1)}{\mathcal{S}}\right)\left(\frac{(\mathcal{N})}{\mathcal{S}}\right).$$

Hence, evidently, the admissibility situation for $\varphi \in \phi_{j,1}[\mho, q]$ in Definition 8 is equivalent to an admissibility situation for $\mathcal{T} \in \mathcal{T}_2[\mho, q]$. Thus, by means of Definition 4 and Lemma 1, we acquire

$$\frac{\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z)}{z} \prec q(z).$$

This completes the desired outcome. □

The next outcome is an expansion of Theorem 4 to the following theorem stated below.

Theorem 5. *Let $\varphi \in \phi_{j,1}[\hslash, q]$. If $\vartheta \in \mathcal{A}$ and $q \in Q_1$ attain the following situations:*

$$\mathcal{R}e\left(\frac{\varsigma q''(\varsigma)}{q'(\varsigma)}\right) \geq 0,\ \left|\frac{\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z)}{z\, q'(\varsigma)}\right| \leq k, \tag{34}$$

and

$$\left(\varphi\left(\frac{\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z)}{z},\ \frac{\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z)}{z},\ \frac{\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+2,\mathcal{S}}\vartheta(z)}{z},\ \frac{\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N}+3,\mathcal{S}}\vartheta(z)}{z}; z \in \mathcal{U}\right)\right) \prec \hslash(z). \tag{35}$$

Then,

$$\frac{\mathcal{D}_z^{\alpha}\mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z)}{z} \prec q(z).\ (z \in \mathcal{U})$$

Using Definition 9 and $q(z) = \mathrm{M}_z$ $(\mathrm{M} > 0)$, the class $\phi_{j,\iota}[\mho, q]$ of admissible functions is expressed as follows:

Definition 10 ([7]). *Let $\mho$ be a set in C and $M > 0$; the admissible class of function $\phi_{j,\iota}[\mho,\ M]$ includes $\varphi : \mathrm{C}^4 \times \mathcal{U} \to \mathrm{C}$ which attain the following:*

$$\varphi\left(\begin{array}{c} Me^{i\theta},\ \left(\frac{\left(K+\left(\frac{\mathcal{N}}{\mathcal{S}}\right)\right)Me^{i\theta}}{\frac{\mathcal{N}}{\mathcal{S}}}\right),\ \frac{\mathrm{L}+\left(\frac{2\mathcal{N}+1}{\mathcal{S}}+1\right)\lambda_2+\left(\frac{\mathcal{N}}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)Me^{i\theta}}{\left(\frac{\mathcal{N}}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)}, \\ \frac{\mathrm{N}+3\left(1+\frac{\mathcal{N}+1}{\mathcal{S}}\right)\ L+\left(\left(\frac{\mathcal{N}}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)+\left(\frac{\mathcal{N}+2}{\mathcal{S}}+1\right)\left(\frac{2\mathcal{N}+1}{\mathcal{S}}+1\right)\right)K+\left(\frac{\mathcal{N}}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+2}{\mathcal{S}}\right)Me^{i\theta}}{\left(\frac{\mathcal{N}}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+1}{\mathcal{S}}\right)\left(\frac{\mathcal{N}+2}{\mathcal{S}}\right)}, z \end{array}\right) \notin \mho, \tag{36}$$

$$\mathcal{R}\mathrm{e}\left(Le^{-i\theta}\right) \geq (k-1)k\ M,$$

and

$$\mathcal{R}e\ \left(\mathrm{N}e^{-i\theta}\right) \geq 0\ (\text{for all real } \theta \in \mathrm{R} \text{ and } k \geq 2).$$

Corollary 6. *Let $\varphi \in \phi_{j,1}[\mho, q]$. If $\vartheta \in \mathcal{A}$ and attains the following situations:*

$$\left|\frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z)}{z}\right| \leq\ kM,\ (z \in \mathcal{U}; k \geq 2; M > 0)$$

and

$$\varphi\left(\frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z)}{z}, \frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z)}{z}, \frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+2,\mathcal{S}}\vartheta(z)}{z}, \frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+3,\mathcal{S}}\vartheta(z)}{z}; z\right) \in \mho,$$

then

$$\left|\frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z)}{z}\right| < M,\ (z \in \mathcal{U}).$$

Corollary 7. *Let $\varphi \in \phi_{j,1}[\mho, q]$. If $\vartheta \in \mathcal{A}$ and $q \in Q_1$ attain the following situations*

$$\left|\frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z)}{q'(\varsigma)}\right| \leq\ kM,\ z \in \mathcal{U} \text{ and } k \geq 2, M > 0,$$

and

$$\left|\varphi\left(\frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N},\mathcal{S}}(\vartheta z)}{z}, \frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+1,\mathcal{S}}\vartheta(z)}{z}, \frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+2,\mathcal{S}}\vartheta(z)}{z}, \frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N}+3,\mathcal{S}}\vartheta(z)}{z}; z \in \mathcal{U}\right)\right| < M,$$

then

$$\left|\frac{\mathcal{D}_z^\alpha \mathfrak{S}_{\mathcal{N},\mathcal{S}}\vartheta(z)}{z}\right| < M\ (z \in \mathcal{U}).$$

3. Conclusions

The investigation of third-order differential subordination and differential superordination for s\analytic functions through the use of fractional differential operators represents a profound and extremely specific domain within the field of mathematics. This integration of fractional differential operators with third-order differential subordination and superordination is a highly specialized and advanced area of mathematical research. It requires a deep understanding of complex analysis, fractional calculus, and the properties of analytic functions. Researchers in this field aim to establish relationships that will help analyze and compare the behavior of analytic functions in complex domains, considering the higher-order derivatives, which can have significant implications in various scientific and engineering applications.

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