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Existence of Solutions for Planar Kirchhoff–Choquard Problems

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Abstract: In this article, we are interested in the study of the following Kirchhoff–Choquard equations: $-\left(a+b\int_{\mathbb{R}^2}|\nabla u|^2 \mathrm{d}x\right)\Delta u+V(x)u=\lambda(\ln|x|*u^2)u+f(u),\ \ x\in\mathbb{R}^2$, where $\lambda>0, a>0, b>0$, V and f are continuous functions with some appropriate assumptions. We prove that when the parameter λ is sufficiently small, the above problem has a mountain pass solution, a least energy solution and a ground state solution by applying the variational methods and building some subtle inequalities.

Keywords: Choquard problem; Kirchhoff-type problems; variational methods; mountain pass solution; ground state solution

MSC: 35B33; 35J20; 35J61

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1. Introduction and Main Results

This paper is dedicated to the study of the existence of solutions to the following Kirchhoff–Choquard problem in $\mathbb{R}^2$:

$$-\left(a+b\int_{\mathbb{R}^2}|\nabla u|^2 \mathrm{d}x\right)\Delta u+V(x)u=\lambda(\ln|x|*u^2)u+f(u),\ \ x\in\mathbb{R}^2, \tag{1}$$

where $\lambda>0$, $a,b>0$, $V\in\mathcal{C}(\mathbb{R}^2,\mathbb{R})$ and $f\in\mathcal{C}(\mathbb{R},\mathbb{R})$. Moreover, the following hypotheses are imposed on V and f:

($\mathcal{V}$) $V\in\mathcal{C}^1(\mathbb{R}^2,[0,\infty))$, $V(x)\geqslant \inf_{x\in\mathbb{R}^2}V(x)\geqslant 0$, and there exists $m>0$, such that the set $\{x\in\mathbb{R}^2: V(x)<m\}$ is nonempty and has finite measure;

($\mathcal{V}_1$) $6V(x)+(\nabla V(x),x)\geqslant 0$;

($\mathcal{V}_2$) for all $x\in\mathbb{R}^2$, the map $t\mapsto \dfrac{(\nabla V(t^{-1}x),t^{-1}x)-2V(t^{-1}x)}{t^6|x|^6}$ is nondecreasing on $(0,+\infty)$;

($\mathcal{F}_1$) there exist constants $C>0$ and $p>5$, such that

$$|f(t)|\leqslant C(1+|t|^{p-1}),\ \ \forall\, t\in\mathbb{R};$$

($\mathcal{F}_2$) $f(t)=o(|t|)$ as $t\to 0$;

($\mathcal{F}_3$) $\lim\limits_{|t|\to\infty}\frac{F(t)}{|t|^5}=\infty$;

($\mathcal{F}_4$) there exist constants $\alpha_0, C>0$ and $q>1$, such that $f(t)t\geqslant 5F(t)$ for all $t\in\mathbb{R}$. Moreover, there holds: $\left|\frac{f(t)}{t}\right|\geqslant\alpha_0$ implies that $\left|\frac{f(t)}{t}\right|^q\leqslant C[f(t)t-5F(t)]$;

($\mathcal{F}_5$) the map $t\mapsto\frac{f(tu)tu-F(tu)}{t^2u^2}$ is nondecreasing on $(0,\infty)$, for all $u\in\mathbb{R}$.

There are some examples of functions V and f:

$$\begin{aligned} V(x) &= r|x|^g,\ r>0,\ g\in[0,2];\\ f_1(u) &= |u|^{p-2}u,\ p\in(5,\infty);\\ f_2(u) &= |u|^{p-2}u+l|u|^{q-2}u,\ l>0, 2<q<5<p<\infty. \end{aligned}$$

One can easily determine that $V_1(x)$ satisfy $(\mathcal{V}),(\mathcal{V}_1),(\mathcal{V}_2)$. Furthermore, $f_1(u)$ satisfies $(\mathcal{F}_1)-(\mathcal{F}_5)$, while $f_2(u)$ satisfies $(\mathcal{F}_1)-(\mathcal{F}_3)$ and $(\mathcal{F}_5)$.

Now, we recall the Choquard equation introduced by Pekar [1]:

$$-\Delta u + u = \left(\frac{1}{|x|} * |u|^2\right)u \text{ in } \mathbb{R}^N, \tag{2}$$

which was derived from the approximation of Hartree–Fock theory, and this problem can be viewed as a model of an electron trapped in its own hole. Later, in [2], the author introduced the above problem as a model of self-gravitating matter. Hence, the Choquard Equation (2) can also be regarded as the Schrödinger–Newton equation, which is used to model the coupling of the Schödinger equation of quantum physics and nonrelativistic Newtonian gravity. We refer to [3–7] for more physical background.

Here, we present some relevant results of Problem (1). In [8], Arora et al. concerned the Kirchhoff problem as follows:

$$\begin{cases} -m(\int_\Omega |\nabla u|^n \mathrm{d}x)\Delta_n u = \left(\int_\Omega \frac{F(x,u)}{|x-y|^\mu}\mathrm{d}y\right)f(x,u), & u>0 \text{ in } \Omega, \\ u=0, \text{ on } \partial\Omega, \end{cases}$$

where Ω is bounded in $\mathbb{R}^n$, $n \geqslant 2$. According to the variational methods, the authors obtained weak solutions to the above problem. In their study, Chen et al. [9] investigated the following problem:

$$\begin{cases} \left(a+b[u]_{s,p}^{p(\theta-1)}\right)(-\Delta_p^s)u = \left(I_\mu * |u|^q\right)|u|^{q-2}u + \frac{|u|^{p_\alpha^*-2}u}{|x|^\alpha}, & u>0 \text{ in } \Omega, \\ u=0, \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is bounded in $\mathbb{R}^N$, $0 \leqslant \alpha < ps < N$ with $0<s<1$, $p>1$, $1<\theta \leqslant p_\alpha^*/p$. The authors proved that there exists a positive weak solution to the above problem which uses the mountain pass theorem and concentration-compactness principle. Böer et al. [10] studied the following Kirchhoff–Choquard problem:

$$-M(\|\nabla u\|_2^2)\Delta u + Q(x)u + \mu(V(|\cdot|) * u^2)u = f(u) \text{ in } \mathbb{R}^2, \tag{3}$$

where $M(t)=a+bt, \mu>0, V$ is a sign-changing potential and Q, f are continuous functions. By applying the variational techniques, the authors proved the existence and multiplicity of solutions to Problem (3). Actually, $V(|x|)$ can be the logarithmic kernel $\ln|x|$ under some special conditions.

Next, we introduce some results with the nonlocal term $\ln|x| * u^2$ on the left side of the problem. Chen et al. in [11] considered the following Schrödinger–Poisson system:

$$\begin{cases} -\Delta\varphi + \varphi + \phi\varphi = f(\varphi), & x \in \mathbb{R}^2, \\ \Delta\phi = \varphi^2, & x \in \mathbb{R}^2. \end{cases} \tag{4}$$

Using the Gagliardo–Nirenberg inequality and the Hardy–Littlewood–Sobolev inequality, they proved the existence of ground state solution and mountain pass solution of Problem (4). In 2021, Alves et al. [12] investigated the following Schrödinger–Poisson system:

$$\begin{cases} -\Delta\varphi + \lambda\varphi + \mu(\ln|\cdot| * |\varphi|^2)\varphi = f(\varphi), & x \in \mathbb{R}^2, \\ \int_{\mathbb{R}^2} |\varphi|^2 \mathrm{d}x = c, & c>0, \end{cases} \tag{5}$$

where $\lambda, \mu \in \mathbb{R}$. The authors showed the existence of normalized solutions of system (5) by using the Hardy–Littlewood–Sobolev inequality and the variational methods. For more related results, we refer to [8,9,12–34] and the references therein.

To the best of our knowledge, there is no result for the existence of solutions for Problem (1). In this article, we consider the existence of solutions to Problem (1). And we can obtain the following corresponding functional for Problem (1):

$$\begin{aligned} I(u) &= \frac{a}{2}\int_{\mathbb{R}^2}|\nabla u|^2 \mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^2}|\nabla u|^2\mathrm{d}x\right)^2 + \frac{1}{2}\int_{\mathbb{R}^2}V(x)u^2\mathrm{d}x \\ &\quad -\frac{\lambda}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(\ln|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^2}F(u)\mathrm{d}x. \end{aligned} \tag{6}$$

Let $H^1(\mathbb{R}^2)$ be the Sobolev space with the following inner product and norm

$$(u,v) = \int_{\mathbb{R}^2}(\nabla u\nabla v + uv)\mathrm{d}x,\ \ \|u\|_{H^1} := \left(\int_{\mathbb{R}^2}(|\nabla u|^2 + u^2)\mathrm{d}x\right)^{\frac{1}{2}}.$$

Moreover, let $E = \{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x)u^2\mathrm{d}x < +\infty\}$ be the Hilbert space with the norm

$$\|u\|^2 := \int_{\mathbb{R}^2}(|\nabla u|^2 + V(x)u^2)\mathrm{d}x.$$

From the assumption$(\mathcal{V})$ and the Poincaré inequality, one can deduce that the embedding $E \hookrightarrow H^1(\mathbb{R}^2)$ is continuous. Similar to Lemma 2.2 in [15], we can conclude that I is of class $\mathcal{C}^1$ on E. Moreover, it is evident that the critical points of I correspond to the weak solutions to Problem (1).

Similar to [35], Lemma 2.4, and [15], Proposition 2.3, we can obtain the Pohožaev funtional of (1):

$$\begin{aligned} P(u) = \frac{1}{2}\int_{\mathbb{R}^2}[2V(x) + (\nabla V(x), x)]u^2\mathrm{d}x - \lambda\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(\ln|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y \\ -\frac{\lambda}{4}\|u\|_2^4 - 2\int_{\mathbb{R}^2}F(u)\mathrm{d}x. \end{aligned} \tag{7}$$

It is standard to find that $P(u) = 0$ if u is the solution of (1). Define

$$\begin{aligned} J(u) &= 2\langle I'(u), u\rangle - P(u) \\ &= 2a\int_{\mathbb{R}^2}|\nabla u|^2\mathrm{d}x + 2b\left(\int_{\mathbb{R}^2}|\nabla u|^2\mathrm{d}x\right)^2 + \frac{1}{2}\int_{\mathbb{R}^2}[2V(x) - (\nabla V(x), x)]u^2\mathrm{d}x \\ &\quad -\lambda\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(\ln|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y + \frac{\lambda}{4}\|u\|_2^4 - 2\int_{\mathbb{R}^2}[f(u)u - F(u)]\mathrm{d}x. \end{aligned} \tag{8}$$

Then, we denote the Nehari–Pohožaev manifold of the functional I as follows:

$$\mathcal{M} := \{u \in E\backslash\{0\} : J(u) = 0\}. \tag{9}$$

Obviously, all nontrivial solutions of (1) are included in $\mathcal{M}$.

Now, we present the main results of this article.

Theorem 1. *Assume that $(\mathcal{V})$, $(\mathcal{V}_1)$ and $(\mathcal{F}_1)$–$(\mathcal{F}_4)$ hold. There exists $\lambda^* > 0$, such that for $0 < \lambda \leqslant \lambda^*$, Equation (1) admits a nontrivial least energy solution in E. Moreover, Equation (1) admits a solution of mountain pass type in E with positive energy.*

Theorem 2. *Assume that $\mathcal{V}$ satisfies $(\mathcal{V})$, $(\mathcal{V}_2)$ and f satisfies $(\mathcal{F}_1)$–$(\mathcal{F}_3)$, $(\mathcal{F}_5)$. There exists $\lambda^{**} > 0$, such that for $0 < \lambda \leqslant \lambda^{**}$, Equation (1) has a ground state solution in E.*

Remark 1. *Compared with [2], we consider the Problem (1) in which the nonlocal term $\ln|x| * u^2$ is on the right-hand side of the equation. Due to the sign-changing property of this nonlocal term, the approach for the nonlocal term on the left side of the problem does not apply with the present* *article. We tested this situation by introducing some subtle analysis. Moreover, we give a gentle assumption* $(\mathcal{V}_2)$*, compared to* [2].

There are two nonlocal terms in this problem, which make the Problem (1) no longer a pointwise identity. It is worth noting that the approach for situation $|x|^{-1} * u^2$ is not often adapted to the Equation (1) since $\ln|x| * u^2$ is sign-changing and is neither bounded from above nor from below. Moreover, compared with the problem where the nonlocal term $\ln|x| * u^2$ is on the left, such as [11,12], we cannot use the Hardy–Littlewood–Sobolev inequality to determine the boundedness of the Cerami sequence. Also, it is difficult to show that the energy functional of Equation (1) satisfies the mountain pass geometry.

In this paper, using the variational methods introduced by [15], we investigate the existence of the least energy solutions. Specifically, we begin by building a Cerami sequence $\{u_n\}$ with $P(u_n) \to 0$, then, by establishing a contradiction and some subtle analytical techniques, we verify that the Cerami sequence is bounded in $H^1(\mathbb{R}^2)$. To prove the existence of ground state solutions, based on the method developed in [36], we construct a important inequality between the corresponding functional with the Pohožaev identity; hence, we can determine the boundedness of the Cerami sequence.

This paper is organized as follows. Section 2 shows the preliminaries. Section 3 gives the proof of Theorem 1, and Section 4 illustrates the proof of Theorem 2.

Throughout this paper, we use the following notations: $L^q(\mathbb{R}^2)$ denotes the Lebesgue space equipped with the norm $\|u\|_s = \left(\int_{\mathbb{R}^2} |u|^s \mathrm{d}x\right)^{1/s}$, where $2 \leqslant s < +\infty$; $\|u\|_*^2 := \int_{\mathbb{R}^2} \ln(1+|x|)u^2(x)\mathrm{d}x$; $B_r(z)$ denotes the open ball centered at z with radius $r > 0$.

2. Preliminaries

In the first part, we define

$$
\begin{aligned}
(w,z) &\mapsto \chi_1(w,z) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)w(x)z(y)\mathrm{d}x\mathrm{d}y, \\
(w,z) &\mapsto \chi_2(w,z) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right)w(x)z(y)\mathrm{d}x\mathrm{d}y, \\
(w,z) &\mapsto \chi_0(w,z) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(|x-y|)w(x)z(y)\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Then, we determine that $\chi_0(w,z) = \chi_1(w,z) - \chi_2(w,z)$. Indeed, all of the above definitions are limited to measurable functions $w, z : \mathbb{R}^2 \to \mathbb{R}$, such that the double integral in the right hand is well defined in the sense of Lebesgue. According to the Hardy–Littlewood–Sobolev inequality [5], we have

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y\right| \\
&\leqslant \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{|x-y|}|u^2(x)u^2(y)|\mathrm{d}x\mathrm{d}y \leqslant C\|u\|_{8/3}^4, \quad \forall u \in L^{\frac{8}{3}}(\mathbb{R}^2).
\end{aligned} \tag{10}
$$

Furthermore, since

$$
\ln(1+|x-y|) \leqslant \ln(1+|x|+|y|) \leqslant \ln(1+|x|) + \ln(1+|y|),
$$

for $w, z, u, v \in E$, one has

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)w(x)z(x)u(y)v(y)\mathrm{d}x\mathrm{d}y\right| \\
\leqslant\ & \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} [\ln(1+|x|) + \ln(1+|y|)]|w(x)z(x)||u(y)v(y)|\mathrm{d}x\mathrm{d}y \\
\leqslant\ & \|w\|_*\|z\|_*\|u\|_2\|v\|_2 + \|w\|_2\|z\|_2\|u\|_*\|v\|_*.
\end{aligned} \tag{11}
$$

Next, to obtain the solutions of (1), the minimax principle (see [37], Proposition 2.8) is presented here.

Lemma 1. *Let Y be a Banach space, and Ω_0 be the closed subspace of the metric space Ω, $\Lambda' \subset \mathcal{C}(\Omega_0, Y)$. Denote*

$$\bar{\Lambda} := \{\theta \in \mathcal{C}(\Omega, Y) : \theta|_{\Omega_0} \in \Lambda'\}.$$

If $\Psi \in \mathcal{C}^1(Y, \mathbb{R})$ satisfies

$$c' := \sup_{\theta' \in \Lambda'} \sup_{u \in \Omega_0} \Psi(\theta'(u)) < c := \inf_{\theta \in \Lambda} \sup_{u \in \Omega} \Psi(\theta(u)) < \infty,$$

then, for $\delta \in (0, (c - c')/2), \xi > 0$ and $\theta \in \bar{\Lambda}$ such that

$$\sup_{\Omega} \Psi \circ \theta \leqslant c + \delta,$$

there is $u \in Y$, such that

(i) $c - 2\delta \leqslant \Psi(u) \leqslant c + 2\delta$;
(ii) $dist(u, \theta(\Omega)) \leqslant 2\xi$;
(iii) $\|\Psi'(u)\| \leqslant \frac{8\delta}{\xi}$.

Inspired by [35] (Lemma 3.2), we find that there exists a Cerami sequence for functional I.

Lemma 2. *Suppose that $(\mathcal{V})$ and $(\mathcal{F}_1) - (\mathcal{F}_3)$ hold. Then, there exists $\lambda^* > 0$, $\{u_n\} \subset E$ such that, for $\lambda \leqslant \lambda^*$,*

$$I(u_n) \to c > 0, \ \ \|I'(u_n)\|_{E^*}(1 + \|u_n\|_E) \to 0, \ \ J(u_n) \to 0, \tag{12}$$

here,

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \ \ \Gamma := \{\gamma \in \mathcal{C}([0,1], E) : \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

Proof. We certify that $0 < c < \infty$ first. For any $t > 0$, set $u_t := u(tx)$ here and in what follows. Then,

$$\begin{aligned}
I(t^2 u_t) \;=\; & \frac{a}{2} t^4 \|\nabla u\|_2^2 + \frac{b}{4} t^8 \|\nabla u\|_2^4 + \frac{t^2}{2} \int_{\mathbb{R}^2} V(t^{-1}x) u^2 \mathrm{d}x \\
& - \frac{\lambda t^4}{4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (\ln|tx - ty| - \ln t) u^2(tx) u^2(ty) \mathrm{d}(tx) \mathrm{d}(ty) - \frac{1}{t^2} \int_{\mathbb{R}^2} F(t^2 u) \mathrm{d}x \\
=\; & \frac{a}{2} t^4 \|\nabla u\|_2^2 + \frac{b}{4} t^8 \|\nabla u\|_2^4 + \frac{t^2}{2} \int_{\mathbb{R}^2} V(t^{-1}x) u^2 \mathrm{d}x - \frac{\lambda t^4}{4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x - y|) u^2(x) u^2(y) \mathrm{d}x \mathrm{d}y \\
& + \frac{\lambda t^4 \ln t}{4} \|u\|_2^4 - \frac{1}{t^2} \int_{\mathbb{R}^2} F(t^2 u) \mathrm{d}x, \ \ \forall t > 0.
\end{aligned} \tag{13}$$

It follows from $(\mathcal{F}_1)$–$(\mathcal{F}_3)$ and (13) that

$$\lim_{t \to 0} I(t^2 u_t) = 0, \ \ \sup_{t > 0} I(t^2 u_t) < \infty, \ \ \lim_{t \to \infty} I(t^2 u_t) = -\infty.$$

Then, we may choose $T > 0$ that is sufficiently large, satisfying $I(T^2 u_T) < 0$. Now, we define $\gamma_T(t) = (tT)^2 u_{tT}$ for $t \in [0,1]$; thus, we can deduce that $\gamma_T(0) = 0, I(\gamma_T(1)) < 0$, $\gamma_T \in \mathcal{C}([0,1], E)$ and $\max_{t \in [0,1]} I(\gamma_T(t)) < \infty$. Consequently, $\Gamma \neq 0, c < \infty$.

According to $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$, for any $\varepsilon > 0, s \in \mathbb{R}$, there exists a constant $C_\varepsilon > 0$, such that

$$f(s)s \leqslant \varepsilon s^2 + C_\varepsilon |s|^p, \ \ F(s) \leqslant \varepsilon s^2 + C_\varepsilon |s|^p, \ \ \forall s \in \mathbb{R}. \tag{14}$$

Fix $\varepsilon = a/4$, then let $\lambda^* > 0$ be sufficiently small such that, for $\lambda \leqslant \lambda^*$,

$$\frac{1}{16}\min\Big\{\min\{a,1\}\|\nabla u_n\|_2^2, b\|\nabla u_n\|_2^4\Big\} - \frac{\lambda}{8}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(1+|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{\lambda}{16}\|u\|_2^4 \geqslant 0. \tag{15}$$

Then, from (6), (10), (14), (15) and the Sobolev imbedding inequality, we know

$$\begin{aligned} I(u) \quad &\geqslant \quad \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 + \frac{1}{2}\int_{\mathbb{R}^2} V(x)u^2\mathrm{d}x - \frac{\lambda}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(|x-y|)w(x)z(y)\mathrm{d}x\mathrm{d}y \\ &\quad -\frac{a}{4}\|u\|_2^2 - C\|u\|_p^p \\ &\geqslant \quad \min\{a,1\}\frac{1}{4}\|u\|^2 - C\|u\|^p, \ \ \forall u \in E. \end{aligned}$$

We can conclude that $\hat{e} > 0$ and $d > 0$, fulfilling

$$I(u) \geqslant 0 \text{ for } \|u\| \leqslant \hat{e}, \ \ I(u) \geqslant l \text{ for } \|u\| = \hat{e}. \tag{16}$$

For any $\gamma \in \Gamma$, we have $\gamma(0) = 0$ and $I(\gamma(1)) < 0$. Thus, it follows from (16) that $\|\gamma(1)\| > \hat{e}$ holds. Since $\gamma(t)$ is continuous, by applying the intermediate value theorem, we can deduce that there is $t_0 \in (0,1)$ satisfying $\|\gamma(t_0)\| = \hat{e}$. Thus,

$$\sup_{t\in[0,1]} I(\gamma(t)) \geqslant I(\gamma(t_0)) \geqslant l > 0, \ \ \forall \gamma \in \Gamma,$$

which means

$$0 < l \leqslant \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)) = c < \infty. \tag{17}$$

Set X as the Banach space endowed with norm

$$\|(t,w)\|_X := (|t|^2 + \|w\|^2)^{\frac{1}{2}}.$$

Next, let $X := \mathbb{R} \times E$; we define

$$P : X \to E, \ \ P(t,w)(x) := e^{2t}w(e^t x) \ \text{ for } t \in \mathbb{R}, w \in E, x \in \mathbb{R}^2,$$

Then,

$$\begin{aligned} \Psi(t,w) \quad &= \quad I(P(t,w)) = \frac{a}{2}\int_{\mathbb{R}^2}|\nabla P(t,w)|^2\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^2}|\nabla P(t,w)|^2\mathrm{d}x\right)^2 + \frac{1}{2}\int_{\mathbb{R}^2}V(x)|P(t,w)|^2\mathrm{d}x \\ &\quad -\frac{\lambda}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(|x-y|)P^2(t,w(x))P^2(t,w(y))\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^2}F(P(t,w))\mathrm{d}x \\ &= \quad \frac{a}{2}e^{4t}\int_{\mathbb{R}^2}|\nabla w|^2\mathrm{d}x + \frac{b}{4}e^{8t}\left(\int_{\mathbb{R}^2}|\nabla w|^2\mathrm{d}x\right)^2 + \frac{e^{2t}}{2}\int_{\mathbb{R}^2}V(e^{-t}x)w^2\mathrm{d}x \\ &\quad -\frac{\lambda e^{4t}}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(|x-y|)w^2(x)w^2(y)\mathrm{d}x\mathrm{d}y + \frac{\lambda t e^{4t}}{4}\left(\int_{\mathbb{R}^2}w^2\mathrm{d}\right)^2 - \frac{1}{e^{2t}}\int_{\mathbb{R}^2}F(e^{2t}w)\mathrm{d}x. \end{aligned}$$

Hence, we get

$$
\begin{aligned}
\partial_t\Psi(t,w) &= 2ae^{4t}\int_{\mathbb{R}^2}|\nabla w|^2\mathrm{d}x+2be^{8t}\left(\int_{\mathbb{R}^2}|\nabla w|^2\mathrm{d}x\right)^2+e^{2t}\int_{\mathbb{R}^2}V(e^{-t}x)w^2\mathrm{d}x\\
&\quad -\frac{e^{2t}}{2}\int_{\mathbb{R}^2}(\nabla V(e^{-t}x),(e^{-t}x))w^2\mathrm{d}x-\lambda e^{4t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(|x-y|)w^2(x)w^2(y)\mathrm{d}x\mathrm{d}y\\
&\quad +\lambda\left(te^{4t}+\frac{e^{4t}}{4}\right)\left(\int_{\mathbb{R}^2}w^2\mathrm{d}x\right)^2+\frac{2}{e^{2t}}\int_{\mathbb{R}^2}F(e^{2t}w)\mathrm{d}x-\frac{2}{e^{2t}}\int_{\mathbb{R}^2}f(e^{2t}v)e^{2t}w\mathrm{d}x\\
&= 2a\|\nabla P(t,w)\|_2^2+2b\|\nabla P(t,w)\|_2^4+\int_{\mathbb{R}^2}V(x)|P(t,w)|^2\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^2}(\nabla V(x),x)|P(t,w)|^2\mathrm{d}x\\
&\quad -\lambda\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(|x-y|)P^2(t,w(x))P^2(t,w(y))\mathrm{d}x\mathrm{d}y+\frac{\lambda}{4}\left(\int_{\mathbb{R}^2}|P(t,w)|^2\mathrm{d}x\right)^2\\
&\quad -2\int_{\mathbb{R}^2}[f(P(t,w))P(t,w)-F(P(t,w))]\mathrm{d}x\\
&= J(P(t,w)),\ \ \forall t\in\mathbb{R}, w\in E,
\end{aligned}
\tag{18}
$$

which shows that Ψ is of class $\mathcal{C}^1$ on X. In addition, as the map $w\mapsto P(t,w)$ is linear, for any $t\in\mathbb{R}, w,z\in E$, one has

$$\partial_w\Psi(t,w)z=I'(P(t,w))P(t,w), \tag{19}$$

Now, define

$$\tilde{c}=\inf_{\tilde{\gamma}\in\tilde{\Gamma}}\max_{t\in[0,1]}I(\tilde{\gamma}(t)),$$

where

$$\tilde{\Gamma}=\{\tilde{\gamma}\in\mathcal{C}([0,1],X):\tilde{\gamma}(0)=0, I(\tilde{\gamma}(1))<0\}.$$

Due to $\Gamma=\{P\circ\tilde{\gamma}:\tilde{\gamma}\in\tilde{\Gamma}\}$, we have $c=\tilde{c}$. With the definition of c, for any $n\in\mathbb{N}$, choosing $\gamma_n\in\Gamma$ such that

$$\max_{t\in[0,1]}\Psi(0,\gamma_n(t))=\max_{[0,1]}I(\gamma_n(t))\leqslant c+\frac{1}{n^2}.$$

From Lemma 1, let $\Omega=[0,1],\Omega_0=\{0,1\}$ and $X,\tilde{\Gamma}$ in place of Y,Γ. Fix $\tilde{\gamma}_n(t)=(0,\gamma_n(t)),\varepsilon_n=\frac{1}{n^2}$ and $\xi_n=\frac{1}{n}$. Using (17), for $n\in\mathbb{N}$ large, $\varepsilon_n=\frac{1}{n^2}\in(0,\frac{c}{2})$. Consequently, in terms of Lemma 1, we deduce that, as $n\to\infty$, there is $(t_n,w_n)\in X$ satisfying

$$\Psi(t_n,w_n)\to c, \tag{20}$$
$$\|\Psi'(t_n,w_n)\|_{X^*}(1+\|(t_n,w_n)\|_X)\to 0, \tag{21}$$
$$\mathrm{dist}((w_n,w_n),\{0\}\times\gamma_n([0,1]))\to 0. \tag{22}$$

By (22), we have

$$t_n\to 0. \tag{23}$$

Noticing that

$$\langle\Psi'(t_n,w_n),(\nu,z)\rangle=\langle I'(P(t_n,w_n)),P(t_n,w_n)\rangle+J(P(t_n,w_n))\mu,\ \ \forall(\nu,z)\in X. \tag{24}$$

Then, from (18) and (19), fix $\nu=1$ and $z=0$ in (24), we have

$$J(P(t_n,w_n))\to 0,\ \ \text{as } n\to\infty. \tag{25}$$

Define $u_n:=P(t_n,w_n)$; combining (20) with (25), one has

$$I(u_n)\to c,\ \ J(u_n)\to 0\ \ \text{as } n\to\infty.$$

Set $\tau_n = e^{-2t_n} w(e^{-t_n} x) \in E$ for $w \in E$. Using (21) and (24), we have

$$(1 + \|u_n\|)|I'(u_n)w| = (1 + \|u_n\|)|I'(u_n)P(t_n, w_n)| = o(1)\|\tau_n\|, \text{ as } n \to \infty.$$

Furthermore, from (23), one has

$$\begin{aligned} \|\tau_n\|^2 &= \|\nabla \tau_n\|_2^2 + \int_{\mathbb{R}^2} V(x)\tau_n^2 \mathrm{d}x \\ &= e^{-4t_n}\|\nabla w\|_2^2 + e^{-2t_n}\int_{\mathbb{R}^2} V(e^{t_n}x)w^2 \mathrm{d}x \\ &= [1 + o(1)]\|\nabla w\|_2^2 + [1 + o(1)]\int_{\mathbb{R}^2} V(x)w^2 \mathrm{d}x \\ &= [1 + o(1)]\|w\|_E^2, \text{ as } n \to \infty. \end{aligned}$$

Hence,

$$(1 + \|u_n\|_E)\|I'(u_n)\|_{E^*} \to 0, \text{ as } n \to \infty.$$

Now, we complete the proof. □

Next, we prove that the Cerami sequence is bounded.

Lemma 3. *Suppose that* $(\mathcal{V})$, $(\mathcal{V}_1)$ *and* $(\mathcal{F}_1)$–$(\mathcal{F}_4)$ *hold. Let* $\{u_n\} \subset E$, *such that* (12) *holds. Then, there is* $\lambda^* > 0$, *such that for* $0 < \lambda \leqslant \lambda^*$, $\{u_n\}$ *is bounded in* $H^1(\mathbb{R}^2)$.

Proof. Using $(\mathcal{F}_4)$, $(\mathcal{V}_1)$, (12), (15), the Gagliardo–Nirenberg inequality (see [38], Theorem 1.3.7), we have

$$\begin{aligned} c + o(1) &= I(u_n) - \frac{1}{8}J(u_n) \\ &= \frac{a}{4}\|\nabla u_n\|_2^2 + \frac{3}{8}\int_{\mathbb{R}^2} V(x)u_n^2 \mathrm{d}x + \frac{1}{16}\int_{\mathbb{R}^2} (\nabla V(x), x)u_n^2 \mathrm{d}x \\ &\quad -\frac{\lambda}{8}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{\lambda}{32}\|u_n\|_2^4 + \frac{1}{4}\int_{\mathbb{R}^2}[f(u_n)u_n - 5F(u_n)]\mathrm{d}x \\ &\geqslant \frac{a}{4}\|\nabla u_n\|_2^2 + \frac{3}{8}\int_{\mathbb{R}^2} V(x)u_n^2 \mathrm{d}x + \frac{1}{16}\int_{\mathbb{R}^2} (\nabla V(x), x)u_n^2 \mathrm{d}x + \frac{\lambda}{32}\|u_n\|_2^4 \\ &\quad -\frac{\lambda}{8}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{\lambda}{16}\|u_n\|_2^4 + \frac{1}{4}\int_{\mathbb{R}^2}[f(u_n)u_n - 5F(u_n)]\mathrm{d}x \\ &\geqslant \frac{\lambda}{32}\|u_n\|_2^4, \end{aligned} \tag{26}$$

which shows

$$\|u_n\|_2 \leqslant C, \quad \int_{\mathbb{R}^2}[f(u_n)u_n - 5F(u_n)]\mathrm{d}x \leqslant C. \tag{27}$$

Now, we illustrate that $\{\|u_n\|\}$ is bounded. With reduction to absurdity, we suppose that $\|u_n\| \to \infty$. Defining $z_n := u_n/\|u_n\|$, by (27), one has $\|z_n\| = 1$, $\|z_n\|_2 \to 0$. Set $r = q/(q-1)$; then, by applying the Gagliardo–Nirenberg inequality, one has

$$\|z_n\|_{2r}^{2r} \leqslant C\|z_n\|_2^2\|\nabla z_n\|_2^{2r-2} = o(1). \tag{28}$$

Let

$$A_n := \left\{x \in \mathbb{R}^2 : \left|\frac{f(u_n)}{u_n}\right| \leqslant \alpha_0\right\}.$$

Thus,

$$\int_{A_n}\left|\frac{f(u_n)}{u_n}\right| z_n^2 \mathrm{d}x \leqslant \alpha_0\|z_n\|_2^2 = o(1). \tag{29}$$

Moreover, it follows from $(\mathcal{F}_4)$, (27), (28) and the Hölder inequality that

$$\begin{aligned}
&\int_{\mathbb{R}^2\setminus A_n}\left|\frac{f(u_n)}{u_n}\right|z_n^2\mathrm{d}x \\
\leqslant\ & \left(\int_{\mathbb{R}^2\setminus A_n}\left|\frac{f(u_n)}{u_n}\right|^q\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^2\setminus A_n}|z_n|^{2r}\mathrm{d}x\right)^{\frac{1}{r}} \\
\leqslant\ & C^{\frac{1}{q}}\left(\int_{\mathbb{R}^2\setminus A_n}[f(u_n)u_n-5F(u_n)]\mathrm{d}x\right)^{\frac{1}{q}}\|z_n\|_{2r}^2=o(1).
\end{aligned}\tag{30}$$

Thus, from (6), (12), (29) and (30), for $\lambda\leqslant\lambda^*$, we have

$$\begin{aligned}
\frac{1}{2}\min\{a,1\}+o(1) &= \frac{\frac{1}{2}\min\{a,1\}\|u_n\|^2-\langle I'(u_n),u_n\rangle}{\|u_n\|^2} \\
&\leqslant \frac{\min\{a,1\}\|u_n\|^2-a\|\nabla u_n\|_2^2-\int_{\mathbb{R}^2}V(x)u_n^2\mathrm{d}x}{\|u_n\|^2} \\
&\quad-\frac{\frac{1}{2}\min\{a,1\}\|u_n\|^2-\lambda\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(1+|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y}{\|u_n\|^2} \\
&\quad+\frac{-\lambda\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\left(1+\frac{1}{|x-y|}\right)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^2}f(u_n)u_n\mathrm{d}x}{\|u_n\|^2} \\
&\leqslant \int_{A_n}\left|\frac{f(u_n)}{u_n}\right|z_n^2\mathrm{d}x+\int_{\mathbb{R}^2\setminus A_n}\left|\frac{f(u_n)}{u_n}\right|z_n^2\mathrm{d}x \\
&= o(1),
\end{aligned}$$

which is a contradiction. Then, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. □

In order to obtain the existence of nontrivial solutions for (1), now we show the following lemma.

Lemma 4 ([15] Lemma 2.1)**.** *Let $\{u_n\}$ be a sequence in $L^2(\mathbb{R}^2)$ satisfying $u_n\to u\in L^2(\mathbb{R}^2)\setminus\{0\}$ a.e. on $\mathbb{R}^2$. If $\{w_n\}$ is a bounded sequence in $L^2(\mathbb{R}^2)$, such that*

$$\sup_{n\in\mathbb{N}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(1+|x-y|)u_n^2(x)w_n^2(y)\mathrm{d}x\mathrm{d}y<\infty,$$

then $\{\|w_n\|_*\}$ is bounded. If, moreover,

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(1+|x-y|)u_n^2(x)w_n^2(y)\mathrm{d}x\mathrm{d}y\to 0,\ \|w_n\|_2\to 0 \text{ as } n\to\infty,$$

then $\|w_n\|_*\to 0$ as $n\to\infty$.

3. Proof of Theorem 1

In this section, we give the proof of Theorem 1.

First of all, in view of Lemmas 2 and 3, for some constant $\rho>0$, there is a sequence $\{u_n\}\subset E$ that satisfies $\|u_n\|^2\leqslant\rho$ and (12). Here, we claim

$$\delta:=\limsup_{n\to\infty}\sup_{y\in\mathbb{R}^2}\int_{B_2(y)}|u_n|^2\mathrm{d}x>0.$$

Actually, if $\delta=0$, using the Lions' concentration compactness principle (see [39], Lemma 1.21), one has $u_n\to 0$ as $n\to\infty$ in $L^s(\mathbb{R}^2)$, $s\in(2,\infty)$. Thus, from (10), we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\left(1+\frac{1}{|x-y|}\right)u_n^2(x)u_n^2(y)\mathrm{d}x\mathrm{d}y\to 0 \text{ as } n\to\infty.$$

From (14), fix $\varepsilon = c/(3\rho)$, we have

$$\int_{\mathbb{R}^2} \left| \frac{1}{2} f(u_n)u_n - F(u_n) \right| \mathrm{d}x \leqslant \frac{3}{2}\varepsilon \|u_n\|_2^2 + C_\varepsilon \|u_n\|_p^p \leqslant \frac{c}{2} + o(1). \tag{31}$$

Using (6), (12), (15) and (31), one has

$$\begin{aligned} c + o(1) &= I(u_n) - \frac{1}{2}\langle I'(u_n), u_n \rangle \\ &= -\frac{b}{4}\|\nabla u_n\|_2^4 + \frac{\lambda}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1 + \frac{1}{|x-y|}\right) u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y \\ &\quad -\frac{\lambda}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1 + \frac{1}{|x-y|})u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^2}\left[\frac{1}{2}f(u_n)u_n - F(u_n)\right]\mathrm{d}x \\ &\leqslant \frac{c}{2} + o(1), \end{aligned}$$

which yields a contradiction; then, $\delta > 0$.

Passing to a subsequence if necessary, we suppose that there is $y_n \in \mathbb{R}^2$, satisfying

$$\int_{B_1(y_n)} |u_n|^2 \mathrm{d}x > \frac{\delta}{2}.$$

Define $\hat{u}_n(x) := u_n(x + y_n)$, then

$$\int_{B_1(0)} |\hat{u}_n|^2 \mathrm{d}x > \frac{\delta}{2}. \tag{32}$$

From (12), we conclude that

$$I(\hat{u}_n) \to c > 0, \ \ \langle I'(\hat{u}_n), \hat{u}_n \rangle \to 0, \ \text{ as } n \to \infty. \tag{33}$$

Passing to a subsequence if necessary, we have $\hat{u}_n \rightharpoonup \hat{u}$ in $H^1(\mathbb{R}^2)$, $\hat{u}_n \to \hat{u}$ in $L^s_{loc}(\mathbb{R}^2)$ for $2 \leqslant s < +\infty$, $\hat{u}_n \to \hat{u}$ a.e. on $\mathbb{R}^2$ as $n \to \infty$. And then, using (32), one has $\hat{u} \neq 0$. By using the boundedness of $\{\|\hat{u}_n\|\}$ in E, going to a subsequence if necessary, we deduce

$$\hat{u}_n \rightharpoonup \hat{u} \text{ in } E, \hat{u}_n \to \hat{u} \text{ in } L^q(\mathbb{R}^2) \text{ for } 2 \leqslant q < +\infty, \hat{u}_n \to \hat{u} \text{ a.e. on } \mathbb{R}^2 \text{ as } n \to \infty. \tag{34}$$

Next, we certify that $I'(\hat{u}) = 0$. To this end, we claim

$$\langle I'(\hat{u}), w \rangle = \lim_{n\to\infty} \langle I'(\hat{u}_n), w \rangle = \lim_{n\to\infty} \langle I'(u_n), w(x - y_n) \rangle = 0, \ \ \forall w \in E. \tag{35}$$

Actually,

$$\left| \langle I'(\hat{u}_n), w \rangle \right| = \left| \langle I'(u_n), w(x - y_n) \rangle \right| \leqslant \|I'(u_n)\|_{E^*} \|w\|_E = o(1), \ \forall w \in E. \tag{36}$$

Thus,

$$\langle I'(\hat{u}_n), \hat{u} \rangle = o(1). \tag{37}$$

From $(\mathcal{F}_1)$, $(\mathcal{F}_2)$, (34) and the Lebesgue's dominated convergence theorem, one can conclude that

$$\int_{\mathbb{R}^2} f(\hat{u}_n)(\hat{u}_n - \hat{u})\mathrm{d}x = o(1). \tag{38}$$

Furthermore, by (10) and (34), one has

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1 + \frac{1}{|x-y|}\right) \hat{u}_n^2(x)(\hat{u}_n(y) - \hat{u}(y))\hat{u}_n(y)\mathrm{d}x\mathrm{d}y \leqslant C\|\hat{u}_n\|_{8/3}^3 \|\hat{u}_n - \hat{u}\|_{8/3} = o(1). \tag{39}$$

Analogously to [15], Lemma 2.6, we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(1+|x-y|)\hat{u}_n^2(x)(\hat{u}_n(y)-\hat{u}(y))w(y)\mathrm{d}x\mathrm{d}y=o(1),\ \ \forall w\in E. \tag{40}$$

Setting $w=\hat{u}_n-\hat{u}$, we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(1+|x-y|)\hat{u}_n^2(x)(\hat{u}_n(y)-\hat{u}(y))^2\mathrm{d}x\mathrm{d}y=o(1). \tag{41}$$

From (33), (34), (37)–(39) and (41), we know

$$\begin{aligned}
o(1) &= \langle I'(\hat{u}_n),\hat{u}_n-\hat{u}\rangle\\
&= a\|\nabla\hat{u}_n\|_2^2-a\|\nabla\hat{u}\|_2^2+b\|\nabla\hat{u}_n\|_2^4-b\|\nabla\hat{u}\|_2^4+\int_{\mathbb{R}^2}V(x)\hat{u}_n^2\mathrm{d}x-\int_{\mathbb{R}^2}V(x)\hat{u}^2\mathrm{d}x\\
&\quad-\lambda\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(1+|x-y|)\hat{u}_n^2(x)(\hat{u}_n(y)-\hat{u}(y))^2\mathrm{d}x\mathrm{d}y\\
&\quad-\lambda\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(1+|x-y|)\hat{u}_n^2(x)(\hat{u}_n(y)-\hat{u}(y))\hat{u}(y)\mathrm{d}x\mathrm{d}y\\
&\quad+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\left(1+\frac{1}{|x-y|}\right)\hat{u}_n^2(x)(\hat{u}_n(y)-\hat{u}(y))\hat{u}_n(y)\mathrm{d}x\mathrm{d}y-\int_{\mathbb{R}^2}f(\hat{u}_n)(\hat{u}_n-\hat{u})\mathrm{d}x\\
&= a\|\nabla\hat{u}_n\|_2^2-a\|\nabla\hat{u}\|_2^2+b\|\nabla\hat{u}_n\|_2^4-b\|\nabla\hat{u}\|_2^4\\
&\quad-\lambda\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(1+|x-y|)\hat{u}_n^2(x)(\hat{u}_n(y)-\hat{u}(y))^2\mathrm{d}x\mathrm{d}y+o(1),
\end{aligned}$$

then, we deduce that as $n\to\infty$

$$\|\hat{u}_n-\hat{u}\|\to 0$$

since $\hat{u}_n\rightharpoonup\hat{u}$ in $H^1(\mathbb{R}^2)$. In terms of Lemma 4, we have $\|\hat{u}_n-\hat{u}\|_*\to 0$. Then,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(1+|x-y|)(\hat{u}_n^2(x)-\hat{u}^2(x))\hat{u}(y)w(y)\mathrm{d}x\mathrm{d}y\right|\\
\leqslant\ &\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}[\ln(1+|x|)+\ln(1+|y|)]|\hat{u}_n^2(x)-\hat{u}^2(x)|\cdot|\hat{u}(y)w(y)|\mathrm{d}x\mathrm{d}y\\
\leqslant\ &\|\hat{u}_n-\hat{u}\|_*\|\hat{u}_n+\hat{u}\|_*\|\hat{u}\|_2\|w\|_2+\|\hat{u}_n-\hat{u}\|_2\|\hat{u}_n+\hat{u}\|_2\|\hat{u}\|_*\|w\|_*\\
=\ &o(1),\ \ \forall w\in E.
\end{aligned} \tag{42}$$

Similar to (38) and (39), one has

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\left(1+\frac{1}{|x-y|}\right)\hat{u}_n^2(x)(\hat{u}_n(y)-\hat{u}(y))w(y)\mathrm{d}x\mathrm{d}y=o(1), \tag{43}$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\left(1+\frac{1}{|x-y|}\right)(\hat{u}_n^2(x)-\hat{u}^2(x))\hat{u}(y)w(y)\mathrm{d}x\mathrm{d}y=o(1) \tag{44}$$

and

$$\int_{\mathbb{R}^2}[f(\hat{u}_n)-f(\hat{u})]w\mathrm{d}x=o(1),\ \ \forall w\in E. \tag{45}$$

From (34), (40)–(45), we have

$$
\begin{aligned}
& \langle I'(\hat{u}_n) - I'(\hat{u}), w\rangle \\
= \; & a(\nabla \hat{u}_n - \nabla \hat{u}, \nabla w) + b\|\nabla \hat{u}_n\|_2^2(\nabla \hat{u}_n, \nabla w) - b\|\nabla \hat{u}\|_2^2(\nabla \hat{u}, \nabla w) \\
& + \int_{\mathbb{R}^2} V(x)\hat{u}_n w \mathrm{d}x - \int_{\mathbb{R}^2} V(x)\hat{u} w \mathrm{d}x - \lambda \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)\hat{u}_n^2(x)(\hat{u}_n(y)-\hat{u}(y))w(y)\mathrm{d}x\mathrm{d}y \\
& -\lambda \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)(\hat{u}_n^2(x)-\hat{u}^2(x))\hat{u}(y)w(y)\mathrm{d}x\mathrm{d}y \\
& +\lambda \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right)\hat{u}_n^2(x)(\hat{u}_n(y)-\hat{u}(y))w(y)\mathrm{d}x\mathrm{d}y \\
& +\lambda \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right)(\hat{u}_n^2(x)-\hat{u}^2(x))\hat{u}(y)w(y)\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^2}[f(\hat{u}_n)-f(\hat{u})]w\mathrm{d}x \\
= \; & o(1), \;\; \forall w \in E.
\end{aligned} \tag{46}
$$

Thus, based on (36) and (46), we can deduce that (35) is true. Consequently, $\hat{u} \in E$ is a nontrivial solution to (1) and $I(\hat{u}) = c > 0$.

Now, we define

$$\mathcal{N} := \{u \in E\backslash\{0\} : I'(u) = 0\}.$$

Note that $\hat{u} \in \mathcal{N}$, one has $\mathcal{N} \neq \emptyset$. From $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$, one has

$$|f(u)u| \leqslant \frac{1}{2}\min\{a,1\}u^2 + C|u|^p, \;\; \forall u \in \mathbb{R}. \tag{47}$$

For $u \in \mathcal{N}$, $\langle I'(u), u\rangle = 0$, in terms of (15), (47) and the Sobolev embedding inequality, we obtain

$$
\begin{aligned}
\min\{a,1\}\|u\|^2 \;\; \leqslant \;\; & a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 + \int_{\mathbb{R}^2} V(x)u^2\mathrm{d}x - \lambda\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(1+|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y \\
& +\lambda\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\left(1+\frac{1}{|x-y|}\right)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y \\
= \;\; & \int_{\mathbb{R}^2} f(u)u\mathrm{d}x \\
\leqslant \;\; & \frac{1}{2}\min\{a,1\}\|u\|^2 + \bar{C}\|u\|^p, \;\; \forall u \in \mathcal{N},
\end{aligned} \tag{48}
$$

which implies

$$\|u\| \geqslant \delta_0 := \left(\frac{1}{2C}\min\{a,1\}\right)^{\frac{1}{p-2}} > 0, \;\; \forall u \in \mathcal{N}. \tag{49}$$

We can conclude that $\inf_{\mathcal{N}} I > -\infty$. Choosing $\{u_n\} \subset \mathcal{N}$ such that $I(u_n) \to \inf_{\mathcal{N}} I$. Then, (12) holds. By applying Lemma 3, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. Now, we claim that $\{u_n\}$ does not vanish. If not, using the Lions' concentration compactness principle (see, for example, [5]), we have $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for $s \in (2,\infty)$. Hence, it follows from (10) and (14) that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\left(1+\frac{1}{|x-y|}\right)u_n^2(x)u_n^2(y)\mathrm{d}x\mathrm{d}y = o(1), \;\; \int_{\mathbb{R}^2} f(u_n)u_n\mathrm{d}x = o(1),$$

which contradicts (48) and (49). Consequently, by applying the same argument as above, we obtain that there exists $u_0 \in \mathcal{N}$ that satisfies $I(u_0) = \inf_{\mathcal{N}} I > -\infty$. Therefore, $u_0 \in E$ is the least energy solution to Problem (1). □

Remark 2. *It is natural to ask whether the nontrivial solution $\hat{u}$ of Problem (1) is equal to the least energy solution u_0. This would be an interesting question to explore in the following work.*

4. Proof of Theorem 2

To prove the existence of ground state solutions for Equation (1), we firstly illustrate some important lemmas.

Lemma 5. *Assume that* $(\mathcal{F}_1), (\mathcal{F}_2)$ *and* $(\mathcal{F}_5)$ *hold. Then*

$$g(t,u) := \frac{1-t^8}{4} f(u)u + \frac{t^8-5}{4} F(u) + \frac{1}{t^2} F(t^2 u) \geqslant 0, \ \ \forall t > 0, u \in \mathbb{R}. \tag{50}$$

Proof. It is clear that (50) holds for $u = 0$. For $u \neq 0$, by $(\mathcal{F}_5)$, we have

$$\begin{aligned} \frac{\mathrm{d}(g(t,u))}{\mathrm{d}t} &= 2t^7 u^2 \left[\frac{f(t^2 u)t^2 u - F(t^2 u)}{t^4 u^2} - \frac{f(u)u - F(u)}{u^2} \right] \\ & \begin{cases} \geqslant 0, t \geqslant 1, \\ \leqslant 0, 0 < t < 1, \end{cases} \end{aligned}$$

then, $g(t,u) \geqslant g(1,u) = 0$ for $t > 0$. □

Lemma 6. *Suppose that* $(\mathcal{V})$ *and* $(\mathcal{V}_2)$ *hold. Then,*

$$h(t) := \frac{1}{2} V(x) - \frac{1}{2} t^2 V(t^{-1}x) - \frac{1-t^8}{8} V(x) + \frac{1-t^8}{16} (\nabla V(x), x) \geqslant 0, \ \ \forall t > 0. \tag{51}$$

Proof. Based on the calculation,

$$\begin{aligned} h'(t) &= \frac{1}{2} t^7 x^6 \left[\frac{(\nabla V(t^{-1}x), t^{-1}x) - 2V(t^{-1}x)}{t^6 x^6} - \frac{(\nabla V(x), x) - 2V(x)}{x^6} \right] \\ & \begin{cases} \geqslant 0, t \geqslant 1, \\ \leqslant 0, 0 < t < 1, \end{cases} \end{aligned}$$

then, $h(t) \geqslant h(1) = 0$ for $t > 0$. □

Lemma 7. *Assume that* $(\mathcal{V})$, $(\mathcal{V}_2)$, $(\mathcal{F}_1)$, $(\mathcal{F}_2)$ *and* $(\mathcal{F}_5)$ *hold. Then, there exists* $\lambda^{**} > 0$ *such that, for* $\lambda \leqslant \lambda^{**}$,

$$I(u) \geqslant I(t^2 u_t) + \frac{1-t^8}{8} J(u), \ \ \forall u \in E, \ t > 0, \tag{52}$$

$$I(u) \geqslant \frac{1}{8} J(u) + \frac{\lambda}{32} \|u\|_2^4, \ \ \forall u \in E. \tag{53}$$

Proof. Choosing $\lambda^{**} > 0$ to be sufficiently small such that, for $\lambda \leqslant \lambda^{**}$,

$$4a(1-t^4)^2 \|\nabla u\|_2^2 - \lambda(1-t^8)\|u\|_2^4 - 8\lambda t^4 \ln t \|u\|_2^4 - 4\lambda(1-t^4)^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x-y|) u^2(x) u^2(y) \mathrm{d}x\mathrm{d}y \geqslant 0 \tag{54}$$

holds. By (50), (51), (54), (8) and (13), one has

$$
\begin{aligned}
I(u)-I(t^2u_t) &= \frac{a}{2}\int_{\mathbb{R}^2}(1-t^4)|\nabla u|^2\mathrm{d}x+\frac{b\|\nabla u\|_2^2}{4}\int_{\mathbb{R}^2}(1-t^8)|\nabla u|^2\mathrm{d}x\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^2}[V(x)-t^2V(t^{-1}x)]u^2\mathrm{d}x-\frac{\lambda}{4}(1-t^4)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y\\
&\quad-\frac{\lambda t^4\ln t}{4}\|u\|_2^4+\int_{\mathbb{R}^2}\left[\frac{1}{t^2}F(t^2u)-F(u)\right]\mathrm{d}x\\
&= \frac{1-t^8}{8}J(u)+\frac{(1-t^4)^2}{4}a\|\nabla u\|_2^2-\frac{\lambda(1-t^8)}{32}\|u\|_2^4-\frac{\lambda t^4\ln t}{8\pi}\|u\|_2^4\\
&\quad-\frac{\lambda(1-t^4)^2}{8}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y+\frac{1}{2}\int_{\mathbb{R}^2}V(x)u^2\mathrm{d}x\\
&\quad-\frac{1}{2}t^2\int_{\mathbb{R}^2}V(t^{-1}x)u^2\mathrm{d}x-\frac{1-t^8}{8}\int_{\mathbb{R}^2}V(x)u^2\mathrm{d}x+\frac{1-t^8}{16}\int_{\mathbb{R}^2}(\nabla V(x),x)u^2\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^2}\left[\frac{1}{t^2}F(t^2u)+\frac{1-t^8}{4}f(u)u+\frac{t^8-5}{4}F(u)\right]\mathrm{d}x\\
&\geqslant \frac{1-t^8}{8}J(u),\ \ \forall u\in E,\ t>0,
\end{aligned}
$$

then, (52) holds. Furthermore, in terms of Lemmas 5 and 6, we have

$$\lim_{t\to 0}g(t,u)=\frac{1}{4}f(u)u-\frac{5}{4}F(u)\geqslant 0,\ u\in\mathbb{R} \tag{55}$$

and

$$\lim_{t\to 0}h(t)=\frac{3}{8}V(x)+\frac{1}{16}(\nabla V(x),x)\geqslant 0,\ x\in\mathbb{R}. \tag{56}$$

Moreover, let $t\to 0$ in (54); one has

$$4a\|\nabla u\|_2^2-\lambda\|u\|_2^4-4\lambda\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y\geqslant 0. \tag{57}$$

Hence, using (55), (56), (57) and (15), we have

$$
\begin{aligned}
I(u)-\frac{1}{8}J(u) &= \frac{a}{4}\|\nabla u\|_2^2+\frac{3}{8}\int_{\mathbb{R}^2}V(x)u^2\mathrm{d}x+\frac{1}{16}\int_{\mathbb{R}^2}(\nabla V(x),x)u^2\mathrm{d}x\\
&\quad-\frac{\lambda}{8}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y+\frac{\lambda}{32}\|u\|_2^4+\frac{1}{4}\int_{\mathbb{R}^2}[f(u)u-5F(u)]\mathrm{d}x\\
&\geqslant \frac{\lambda}{32}\|u\|_2^4,\ \ \forall u\in E.
\end{aligned}
$$

Thus, (53) holds. □

By virtue of Lemma 7, we deduce the following corollary.

Corollary 1. *Assume that $(\mathcal{V})$, $(\mathcal{V}_2)$, $(\mathcal{F}_1)$, $(\mathcal{F}_2)$ and $(\mathcal{F}_5)$ hold. Then, there is $\lambda^{**}>0$ such that, for $\lambda\leqslant\lambda^{**}$,*

$$I(u)=\max_{t>0}I(t^2u_t),\ \ \forall u\in\mathcal{M}. \tag{58}$$

Lemma 8. *Assume that $(\mathcal{V})$, $(\mathcal{V}_2)$, $(\mathcal{F}_1)$–$(\mathcal{F}_3)$ and $(\mathcal{F}_5)$ hold. Then, for $u\in E\backslash\{0\}$, there is a constant $t(u)>0$, such that $[t(u)]^2u_{t(u)}\in\mathcal{M}$.*

Proof. Let $u\in E\backslash\{0\}$, define $\eta(t):=I(t^2u_t)$ for $t\in(0,\infty)$, then we have

$$\begin{aligned}\eta'(t)=0 \quad &\Leftrightarrow \quad 2at^3\|\nabla u\|_2^2+2bt^7\|\nabla u\|_2^4+t\int_{\mathbb{R}^2}V(t^{-1}x)u^2\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^2}(\nabla V(t^{-1}x),x)u^2\mathrm{d}x\\ &\quad -t^3\lambda\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y+\frac{4t^3\ln t+t^3}{4}\lambda\|u\|_2^4+\frac{2}{t^3}\int_{\mathbb{R}^2}F(t^2u)\mathrm{d}x\\ &\quad -\frac{2}{t}\int_{\mathbb{R}^2}f(t^2u)u\mathrm{d}x=0\\ &\Leftrightarrow \quad J(t^2u_t)=0\\ &\Leftrightarrow \quad t^2u_t\in\mathcal{M},\ \ \forall t>0.\end{aligned}$$

From $(\mathcal{F}_1)$–$(\mathcal{F}_3)$, one can easily determine that $\lim\limits_{t\to 0}\eta(t)=0$, $\eta(t)>0$ for t sufficiently small and $\eta(t)<0$ for t large enough. Therefore, there is a constant $t(u)>0$, such that $\eta(t(u))=\max_{t>0}\eta(t)$. Hence, $\eta'(t(u))=0$, and then, $t(u)^2u_{t(u)}\in\mathcal{M}$. □

By applying Corollary 1 and Lemma 8, we obtain the following lemma.

Lemma 9. *Assume that $(\mathcal{V})$, $(\mathcal{V}_2)$, $(\mathcal{F}_1)$–$(\mathcal{F}_3)$ and $(\mathcal{F}_5)$ hold. Then,*

$$\inf_{u\in\mathcal{M}}I(u):=m_1=\inf_{u\in E\setminus\{0\}}\max_{t>0}I(t^2u_t).$$

Lemma 10. *Assume that $(\mathcal{V}_2)$, $(\mathcal{F}_1)$–$(\mathcal{F}_3)$ and $(\mathcal{F}_5)$ hold. Then,*

(i) there is a constant $\varrho>0$ that satisfies $\|u\|\geqslant\varrho,\forall u\in\mathcal{M}$;
(ii) $m_1=\inf\limits_{u\in\mathcal{M}}I(u)>0$.

Proof. (i) It follows from $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$ that

$$|f(u)u|+|F(u)|\leqslant\frac{\min\{a,1\}}{4}u^2+C|u|^p,\ \ \forall u\in\mathbb{R}. \tag{59}$$

For any $u\in\mathcal{M}$, we have $J(u)=0$. From (54), (8), (59), Hardy–Littlewood–Sobolev inequality and Sobolev emmbedding inequality, we know

$$\begin{aligned}\min\{a,1\}\|u\|^2 &\leqslant 2a\|\nabla u\|_2^2+2b\|\nabla u\|_2^4+\frac{5}{2}\int_{\mathbb{R}^2}V(x)u^2\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^2}(\nabla V(x),x)u^2\mathrm{d}x\\ &\quad -\lambda\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y+\frac{\lambda}{4}\|u\|_2^4\\ &= 2\int_{\mathbb{R}^2}[f(u)u-F(u)]\mathrm{d}x\\ &\leqslant \frac{1}{2}\min\{a,1\}\|u\|^2+C\|u\|^p,\ \ \forall u\in\mathcal{M},\end{aligned}$$

which implies

$$\|u\|\geqslant\varrho:=\left(\frac{1}{2C}\min\{a,1\}\right)^{\frac{1}{p-2}},\ \ \forall u\in\mathcal{M}. \tag{60}$$

(ii) We may choose $\{u_n\}\subset\mathcal{M}$, satisfying $I(u_n)\to m_1$. There are two cases that need to be distinguished: $\inf\limits_{n\in\mathbb{N}}\|u_n\|_2>0$ and $\inf\limits_{n\in\mathbb{N}}\|u_n\|_2=0$.

Case 1: $\inf\limits_{n\in\mathbb{N}}\|u_n\|_2:=\varrho_1>0$, from (53), one has

$$m_1+o(1)=I(u_n)\geqslant\frac{\lambda}{32}\|u_n\|_2^4\geqslant\frac{\lambda}{32}\varrho_1^4.$$

Case 2: $\inf\limits_{n\in\mathbb{N}} \|u_n\|_2 = 0$, by (60), passing if necessary to a subsequence, one has

$$\|u_n\|_2 \to 0, \ \ \|\nabla u_n\|_2 \geqslant \varrho. \tag{61}$$

Then,

$$\frac{|\ln(\|\nabla u_n\|_2)|}{\|\nabla u_n\|_2^2} \leqslant C. \tag{62}$$

Set $t_n = \|\nabla u_n\|_2^{-\frac{1}{2}}$, for any $u_n \in \mathcal{M}$, by (10), (13) and (14) in Lemma 2, together with (57), (58), (61), (62) and the Gagliardo–Nirenberg inequality, we have

$$\begin{aligned}
m_1 + o(1) &= I(u_n) \geqslant I(t_n^2 (u_n)_{t_n}) \\
&= \frac{a}{2} t_n^4 \|\nabla u_n\|_2^2 + \frac{b}{4} t_n^8 \|\nabla u_n\|_2^4 + \frac{t_n^2}{2} \int_{\mathbb{R}^2} V(t_n^{-1} x) u_n^2 \mathrm{d}x \\
&\quad - \frac{\lambda t_n^4}{4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x-y|) u_n^2(x) u_n^2(y) \mathrm{d}x\mathrm{d}y + \frac{\lambda t_n^4 \ln t_n}{4} \|u_n\|_2^4 - \frac{1}{t_n^2} \int_{\mathbb{R}^2} F(t_n^2 u_n) \mathrm{d}x \\
&\geqslant \frac{a}{2} t_n^4 \|\nabla u_n\|_2^2 + \frac{\lambda t_n^4 \ln t_n}{4} \|u_n\|_2^4 - t_n^2 \|u_n\|_2^2 - C t_n^{2p-2} \|u_n\|_2^2 \|\nabla u_n\|_2^{p-2} + o(1) \\
&= \frac{a}{2} - \frac{\lambda \ln(\|\nabla u_n\|_2)}{8 \|\nabla u_n\|_2^2} \|u_n\|_2^4 - \frac{\|u_n\|_2^2}{\|\nabla u_n\|_2} - C \frac{\|u_n\|_2^2}{\|\nabla u_n\|_2} + o(1) \\
&= \frac{a}{2} + o(1).
\end{aligned}$$

Combining the above two cases, we conclude that $m_1 = \inf\limits_{u\in\mathcal{M}} I(u) > 0$. □

Motivated by [11] (Lemma 4.7), we verify that the Cerami sequence given in Lemma 2 is also a minimizing sequence.

Lemma 11. *Suppose that* $(\mathcal{F}_1)$*–*$(\mathcal{F}_3)$ *and* $(\mathcal{F}_5)$ *are satisfied. Then, there exists a sequence* $\{u_n\} \subset E$*, such that*

$$I(u_n) \to c \in (0, m_1], \ \ \|I'(u_n)\|_{E^*}(1 + \|u_n\|) \to 0, \ \ J(u_n) \to 0. \tag{63}$$

Proof. By means of Lemmas 9 and 10, we choose $v_n \in \mathcal{M}$, such that

$$0 < m_1 \leqslant I(v_n) < m_1 + \frac{1}{n}, \ \ \forall n \in \mathbb{N}. \tag{64}$$

In terms of Lemma 2, there is a sequence $\{u_n\} \subset E$ that satisfies (12) for $n \in \mathbb{N}$. And then, we can choose $T_n > 0$, such that $I(T_n^2 (v_n)_{T_n}) < 0$. Next, we define $\gamma_n(t) = (tT_n)^2 (v_n)_{tT_n}$ for $t \in [0,1]$. Then, $\gamma_n \in \Gamma$. Moreover, according to (16), one has

$$c \in [d, \sup_{t>0} I(t^2 (v_n)_t)].$$

According to Corollary 1, one has

$$I(v_n) = \sup_{t>0} I(t^2 (v_n)_t).$$

Hence, using (64), one has

$$c \leqslant \sup_{t>0} I(t^2 (v_n)_t) < m_1 + \frac{1}{n}, \ \ \forall n \in \mathbb{N}. \tag{65}$$

Let $n \to \infty$ in (65); in terms of Lemma 2, we obtain the desired conclusion. □

Proof of Theorem 2. By virtue of Lemma 11, there is a sequence $\{u_n\} \subset E$ that satisfies (63). From (53) and (63), one has

$$c + o(1) = I(u_n) - \frac{1}{8}J(u_n) \geqslant \frac{\lambda}{32}\|u_n\|_2^4, \tag{66}$$

which yields the boundedness of $\{\|u_n\|_2\}$. And then, we verify that $\{\|\nabla u_n\|_2\}$ is bounded. With reduction to absurdity, we suppose $\|\nabla u_n\|_2 \to \infty$. Set $t_n = \left(\frac{4\sqrt{m_1}}{\sqrt{a}\|\nabla u_n\|_2}\right)^{\frac{1}{2}}$, we have $t_n \to 0$ and $t_n^4 \ln t_n \to 0$ as $n \to \infty$. Then, according to (10), (13) and (14) in Section 2, together with (52), (54), (63), (66) and the Gagliardo–Nirenberg inequality, one has

$$\begin{aligned}
m_1 + o(1) &\geqslant c + o(1) = I(u_n) \\
&\geqslant I(t_n^2 (u_n)_{t_n}) + \frac{1 - t_n^8}{8}J(u_n) \\
&= \frac{a}{2}t_n^4\|\nabla u_n\|_2^2 + \frac{b}{4}t_n^8\|\nabla u_n\|_2^4 + \frac{t_n^2}{2}\int_{\mathbb{R}^2} V(t_n^{-1}x)u_n^2 \mathrm{d}x \\
&\quad - \frac{\lambda t_n^4}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(|x-y|)u_n^2(x)u_n^2(y)\mathrm{d}x\mathrm{d}y + \frac{\lambda t_n^4 \ln t_n}{8\pi}\|u_n\|_2^4 - \frac{1}{t^2}\int_{\mathbb{R}^2} F(t_n^2 u_n)\mathrm{d}x \\
&\geqslant \frac{a}{4}t_n^4\|\nabla u_n\|_2^2 - t_n^2\|u_n\|_2^2 - Ct_n^{2p-2}\|u_n\|_p^p + o(1) \\
&\geqslant \frac{a}{4}t_n^4\|\nabla u_n\|_2^2 - t_n^2\|u_n\|_2^2 - Ct_n^{2p-2}\|u_n\|_2^2\|\nabla u_n\|_2^{p-2} + o(1) \\
&= 4m_1 - \frac{4\sqrt{m_1}}{\sqrt{a}\|\nabla u_n\|_2}\|u_n\|_2^2 - \frac{C(\sqrt{m_1})^{p-1}}{(\sqrt{a})^{p-1}\|\nabla u_n\|_2}\|u_n\|_2^2 + o(1) \\
&= 4m_1 + o(1),
\end{aligned}$$

which is impossible; hence, $\{\|\nabla u_n\|_2\}$ is bounded. Consequently, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. By applying the similar arguments as those in the proof of Theorem 1, we conclude that there exists $\tilde{u} \in E \backslash \{0\}$, such that

$$I'(\tilde{u}) = 0, \quad I(\tilde{u}) = c \in (0, m_1].$$

Moreover, from $\tilde{u} \in \mathcal{M}$, one has $I(\tilde{u}) \geqslant m_1$. Thus, $\tilde{u} \in E$ is a ground state solution of (1). This completes the proof. □

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