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Rights Systems and Aggregation Functions on Property Spaces

Marta Cardin

Department of Economics, Ca'Foscari University of Venice, Sestiere Cannaregio 873, 30121 Venezia, Italy; mcardin@unive.it

Abstract: The notion of decisive coalitions of voters with different grades of decisiveness is a part of the mathematical framework for many models in social choice theory. More generally, we study aggregation problems in which a subgroup of decision makers have the right to determine the properties of the aggregate. Then, we introduce property spaces and rights to properties and characterize aggregation operators that are consistent with rights to properties. Moreover, we define congruences in property spaces, and we propose a generalization of the Sugeno integral in our framework.

Keywords: property spaces; rights systems; aggregation functions; congruences; compatibility

MSC: 06A15 ; 06A75 ; 54A05

1. Introduction

Multicriteria choice problems are characterized by a plurality of points of view, and there are different dimensions from which the alternatives can be viewed. In order to solve a decision problem, we have to compare and rank a set of alternatives, and each alternative is often defined by its attributes or properties. We consider the abstract aggregation model recently studied in [1] that represents a decision problem in terms of a set of Boolean properties specifying a list of properties that are satisfied for every alternative. The "property space" model has received attention in the literature on social choice, judgment aggregation and aggregation of preferences (see [2,3]).

In this paper, we consider a property space as a convex space since every property space defines a convex space. As it is well known, the notion of convexity is a basic mathematical structure, and in the literature, there are various kinds of generalized, topological or axiomatically defined convex structures. Van De Vel analyzed the theory of convexity systematically in [4].

We consider abstract convex structures that are combinatorial objects studied in various areas of mathematics. Abstract notions of convexity are considered in different environments, such as vector spaces, metric spaces, graphs, matroids, median algebras, lattices and so on.

There are also many applications in different fields; for example, in economic theory and in rough set theory. Then, we can refer to the general definition of convex preferences proposed in [5] where abstract convex spaces are considered underlying universes with an abstract convex structure. We can also refer to the important notion of abstract convex geometry due to [6]; that is, an abstraction of the standard notion of convexity in a linear space that is at the root of the convex rough sets defined recently in [7]. Then, we consider aggregation operators that are compatible with rights systems; that is to say, they allow subgroups of decision makers to enforce a property to which they hold a right.

Moreover, we define and study congruences in property spaces as convex spaces, and then we prove that the aggregation operators in property spaces considered in [1,2] can be viewed as Sugeno integrals in property spaces since the operators are compatible with congruences (see [8–10]).

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The current paper is organized as follows. In Section 2, we provide the basic notations and definitions we work with. In Section 3, we consider rights-consistent aggregation functions, while in Section 4, we define congruences in convexity spaces and in property spaces, and some basic information concerning Sugeno integrals is given. Then, we study aggregation operators that are compatible with congruences. Finally, some concluding remarks are added.

2. Model and Preliminaries

In this section, we recall some basic definitions of lattice theory. Then, we introduce the notion of property space and define rights to properties.

2.1. Preliminaries on Lattices

In this section, we briefly recall basic definitions needed for our purposes. We assume that the reader is familiar with the basic notions of lattice theory and refer the reader to [11,12].

Throughout this paper, we assume that all lattices are distributive and bounded with meet and join operations denoted by $\wedge$ and $\vee$, respectively. We denote the greatest and the least element with $0, 1$, respectively, and we assume that $0 \neq 1$.

A *filter* of a finite lattice L is a subset F of L such that

(i) if $x \in F$ and $x \leq y$, then $y \in F$;
(ii) if $x, y \in F$, then $x \vee y \in F$.

Any ordered set subsets satisfying (i) are called upsets.

The dual notion is that of an ideal. An *ideal* of a lattice L is a subset I of L such that

(i) if $x \in I$ and $y \leq x$, then $y \in I$;
(ii) if $x, y \in I$, then $x \wedge y \in I$.

A *proper filter* is a filter that is neither empty nor the whole lattice, while a *prime filter* is a proper filter F such that whenever $x \vee y \in F$, we have $x \in F$ or $y \in F$.

A *proper ideal* is an ideal that is neither empty nor the whole lattice, while a *prime ideal* is a proper ideal I such that whenever $x \wedge y \in I$, we have $x \in I$ or $y \in I$.

In any lattice L, $F \subset L$ is a prime filter if an only if its complement F^c is a prime ideal. Moreover, it can be proved that if x, y are two elements of a finite distributive lattice L and $x \not\geq y$, there exists a prime filter F where $y \in F$ and $x \notin F$. If $x \in L$, let $\uparrow x = \{y \in L : y \geq x\}$. It is easy to prove that $\uparrow x$ is a filter for every $x \in L$ that is called the *principal filter* generated by x. It can be proved that in a finite lattice, each filter and each ideal are principal.

If L is a finite lattice, $p \in L$ is said to be *join prime* if $p \leq x \vee y$, and then $p \leq x$ or $p \leq y$. A filter F of a finite distributive lattice is prime if and only if $F = \uparrow p$ for some join prime element p of L.

2.2. Property Spaces

In this section, we consider the model of abstract aggregation studied in [1,2,13,14] that describes a decision problem in which we consider a set of alternatives, and each alternative is defined by its attributes or properties. Note that in this framework, properties are extensionally defined as subsets of alternatives. Let X be a finite set of objects or alternatives such that $|X| > 2$.

Definition 1. *A property space is a pair $(X, \mathcal{P})$ where $\mathcal{P} \subseteq 2^X$, which satisfies the following properties:*
P1 (Nontriviality) $P \in \mathcal{P} \Rightarrow P \neq \emptyset$
P2 (Closedness under negation) $P \in \mathcal{P} \Rightarrow P^c \in \mathcal{P}$
P3 (Separation) if $x, y \in X$ and $x \neq y$, there exists $P \in \mathcal{P}$ such that $x \in P$ and $y \notin P$.

Then, the elements of $\mathcal{P}$ are referred to as properties, and if $x \in P$, we say that x has property represented by the subset P. We can say that for every property P, there is an alternative that has this property. In addition, property membership is binary, and two alternatives can be distinguished by at least one property. The "property space" model provides a very general framework for representing different structures and preferences and then the aggregation of preferences. Here are some important examples:

Example 1. *If $(X, \leq)$, where $\leq$ is a total order in the set X, the set X is a property space with respect to the family of subsets $\{P_t : t \in X\} \cup \{Q_t : t \in X\}$, where $P_t = \{x \in X : x \geq t\}$ and $Q_t = \{x \in X : x < t\}$.*

Example 2. *If X is a finite distributive lattice, the family $\mathcal{P}$ of the prime filters and prime ideals of X defines a property space.*

Example 3. *We consider a finite set of alternatives A and a set $\mathcal{R}$ of linear orders in A.*
If we define for each pair $a, b \in A$, the set

$$P_{a,b} = \{R \in \mathcal{R} : aRb\}$$

the family $\mathcal{P}_{\mathcal{R}}$ of all the sets $P_{a,b}$ and their complement defines a property space structure on the set $\mathcal{R}$. See [2,3] for more details on the Arrowian framework.

Example 4. *The geometric notion of a point lying between two given points on a geometric line or a totally ordered set has strong intuitive appeal and has been generalized in a number of directions. In all of these, betweenness is taken to be a ternary relation that satisfies certain conditions. The modern axiomatic definition of betweenness is due to Hedlíková [15], who introduced the ternary representation of betweenness. Recently, concepts of betweenness were developed in [16,17]. Moreover, a notion of betweenness has recently been encoded in choice theory [18]. Betweenness has been introduced in the context of abstract convexity in [4] and in the context of property spaces in [2,19].*
In a. lattice L is defined as a ternary betweenness relation:

$$B = \{(x, z, y) \in L^3 : x \wedge y \leq z \leq x \vee y\}.$$

This ternary relation satisfies the following properties:
***[B1] (Reflexivity)** If $z \in \{x, y\}$, then $B(x, z, y)$*
***[B2] (Symmetry)** If $B(x, z, y)$, then $B(y, z, x)$*
***[B3] (Transitivity)** If $B(x, x', y)$, $B(x, y', y)$ and $B(x', z, y')$, then $B(x, z, y)$.*
***[B4] (Antisymmetry)** If $B(x, z, y)$ and $B(x, y, z)$, then $y = z$.*
Let us assume that these properties characterize a ternary betweenness relation.
We say that a set $A \subseteq X$ is convex if and only if

$$\text{if} \quad x, y \in A \quad \text{and} \quad B(x, z, y), \quad \text{then} \quad z \in A.$$

Now we consider the following property of a ternary betweenness relation:
***[B5] (Separation)** If the relation $B(x, z, y)$ is not satisfied, then there exists a subset $H \subseteq X$ such that the set H and H^c are convex and $x, y \in H, z \notin H$.*
*It can be proved (see [19]) that if a ternary betweenness relation satisfies the conditions **[B1]–[B5]**, then there exists a property space $(X, \mathcal{P})$ such that*

$$\text{if} \quad x, y \in P \in \mathcal{P} \quad \text{and} \quad B(x, z, y), \quad \text{then} \quad z \in P.$$

Conversely, if $(X, \mathcal{P})$ is a property space, the ternary relation is defined by

$$B(x, z, y) \quad \text{if and only if} \qquad x, y \in P \in \mathcal{P}, \quad \text{then} \quad z \in P$$

*which is a betweenness relation that satisfies conditions **[B1]–[B5]** (see [19]).*

2.3. Rights Systems

Let $N = \{1, \ldots, n\}$ be a set of agents or decision makers or voters. We consider a panel of experts each making judgments about a given set of properties, according to their expertise. We might consider many situations in which a subgroup of members could be decisive even though they do not constitute a majority. As an example, we could consider that a hiring committee or university department can be tasked with evaluating some applicants for an open faculty position in terms of research, teaching and service. We suppose that there are different subcommittees that have the right to evaluate the respective qualifications.

Definition 2. *A rights system with respect to the set N and to the property space $(X, \mathcal{P})$ is a correspondence $r\colon \mathcal{P} \rightrightarrows 2^N$. A rights system r is said to be exhaustive if for all $P \in \mathcal{P}$ and $A \subseteq N$, either $A \in r(P)$ or $N \setminus A \in r(P^c)$.*

Then, $P \in \mathcal{P}, r(P)$ is the set of subsets of N that have the right to property P. Note that our definition is more general than that of [1] since in [1], it is assumed that $r(P)$ is an upset. Therefore, in our model, it is not assumed that the function r is monotone with respect to inclusion and that there are closed groups of agents that want to make collective judgments on certain properties.

3. Rights-Consistent Aggregation Functions

In our model, we assume that a description of alternatives is given in terms of their properties, so a natural way to generate an aggregation procedure is to determine the final outcome via its properties. Then, we consider the following condition that the aggregation function $F\colon X^n \to X$ may satisfy, with $(X, \mathcal{P})$ being a property space.

Definition 3. *The function $F\colon X^n \to X$ is compatible with the rights system $r\colon \mathcal{P} \to 2^N$ if and only if for all $\mathbf{x} = (x_1, \ldots, x_n)$ and all $P \in \mathcal{P}$,*

$$F(\mathbf{x}) \in P \quad \text{if and only if} \quad \{i \in N : x_i \in P\} \in r(P).$$

If the function $F\colon X^n \to X$ is compatible with the rights system $r\colon \mathcal{P} \to 2^N$, the collective choice $F(x_1, \ldots, x_n)$ follows each agent in its respective area of competence or guarantees all properties P that are endorsed by a subgroup of individuals with a right to P.

We consider now some properties that an aggregation function $F\colon X^n \to X$ may satisfy, with $(X, \mathcal{P})$ being a property space (see [2]).

Definition 4. *F is a monotone aggregation operator if and only if whenever $F(x_1, \ldots x_i \ldots, x_n) \in P$ for $P \in \mathcal{P}$ and $y_i \in P$, then $F(x_1, \ldots, y_i \ldots, y_n) \in P$.*

Note that F is a monotone aggregation operator if and only if when $\{i \in N : x_i \in P\} \subseteq \{i \in N : y_i \in P\}$ if $F(\mathbf{x}) \in P$, then $F(\mathbf{x}) \in P$, so monotonicity requires consistency with the property space structure. Monotonicity is a very natural property of an aggregation operator; for example, every aggregation rule studied in the Arrowian framework is monotone.

Definition 5. *F is an independent aggregation operator if for $P \in \mathcal{P}$, when $F(x_1, \ldots, x_n) \in P$ for $P \in \mathcal{P}$ and for every $i, 1 \leq i \leq n$, $[x_i \in P \Leftrightarrow y_i \in P]$, then $F(x_1, \ldots, y_n) \in P$.*

We can say that the aggregation operator F is independent if and only if $F(\mathbf{x}) \in P$ if and only if $F(\mathbf{y}) \in P$ when $\{i \in N : x_i \in P\} = \{i \in N : y_i \in P\}$. This definition implies that the fact that whether the outcome $F(\mathbf{x})$ satisfies a property depends only on the pattern of the considered property at the vector $\mathbf{x}$ and not on the patterns of other properties at $\mathbf{x}$. Independence is a characteristic property of aggregation operators that considers only an ordinal point of view. So, if we consider an aggregation operator on the real line, the

median is an independent aggregation operator that considers only the ordinal structure while the mean also considers the cardinal structure of the real line
The following property can also be called an unanimity condition:

Definition 6. *F is an idempotent aggregation operator if and only if $F(x, x, \ldots, \ldots, x) = x$ for every $x \in X$.*

In a multicriteria decision-making approach, this property can be read as follows: if all criteria are satisfied to the same degree x, then the global score should also be x.
We can easily prove the following result:

Proposition 1. *Let $F\colon X^n \to X$ be an aggregation function compatible with the rights system $r\colon \mathcal{P} \to 2^N$:*

(i) Fis independent;
(ii) if r is exhaustive, then F is idempotent;
(iii) if for every $p \in \mathcal{P}$ $r(P)$ is an upset, then F is monotone.

We observe that a function $F\colon X^n \to X$ is compatible with the rights system $r\colon \mathcal{P} \to 2^N$ if and only if

$$F(x_1, \ldots, x_n) \in \bigcap\{P \in \mathcal{P} : \{i \in N : x_i \in P\} \in r(P)\}. \tag{1}$$

Furthermore, if we define $P_r(\mathbf{x}) = \bigcap\{P \in \mathcal{P} : \{i \in N : x_i \in P\} \in r(P)\}$, we could see that for some $\mathbf{x} \in X^n$, $P_r(\mathbf{x}) = \emptyset$ and then does not exist, and a function $F\colon X^n \to X$ is compatible with the rights system $r\colon \mathcal{P} \to 2^N$.

We consider in our framework a condition similar to the Intersection Property over Critical Families condition in [1]:

Proposition 2. *There exists a function $F\colon X^n \to X$ compatible with the rights system $r\colon \mathcal{P} \to 2^N$ if and only if for every $\mathcal{C} = \{P_1, \ldots, P_n\} \subseteq \mathcal{P}$ such that $\bigcap\{P_i,\ \ 1 \le i \le n\} = \emptyset$ and for all $k, 1 \le k \le n$, $\bigcap\{P_i \in 1 \le i \le n, i \neq k\} \neq \emptyset$, if $\mathbf{x} \in X^n$ exists such that $A_i = \{i \in N : x_i \in P_i\} \in r(P_i)$ for every $i \in N$, then $\bigcap\{A_i,\ \ 1 \le i \le n\} \neq \emptyset$.*

Proof. Suppose that there does not exist a function $F\colon X^n \to X$ compatible with the rights system $r\colon \mathcal{P} \to 2^N$. This implies that some $\mathbf{x} \in X^n$ by Equation (1) $\bigcap\{P \in \mathcal{P} : \{i \in N : x_i \in P\} \in r(P)\} = \emptyset$. Then, there exists $\mathcal{C} = \{P_1, \ldots, P_n\} \subseteq \mathcal{P}$ such that $\bigcap\{P_i,\ \ 1 \le i \le n\} = \emptyset$, and for all $k, 1 \le k \le n$, $\bigcap\{P_i \in 1 \le i \le n, i \neq k\} \neq \emptyset$, such that $A_i = \{i \in N : x_i \in P_i\} \in r(P_i)$ for every $i \in N$. Hence, we have that $\bigcap\{A_i,\ \ 1 \le i \le n\} \neq \emptyset$, but if $i \in \bigcap\{A_i,\ \ 1 \le i \le n\}$, then $x_i \in \bigcap\{P \in \mathcal{P} : \{i \in N : x_i \in P\} \in r(P)\}$, and this is a contradiction.
Conversely, suppose that there exists a subset $\mathcal{C}$ of $\mathcal{P}$, $\mathcal{C} = \{P_1, \ldots, P_n\} \subseteq \mathcal{P}$ such that $\bigcap\{P_i,\ \ 1 \le i \le n\} = \emptyset$, and for all $k, 1 \le k \le n$, $\bigcap\{P_i \in 1 \le i \le n, i \neq k\} \neq \emptyset$ and $\mathbf{x} \in X^n$ such that $A_i = \{i \in N : x_i \in P_i\} \in r(P_i)$ for every $i \in N$. Now suppose that there exists a function $F\colon X^n \to X$ compatible with the rights system $r\colon \mathcal{P} \to 2^N$; then, $F(\mathbf{x}) \in \bigcap\{P_i,\ \ 1 \le i \le n\} = \emptyset$, a contradiction. □

4. Congruences

In this section, we consider congruences that are equivalence relations that preserve the structure of the space. We define congruences in convexity spaces and in lattices. In [20], the authors studied the notion of congruence on a choice space.

4.1. Congruences in Convexity Spaces

The notion of convexity is a basic mathematical structure that is used to analyze many different models, and in the literature, there are various kinds of generalized, topological or axiomatically defined convexities. In this paper, the general notion of an abstract convexity structure that is studied in [4] is considered. This definition is based on the properties of a family of sets.

Definition 7. *If X is a set and $\mathcal{C}$ is a subset of 2^X, $(X, \mathcal{C})$ is a convexity space if the following conditions are satisfied:*

(1) $\mathcal{C}$ is closed under arbitrary intersections;
(2) if $\{X_j : j \in J\}$ is a totally ordered subset of $\mathcal{C}$ with respect to inclusion, then $\bigcup_{j \in J} X_j \in \mathcal{C}$.

The elements of $\mathcal{C}$ are called convex sets of X.

Moreover, the convexity notion allows us to define the notion of the convex hull operator, which is similar to that of the closure operator in topology.

If X is a set with convexity $\mathcal{C}$ and A is a subset of X, then the convex hull of $A \subseteq X$ is the set

$$\mathrm{co}A = \bigcap\{C \in \mathcal{C} : A \subseteq C\}.$$

This operator enjoys certain properties that are identical to those of usual convexity: for instance, co$A$ is the smallest convex set that contains set A. It is also clear that C is convex if and only if $\mathrm{co}\, C = C$.

We consider equivalence relations that are compatible with the structure of a convexity space as defined in [4]. If $A \subseteq X$, we define $E(A)$, which is called the saturation of the set A, as $E(A) = \{y \in X : (x, y) \in E \text{ for some } x \in A\}$, and then we denote the equivalence class of $x \in X$ with $E(x)$.

Definition 8. *If $(X, \mathcal{C})$ is a convexity space, an equivalence relation E in X is called a congruence in X if and only if*

$$co(E(A)) \subseteq E(co(A))$$

for every $A \subseteq X$.

Proposition 3. *Let $(X, \mathcal{C})$ be a convex space. An equivalence relation E in X is a congruence if and only if $E(C)$ is convex if C is convex.*

Proof. Let E be a congruence in a convex space $(X, \mathcal{C})$ and $C \subseteq X$ be a convex set. By definition, $\mathrm{co}(C) = C$, then $\mathrm{co}(E(C)) \subseteq E(C)$, and this implies that $E(C)$ is a convex set. Conversely, suppose that the saturation of a convex set is a convex set. If we consider a subset $A \subseteq X$, since $\mathrm{co}(A)$ is a convex set, we have that $E(\mathrm{co}(A))$ is a convex set. Moreover, $E(A) \subseteq E(\mathrm{co}(A))$, and then $\mathrm{co}(E(A)) \subseteq E(\mathrm{co}(A))$. □

The following result follows from Proposition 3.

Proposition 4. *A property space $(X, \mathcal{P})$ is a convex space with respect to the convexity $\mathcal{C}$ such that if $C \subseteq X$, then $C \in \mathcal{C}$ if $C = \emptyset$ or C is an intersection of the elements of $\mathcal{P}$.*
An equivalence relation E in X is a congruence with respect to the convexity $\mathcal{C}$ if and only if C is an intersection of the elements of $\mathcal{P}$; then, $E(C)$ is an intersection of the elements of $\mathcal{P}$.

4.2. Congruences in a Lattice

One of the important tools of lattice theoretical research is the study of lattice congruences. In this section, we consider congruences in a finite and distributive lattice L. The definition of a congruence in a lattice as a property space (or as a convexity space) is the same definition of a congruence in a lattice considered, for example, in [8–10] as it is proved with the following result.

Proposition 5. *An equivalence relation E on a finite distributive lattice L is a congruence with respect to the property space structure associated with the lattice structure if and only if it satisfies the following properties:*

(i) *if* $(x, y) \in E$, *then* $(x \wedge z, y \wedge z) \in E$ *for each* $z \in L$;
(ii) *if* $(x, y) \in E$, *then* $(x \vee z, y \vee z) \in E$ *for each* $z \in L$.

Proof. If $\mathcal{J}$ is the set of join prime elements of a lattice L, then for every $x \in L$, there exists $p_1, \ldots p_n$ elements of $\mathcal{J}$ such that

$$x = p_1 \vee \ldots \vee p_n \quad \text{and} \quad \uparrow x = \uparrow p_1 \cap \ldots \cap \uparrow p_n.$$

Consequently, a subset of L is an intersection of prime filters (and then an element of the convexity space associated with L) if and only if it is a principal filter.
Then, for every $a \in \mathcal{J}$, we can define a relation E_a on L as follows: for all $x, y \in L$, set

$$(x, y) \in E_a \quad \Longleftrightarrow \quad [x \leq a, y \leq a] \text{ or } [x \nleq a, y \nleq a]$$

that is,

$$x, y \in L \ \ (x, y) \in E_a \quad \Longleftrightarrow \quad [x, y \in \uparrow a,] \text{ or } [x, y \notin \ \uparrow a].$$

We can prove that this relation is a congruence in L as a property space since the two equivalence classes are the prime filter $\uparrow a$ and the ideal $X \setminus \uparrow a$.

Because a is a join prime element of L, we can also prove that E_a is compatible with the lattice operations of L. An equivalence relation on a finite distributive lattice L is a congruence with respect to the property space structure associated with the lattice structure if and only if the equivalence classes are intersections of the prime ideals. Then, it is easy to see that E is a congruence in L if and only it is an intersection of the relation $E_a : a \in A$ and $A \subseteq \mathcal{J}$; then, the thesis follows. □

4.3. Sugeno Integral

The process of combining sets of numerical or qualitative inputs into a single one is usually achieved by the aggregation functionals; see [21] for a comprehensive overview of aggregation theory. The importance of aggregation functionals is made apparent by their wide use in several fields such as decision sciences, computer and information sciences, economics and social sciences. There are a large number of different aggregation operators that differ in the assumptions on the inputs and on the information that you want to consider in the model.

One of the most noteworthy aggregation functionals making sense in a qualitative framework is the Sugeno integral, which is very useful across a range of decision contexts. The Sugeno integral was proposed in 1972 by M. Sugeno in a paper written in Japanese, and it became a very important study due to Sugeno's PhD dissertation [22]. The definition of the Sugeno integral primarily introduced on real intervals can be extended to bounded distributive lattices (see [23]). The Sugeno integral is useful when combining values on an ordinal scale, where the usual sums and products are not defined since its calculation requires only the lattice minimum and maximum operations.

We consider the set of inputs $N = \{1, 2, ..., n\}, n \geq 2$, which can be either decision criteria, optimization objectives or any other aggregation variables, and let $\mathcal{P}(N)$ be its power set. Here, we only provide some basic definitions; the rest can be found in [8–10,23].
A set function $m \colon \mathcal{P}(N) \to [0, 1]$ satisfying $m(\emptyset) = 0, m(N) = 1$ and being monotone, that is to say that $m(A) \subseteq m(B)$ whenever $A \subseteq B$, is called a capacity or a fuzzy measure. The value of $m(A)$ reflects the importance of each subset A in the considered problem.
The discrete Sugeno integral with respect to the capacity m is given by

$$S(x_1, \ldots, x_n) = \bigvee_{t \in [0,1]} (t \wedge m(\{i \in N : x_i \geq t\})).$$

Several equivalent definitions of the Sugeno integral have been proposed (see, for example, [8,9,23]).

The Sugeno integral is a basic tool for many models in applied and theoretical studies in several domains, especially in measure theory, decision making, probability, finance and also in scientometrics. Torra and Narukawa showed in [24] that the h-index is a particular case of some Sugeno integral, see also [25].

4.4. Compatible Aggregation Functions in Property Spaces

It has been proved that Sugeno integrals are aggregation functions preserving congruences in $[0,1]$ (see [9]) and in a bounded distributive lattice (see [8,10]). If E is a congruence in a property space $(X,\mathcal{P})$, we denote the fact that x,y belong to the same congruence class; that is to say that $E(x)=E(y)$, where $x\cong y$.

Definition 9. *Let $(X,\mathcal{P})$ be a property space and F a function $F\colon X^n\to X$. F is said to be compatible with the congruence E if when $\mathbf{x},\mathbf{y}\in X^n$ and for every i, $1\le i\le n$, $x_i\cong y_i$, then $F(\mathbf{x})\cong F(\mathbf{y})$.*

We want to characterize the class of multivariate functions compatible with every congruence of a property space.

Proposition 6. *Let $(X,\mathcal{P})$ be a property space and F a function $F\colon X^n\to X$ and $N=\{1,\dots,n\}$. If the function F is compatible with a rights system with respect to the set N, $r\colon\mathcal{P}\to 2^N$, then F is compatible with every congruence in $(X,\mathcal{P})$.*

Proof. Let the function F be compatible with a rights system with respect to the set N, $r\colon\mathcal{P}\to 2^N$. If P is an element of $\mathcal{P}$, we can consider the congruence E_P in X defined by

$$x\cong_P y \ \text{ when } \ x,y\in P \ \text{ or } \ x,y\notin P.$$

If we consider two elements $\mathbf{x},\mathbf{y}\in X^n$ such that for every $i\in N$, $x_i\in P$ if and only if $y_i\in P$, then $F(\mathbf{x})\in P$ if and only if $F(\mathbf{y})\in P$, and then the function F is compatible with the congruence E_P. Moreover, by Proposition 4, every congruence E in $(X,\mathcal{P})$ is such that the equivalence classes are intersections of the elements of $\mathcal{P}$, and so we can say that it is an intersection of the elements E_P with $P\in\mathcal{Q}$, $\mathcal{Q}\subseteq\mathcal{P}$, and so we can say that F is compatible with every congruence in $(X,\mathcal{P})$. □

Proposition 7. *Let $(X,\mathcal{P})$ be a property space and F a function $F\colon X^n\to X$.
If the function F is a monotone and idempotent aggregation functional that is compatible with every congruence in X, there exists an exhaustive rights system $r\colon\mathcal{P}\to 2^X$ such that*

$$F(x_1,\dots,x_n)=\bigcap\{P\in\mathcal{P}:\{i\in N: x_i\in P\}\in r(P)\}. \tag{2}$$

Proof. Let F be a monotone and idempotent aggregation functional that is compatible with every congruence in X. If P is an element of $\mathcal{P}$, we can consider the congruence E_P in X defined by

$$x\cong_P y \ \text{ when } \ x,y\in P \ \text{ or } \ x,y\notin P.$$

It can be proved that the function F is compatible with the congruence E_P, so if we consider two elements $\mathbf{x},\mathbf{y}\in X^N$ such that for every $i\in N$, $x_i\in P$ if and only if $y_i\in P$, then $F(\mathbf{x})\in P$ if and only if $F(\mathbf{y})\in P$. Then, F satisfies the independence property; hence, it satisfies monotone independence as defined in [1] and is idempotent. Then, according to Fact 2 in [1], there exists an exhaustive rights system r such that

$$f(x_1,\dots,x_n)=\bigcap\{P\in\mathcal{P}:\{i\in N: x_i\in P\}\in r(P)\}.$$

□

Example 5. *As an important example, we consider the real unit interval $X = [0,1]$ with the usual convexity, and we suppose that $F\colon X^N \to X$ is a functional that satisfies the hypothesis of Proposition 7 and Equation (2). We define a set function $m\colon 2^N \to [0,1]$ by $m(A) = F(1_A)$ where 1_A is the characteristic function of the set $A \subseteq N$. Because F is monotone and idempotent, we can easily prove that m is a capacity. We consider the sets $P_t, 0 \le t \le 1$ defined by $P_t = \{x \in X : x \ge t\}$. So $m(A) = F(1_A) \ge t$ if and only if $A \in r(P_t)$. Then, we find that $F(\mathbf{x}) \in P_t$; that is, $F(\mathbf{x}) \ge t$ if and only if $m(\{i \in N : x_i \ge t\}) \ge t$. We can conclude that*

$$F(\mathbf{x}) = \bigvee_{t \in [0,1]} (t \wedge m(\{i \in N : x_i \ge t\}))$$

is the usual form of the discrete Sugeno integral in $[0,1]$.

5. Concluding Remarks

We proposed the definition of a property space, and we showed that this definition can be considered as a particular case of the definition of abstract convex space. Moreover, we studied the notion of congruence in a property space.

Then, we introduced aggregation functions in property spaces, and we proposed a definition of the Sugeno integral operator in our framework. This definition is related to the property of compatibility with congruences that is introduced in [9] for aggregation functions in $[0,1]$ and in [8] for aggregation operators in bounded and distributive lattices.

We plan to consider other classes of aggregation functions and to study a wide range of applications of our results in future work.

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