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Exploring Hybrid H-bi-Ideals in Hemirings: Characterizations and Applications in Decision Making

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Abstract: The concept of the hybrid structure, as an extension of both soft sets and fuzzy sets, has gained significant attention in various mathematical and decision-making domains. In this paper, we delve into the realm of hemirings and investigate the properties of hybrid h-bi-ideals, including prime, strongly prime, semiprime, irreducible, and strongly irreducible ones. By employing these hybrid h-bi-ideals, we provide insightful characterizations of h-hemiregular and h-intra-hemiregular hemirings, offering a deeper understanding of their algebraic structures. Beyond theoretical implications, we demonstrate the practical value of hybrid structures and decision-making theory in handling real-world problems under imprecise environments. Using the proposed decision-making algorithm based on hybrid structures, we have successfully addressed a significant real-world problem, showcasing the efficacy of this approach in providing robust solutions.

Keywords: prime; strongly prime; semiprime; irreducible and strongly irreducible hybrid h-bi-ideals; decision making

MSC: 94D05

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Citation: Hadi, A.; Khan, A.; Faiz, N.; Khan, D.M.; Bantan, R.A.R.; Elgarhy, M. Exploring Hybrid H-bi-Ideals in Hemirings: Characterizations and Applications in Decision Making. *Mathematics* **2023**, *11*, 3683. https://doi.org/10.3390/math11173683

Academic Editors: Oscar Valero Sierra, Rajeev Kumar and Michael Voskoglou

Received: 10 July 2023
Revised: 1 August 2023
Accepted: 19 August 2023
Published: 26 August 2023

1. Introduction

L. A. Zadeh in 1965 [1] initiated the concept of fuzzy sets, the best framework for addressing uncertainties and imprecise information. Fuzzy set is defined by its membership function, whose values are defined on the closed interval [0, 1]. This approach extends the generalized theory of uncertainty described in [2] to a broader context. Numerous works [3–5] are based on the idea of fuzzy set theory, its extensions, and applications.

Modelling uncertain data is a challenge for researchers in a variety of disciplines, including economics, engineering, environmental science, sociology, and medical science. Classical methods might not always be adequate to deal with uncertainties appearing in these domains. Although mathematical methods like rough sets [6], fuzzy sets [1], and other mathematical tools [7] are frequently employed to express uncertainty, Molodtsov [8] highlighted that each has its own challenges. Molodtsov offered a novel method to model ambiguity and uncertainty as a result [8]. Many researchers contributed to extending soft sets with fuzzy set theory [9–11]. Fuzzy soft sets are an extension introduced by Maji et al. [12] and provide a more flexible approach to handling uncertainty. Since then there has been a rapid growth of interest in soft sets and their various applications to algebraic systems [13–20], data analysis [21], and decision making under uncertainty [22–26].

Hemirings are thought of as a generalization of rings and refer to an additively commutative semiring with zero members. Ideals of hemirings play a significant part in the theories of algebraic structure. K-ideals are a special type of ideal, studied by Henriksen [27], while h-ideals are a more restrictive class of ideals, proposed by Iizuka [28]. H-ideals and k-ideals have been implemented in hemirings by [29]. The generalization of the theory of h-ideals is called fuzzy h-ideals of hemirings [30]. By utilizing the fuzzy h-ideals, Zhan et al. [31] explored the h-hemiregular hemirings. Furthermore, the concepts of fuzzy h-bi-ideals and fuzzy h-quasi-ideals of hemirings are extended in [32]. A prime h-bi-ideal [33] is an extension of h-bi-ideals that has an additional property (defined in Section 2). The concept of fuzzy prime h-bi-ideals [33] refers to a fuzzified variant of prime h-bi-ideals, where degrees of membership are used to define how much an element belongs to the fuzzy prime h-bi-ideal rather than strict inclusion.

Jun et al. [34] developed the idea of a hybrid structure as a parallel circuit of fuzzy sets and soft sets by integrating the notions of soft sets and fuzzy sets. In hybrid structures, fuzzy sets are utilized to represent negative membership, while set-valued mappings are employed to represent positive membership. Algebric structures, for example, BCK/BCI algebras and semigroups are studied in relation to the concept of the hybrid structure [35–39]. To deal with uncertainty and imprecision, hybrid structures are implemented in hemirings by incorporating the concepts of soft sets and fuzzy sets. In order to apply the hemiring framework, it is necessary to define operations that take into account both fuzzy and soft sets [40,41]. This enables a more thorough approach to knowledge representation, problem-solving, and decision-making in uncertain situations. Asmat et al. [41] assessed the hybrid structure in hemirings and investigated several properties of hybrid h-ideals, hybrid h-bi-ideals, and hybrid h-quasi-ideals. The characterizations of h-hemiregular hemirings are discussed and several important results of a h-hemiregular hemirings are provided. This paper proposes prime hybrid h-bi-ideals which is an extension of hybrid h-bi-ideals [41] and we aim to investigate various aspects of hemirings. Also, this work has conducted a characterization of certain classes of hemirings based on these prime hybrid h-bi-ideals. Furthermore, we illustrate the importance of the defined hybrid structure in the decision-making process with the help of examples from real-world situations to explore new directions in algebraic development and tackle practical problems with improved uncertainty-handling abilities by utilising hybrid structures in hemirings.

The arrangement of paper is given as follows. The basic concepts and preliminary results concerning hemirings and hybrid structures, which will be used throughout this paper, are provided in Section 2. Section 3 provides the concepts of prime (semiprime, strongly prime) hybrid h-bi-ideals and the characterization of some classes of hemings in terms of these hybrid h-bi-ideals has been carried out. In the final part of the paper, we present a hybrid structure-based decision-making algorithm and use it to solve a problem that exists in the real world.

2. Preliminaries

We give a concise overview of the fundamental ideas and concepts employed in hemirings.

A nonempty set $\mathbb{N}$ with "$\dotplus$" and "$\diamond$" as binary operations on $\mathbb{N}$ is said to be a semiring if $(\mathbb{N}, \dotplus)$ and $(\mathbb{N}, \diamond)$ are semigroups and the following laws

$$\varphi_L \diamond (\varphi_M \dotplus \varphi_N) = \varphi_L \diamond \varphi_M \dotplus \varphi_L \diamond \varphi_N \text{ and } (\varphi_L \dotplus \varphi_M) \diamond \varphi_N = \varphi_L \diamond \varphi_N \dotplus \varphi_M \diamond \varphi_N$$

are satisfied for each $\varphi_L, \varphi_M, \varphi_N \in \mathbb{N}$,

A member 0 of a semiring is said to be zero if and only if all of its members satisfy the conditions $0 \dotplus \varphi_P = \varphi_P \dotplus 0 = \varphi_P$ and $0 \diamond \varphi_P = \varphi_P \diamond 0 = 0$. Hemirings are semirings $(\mathbb{N}, \dotplus, \diamond)$ that contain zero members and are commutative with regard to addition "$\dotplus$".

The sum and product of $\Bbbk$ and $\mathbb{R}$ where $\emptyset \neq \Bbbk \subseteq \mathbb{N}$ and $\emptyset \neq \mathbb{R} \subseteq \mathbb{N}$ are provided in a hemiring $(\mathbb{N}, \dotplus, \diamond)$ by

$$\Bbbk \dotplus \mathbb{R} = \{\varphi_P \dotplus \varphi_Q : \varphi_P \in \Bbbk \text{ and } \varphi_Q \in \mathbb{R}\}$$
$$\Bbbk\mathbb{R} = \{\varphi_P \varphi_Q : \varphi_P \in \Bbbk \text{ and } \varphi_Q \in \mathbb{R}\}.$$

When a subset $\mathbb{Q}$ of a hemiring $\mathbb{N}$ is closed under addition and multiplication while $\mathbb{Q}\mathbb{N}\mathbb{Q} \subseteq \mathbb{Q}$, the subset is said to be a bi-ideal.

Let $\emptyset \neq \mathbb{Q} \subseteq \mathbb{N}$, the set $\overline{\mathbb{Q}} = \{\phi_P \in \mathbb{N} : \phi_P \dotplus \varphi_L \dotplus \phi_R = \varphi_M \dotplus \phi_R$ for some $\varphi_L, \varphi_M \in \mathbb{Q}, \phi_R \in \mathbb{N}\}$ is referred to as h-closure of $\mathbb{Q}$.

For a bi-ideal $\mathbb{Q}$ of a hemiring $\mathbb{N}$, if $\phi_P, \phi_R \in \mathbb{N}$, $\varphi_L, \varphi_M \in \mathbb{Q}$ and $\phi_P \dotplus \varphi_L \dotplus \phi_R = \varphi_M \dotplus \phi_R$ implies $\phi_P \in \mathbb{Q}$ then $\mathbb{Q}$ is said to be an h-bi-ideal (H-BI) of $\mathbb{N}$.

An ideal $\mathbb{Q}$ in $\mathbb{N}$ satisfying $\mathbb{Q} = \overline{\mathbb{Q}^2}$ is referred to as an h-idempotent ideal of a hemiring $\mathbb{N}$.

Proposition 1 ([33]). *If $\Bbbk$ and $\mathbb{R}$ are the H-BI of a hemiring $\mathbb{N}$, following that $\overline{\Bbbk\mathbb{R}}$ is an H-BI of $\mathbb{N}$.*

Definition 1 ([33]). *If $\overline{\Bbbk\mathbb{R}} \subseteq \mathbb{Q}$ ($\overline{\Bbbk^2} \subseteq \mathbb{Q}$) implies $\Bbbk \subseteq \mathbb{Q}$ or $\mathbb{R} \subseteq \mathbb{Q}$ ($\Bbbk \subseteq \mathbb{Q}$) for all H-BI $\Bbbk$ and $\mathbb{R}$ of $\mathbb{N}$, following that H-BI $\mathbb{Q}$ of a hemiring $\mathbb{N}$ is known as a prime (semiprime) H-BI of $\mathbb{N}$.*

Definition 2 ([33]). *If $\overline{\Bbbk\mathbb{R}} \cap \overline{\mathbb{R}\Bbbk} \subseteq \mathbb{Q}$ indicating $\Bbbk \subseteq \mathbb{Q}$ or $\mathbb{R} \subseteq \mathbb{Q}$ for all H-BI $\Bbbk$ and $\mathbb{R}$ of a hemiring $\mathbb{N}$ then the H-BI $\mathbb{Q}$ of $\mathbb{N}$ is said to be strongly prime.*

A hemiring $\mathbb{N}$ is h-hemiregular (H-HemiR) if there exists $\varphi_L, \varphi_M, \varphi_N \in \mathbb{N}$ satisfying $\phi_P \dotplus \phi_P\varphi_L\phi_P \dotplus \varphi_N = \phi_P\varphi_M\phi_P \dotplus \varphi_N$ for any $\phi_P \in \mathbb{N}$.

If there exist $\varphi_{A_i}, \varphi_{M_j}, \varphi_{B_i}, \varphi_{N_j}$ and $\phi_R \in \mathbb{N}$ such that $\phi_P \dotplus \sum_{i=1}^{m} \varphi_{A_i}\phi_P^2\varphi_{B_i} \dotplus \phi_R = \sum_{j=1}^{n} \varphi_{M_j}\phi_P^2\varphi_{N_j} \dotplus \phi_R$, then a hemiring $\mathbb{N}$ is said to be h-intra-hemiregular (H-IHemiR) for each $\phi_P \in \mathbb{N}$.

By a fuzzy subset $\mathcal{L}$ of a non-empty set $\mathbb{N}$, we mean a mapping $\mathcal{L} : \mathbb{N} \longrightarrow [0,1]$, from a non-empty set $\mathbb{N}$ within $[0,1]$ unit interval.

If $\mathcal{L}$ and $\mathcal{F}$ are fuzzy subsets of $\mathbb{N}$ then the fuzzy subsets $\mathcal{L} \barwedge \mathcal{F} and \mathcal{L} \veebar \mathcal{F}$ are defined as: $(\mathcal{L} \barwedge \mathcal{F})(\phi_P) = \mathcal{L}(\phi_P) \barwedge \mathcal{F}(\phi_P)$ and $(\mathcal{L} \veebar \mathcal{F})(\phi_P) = \mathcal{L}(\phi_P) \veebar \mathcal{F}(\phi_P)$ for all $\phi_P \in \mathbb{N}$.

The term "soft set" (£, ¥) over $\mathbb{C}$ is a mapping of £ into the set of all subsets of $\mathbb{C}$ i.e., $\mathcal{L}$: ¥ $\longrightarrow \wp(\mathbb{C})$, where $\mathbb{C}$ is the initial universe set, ¥ is a collection of attributes that the entities in $\mathbb{C}$ hold, and $\wp(\mathbb{C})$ is the power set of $\mathbb{C}$.

Basic Operations of Hybrid Structures

Definition 3 ([34]). *A hybrid structure ($\ddot{H}$yS) in a set of parameters $\mathbb{Q}$ over an initial universe set $\mathbb{C}$ is defined as:*

$$\mathbb{Z}_{\tilde{1}} = (\xi^S_{1z}, \eta^F_{1z}) : \mathbb{Q} \longrightarrow \wp(\mathbb{C}) \times I, \phi_P \longmapsto \left(\xi^S_{1z}(\phi_P), \eta^F_{1z}(\phi_P)\right)$$

where, $\xi^S_{1z} : \mathbb{Q} \longrightarrow \wp(\mathbb{C})$ and $\eta^F_{1z} : \mathbb{Q} \longrightarrow I$ are mappings, $\wp(\mathbb{C})$ represents the set of all subsets of $\mathbb{C}$ and $I = [0,1]$.

Definition 4. *Let us represent the set of all $\ddot{H}$yS in $\mathbb{Q}$ over $\mathbb{C}$ by $H(\mathbb{Q})$. Here, we define an order $\widetilde{\subseteq}$ in $H(\mathbb{Q})$:*

$$(\forall \mathbb{Z}_{\tilde{1}}, \mathbb{Z}_{\tilde{2}} \in H(\mathbb{Q}))\ \mathbb{Z}_{\tilde{1}} \widetilde{\subseteq} \mathbb{Z}_{\tilde{2}} \text{ means } \xi^S_{1z} \sqsubseteq \xi^S_{2z} \text{ and } \eta^F_{1z} \succcurlyeq \eta^F_{2z}.$$

The hybrid intersection of two hybrid structures $\mathbb{Z}_{\tilde{1}}$ and $\mathbb{Z}_{\tilde{2}}$ in $\mathbb{Q}$ over $\mathbb{C}$ is defined as:

$$\mathbb{Z}_{\tilde{1}} \widetilde{\cap} \mathbb{Z}_{\tilde{2}} = \left\{\left\langle \varphi_A, (\xi^S_{1z} \sqcap \xi^S_{2z})(\varphi_A), (\eta^F_{1z} \veebar \eta^F_{1z})(\varphi_A)\right\rangle : \varphi_A \in \mathbb{Q}\right\}.$$

Definition 5. *The hybrid union of two hybrid structures $\mathbb{Z}_{\tilde{1}}$ and $\mathbb{Z}_{\tilde{2}}$ in $\mathbb{Q}$ over $\mathbb{C}$ is defined as:*

$$\mathbb{Z}_{\tilde{1}} \widetilde{\cup} \mathbb{Z}_{\tilde{2}} = \left\{\left\langle \varphi_A, (\xi^S_{1z} \sqcup \xi^S_{2z})(\varphi_A), (\eta^F_{1z} \barwedge \eta^F_{2z})(\varphi_A)\right\rangle : \varphi_A \in \mathbb{Q}\right\}.$$

The hybrid framework described by

$$\mathtt{C} = \left\langle \xi^S_{\mathtt{C}}, \eta^F_{\mathtt{C}} \right\rangle \text{ where } \xi^S_{\mathtt{C}}(\phi_R) = \mathbb{C} \text{ and } \eta^F_{\mathtt{C}}(\phi_R) = 0, \forall \phi_R \in \mathbb{N}''$$

is called identity hybrid mapping in $\mathbb{Q}$ over $\mathbb{C}$.

Definition 6 ([41]). *Let $\mathbb{Z}_{\tilde{1}}$ and $\mathbb{Z}_{\tilde{2}}$ be two $\ddot{H}yS$ in $\mathbb{Q}$ over $\mathbb{C}$. The hybrid h-sum $\mathbb{Z}_{\tilde{1}} \boxplus \mathbb{Z}_{\tilde{2}}$ is defined as:*

$$\mathbb{Z}_{\tilde{1}} \boxplus \mathbb{Z}_{\tilde{2}} = \{\langle \varphi_A, (\xi^S_{1z} \oplus_H \xi^S_{2z})(\varphi_A), (\eta^F_{1z} +_H \eta^F_{2z})(\varphi_A)\rangle : \varphi_A \in \mathbb{N}\}$$

$$(\xi^S_{1z} \oplus_H \xi^S_{2z})(\varphi_A) = \biguplus_{\varphi_A+(\varphi_M+\varphi_N)+\varphi_Y=(\varphi_S+\varphi_T)+\varphi_Y} (\xi^S_{1z}(\varphi_M) \sqcap \xi^S_{2z}(\varphi_N) \sqcap \xi^S_{1z}(\varphi_S) \sqcap \xi^S_{2z}(\varphi_T)),$$
$$(\eta^F_{1z} +_H \eta^F_{2z})(\varphi_A) = \prod_{\varphi_A+(\varphi_M+\varphi_N)+\varphi_Y=(\varphi_S+\varphi_T)+\varphi_Y} (\eta^F_{1z}(\varphi_M) \veebar \eta^F_{2z}(\varphi_N) \veebar \eta^F_{1z}(\varphi_S) \veebar \eta^F_{2z}(\varphi_T)),$$

Each $\varphi_A, \varphi_M, \varphi_N, \varphi_S, \varphi_T, \varphi_Y \in \mathbb{N}$, where, the symbols for supremum and infimum are $\biguplus$ and $\prod$, respectively.

Definition 7. *Let $\mathbb{Z}_{\tilde{1}}$ and $\mathbb{Z}_{\tilde{2}}$ be two $\ddot{H}yS$ in a hemiring $\mathbb{N}$ over $\mathbb{C}$.The hybrid h-product $\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}}$ is defined as*

$$\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}}(\varphi_A) = \{\langle \varphi_A, (\xi^S_{1z} \circledast_H \xi^S_{2z})(\varphi_A), (\eta^F_{1z} \odot_H \eta^F_{2z})(\varphi_A)\rangle : \varphi_A \in \mathbb{N}\}$$

which is concisely represented by $\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}} = \langle \xi^S_{1z} \circledast_H \xi^S_{2z}, \eta^F_{1z} \odot_H \eta^F_{2z}\rangle$, where, the definitions of $\xi^S_{1z} \circledast_H \xi^S_{2z}$ and $\eta^F_{1z} \odot_H \eta^F_{2z}$ are given as:

$$(\xi^S_{1z} \circledast_H \xi^S_{2z})(\varphi_A) = \begin{cases} \biguplus_{\varphi_A+(\varphi_M\varphi_N)+\varphi_Y=(\varphi_S\varphi_T)+\varphi_Y} (\xi^S_{1z}(\varphi_M) \sqcap \xi^S_{2z}(\varphi_N) \sqcap \xi^S_{1z}(\varphi_S) \sqcap \xi^S_{2z}(\varphi_T)) \text{ if } \varphi_A \text{ is} \\ \text{can be expressed as } \varphi_A + (\varphi_M\varphi_N) + \varphi_Y = (\varphi_S\varphi_T) + \varphi_Y \\ \emptyset \text{ alternatively} \end{cases}$$

and

$$(\eta^F_{1z} \odot_H \eta^F_{2z})(\varphi_A) = \begin{cases} \prod_{\varphi_A+(\varphi_M\varphi_N)+\varphi_Y=(\varphi_S\varphi_T)+\varphi_Y} (\eta^F_{1z}(\varphi_M) \veebar \eta^F_{2z}(\varphi_N) \veebar \eta^F_{1z}(\varphi_S) \veebar \eta^F_{2z}(\varphi_T)) \text{ if } \varphi_A \text{ is} \\ \text{can be expressed as } \varphi_A + (\varphi_M\varphi_N) + \varphi_Y = (\varphi_S\varphi_T) + \varphi_Y \\ 1 \text{ alternatively} \end{cases}$$

where, $\varphi_A, \varphi_M, \varphi_N, \varphi_S, \varphi_T, \varphi_Y \in \mathbb{N}$.

Definition 8 ([41]). *A hybrid h-bi-ideal ($\ddot{H}yH$-BI) in $\mathbb{N}$ upon $\mathbb{C}$ is defined to be a $\mathbb{Z}_{\tilde{1}} = (\xi^S_{1z}, \eta^F_{1z})$ if $\forall\ \varphi_A, \varphi_B, \varphi_C, \varphi_L, \varphi_M \in \mathbb{N}$, we have*

(1) $\xi^S_{1z}(\varphi_A \dot{+} \varphi_B) \sqsupseteq \xi^S_{1z}(\varphi_A) \sqcap \xi^S_{1z}(\varphi_B)$,
(2) $\eta^F_{1z}(\varphi_A \dot{+} \varphi_B) \preccurlyeq \eta^F_{1z}(\varphi_A) \veebar \eta^F_{1z}(\varphi_B)$,
(3) $\xi^S_{1z}(\varphi_A\varphi_B) \sqsupseteq \xi^S_{1z}(\varphi_A) \sqcap \xi^S_{1z}(\varphi_B)$,
(4) $\eta^F_{1z}(\varphi_A\varphi_B) \preccurlyeq \eta^F_{1z}(\varphi_A) \veebar \eta^F_{1z}(\varphi_B)$,
(5) $\xi^S_{1z}(\varphi_A\varphi_B\varphi_C) \sqsupseteq \xi^S_{1z}(\varphi_A) \sqcap \xi^S_{1z}(\varphi_C)$,
(6) $\eta^F_{1z}(\varphi_A\varphi_B\varphi_C) \preccurlyeq \eta^F_{1z}(\varphi_A) \veebar \eta^F_{1z}(\varphi_C)$,
(7) $\varphi_A \dot{+} \varphi_L \dot{+} \varphi_C = \varphi_M \dot{+} \varphi_C \longrightarrow \xi^S_{1z}(\varphi_A) \sqsupseteq \xi^S_{1z}(\varphi_L) \sqcap \xi^S_{1z}(\varphi_M)$,
(8) $\varphi_A \dot{+} \varphi_L \dot{+} \varphi_C = \varphi_M \dot{+} \varphi_C \longrightarrow \eta^F_{1z}(\varphi_A) \preccurlyeq \eta^F_{1z}(\varphi_L) \veebar \eta^F_{1z}(\varphi_M)$.

3. Prime Hybrid *H*-bi-Ideals

In this section, the concept of prime, strongly prime, semiprime, irreducible and strongly irreducible ΗyH-BI are provided with examples. The characterization of H-HemiR and H-IHemiR hemirings by these ΗyH-BI is also discussed.

Definition 9. *A $\ddot{H}$yH-BI $\mathbb{Z}_{\tilde{1}}$ of $\mathbb{N}$ over $\mathbb{C}$ is called a prime hybrid h-bi-ideal (P$\ddot{H}$yH-BI) if $\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{3}} \widetilde{\subseteq} \mathbb{Z}_{\tilde{1}}$ suggests $\mathbb{Z}_{\tilde{2}} \widetilde{\subseteq} \mathbb{Z}_{\tilde{1}}$ or $\mathbb{Z}_{\tilde{3}} \widetilde{\subseteq} \mathbb{Z}_{\tilde{1}}$ for every $\ddot{H}$yH-BI $\mathbb{Z}_{\tilde{2}}$ and $\mathbb{Z}_{\tilde{3}}$ of $\mathbb{N}$ upon $\mathbb{C}$.*

Example: Suppose that there are five houses in the initial universe set $\mathbb{C}$ given by $\mathbb{C} = \{H_1, H_2, H_3, H_4, H_5\}$. Let a set of parameters $\mathbb{N} = \{\varphi_A, \varphi_B, \varphi_C, \varphi_D\}$ be a set of status of houses in which φ_A stands for the parameter "beautiful", φ_B stands for the parameter "cheap", φ_C stands for the parameter "in good location", φ_D stands for the parameter "in green surrounding". We define the binary operation $\diamond$ and $\dotplus$ on $\mathbb{N}$ by the Cayley table in Table 1.

Table 1. Cayley table for the binary operations $\dotplus$ and $\diamond$.

$\dotplus$	φ_A	φ_B	φ_C	φ_D
φ_A	φ_A	φ_B	φ_C	φ_D
φ_B	φ_B	φ_A	φ_B	φ_C
φ_C	φ_C	φ_B	φ_B	φ_C
φ_D	φ_D	φ_C	φ_C	φ_B

$\diamond$	φ_A	φ_B	φ_C	φ_D
φ_A	φ_A	φ_A	φ_A	φ_A
φ_B	φ_A	φ_B	φ_B	φ_B
φ_C	φ_A	φ_B	φ_B	φ_B
φ_D	φ_A	φ_B	φ_B	φ_B

Then $(\mathbb{N}, \dotplus, \diamond)$ is a hemiring. Let $\mathbb{Z}_{\tilde{1}} = (\xi^S_{1z}, \eta^F_{1z}), \mathbb{Z}_{\tilde{2}} = (\xi^S_{2z}, \eta^F_{2z})$ and $\mathbb{Z}_{\tilde{3}} = (\xi^S_{3z}, \eta^F_{3z})$ be a any $\ddot{H}$yH-BI in $\mathbb{N}$ over $\mathbb{C}$ which is given by Tables 2–4.

Table 2. Tabular representation of $\ddot{H}$yH-BI $\mathbb{Z}_{\tilde{1}} = (\xi^S_{1z}, \eta^F_{1z})$.

$\mathbb{N}$	ξ^S_{1z}	η^F_{1z}
φ_A	$\{H_1, H_2, H_3, H_4, H_5\}$	0.2
φ_B	$\{H_1, H_2, H_3, H_4\}$	0.4
φ_C	$\{H_1, H_2\}$	0.7
φ_D	$\{H_1\}$	0.8

Table 3. Tabular representation of $\ddot{H}$yH-BI $\mathbb{Z}_{\tilde{2}} = (\xi^S_{2z}, \eta^F_{2z})$.

$\mathbb{N}$	ξ^S_{2z}	η^F_{2z}
φ_A	$\{H_1, H_2, H_3, H_4\}$	0.3
φ_B	$\{H_1, H_2, H_3\}$	0.5
φ_C	$\{H_1, H_2\}$	0.7
φ_D	$\{H_1\}$	0.9

Table 4. Tabular representation of $\ddot{H}$yH-BI $\mathbb{Z}_{\tilde{3}} = (\xi^S_{3z}, \eta^F_{3z})$.

$\mathbb{N}$	ξ^S_{3z}	η^F_{3z}
φ_A	$\{H_1, H_3, H_4, H_5\}$	0.2
φ_B	$\{H_3, H_4, H_5\}$	0.3
φ_C	$\{H_4, H_5\}$	0.6
φ_D	$\{H_4\}$	0.8

It is routine calculation to verify that if $\left(\xi^S_{2z} \circledast_H \xi^S_{3z}\right) \sqsubseteq \xi^S_{1z}$ and $\left(\eta^F_{2z} \odot_H \eta^F_{3z}\right) \succcurlyeq \eta^F_{1z}$ implies $\xi^S_{2z} \sqsubseteq \xi^S_{1z}$ or $\xi^S_{3z} \sqsubseteq \xi^S_{1z}$ and $\eta^F_{2z} \succcurlyeq \eta^F_{1z}$ or $\eta^F_{3z} \succcurlyeq \eta^F_{1z}$ for all $\varphi_A, \varphi_B, \varphi_C, \varphi_D$ in $\mathbb{N}$. This implies that $\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{3}} \widetilde{\subseteq} \mathbb{Z}_{\tilde{1}}$ gives $\mathbb{Z}_{\tilde{2}} \widetilde{\subseteq} \mathbb{Z}_{\tilde{1}}$ or $\mathbb{Z}_{\tilde{3}} \widetilde{\subseteq} \mathbb{Z}_{\tilde{1}}$. Thus $\mathbb{Z}_{\tilde{1}}$ is a prime $\ddot{H}$yH-BI of $\mathbb{N}$ over $\mathbb{C}$.

Definition 10. *A $\ddot{H}$yH-BI $\mathbb{Z}_{\tilde{1}}$ of $\mathbb{N}$ over $\mathbb{C}$ is said to be strongly prime hybrid h-bi-ideal (StP$\ddot{H}$yH-BI) if for all $\ddot{H}$yH-BI $\mathbb{Z}_{\tilde{2}}$ and $\mathbb{Z}_{\tilde{3}}$ of $\mathbb{N}$ over $\mathbb{C}$ $(\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{3}}) \widetilde{\cap} (\mathbb{Z}_{\tilde{3}} \boxtimes \mathbb{Z}_{\tilde{2}}) \widetilde{\subseteq} \mathbb{Z}_{\tilde{1}}$ implies $\mathbb{Z}_{\tilde{2}} \widetilde{\subseteq} \mathbb{Z}_{\tilde{1}}$ or $\mathbb{Z}_{\tilde{3}} \widetilde{\subseteq} \mathbb{Z}_{\tilde{1}}$ (see the related example in Appendix A i.e., Example A1).*

Definition 11. *A ḦyH-BI $\mathbb{Z}_{\tilde{2}}$ of $\mathbb{N}$ over $\mathbb{C}$ is idempotent if $\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{2}} = \mathbb{Z}_{\tilde{2}}$.*

Definition 12. *A semiprime hybrid h-bi-ideal (briefly SPḦyH-BI) is defined to be a ḦyH-BI $\mathbb{Z}_{\tilde{1}}$ in $\mathbb{N}$ over $\mathbb{C}$ satisfying $\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{2}} \widetilde{\subseteq} \mathbb{Z}_{\tilde{1}}$ means $\mathbb{Z}_{\tilde{2}} \widetilde{\subseteq} \mathbb{Z}_{\tilde{1}}$ for every single ḦyH-BI $\mathbb{Z}_{\tilde{2}}$ of $\mathbb{N}$ upon $\mathbb{C}$ (see the related example in Appendix A i.e., Example A2).*

Further, we demonstrate that the hybrid h-product of any two ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$ is similar to ḦyH-BI in the following proposition.

Proposition 2. *$\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}}$ is an ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$, whereas, $\mathbb{Z}_{\tilde{1}} = (\xi^S_{1z}, \eta^F_{1z})$ and $\mathbb{Z}_{\tilde{2}} = (\xi^S_{2z}, \eta^F_{2z})$ be any ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$.*

Proof. Let $\mathbb{Z}_{\tilde{1}}$ and $\mathbb{Z}_{\tilde{2}}$ be any ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$ and $\varphi_A, \varphi_B \in \mathbb{N}$. Then

$$\left(\xi^S_{1z} \circledast_H \xi^S_{2z}\right)(\varphi_A) = \biguplus_{\varphi_A \dotplus (\varphi_E \varphi_F) \dotplus \varphi_C = (\varphi_U \varphi_V) \dotplus \varphi_C} \left\{\xi^S_{1z}(\varphi_E) \sqcap \xi^S_{2z}(\varphi_F) \sqcap \xi^S_{1z}(\varphi_U) \sqcap \xi^S_{2z}(\varphi_V)\right\},$$
$$\left(\eta^F_{1z} \odot_H \eta^F_{2z}\right)(\varphi_A) = \prod_{\varphi_A \dotplus (\varphi_E \varphi_F) \dotplus \varphi_C = (\varphi_U \varphi_V) \dotplus \varphi_C} \left\{\eta^F_{1z}(\varphi_E) \veebar \eta^F_{2z}(\varphi_F) \veebar \eta^F_{1z}(\varphi_U) \veebar \eta^F_{2z}(\varphi_V)\right\}$$

and

$$\left(\xi^S_{1z} \circledast_H \xi^S_{2z}\right)(\varphi_B) = \biguplus_{\varphi_B + (\varphi_P \varphi_Q) + \varphi_C = (\varphi_S \varphi_T) + \varphi_C} \left\{\xi^S_{1z}(\varphi_P) \sqcap \xi^S_{2z}(\varphi_Q) \sqcap \xi^S_{1z}(\varphi_S) \sqcap \xi^S_{2z}(\varphi_T)\right\},$$
$$\left(\eta^F_{1z} \odot_H \eta^F_{2z}\right)(\varphi_B) = \prod_{\varphi_B + (\varphi_P \varphi_Q) + \varphi_C = (\varphi_S \varphi_T) + \varphi_C} \left\{\eta^F_{1z}(\varphi_P) \veebar \eta^F_{2z}(\varphi_Q) \veebar \eta^F_{1z}(\varphi_S) \veebar \eta^F_{2z}(\varphi_T)\right\},$$

whereas, $\varphi_E, \varphi_F, \varphi_U, \varphi_V, \varphi_P, \varphi_Q, \varphi_S, \varphi_T \in \mathbb{N}$. At this point

$$\begin{aligned}
&\left(\xi^S_{1z} \circledast_H \xi^S_{2z}\right)(\varphi_A \dotplus \varphi_B) \\
&= \biguplus_{\varphi_A \dotplus \varphi_B \dotplus (\varphi_L \varphi_M) \dotplus \varphi_C = (\varphi_X \varphi_Y) \dotplus \varphi_C} \left\{\xi^S_{1z}(\varphi_L) \sqcap \xi^S_{2z}(\varphi_M) \sqcap \xi^S_{1z}(\varphi_X) \sqcap \xi^S_{2z}(\varphi_Y)\right\} \\
&\sqsupseteq \biguplus_{\varphi_A + (\varphi_E \varphi_F) + \varphi_C = (\varphi_U \varphi_V) + \varphi_C} \left[\biguplus_{\varphi_B + (\varphi_P \varphi_Q) + \varphi_C = (\varphi_S \varphi_T) + \varphi_C} \left\{\begin{array}{c} \xi^S_{1z}(\varphi_E) \sqcap \xi^S_{2z}(\varphi_F) \sqcap \xi^S_{1z}(\varphi_U) \sqcap \xi^S_{2z}(\varphi_V) \sqcap \\ \xi^S_{1z}(\varphi_P) \sqcap \xi^S_{2z}(\varphi_Q) \sqcap \xi^S_{1z}(\varphi_S) \sqcap \xi^S_{2z}(\varphi_T) \end{array}\right\}\right] \\
&= \biguplus_{\varphi_A + (\varphi_E \varphi_F) + \varphi_C = (\varphi_U \varphi_V) + \varphi_C} \left\{\xi^S_{1z}(\varphi_E) \sqcap \xi^S_{2z}(\varphi_F) \sqcap \xi^S_{1z}(\varphi_U) \sqcap \xi^S_{2z}(\varphi_V)\right\} \sqcap \\
&\biguplus_{\varphi_B + (\varphi_P \varphi_Q) + \varphi_C = (\varphi_S \varphi_T) + \varphi_C} \left\{\xi^S_{1z}(\varphi_P) \sqcap \xi^S_{2z}(\varphi_Q) \sqcap \xi^S_{1z}(\varphi_S) \sqcap \xi^S_{2z}(\varphi_T)\right\} \\
&= \left(\xi^S_{1z} \circledast_H \xi^S_{2z}\right)(\varphi_A) \sqcap \left(\xi^S_{1z} \circledast_H \xi^S_{2z}\right)(\varphi_B)
\end{aligned}$$

and

$$
\begin{aligned}
&\left(\eta^F_{1z} \odot_H \eta^F_{2z}\right)(\varphi_A \dotplus \varphi_B) \\
&= \prod_{\varphi_A \dotplus \varphi_B \dotplus (\varphi_L\varphi_M) \dotplus \varphi_C = (\varphi_X\varphi_Y) \dotplus \varphi_C} \left\{ \eta^F_{1z}(\varphi_L) \veebar \eta^F_{2z}(\varphi_M) \veebar \eta^F_{1z}(\varphi_X) \veebar \eta^F_{2z}(\varphi_Y) \right\} \\
&\preccurlyeq \prod_{\varphi_A + (\varphi_E\varphi_F) + \varphi_C = (\varphi_U\varphi_V) + \varphi_C} \left[\prod_{\varphi_B + (\varphi_P\varphi_Q) + \varphi_C = (\varphi_S\varphi_T) + \varphi_C} \left\{ \begin{array}{c} \eta^F_{1z}(\varphi_E) \veebar \eta^F_{2z}(\varphi_F) \veebar \eta^F_{1z}(\varphi_U) \veebar \eta^F_{2z}(\varphi_V), \\ \eta^F_{1z}(\varphi_P) \veebar \eta^F_{2z}(\varphi_Q) \veebar \eta^F_{1z}(\varphi_S) \veebar \eta^F_{2z}(\varphi_T) \end{array} \right\} \right] \\
&= \prod_{\varphi_A + (\varphi_E\varphi_F) + \varphi_C = (\varphi_U\varphi_V) + \varphi_C} \left\{ \eta^F_{1z}(\varphi_E) \veebar \eta^F_{2z}(\varphi_F) \veebar \eta^F_{1z}(\varphi_U) \veebar \eta^F_{2z}(\varphi_V) \right\} \veebar \\
&\prod_{\varphi_B + (\varphi_P\varphi_Q) + \varphi_C = (\varphi_S\varphi_T) + \varphi_C} \left\{ \eta^F_{1z}(\varphi_P) \veebar \eta^F_{2z}(\varphi_Q) \veebar \eta^F_{1z}(\varphi_S) \veebar \eta^F_{2z}(\varphi_T) \right\} \\
&= \left(\eta^F_{1z} \odot_H \eta^F_{2z}\right)(\varphi_A) \veebar \left(\eta^F_{1z} \odot_H \eta^F_{2z}\right)(\varphi_B).
\end{aligned}
$$

To prove that $\varphi_J \dotplus \varphi_E \dotplus \varphi_K = \varphi_F \dotplus \varphi_K$, implies $\left(\xi^S_{1z} \circledast_H \xi^S_{2z}\right)(\varphi_J) \sqsupseteq \left(\xi^S_{1z} \circledast_H \xi^S_{2z}\right)(\varphi_E) \sqcap \left(\xi^S_{1z} \circledast_H \xi^S_{2z}\right)(\varphi_F)$ and $\left(\eta^F_{1z} \odot_H \eta^F_{2z}\right)(\varphi_J) \preccurlyeq \left(\eta^F_{1z} \odot_H \eta^F_{2z}\right)(\varphi_E) \veebar \left(\eta^F_{1z} \odot_H \eta^F_{2z}\right)(\varphi_F)$. Similarly, $\varphi_J \dotplus \varphi_E \dotplus (\varphi_L\varphi_M) \dotplus \varphi_K = \varphi_F \dotplus (\varphi_L\varphi_M) \dotplus \varphi_K$ is obtained by combining $\varphi_E \dotplus (\varphi_L\varphi_M) \dotplus \varphi_K = (\varphi_P\varphi_Q) \dotplus \varphi_K$, $\varphi_F \dotplus (\varphi_S\varphi_T) \dotplus \varphi_K = (\varphi_G\varphi_H) \dotplus \varphi_K$ and $\varphi_J \dotplus \varphi_E \dotplus \varphi_K = \varphi_F \dotplus \varphi_K$. This results in the equation $\varphi_J \dotplus (\varphi_P\varphi_Q) \dotplus \varphi_K = \varphi_F \dotplus (\varphi_L\varphi_M) \dotplus \varphi_K$ and $\varphi_J \dotplus (\varphi_P\varphi_Q) \dotplus (\varphi_S\varphi_T) \dotplus \varphi_K = \varphi_F \dotplus (\varphi_S\varphi_T) \dotplus (\varphi_L\varphi_M) \dotplus \varphi_K = (\varphi_L\varphi_M) \dotplus (\varphi_G\varphi_H) \dotplus \varphi_K$. Due to this, $\varphi_J \dotplus (\varphi_P\varphi_Q) \dotplus (\varphi_S\varphi_T) \dotplus \varphi_K = (\varphi_L\varphi_M) \dotplus (\varphi_G\varphi_H) \dotplus \varphi_K$.

$$
\begin{aligned}
&\left(\xi^S_{1z} \circledast_H \xi^S_{2z}\right)(\varphi_E) \sqcap \left(\xi^S_{1z} \circledast_H \xi^S_{2z}\right)(\varphi_F) \\
&= \biguplus_{\varphi_E + (\varphi_L\varphi_M) + \varphi_C = (\varphi_P\varphi_Q) + \varphi_C} \left\{ \xi^S_{1z}(\varphi_L) \sqcap \xi^S_{2z}(\varphi_M) \sqcap \xi^S_{1z}(\varphi_P) \sqcap \xi^S_{2z}(\varphi_Q) \right\} \sqcap \\
&\biguplus_{\varphi_F + (\varphi_S\varphi_T) + \varphi_C = (\varphi_G\varphi_H) + \varphi_C} \left\{ \xi^S_{1z}(\varphi_S) \sqcap \xi^S_{2z}(\varphi_T) \sqcap \xi^S_{1z}(\varphi_G) \sqcap \xi^S_{2z}(\varphi_H) \right\} \\
&= \biguplus_{\varphi_E + (\varphi_L\varphi_M) + \varphi_C = (\varphi_P\varphi_Q) + \varphi_C} \left[\biguplus_{\varphi_F + (\varphi_S\varphi_T) + \varphi_C = (\varphi_G\varphi_H) + \varphi_C} \left\{ \begin{array}{c} \xi^S_{1z}(\varphi_L) \sqcap \xi^S_{2z}(\varphi_M) \sqcap \xi^S_{1z}(\varphi_P) \sqcap \xi^S_{2z}(\varphi_Q) \\ \sqcap \xi^S_{1z}(\varphi_S) \sqcap \xi^S_{2z}(\varphi_T) \sqcap \xi^S_{1z}(\varphi_G) \sqcap \xi^S_{2z}(\varphi_H) \end{array} \right\} \right] \\
&\sqsubseteq \biguplus_{\varphi_J + (\varphi_A\varphi_B) + \varphi_K = (\varphi_X\varphi_Y) + \varphi_K} \left\{ \xi^S_{1z}(\varphi_A) \sqcap \xi^S_{2z}(\varphi_B) \sqcap \xi^S_{1z}(\varphi_X) \sqcap \xi^S_{2z}(\varphi_Y) \right\} \\
&= \left(\xi^S_{1z} \circledast_H \xi^S_{2z}\right)(\varphi_J)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\eta^F_{1z} \odot_H \eta^F_{2z}\right)(\varphi_E) \vee \left(\eta^F_{1z} \odot_H \eta^F_{2z}\right)(\varphi_F) \\
&= \prod_{\varphi_E+(\varphi_L\varphi_M)+\varphi_C=(\varphi_P\varphi_Q)+\varphi_C} \left\{\eta^F_{1z}(\varphi_L) \veebar \eta^F_{2z}(\varphi_M) \veebar \eta^F_{1z}(\varphi_P) \veebar \eta^F_{2z}(\varphi_Q)\right\} \veebar \\
&\prod_{\varphi_F+(\varphi_S\varphi_T)+\varphi_C=(\varphi_G\varphi_H)+\varphi_C} \left\{\eta^F_{1z}(\varphi_S) \veebar \eta^F_{2z}(\varphi_T) \veebar \eta^F_{1z}(\varphi_G) \veebar \eta^F_{2z}(\varphi_H)\right\} \\
&= \prod_{\varphi_E+(\varphi_L\varphi_M)+\varphi_C=(\varphi_P\varphi_Q)+\varphi_C} \left[\prod_{\varphi_F+(\varphi_S\varphi_T)+\varphi_C=(\varphi_G\varphi_H)+\varphi_C} \left\{\begin{array}{c} \eta^F_{1z}(\varphi_L) \veebar \eta^F_{2z}(\varphi_M) \veebar \eta^F_{1z}(\varphi_P) \veebar \eta^F_{2z}(\varphi_Q), \\ \eta^F_{1z}(\varphi_S) \veebar \eta^F_{2z}(\varphi_T) \veebar \eta^F_{1z}(\varphi_G) \veebar \eta^F_{2z}(\varphi_H) \end{array}\right\}\right] \\
&\succeq \prod_{\varphi_J+(\varphi_A\varphi_B)+\varphi_K=(\varphi_X\varphi_Y)+\varphi_K} \left\{\eta^F_{1z}(\varphi_A) \veebar \eta^F_{2z}(\varphi_B) \veebar \eta^F_{1z}(\varphi_X) \veebar \eta^F_{2z}(\varphi_Y)\right\} \\
&= \left(\eta^F_{1z} \odot_H \eta^F_{2z}\right)(\varphi_J).
\end{aligned}
$$

Now

$$
\begin{aligned}
\left(\zeta^S_{1z} \circledast_H \zeta^S_{2z}\right) \circledast_H \left(\zeta^S_{1z} \circledast_H \zeta^S_{2z}\right) &= (\zeta^S_{1z} \circledast_H \zeta^S_{2z} \circledast_H \zeta^S_{1z}) \circledast_H \zeta^S_{2z} \\
&\sqsubseteq (\zeta^S_{1z} \circledast_H \zeta^S_{\complement} \circledast_H \zeta^S_{1z}) \circledast_H \zeta^S_{2z} \\
&\sqsubseteq \left(\zeta^S_{1z} \circledast_H \zeta^S_{2z}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\eta^F_{1z} \odot_H \eta^F_{2z}\right) \odot_H \left(\eta^F_{1z} \odot_H \eta^F_{2z}\right) &= (\eta^F_{1z} \odot_H \eta^F_{2z} \odot_H \eta^F_{1z}) \odot_H \eta^F_{2z} \\
&\succcurlyeq (\eta^F_{1z} \odot_H \eta^F_{\complement} \odot_H \eta^F_{1z}) \odot_H \eta^F_{2z} \\
&\succcurlyeq \left(\eta^F_{1z} \odot_H \eta^F_{2z}\right).
\end{aligned}
$$

Also,

$$
\begin{aligned}
\left(\zeta^S_{1z} \circledast_H \zeta^S_{2z}\right) \circledast_H \zeta^S_{\complement} \circledast_H \left(\zeta^S_{1z} \circledast_H \zeta^S_{2z}\right) &= \zeta^S_{1z} \circledast_H (\zeta^S_{2z} \circledast_H \zeta^S_{\complement} \circledast_H \zeta^S_{1z}) \circledast_H \zeta^S_{2z} \\
&\sqsubseteq (\zeta^S_{1z} \circledast_H \zeta^S_{\complement} \circledast_H \zeta^S_{1z}) \circledast_H \zeta^S_{2z} \\
&\sqsubseteq \left(\zeta^S_{1z} \circledast_H \zeta^S_{2z}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\eta^F_{1z} \odot_H \eta^F_{2z}\right) \odot_H \eta^F_{\complement} \odot_H \left(\eta^F_{1z} \odot_H \eta^F_{2z}\right) &= \eta^F_{1z} \odot_H (\eta^F_{2z} \odot_H \eta^F_{\complement} \odot_H \eta^F_{1z}) \odot_H \eta^F_{2z} \\
&\succcurlyeq (\eta^F_{1z} \odot_H \eta^F_{\complement} \odot_H \eta^F_{1z}) \odot_H \eta^F_{2z} \\
&\succcurlyeq \left(\eta^F_{1z} \odot_H \eta^F_{2z}\right).
\end{aligned}
$$

Consequently, $\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}}$ is likewise a ЁyH-BI of $\mathbb{N}$ over $\mathbb{C}$. □

The intersection of any collection of ЁyH-BI of $\mathbb{N}$ over $\mathbb{C}$ is shown to be a ЁyH-BI in the ensuing Lemma 1.

Lemma 1. *For a collection* $\{(\mathbb{Z}_{\tilde{1}})_{\hat{o}} : \hat{o} \in \Omega\}$ *of ЁyH-BI of* $\mathbb{N}$ *over* $\mathbb{C}$*, their intersection is also a ЁyH-BI of* $\mathbb{N}$ *over* $\mathbb{C}$*.*

Proof. $\{(\mathbb{Z}_{\tilde{1}})_{\hat{o}} : \hat{o} \in \Omega\}$ is a collection of ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$. We have to prove that $\widetilde{\bigcap\limits_{\hat{o}\in\Omega}}(\mathbb{Z}_{\tilde{1}})_{\hat{o}} = (\bigsqcap\limits_{\hat{o}\in\Omega}(\xi^S_{1z})_{\hat{o}}, \veebar\limits_{\hat{o}\in\Omega}(\eta^F_{1z})_{\hat{o}})$ is a ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$. Let $\varphi_C, \varphi_D \in \mathbb{N}$, then

$$\begin{aligned}
&\left(\bigsqcap_{\hat{o}\in\Omega}(\xi^S_{1z})_{\hat{o}}\right)(\varphi_C\varphi_D) \\
&= \bigsqcap_{\hat{o}\in\Omega}\{(\xi^S_{1z})_{\hat{o}}(\varphi_C\varphi_D) : \hat{o} \in \Omega \text{ and } \varphi_C, \varphi_D \in \mathbb{N}\} \\
&\sqsupseteq \bigsqcap_{\hat{o}\in\Omega}\{(\xi^S_{1z})_{\hat{o}}(\varphi_C) \sqcap (\xi^S_{1z})_{\hat{o}}(\varphi_D)\} \\
&= \{\bigsqcap_{\hat{o}\in\Omega}((\xi^S_{1z})_{\hat{o}}(\varphi_C))\} \sqcap \{\bigsqcap_{\hat{o}\in\Omega}((\xi^S_{1z})_{\hat{o}}(\varphi_D))\} \\
&= \{\bigsqcap_{\hat{o}\in\Omega}(\xi^S_{1z})_{\hat{o}}(\varphi_C)\} \sqcap \{\bigsqcap_{\hat{o}\in\Omega}(\xi^S_{1z})_{\hat{o}}(\varphi_D)\} \\
&= (\bigsqcap_{\hat{o}\in\Omega}(\xi^S_{1z})_{\hat{o}})(\varphi_C) \sqcap (\bigsqcap_{\hat{o}\in\Omega}(\xi^S_{1z})_{\hat{o}})(\varphi_D)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\veebar_{\hat{o}\in\Omega}(\eta^F_{1z})_{\hat{o}}\right)(\varphi_C\varphi_D) \\
&= \veebar_{\hat{o}\in\Omega}\{(\eta^F_{1z})_{\hat{o}}(\varphi_C\varphi_D) : \hat{o} \in \Omega \text{ and } \varphi_C, \varphi_D \in \mathbb{N}\} \\
&\preccurlyeq \veebar_{\hat{o}\in\Omega}\{(\eta^F_{1z})_{\hat{o}}(\varphi_C) \veebar (\eta^F_{1z})_{\hat{o}}(\varphi_D)\} \\
&= \{\veebar_{\hat{o}\in\Omega}((\eta^F_{1z})_{\hat{o}}(\varphi_C))\} \veebar \{\veebar_{\hat{o}\in\Omega}((\eta^F_{1z})_{\hat{o}}(\varphi_D))\} \\
&= \{\veebar_{\hat{o}\in\Omega}(\eta^F_{1z})_{\hat{o}}(\varphi_C)\} \veebar \{\veebar_{\hat{o}\in\Omega}(\eta^F_{1z})_{\hat{o}}(\varphi_D)\} \\
&= (\veebar_{\hat{o}\in\Omega}(\eta^F_{1z})_{\hat{o}})(\varphi_C) \veebar (\veebar_{\hat{o}\in\Omega}(\eta^F_{1z})_{\hat{o}})(\varphi_D).
\end{aligned}$$

$$\begin{aligned}
&\left(\bigsqcap_{\hat{o}\in\Omega}(\xi^S_{1z})_{\hat{o}}\right)(\varphi_C \dotplus \varphi_D) \\
&= \bigsqcap_{\hat{o}\in\Omega}\{(\xi^S_{1z})_{\hat{o}}(\varphi_C \dotplus \varphi_D) : \hat{o} \in \Omega \text{ and } \varphi_C, \varphi_D \in \mathbb{N}\} \\
&\sqsupseteq \bigsqcap_{\hat{o}\in\Omega}\{(\xi^S_{1z})_{\hat{o}}(\varphi_C) \sqcap (\xi^S_{1z})_{\hat{o}}(\varphi_D)\} \\
&= \{\bigsqcap_{\hat{o}\in\Omega}\{(\xi^S_{1z})_{\hat{o}}(\varphi_C)\}\} \sqcap \{\bigsqcap_{\hat{o}\in\Omega}\{(\xi^S_{1z})_{\hat{o}}(\varphi_D)\}\} \\
&= \{\bigsqcap_{\hat{o}\in\Omega}(\xi^S_{1z})_{\hat{o}}(\varphi_C)\} \sqcap \{\bigsqcap_{\hat{o}\in\Omega}(\xi^S_{1z})_{\hat{o}}(\varphi_D)\} \\
&= (\bigsqcap_{\hat{o}\in\Omega}(\xi^S_{1z})_{\hat{o}})(\varphi_C) \sqcap (\bigsqcap_{\hat{o}\in\Omega}(\xi^S_{1z})_{\hat{o}})(\varphi_D)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\veebar_{\hat{o}\in\Omega}(\eta^F_{1z})_{\hat{o}}\right)(\varphi_C \dotplus \varphi_D) \\
&= \veebar_{\hat{o}\in\Omega}\{(\eta^F_{1z})_{\hat{o}}(\varphi_C \dotplus \varphi_D) : \hat{o} \in \Omega \text{ and } \varphi_C, \varphi_D \in \mathbb{N}\} \\
&\preccurlyeq \veebar_{\hat{o}\in\Omega}\{(\eta^F_{1z})_{\hat{o}}(\varphi_C) \veebar (\eta^F_{1z})_{\hat{o}}(\varphi_D)\} \\
&= \{left(\veebar_{\hat{o}\in\Omega}\{(\eta^F_{1z})_{\hat{o}}(\varphi_C)\}\} \veebar \{\veebar_{\hat{o}\in\Omega}\{(\eta^F_{1z})_{\hat{o}}(\varphi_D)\}\} \\
&= \{\veebar_{\hat{o}\in\Omega}\{(\eta^F_{1z})_{\hat{o}}(\varphi_C)\} \veebar \{\veebar_{\hat{o}\in\Omega}\{(\eta^F_{1z})_{\hat{o}}(\varphi_D))\} \\
&= (\veebar_{\hat{o}\in\Omega}\{(\eta^F_{1z})_{\hat{o}})(\varphi_C) \veebar (\veebar_{\hat{o}\in\Omega}\{(\eta^F_{1z})_{\hat{o}})(\varphi_D).
\end{aligned}$$

$$\begin{aligned}
&\left(\underset{\hat{o}\in\Omega}{\sqcap}(\xi_{1z}^{S})_{\hat{o}}\right)(\varphi_L\varphi_M\varphi_N)\\
&=\underset{\hat{o}\in\Omega}{\sqcap}\{(\xi_{1z}^{S})_{\hat{o}}(\varphi_L\varphi_M\varphi_N):\hat{o}\in\Omega \text{ and } \varphi_L,\varphi_M,\varphi_N\in\mathbb{N}\}\\
&\sqsupseteq\underset{\hat{o}\in\Omega}{\sqcap}\{(\xi_{1z}^{S})_{\hat{o}}(\varphi_L)\sqcap((\xi_{1z}^{S})_{\hat{o}}(\varphi_N)\}\\
&=\{\underset{\hat{o}\in\Omega}{\sqcap}((\xi_{1z}^{S})_{\hat{o}}(\varphi_L))\}\sqcap\{\underset{\hat{o}\in\Omega}{\sqcap}((\xi_{1z}^{S})_{\hat{o}}(\varphi_N))\}\\
&=\{\underset{\hat{o}\in\Omega}{\sqcap}(\xi_{1z}^{S})_{\hat{o}}(\varphi_L)\}\sqcap\{\underset{\hat{o}\in\Omega}{\sqcap}(\xi_{1z}^{S})_{\hat{o}}(\varphi_N)\}\\
&=(\underset{\hat{o}\in\Omega}{\sqcap}(\xi_{1z}^{S})_{\hat{o}}(\varphi_L)\sqcap(\underset{\hat{o}\in\Omega}{\sqcap}(\xi_{1z}^{S})_{\hat{o}}(\varphi_N)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\underset{\hat{o}\in\Omega}{\veebar}(\eta_{1z}^{F})_{\hat{o}}\right)(\varphi_L\varphi_M\varphi_N)\\
&=\underset{\hat{o}\in\Omega}{\veebar}\{(\eta_{1z}^{F})_{\hat{o}}(\varphi_L\varphi_M\varphi_N):\hat{o}\in\Omega \text{ and } \varphi_L,\varphi_M,\varphi_N\in\mathbb{N}\}\\
&\preccurlyeq\underset{\hat{o}\in\Omega}{\veebar}\{(\eta_{1z}^{F})_{\hat{o}}(\varphi_L)\veebar(\eta_{1z}^{F})_{\hat{o}}(\varphi_N)\}\\
&=\{\underset{\hat{o}\in\Omega}{\veebar}((\eta_{1z}^{F})_{\hat{o}}(\varphi_L))\}\veebar\{\underset{\hat{o}\in\Omega}{\veebar}((\eta_{1z}^{F})_{\hat{o}}(\varphi_N))\}\\
&=\{\underset{\hat{o}\in\Omega}{\veebar}(\eta_{1z}^{F})_{\hat{o}}(\varphi_L)\}\veebar\{\underset{\hat{o}\in\Omega}{\veebar}(\eta_{1z}^{F})_{\hat{o}}(\varphi_N)\}\\
&=(\underset{\hat{o}\in\Omega}{\veebar}(\eta_{1z}^{F})_{\hat{o}}(\varphi_L)\veebar(\underset{\hat{o}\in\Omega}{\veebar}(\eta_{1z}^{F})_{\hat{o}})(\varphi_N).
\end{aligned}$$

$$\begin{aligned}
\left(\underset{\hat{o}\in\Omega}{\sqcap}(\xi_{1z}^{S})_{\hat{o}}\right)(\varphi_R)&=\underset{\hat{o}\in\Omega}{\sqcap}\{\{(\xi_{1z}^{S})_{\hat{o}}(\varphi_R):\hat{o}\in\Omega \text{ and } \varphi_R\in\mathbb{N}\}\\
&\sqsupseteq\underset{\hat{o}\in\Omega}{\sqcap}\{(\xi_{1z}^{S})_{\hat{o}}(\varphi_S)\sqcap\underset{\hat{o}\in\Omega}{\sqcap}(\xi_{1z}^{S})_{\hat{o}}(\varphi_T)\}\\
&=\{\underset{\hat{o}\in\Omega}{\sqcap}(\xi_{1z}^{S})_{\hat{o}}(\varphi_S)\}\sqcap\{\underset{\hat{o}\in\Omega}{\sqcap}(\xi_{1z}^{S})_{\hat{o}}(\varphi_T)\}\\
&=\{\underset{\hat{o}\in\Omega}{\sqcap}(\xi_{1z}^{S})_{\hat{o}}(\varphi_S)\}\sqcap\{\underset{\hat{o}\in\Omega}{\sqcap}((\xi_{1z}^{S})_{\hat{o}}(\varphi_T))\}\\
&=(\underset{\hat{o}\in\Omega}{\sqcap}(\xi_{1z}^{S})_{\hat{o}})(\varphi_S)\sqcap(\underset{\hat{o}\in\Omega}{\sqcap}(\xi_{1z}^{S})_{\hat{o}})(\varphi_T)
\end{aligned}$$

and

$$\begin{aligned}
\left(\underset{\hat{o}\in\Omega}{\veebar}(\eta_{1z}^{F})_{\hat{o}}\right)(\varphi_R)&=\underset{\hat{o}\in\Omega}{\veebar}\{(\eta_{1z}^{F})_{\hat{o}}(\varphi_R):\hat{o}\in\Omega \text{ and } \varphi_R\in\mathbb{N}\}\\
&\preccurlyeq\underset{\hat{o}\in\Omega}{\veebar}\{(\eta_{1z}^{F})_{\hat{o}}(\varphi_S)\veebar(\eta_{1z}^{F})_{\hat{o}}(\varphi_T)\}\\
&=\{\underset{\hat{o}\in\Omega}{\veebar}((\eta_{1z}^{F})_{\hat{o}}(\varphi_S))\}\veebar\{\underset{\hat{o}\in\Omega}{\veebar}((\eta_{1z}^{F})_{\hat{o}}(\varphi_T))\}\\
&=\{\underset{\hat{o}\in\Omega}{\veebar}(\eta_{1z}^{F})_{\hat{o}}(\varphi_S)\}\veebar\{\underset{\hat{o}\in\Omega}{\veebar}(\eta_{1z}^{F})_{\hat{o}}(\varphi_T)\}\\
&=(\underset{\hat{o}\in\Omega}{\veebar}(\eta_{1z}^{F})_{\hat{o}})(\varphi_S)\veebar(\underset{\hat{o}\in\Omega}{\veebar}(\eta_{1z}^{F})_{\hat{o}})(\varphi_T).
\end{aligned}$$

□

The result given below tells us that the intersection of a family of prime hybrid h-bi-ideals in a hemring $\mathbb{N}$ over $\mathbb{C}$ is semiprime.

Lemma 2. *If $\mathbb{Z}_{\tilde{1}}$ is a member of the family of PḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$ and subsequently $\underset{\iota\in\zeta}{\tilde{\cap}}(\mathbb{Z}_{\tilde{1}})_{\iota}$ is a SPḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$.*

Proof. Assume that $\mathbb{Z}_{\tilde{1}}$ is an element of a collection $\{(\mathbb{Z}_{\tilde{1}})_\iota : \iota \in \zeta\}$ of PḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$, Lemma 1 states that $\underset{\iota\in\zeta}{\widetilde{\cap}}(\mathbb{Z}_{\tilde{1}})_\iota$ is a ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$. Consequently, we acquire that $(\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{2}})\widetilde{\subseteq}(\mathbb{Z}_{\tilde{1}})_\iota$, for all $\iota \in \zeta$ if $(\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{2}})\widetilde{\subseteq} \underset{\iota\in\zeta}{\widetilde{\cap}}(\mathbb{Z}_{\tilde{1}})_\iota$ for any ḦyH-BI $\mathbb{Z}_{\tilde{2}}$ of $\mathbb{N}$. Since each $(\mathbb{Z}_{\tilde{1}})_\iota$ is a PḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$ eventually $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}(\mathbb{Z}_{\tilde{1}})_\iota$ for all $\iota \in \zeta$. Therefore $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq} \underset{\iota\in\zeta}{\widetilde{\cap}}(\mathbb{Z}_{\tilde{1}})_\iota$. □

Definition 13. *An irreducible (strongly irreducible) hybrid h-bi-ideal Ir(SIr)ḦyH-BI of* $\mathbb{N}$ *over* $\mathbb{C}$ *is an ḦyH-BI* $\mathbb{Z}_{\tilde{3}}$ *in such a manner that* $\mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}} = \mathbb{Z}_{\tilde{3}}$ $(\mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{3}})$ *implies* $\mathbb{Z}_{\tilde{1}} = \mathbb{Z}_{\tilde{3}}$ *or* $\mathbb{Z}_{\tilde{2}} = \mathbb{Z}_{\tilde{3}}(\mathbb{Z}_{\tilde{1}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{3}}$ *or* $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{3}}$ *), whereas* $\mathbb{Z}_{\tilde{1}}$, $\mathbb{Z}_{\tilde{2}}$ *are ḦyH-BI of* $\mathbb{N}$ *over* $\mathbb{C}$.

We demonstrate that every strongly irreducible semiprime ḦyH-BI of $\mathbb{N}$ upon $\mathbb{C}$ is a StPḦyH-BI in the paragraph that follows.

Theorem 1. *Every strongly irreducible, SPḦyH-BI of* $\mathbb{N}$ *over* $\mathbb{C}$ *is a StPḦyH-BI.*

Proof. Assume that $\mathbb{Z}_{\tilde{1}}$ be a strongly irreducible, SPḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$. Let $\mathbb{Z}_{\tilde{2}}$ and $\mathbb{Z}_{\tilde{3}}$ are ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$ such that $(\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{3}})\widetilde{\cap}(\mathbb{Z}_{\tilde{3}} \boxtimes \mathbb{Z}_{\tilde{2}})\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$. Since $\mathbb{Z}_{\tilde{2}}\widetilde{\cap}\mathbb{Z}_{\tilde{3}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{2}}$ and $\mathbb{Z}_{\tilde{2}}\widetilde{\cap}\mathbb{Z}_{\tilde{3}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{3}}$, so $(\mathbb{Z}_{\tilde{2}}\widetilde{\cap}\mathbb{Z}_{\tilde{3}}) \boxtimes (\mathbb{Z}_{\tilde{2}}\widetilde{\cap}\mathbb{Z}_{\tilde{3}})\widetilde{\subseteq}\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{3}}$ and $(\mathbb{Z}_{\tilde{2}}\widetilde{\cap}\mathbb{Z}_{\tilde{3}}) \boxtimes (\mathbb{Z}_{\tilde{2}}\widetilde{\cap}\mathbb{Z}_{\tilde{3}})\widetilde{\subseteq}\mathbb{Z}_{\tilde{3}} \boxtimes \mathbb{Z}_{\tilde{2}}$. Thus $(\mathbb{Z}_{\tilde{2}}\widetilde{\cap}\mathbb{Z}_{\tilde{3}}) \boxtimes (\mathbb{Z}_{\tilde{2}}\widetilde{\cap}\mathbb{Z}_{\tilde{3}})$ $\widetilde{\subseteq}(\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{3}})\widetilde{\cap}(\mathbb{Z}_{\tilde{3}} \boxtimes \mathbb{Z}_{\tilde{2}})\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$. This implies that $\mathbb{Z}_{\tilde{2}}\widetilde{\cap}\mathbb{Z}_{\tilde{3}}\ \widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$, because $\mathbb{Z}_{\tilde{1}}$ is a SPḦyH-BI of $\mathbb{N}$. Since $\mathbb{Z}_{\tilde{1}}$ is SIrḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$, $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$ or $\mathbb{Z}_{\tilde{3}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$. □

4. Hemirings in Which Each Hybrid H-bi-Ideal Is Strongly Prime

The hemirings in which each ḦyH-BI is semiprime are studied in this part of the paper. Furthermore, we talk about the hemirings in which each ḦyH-BI is strongly prime.

Proposition 3. *Let* $\varphi_E \in \mathbb{N}$ *and* $\omega \in \wp(\mathbb{C}), m \in (0,1]$ *and* $\mathbb{Z}_{\tilde{1}}$ *be a ḦyH-BI of* $\mathbb{N}$ *over* $\mathbb{C}$ *defined by* $\mathbb{Z}_{\tilde{1}}(\varphi_E) = (\xi^S_{1z}(\varphi_E), \eta^F_{1z}(\varphi_E)) = (\omega, m)$. *Then* $\exists$ *an IrḦyH-BI* $\mathbb{Z}_{\tilde{2}}$ *of* $\mathbb{N}$ *upon* $\mathbb{C}$ *with* $\mathbb{Z}_{\tilde{1}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{2}}$ *and defined by* $\mathbb{Z}_{\tilde{2}}(\varphi_E) = (\xi^S_{2z}(\varphi_E), \eta^F_{2z}(\varphi_E)) = (\omega, m)$.

Proof. Let $\mathbb{Z}_{\tilde{3}}$ is a ḦyH-BI of $\mathbb{N}$ defined by $\mathbb{Z}_{\tilde{3}}(\varphi_E) = (\xi^S_{3z}(\varphi_E), \eta^F_{3z}(\varphi_E)) = (\omega, m)$ with $\mathbb{Z}_{\tilde{1}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{3}}$. Now let $\Im \neq \emptyset$, is a collection of ḦyH-BI $\mathbb{Z}_{\tilde{3}}$ of $\mathbb{N}$ and the elements of this collection are partially ordered. Let $\mathcal{L}\widetilde{\subseteq}\Im$ and is totally ordered. Then $\underset{\iota\in\Omega}{\widetilde{\cup}}(\mathbb{Z}_{\tilde{3}})_\iota = \left\langle \underset{\iota\in\Omega}{\sqcup}(\xi^S_{3z})_\iota, \underset{\iota\in\Omega}{\barwedge}(\eta^F_{3z})_\iota \right\rangle$ is a ḦyH-BI of $\mathbb{N}$ such that $\mathbb{Z}_{\tilde{1}}\widetilde{\subseteq} \underset{\iota\in\Omega}{\sqcup}(\mathbb{Z}_{\tilde{3}})_\iota$. Clearly, when we take $\varphi_A, \varphi_B, \varphi_C \in \mathbb{N}$, we get

$$\begin{aligned}
(\underset{\iota\in\Omega}{\sqcup}(\xi^S_{3z}))(\varphi_A\varphi_B) &= \underset{\iota\in\Omega}{\sqcup}((\xi^S_{3z})(\varphi_A\varphi_B) \\
&\sqsupseteq \underset{\iota\in\Omega}{\sqcup}\{(\xi^S_{3z})(\varphi_A) \sqcap (\xi^S_{3z})(\varphi_B)\} \\
&= \{\underset{\iota\in\Omega}{\sqcup}(\xi^S_{3z})(\varphi_A) \sqcap \underset{\iota\in\Omega}{\sqcup}(\xi^S_{3z})(\varphi_B)\}
\end{aligned}$$

and

$$\begin{aligned}
(\underset{\iota\in\Omega}{\barwedge}(\eta^F_{3z})_\iota)(\varphi_A\varphi_B) &= \underset{\iota\in\Omega}{\barwedge}((\eta^F_{3z})_\iota(\varphi_A\varphi_B)) \\
&\preccurlyeq \underset{\iota\in\Omega}{\barwedge}\{(\eta^F_{3z})_\iota(\varphi_A) \veebar (\eta^F_{3z})_\iota(\varphi_B)\} \\
&= \{\underset{\iota\in\Omega}{\barwedge}(\eta^F_{3z})_\iota(\varphi_A) \veebar \underset{\iota\in\Omega}{\barwedge}(\eta^F_{3z})_\iota(\varphi_B)\}.
\end{aligned}$$

$$
\begin{aligned}
(\bigsqcup_{\iota\in\Omega}(\xi^S_{3z}))(\varphi_A \dotplus \varphi_B) &= \bigsqcup_{\iota\in\Omega}((\xi^S_{3z})(\varphi_A \dotplus \varphi_B)) \\
&\sqsupseteq \bigsqcup_{\iota\in\Omega}\{(\xi^S_{3z})(\varphi_A) \sqcap (\xi^S_{3z})(\varphi_B)\} \\
&= \{\bigsqcup_{\iota\in\Omega}(\xi^S_{3z})(\varphi_A) \sqcap \bigsqcup_{\iota\in\Omega}(\xi^S_{3z})(\varphi_B)\}
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathop{\overline{\wedge}}_{\iota\in\Omega}(\eta^F_{3z})_\iota)(\varphi_A \dotplus \varphi_B) &= \mathop{\overline{\wedge}}_{\iota\in\Omega}((\eta^F_{3z})_\iota(\varphi_A \dotplus \varphi_B)) \\
&\preccurlyeq \mathop{\overline{\wedge}}_{\iota\in\Omega}\{(\eta^F_{3z})_\iota(\varphi_A) \veebar (\eta^F_{3z})_\iota(\varphi_B)\} \\
&= \{\mathop{\overline{\wedge}}_{\iota\in\Omega}(\eta^F_{3z})_\iota(\varphi_A) \veebar \mathop{\overline{\wedge}}_{\iota\in\Omega}(\eta^F_{3z})_\iota(\varphi_B)\}.
\end{aligned}
$$

$$
\begin{aligned}
(\bigsqcup_{\iota\in\Omega}(\xi^S_{3z}))(\varphi_C) &= \bigsqcup_{\iota\in\Omega}((\xi^S_{3z})(\varphi_C)) \\
&\sqsupseteq \bigsqcup_{\iota\in\Omega}\{(\xi^S_{3z})(\varphi_A) \sqcap (\xi^S_{3z})(\varphi_B)\} \\
&= \{\bigsqcup_{\iota\in\Omega}(\xi^S_{3z})(\varphi_A) \sqcap \bigsqcup_{\iota\in\Omega}(\xi^S_{3z})(\varphi_B)\}
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathop{\overline{\wedge}}_{\iota\in\Omega}(\eta^F_{3z})_\iota)(\varphi_C) &= \mathop{\overline{\wedge}}_{\iota\in\Omega}((\eta^F_{3z})_\iota(\varphi_C)) \\
&\preccurlyeq \mathop{\overline{\wedge}}_{\iota\in\Omega}\{(\eta^F_{3z})_\iota(\varphi_A) \veebar (\eta^F_{3z})_\iota(\varphi_B)\} \\
&= \{\mathop{\overline{\wedge}}_{\iota\in\Omega}(\eta^F_{3z})_\iota(\varphi_A) \veebar \mathop{\overline{\wedge}}_{\iota\in\Omega}(\eta^F_{3z})_\iota(\varphi_B)\}
\end{aligned}
$$

$$
\begin{aligned}
(\bigsqcup_{\iota\in\Omega}(\xi^S_{3z}))(\varphi_A\varphi_B\varphi_C) &= \bigsqcup_{\iota\in\Omega}((\xi^S_{3z})(\varphi_A\varphi_B\varphi_C)) \\
&\sqsupseteq \bigsqcup_{\iota\in\Omega}\{(\xi^S_{3z})(\varphi_A) \sqcap (\xi^S_{3z})(\varphi_C)\} \\
&= \{\bigsqcup_{\iota\in\Omega}(\xi^S_{3z})(\varphi_A) \sqcap \bigsqcup_{\iota\in\Omega}(\xi^S_{3z})(\varphi_C)\}
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathop{\overline{\wedge}}_{\iota\in\Omega}(\eta^F_{3z})_\iota)(\varphi_A\varphi_B\varphi_C) &= \mathop{\overline{\wedge}}_{\iota\in\Omega}((\eta^F_{3z})_\iota(\varphi_A\varphi_B\varphi_C)) \\
&\preccurlyeq \mathop{\overline{\wedge}}_{\iota\in\Omega}\{(\eta^F_{3z})_\iota(\varphi_A) \veebar (\eta^F_{3z})_\iota(\varphi_C)\} \\
&= \{\mathop{\overline{\wedge}}_{\iota\in\Omega}(\eta^F_{3z})_\iota(\varphi_A) \veebar \mathop{\overline{\wedge}}_{\iota\in\Omega}(\eta^F_{3z})_\iota(\varphi_C)\}.
\end{aligned}
$$

Thus $\widetilde{\bigcup}_{\iota\in\Omega}(\mathbb{Z}_{\tilde{3}})_\iota$ is a ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$. Since $\mathbb{Z}_{\tilde{1}}\widetilde{\subseteq}(\mathbb{Z}_{\tilde{3}})_\iota$ for all $\iota \in \Omega$, we have $\mathbb{Z}_{\tilde{1}}\widetilde{\subseteq}\widetilde{\bigcup}_{\iota\in\Omega}(\mathbb{Z}_{\tilde{3}})_\iota$. Also $(\widetilde{\bigcup}_{\iota\in\Omega}(\mathbb{Z}_{\tilde{3}})_\iota)(\varphi_E) = \widetilde{\bigcup}_{\iota\in\Omega}(\mathbb{Z}_{\tilde{3}})_\iota(\varphi_E) = (\omega, m)$. Thus $\widetilde{\bigcup}_{\iota\in\Omega}(\mathbb{Z}_{\tilde{3}})_\iota \in \mathcal{L}$ and $\widetilde{\bigcup}_{\iota\in\Omega}(\mathbb{Z}_{\tilde{3}})_\iota$ is an upper bound of $\mathbb{N}$. The existence of a maximal ḦyH-BI $\mathbb{Z}_{\tilde{2}}$ of $\mathbb{N}$ is declared by Zorn's Lemma and is defined by $\mathbb{Z}_{\tilde{2}}(\varphi_E) = (\omega, m)$, also $\mathbb{Z}_{\tilde{1}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{2}}$. Let $\mathbb{Z}_{\tilde{a}}\widetilde{\cap}\mathbb{Z}_{\tilde{b}} = \mathbb{Z}_{\tilde{2}}$ for any ḦyH-BI $\mathbb{Z}_{\tilde{a}}, \mathbb{Z}_{\tilde{b}}$ of $\mathbb{N}$. We get $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{a}}$ and $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{b}}$. It is obvious that $\mathbb{Z}_{\tilde{2}} = \mathbb{Z}_{\tilde{a}}$ or $\mathbb{Z}_{\tilde{2}} = \mathbb{Z}_{\tilde{b}}$. We claim that $\mathbb{Z}_{\tilde{2}} \neq \mathbb{Z}_{\tilde{a}}$ and $\mathbb{Z}_{\tilde{2}} \neq \mathbb{Z}_{\tilde{b}}$. Then $\mathbb{Z}_{\tilde{a}}(\varphi_E) \neq (\delta, t)$, $\mathbb{Z}_{\tilde{b}}(\varphi_E) \neq (\omega, m)$. Thus, $(\mathbb{Z}_{\tilde{a}}\widetilde{\cap}\mathbb{Z}_{\tilde{b}})(\varphi_E) = (\mathbb{Z}_{\tilde{a}}(\varphi_E)\widetilde{\cap}\mathbb{Z}_{\tilde{b}}(\varphi_E)) \neq (\omega, m)$, and a contradiction arises to our supposition that $(\mathbb{Z}_{\tilde{a}}(\varphi_E)\widetilde{\cap}\mathbb{Z}_{\tilde{b}}(\varphi_E)) = \mathbb{Z}_{\tilde{2}}(\varphi_E) = (\omega, m)$. Hence, $\mathbb{Z}_{\tilde{2}} = \mathbb{Z}_{\tilde{a}}$ or $\mathbb{Z}_{\tilde{2}} = \mathbb{Z}_{\tilde{b}}$. Therefore, $\mathbb{Z}_{\tilde{2}}$ is an IrḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$.

The following theorem investigates the H-HemiR and H-IHemiR hemirings for which each ḦyH-BI is semiprime. □

Theorem 2. *The following statements have similar results in* $\mathbb{N}$*:*

(1) $\mathbb{N}$ is H-HemiR and H-IHemiR hemiring at the same time.
(2) For each single ḦyH-BI $\mathbb{Z}_{\tilde{1}}$ of $\mathbb{N}$ over $\mathbb{C}$, $\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{1}} = \mathbb{Z}_{\tilde{1}}$.
(3) With $\mathbb{Z}_{\tilde{1}}$ and $\mathbb{Z}_{\tilde{2}}$ selected at random ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$, $\mathbb{Z}_{\tilde{1}} \tilde{\cap} \mathbb{Z}_{\tilde{2}} = (\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}}) \tilde{\cap} (\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{1}})$.
(4) Every ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$ results in an SPḦyH-BI.
(5) Every proper ḦyH-BI of $\mathbb{N}$ is the intersection of irreducible SPḦyH-BI of $\mathbb{N}$ and the proper ḦyH-BI is likewise included in the intersection.

Proof. (1)$\Longrightarrow$ (2) Understood.

(2)$\Longrightarrow$ (3) Let $\mathbb{Z}_{\tilde{1}}, \mathbb{Z}_{\tilde{2}}$ be ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$. Lemma 2 suggests that $\mathbb{Z}_{\tilde{1}} \tilde{\cap} \mathbb{Z}_{\tilde{2}}$ is a ḦyH-BI. Hypothesis gives the result that $\mathbb{Z}_{\tilde{1}} \tilde{\cap} \mathbb{Z}_{\tilde{2}} = \left(\mathbb{Z}_{\tilde{1}} \tilde{\cap} \mathbb{Z}_{\tilde{2}}\right) \boxtimes \left(\mathbb{Z}_{\tilde{1}} \tilde{\cap} \mathbb{Z}_{\tilde{2}}\right)$. Also we know that $\left(\mathbb{Z}_{\tilde{1}} \tilde{\cap} \mathbb{Z}_{\tilde{2}}\right) \boxtimes \left(\mathbb{Z}_{\tilde{1}} \tilde{\cap} \mathbb{Z}_{\tilde{2}}\right) \tilde{\subseteq} \mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}}$.

In the same way we can write $\left(\mathbb{Z}_{\tilde{1}} \tilde{\cap} \mathbb{Z}_{\tilde{2}}\right) \tilde{\subseteq} \mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{1}}$. Thereby $\left(\mathbb{Z}_{\tilde{1}} \tilde{\cap} \mathbb{Z}_{\tilde{2}}\right) \tilde{\subseteq} (\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}}) \tilde{\cap} (\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{1}})$ is obtained.

$\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}}$ and $\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{1}}$ are ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$ according to the Proposition 2. As a result, $(\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}}) \tilde{\cap} (\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{1}})$ is a ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$. In the light of this, we can it write as:

$$\begin{aligned}
&\left((\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}}) \tilde{\cap} (\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{1}})\right) \\
&= \left((\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}}) \tilde{\cap} (\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{1}})\right) \boxtimes \\
&\left((\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}}) \tilde{\cap} (\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{1}})\right) \\
&\tilde{\subseteq} (\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}}) \boxtimes (\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{1}}) \\
&\tilde{\subseteq} \mathbb{Z}_{\tilde{1}} \boxtimes (\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{2}}) \boxtimes \mathbb{Z}_{\tilde{1}} \\
&\tilde{\subseteq} \mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{1}} \\
&\tilde{\subseteq} \mathbb{Z}_{\tilde{1}} \boxtimes \mathtt{C} \boxtimes \mathbb{Z}_{\tilde{1}} \\
&\tilde{\subseteq} \mathbb{Z}_{\tilde{1}}.
\end{aligned}$$

Analogously, we obtain $\left((\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}}) \tilde{\cap} (\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{1}})\right) \tilde{\subseteq} \mathbb{Z}_{\tilde{2}}$. Hence $(\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}}) \tilde{\cap} (\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{1}}) \tilde{\subseteq} \mathbb{Z}_{\tilde{1}} \tilde{\cap} \mathbb{Z}_{\tilde{2}}$. Therefore, $(\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}}) \tilde{\cap} (\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{1}}) = \mathbb{Z}_{\tilde{1}} \tilde{\cap} \mathbb{Z}_{\tilde{2}}$.

(3)$\Longrightarrow$ (2) Understood.

(2)$\Longrightarrow$ (4) Let $\mathbb{Z}_{\tilde{1}}$, $\mathbb{Z}_{\tilde{2}}$ be ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$ be such that $\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{1}} \tilde{\subseteq} \mathbb{Z}_{\tilde{2}}$. Since by (2) $\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{1}} = \mathbb{Z}_{\tilde{1}}$, so $\mathbb{Z}_{\tilde{1}} \tilde{\subseteq} \mathbb{Z}_{\tilde{2}}$. Thus $\mathbb{Z}_{\tilde{2}}$ is SpḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$.

(4)$\Longrightarrow$ (2) Understood.

(4)$\Longrightarrow$ (5) Let $\mathbb{Z}_{\tilde{1}}(\varphi_S) = (\delta, t)$ be a proper ḦyH-BI of $\mathbb{N}$ for $\varphi_S \in \mathbb{N}, \delta \subseteq \wp(\mathbb{C}), t \in (0,1]$, respectively. An IrḦyH-BI $\mathbb{Z}_{\tilde{2}}$ of $\mathbb{N}$ exists under the assumption that $\mathbb{Z}_{\tilde{1}} \tilde{\subseteq} (\mathbb{Z}_{\tilde{2}})_\alpha = (\delta, t)$, which produces $\mathbb{Z}_{\tilde{1}} \tilde{\subseteq} \tilde{\cap} (\mathbb{Z}_{\tilde{2}})_\alpha$, in accordance with Proposition 3. This means that $\mathbb{Z}_{\tilde{1}}$ is the intersection of all IrḦyH-BI of $\mathbb{N}$ which include $\mathbb{Z}_{\tilde{1}}$. Based on the assumption we have, every ḦyH-BI is a StPḦyH-BI. Finally, $\mathbb{Z}_{\tilde{1}}$ is the intersection of all irreducible SPḦyH-BI of $\mathbb{N}$ containing $\mathbb{Z}_{\tilde{1}}$.

(5)$\Longrightarrow$ (2) Suppose $\mathbb{Z}_{\tilde{1}}$ is ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$, in that case, $\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{1}}$ is a ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$ as well. Hence, $\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{1}} = \underset{\iota \in \Omega}{\tilde{\cap}} (\mathbb{Z}_{\tilde{1}})_i$ wherein every $(\mathbb{Z}_{\tilde{1}})_\iota$ is an irreducible, StPḦyH-BI of $\mathbb{N}$ which may be stated in the manner of $\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{1}} \tilde{\subseteq} (\mathbb{Z}_{\tilde{1}})_\iota \; \forall \; \iota \in \Omega$. In the light of the fact that each $(\mathbb{Z}_{\tilde{1}})_\iota$ is semiprime, we can infer that $\mathbb{Z}_{\tilde{1}} \tilde{\subseteq} (\mathbb{Z}_{\tilde{1}})_\iota$ for all ι. As a result $\mathbb{Z}_{\tilde{1}} \tilde{\subseteq} \underset{\iota \in \Omega}{\tilde{\cap}} (\mathbb{Z}_{\tilde{1}})_\iota = \mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{1}}$. However, $\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{1}} \tilde{\subseteq} \mathbb{Z}_{\tilde{1}}$ is definitely valid. Consequently $\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{1}} = \mathbb{Z}_{\tilde{1}}$. □

Theorem 3. *For an H-HemiR and H-IHemiR hemiring* $\mathbb{N}$*, the subsequent results are equivalent.*

(1) $\mathbb{Z}_{\tilde{1}}$ has the form SIrḦyH-BI.
(2) $\mathbb{Z}_{\tilde{1}}$ exhibits StPḦyH-BI.

Proof. (1)$\Longrightarrow$ (2) Considering the fact that $\mathbb{N}$ is an H-HemiR and H-IHemiR-hemiring, we can come up with $\mathbb{Z}_{\tilde{2}}\widetilde{\cap}\mathbb{Z}_{\tilde{3}} = (\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{3}})\widetilde{\cap}(\mathbb{Z}_{\tilde{3}} \boxtimes \mathbb{Z}_{\tilde{2}})$ by applying Theorem 2. $\mathbb{Z}_{\tilde{1}}$ being SIrḦyH-BI, $\mathbb{Z}_{\tilde{2}}\widetilde{\cap}\mathbb{Z}_{\tilde{3}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$ points to $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$ or $\mathbb{Z}_{\tilde{3}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$ with any ḦyH-BI i.e.,$\mathbb{Z}_{\tilde{2}}$and $\mathbb{Z}_{\tilde{3}}$. As a result, $(\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{3}})\widetilde{\cap}(\mathbb{Z}_{\tilde{3}} \boxtimes \mathbb{Z}_{\tilde{2}})\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$ suggests either $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$ or $\mathbb{Z}_{\tilde{3}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$. $\mathbb{Z}_{\tilde{1}}$ is therefore a StPḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$.

(2)$\Longrightarrow$ (1) Considering that $\mathbb{Z}_{\tilde{1}}$ is a StPḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$, the statement $\mathbb{Z}_{\tilde{2}}\widetilde{\cap}\mathbb{Z}_{\tilde{3}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$ indicates that $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$ or $\mathbb{Z}_{\tilde{3}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$ correspond to any ḦyH-BI, $\mathbb{Z}_{\tilde{2}}$ and $\mathbb{Z}_{\tilde{3}}$, respectively. We can deduce from Theorem 2 that $(\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{3}})\widetilde{\cap}(\mathbb{Z}_{\tilde{3}} \boxtimes \mathbb{Z}_{\tilde{2}}) = \mathbb{Z}_{\tilde{2}}\widetilde{\cap}\mathbb{Z}_{\tilde{3}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$ since $\mathbb{N}$ is both H-HemiR and H-IHemiR. In light of the fact that $\mathbb{Z}_{\tilde{1}}$ is StPḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$, the outcome is either $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$ or $\mathbb{Z}_{\tilde{3}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$. □

In the following Theorem, it is shown that every ḦyH-BI of a totally ordered, H-HemiR and H-IHemiR hemiring $\mathbb{N}$ is strongly prime.

Theorem 4. *ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$ satisfies total order and $\mathbb{N}$ is H-HemiR and H-IHemiR if and only if each ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$ is StPḦyH-BI.*

Proof. Let $\mathbb{Z}_{\tilde{1}}, \mathbb{Z}_{\tilde{2}}$ and $\mathbb{Z}_{\tilde{3}}$ be any ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$ arranged in the way $(\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}})\widetilde{\cap}(\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{1}})\widetilde{\subseteq}\mathbb{Z}_{\tilde{3}}$. Consider the case where $\mathbb{N}$ is H-HemiR and H-IHemiR and its elements are in total order. According to Theorem 2,

$$\begin{aligned}\mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}} &= (\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}})\widetilde{\cap}(\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{1}}) \\ &\widetilde{\subseteq}\mathbb{Z}_{\tilde{3}}.\end{aligned}$$

$\mathbb{Z}_{\tilde{1}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{2}}$ or $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$, implies $\mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}} = \mathbb{Z}_{\tilde{1}}$ or $\mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}} = \mathbb{Z}_{\tilde{2}}$. Further, $\mathbb{Z}_{\tilde{1}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{3}}$ or $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{3}}$ concludes as a result of assumption. Hence $\mathbb{Z}_{\tilde{3}}$ is StPḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$.

Contrary to this, every ḦyH-BI of $\mathbb{N}$ is SPḦyH-BI since, by presumption, each ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$ is StPḦyH-BI. $\mathbb{N}$ is H-HemiR and H-IHemiR, as proven by Theorem 2. Additionally, take arbitrary ḦyH-BI $\mathbb{Z}_{\tilde{1}}$ and $\mathbb{Z}_{\tilde{2}}$ of $\mathbb{N}$ over $\mathbb{C}$. It follows from Theorem 2 that $(\mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}}) = (\mathbb{Z}_{\tilde{1}} \boxtimes \mathbb{Z}_{\tilde{2}})\widetilde{\cap}(\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{1}})$. $\mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}}$ is StPḦyH-BI since every single ḦyH-BI is StPḦyH-BI. Consequently, $\mathbb{Z}_{\tilde{1}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}}$ or $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}}$. Furthermore, if $\mathbb{Z}_{\tilde{1}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}}$, then $\mathbb{Z}_{\tilde{1}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{2}}$ and if $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}}$, in turn $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$. □

Theorem 5. *In a hemiring $\mathbb{N}$, the statements provided here are identical.*

(1) Set of ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$ satisfies total order (TO).
(2) Each ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$ is SIrḦyH-BI.
(3) Each ḦyH-BI of $\mathbb{N}$ upon $\mathbb{C}$ is IrḦyH-BI.

Proof. (1) $\Longrightarrow$ (2) Assume that $(\mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}})\widetilde{\subseteq}\mathbb{Z}_{\tilde{3}}$ for any ḦyH-BI $\mathbb{Z}_{\tilde{1}}$, $\mathbb{Z}_{\tilde{2}}$ and $\mathbb{Z}_{\tilde{3}}$ of $\mathbb{N}$ over $\mathbb{C}$. Now $\mathbb{Z}_{\tilde{1}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{2}}$ or $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$ by our assumption that the set of ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$ is TO .This implies that either $(\mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}}) = \mathbb{Z}_{\tilde{1}}$ or $(\mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}}) = \mathbb{Z}_{\tilde{2}}$. Thus $\mathbb{Z}_{\tilde{1}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{3}}$ or $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{3}}$. Hence $\mathbb{Z}_{\tilde{3}}$ is a SIrḦyH-BI of $\mathbb{N}$ upon $\mathbb{C}$.

(2)$\Longrightarrow$ (3) Let $(\mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}}) = \mathbb{Z}_{\tilde{3}}$ where $\mathbb{Z}_{\tilde{1}}, \mathbb{Z}_{\tilde{2}}$ and $\mathbb{Z}_{\tilde{3}}$ are ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$. We obtain that $\mathbb{Z}_{\tilde{3}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$ and $\mathbb{Z}_{\tilde{3}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{2}}$. Again by hypothesis, either $\mathbb{Z}_{\tilde{1}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{3}}$ or $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{3}}$. Thus, we can see that either $\mathbb{Z}_{\tilde{1}} = \mathbb{Z}_{\tilde{3}}$ or $\mathbb{Z}_{\tilde{2}} = \mathbb{Z}_{\tilde{3}}$. Hence, $\mathbb{Z}_{\tilde{3}}$ is IrḦyH-BI of $\mathbb{N}$ upon $\mathbb{C}$.

(3)$\Longrightarrow$ (1) Let $\mathbb{Z}_{\tilde{1}}$ and $\mathbb{Z}_{\tilde{2}}$ be any ḦyH-BI of $\mathbb{N}$ upon $\mathbb{C}$ then $\mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}}$ is a ḦyH-BI. We can write $\mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}} = \mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}}$. Thus either $\mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}} = \mathbb{Z}_{\tilde{1}}$ or $\mathbb{Z}_{\tilde{1}}\widetilde{\cap}\mathbb{Z}_{\tilde{2}} = \mathbb{Z}_{\tilde{2}}$ by hypothesis, that is $\mathbb{Z}_{\tilde{1}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{2}}$ or $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$. As a result, the set of ḦyH-BI of $\mathbb{N}$ upon $\mathbb{C}$ is totally ordered. □

5. Proposed Hybrid Structure-Based Algorithm in Decision Making

This section introduces a novel algorithm utilizing hybrid structures to handle uncertain information in real-world decision-making scenarios. By showcasing its practical application, we aim to demonstrate the algorithm's effectiveness and versatility, offering valuable insights for enhancing decision-making processes amid complex data uncertainties.

5.1. Tabular Representation of Hybrid Structure

Here, we describe the general tabular form before proposing a hybrid structure-based algorithm.

Let $\mathbb{Z}_{\tilde{1}} = (\xi^S_{1z}, \eta^F_{1z})$ be any ḦyS in a hemiring $\mathbb{N}$ over an initial universal set $\mathbb{C}$. Then, $\mathbb{Z}_{\tilde{1}}$ is presented as:

$\mathbb{Z}_{\tilde{1}} = \{\langle \varphi_A, (\xi^S_{1z})(\varphi_A), (\eta^F_{1z})(\varphi_A)\rangle : \varphi_A \in \mathbb{N}\}$, where $\xi^S_{1z} : \mathbb{N} \longrightarrow \wp(\mathbb{C})$ and $\eta^F_{1z} : \mathbb{N} \longrightarrow I$ are mappings, $\wp(\mathbb{C})$ represents the set of all subsets of $\mathbb{C}$ and $I = [0,1]$.

In general, let $\mathbb{C} = \{S_1, S_2, S_3, S_5, ..., S_m\}$ an initial universal set and $\mathbb{Q} = \{\varphi_{A_1}, \varphi_{A_2}, \varphi_{A_3}, ..., \varphi_{A_n}\} \subseteq \mathbb{N}$, the ḦyS $\mathbb{Z}_{\tilde{1}}$ of $\mathbb{Q}$ over $\mathbb{C}$ is given by

$$\mathbb{Z}_{\tilde{1}} = (\xi^S_{1z}, \eta^F_{1z}) = \left\{ \varphi_{A_i} : \begin{array}{c} (\xi^S_{1z}, \eta^F_{1z})(\varphi_{A_1}) = (\xi^S_{1z}(\varphi_{A_1}), \eta^F_{1z}(\varphi_{A_1})), \\ (\xi^S_{1z}, \eta^F_{1z})(\varphi_{A_2}) = (\{\xi^S_{1z}(\varphi_{A_2}), \eta^F_{1z}(\varphi_{A_2})), \\ (\xi^S_{1z}, \eta^F_{1z})(\varphi_{A_3}) = (\xi^S_{1z}(\varphi_{A_1}), \eta^F_{1z}(\varphi_{A_1})), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ (\xi^S_{1z}, \eta^F_{1z})(\varphi_{A_n}) = (\xi^S_{1z}(\varphi_{A_n}), \eta^F_{1z}(\varphi_{A_n})), \end{array} \forall \varphi_{A_i} \in \mathbb{Q} \right\}.$$

The tabular representation of $\mathbb{Z}_{\tilde{1}}$ is signified in Table 5 [35].

Table 5. The tabular description of hybrid structure.

$\mathbb{N}$	$\boldsymbol{\xi^S_{1z}}$	$\boldsymbol{\eta^F_{1z}}$
φ_{A_1}	$\xi^S_{1z}(\varphi_{A_1})$	$\eta^F_{1z}(\varphi_{A_1})$
φ_{A_2}	$\xi^S_{1z}(\varphi_{A_2})$	$\eta^F_{1z}(\varphi_{A_2})$
.	.	.
.	.	.
φ_{A_n}	$\xi^S_{1z}(\varphi_{A_n})$	$\eta^F_{1z}(\varphi_{A_n})$

5.2. Proposed Algorithm

In this section, we propose a hybrid structure-based algorithm (Algorithm 1). The proposed algorithm combines both soft sets and fuzzy sets to effectively handle uncertainties and imprecision in a variety of domains, resulting in a more stable and flexible modelling framework. The proposed algorithm consists of the following steps:

Algorithm 1 Hybrid structure-based algorithm

Step 1. Input the hybrid structure in tabular form (defined in Section 5.1).

Step 2. Represent the tabular form of the soft set $\xi^S_{1z} : \mathbb{N} \longrightarrow \wp(\mathbb{C})$ as given in [9] and fuzzy set $\eta^F_{1z} : \mathbb{N} \longrightarrow \wp(\mathbb{C})$ in separate tables.

Step 3. Construct the priority table (PT). This table can be achieved by multiplying the tabular values of the soft set with the corresponding values of the fuzzy membership of parameters. Also calculate the row sum of each row in priority table.

Step 4. Construct the comparison table (CT) according to [42]. This can be achieved by finding the entries as differences of each row sum in priority table with those of all other rows.

Step 5. Find the row sum of each row in the comparison table to obtain the score.

Step 6. Finally, the highest score is chosen.

The flowchart in Figure 1 summarizes the step-by-step process of the proposed algorithm and logical flow towards achieving its objectives.

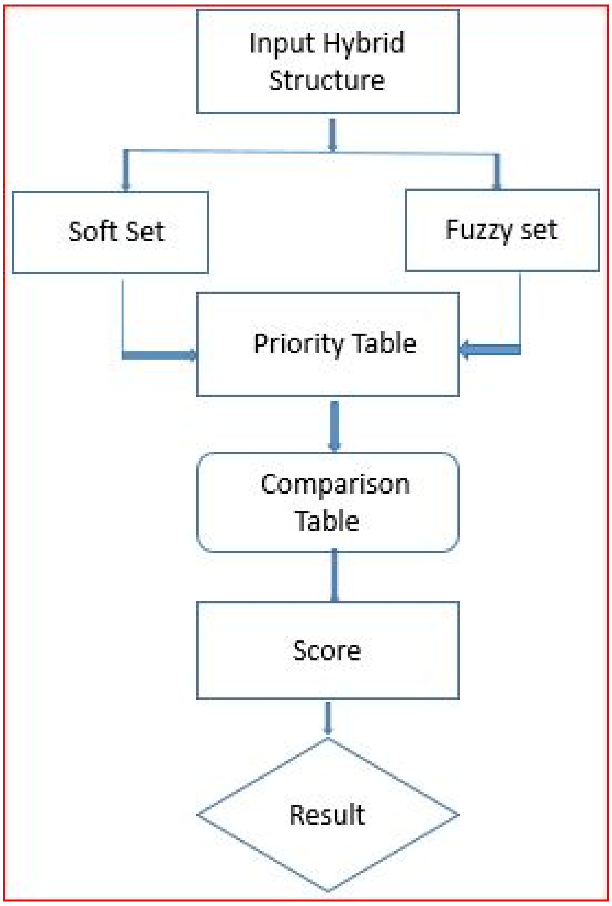

Figure 1. Flow diagram of the proposed algorithm.

5.3. Example

To empirically evaluate the effectiveness of our proposed algorithm, we give an example of the selection of a school for a cochlear-implanted child from a real-life scenario. The decision of selecting a school for a child with a cochlear implant is a critical and common decision that parents of such children often face.

A cochlear implant is an electronic device that can provide partial hearing to individuals with severe to profound hearing loss. Advances in early identification, implant technology, and early intensive therapy have enabled the implanted child to study in mainstream schools. Visual distraction, background noise, or any other environmental sounds may interfere with the understanding speech for a child with an implant. So, the decision of the selection of a school for an implanted child is based on the individual needs of the child, their capacity to learn in a spoken language environment, the environment of the school, and cooperation of the teaching staff.

Suppose Mr and Mrs Ali are in search of a school for their implanted child. To complete this task, the parents visit some schools and collect the required information about different schools in their area. They choose the five schools, $\mathbb{C} = \{S_1, S_2, S_3, S_4, S_5\}$ namely, "Beacon House" (S_1) " The city school" (S_2) "Pak-American" (S_3), "Educators"(S_4), and "Superior Montessori" (S_5) who are willing to give admission to the child. The parents configure five attributes $\mathbb{N} = \{\varphi_A, \varphi_B, \varphi_C, \varphi_D, \varphi_E\}$, where "access" ($\varphi_A$),"environment" ($\varphi_B$), "learning" ($\varphi_C$), "staff cooperation" ($\varphi_D$), and "no of students in class" (φ_E) as a set of parameters which they think are crucial for making the best option and ensuring their child's adequate education. A hierarchical structure is shown in Figure 2, presenting the five selection criteria (i.e., access, environment, learning, staff cooperation, and the number of pupils in the class).

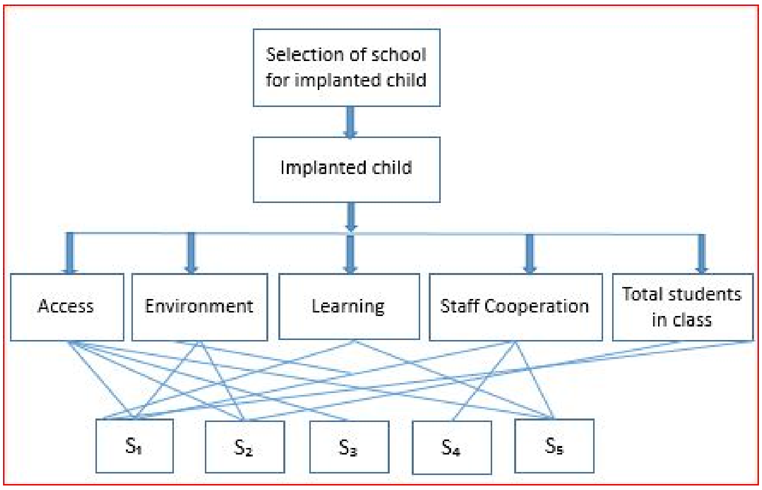

Figure 2. Hierarchical structure for the selection of school for implanted child.

Based on these criteria, the family wants to select the best school. The following hybrid structure illustrates the information of the schools based on the chosen criteria.

$$(\xi^S_{1z}, \eta^F_{1z}) = \begin{Bmatrix} (\xi^S_{1z}, \eta^F_{1z})(\varphi_A) = (\{S_1, S_2, S_3, S_5\}, 0.5), \\ (\xi^S_{1z}, \eta^F_{1z})(\varphi_B) = (\{S_1, S_2, S_3\}, 0.8), \\ (\xi^S_{1z}, \eta^F_{1z})(\varphi_C) = (\{S_1, S_5\}, 0.3), \\ (\xi^S_{1z}, \eta^F_{1z})(\varphi_D) = (\{S_3, S_4, S_5\}, 0.4), \\ (\xi^S_{1z}, \eta^F_{1z})(\varphi_E) = (\{S_1, S_2\}, 0.7) \end{Bmatrix}$$

We define the binary operation $\diamond$ and $\dotplus$ on $\mathbb{N}$ by the Cayley table in Table 6.

Table 6. Cayley tableof the binary operations $\diamond$ and $\dotplus$.

$\dotplus$	φ_A	φ_B	φ_C	φ_D	φ_E
φ_A	φ_A	φ_B	φ_C	φ_D	φ_E
φ_B	φ_B	φ_D	φ_B	φ_C	φ_B
φ_C	φ_C	φ_B	φ_C	φ_D	φ_C
φ_D	φ_D	φ_C	φ_D	φ_B	φ_D
φ_E	φ_E	φ_B	φ_C	φ_D	φ_E

$\diamond$	φ_A	φ_B	φ_C	φ_D	φ_E
φ_A	φ_A	φ_A	φ_A	φ_A	φ_A
φ_B	φ_A	φ_B	φ_C	φ_D	φ_E
φ_C	φ_A	φ_C	φ_C	φ_C	φ_E
φ_D	φ_A	φ_D	φ_C	φ_B	φ_E
φ_E	φ_A	φ_E	φ_E	φ_E	φ_A

Then $(\mathbb{N}, \dotplus, \diamond)$ is a hemiring. For the implementation of our proposed Algorithm 1, the following steps are used.

Step 1: Based on the hybrid structure, all the possible values are estimated for the attributes, given in Table 7.

Table 7. Tabular representation of hybrid structure.

$\mathbb{N}$	ξ^S_{1z}	η^F_{1z}
φ_A	$\{S_1, S_2, S_3, S_5\}$	0.5
φ_B	$\{S_1, S_2, S_3\}$	0.8
φ_C	$\{S_1, S_5\}$	0.3
φ_D	$\{S_3, S_4, S_5\}$	0.4
φ_E	$\{S_1, S_2\}$	0.7

Step 2. Construct seperate tables for the soft set ξ^S_{1z} and fuzzy set η^F_{1z}.

Step 3. By multiplying the corresponding values of ξ^S_{1z} in Table 8 and η^F_{1z} in Table 9, Table 10 computes the priority table.

Table 8. Tabular representation of $\xi_{1z}^{S} : \mathbb{N} \longrightarrow \wp(\mathbb{C})$.

	S_1	S_2	S_3	S_4	S_5
φ_A	1	1	1	0	1
φ_B	1	1	1	0	0
φ_C	1	0	0	0	1
φ_D	0	0	1	1	1
φ_E	1	1	0	0	0

Table 9. Tabular representation of $\eta_{1z}^{F} : \mathbb{N} \longrightarrow [0,1]$.

	η_{1z}^{F}
φ_A	0.5
φ_B	0.8
φ_C	0.3
φ_D	0.4
φ_E	0.7

Table 10. Priority table (PT).

	φ_A	φ_B	φ_C	φ_D	φ_E	**Row-Sum**
S_1	0.5	0.8	0.3	0	0.7	2.3
S_2	0.5	0.8	0.3	0	0	1.6
S_3	0.5	0	0	0	0.7	1.2
S_4	0	0	0.3	0.4	0.7	1.4
S_5	0.5	0.8	0	0	0	1.3

Step 4. In Table 11, each attribute is obtained as the difference of row sum with the all the other rows.

Table 11. Comprison Table (CT).

	S_1	S_2	S_3	S_4	S_5
S_1	0	0.7	1.1	0.9	1.0
S_2	−0.7	0	0.4	0.2	0.3
S_3	−1.1	−0.4	0	−0.2	−0.1
S_4	−0.9	−0.2	0.2	0	0.1
S_5	−1.0	−0.3	0.1	−0.1	0

Step 5. Calculate the sum of each row in the comparison table to obtain the score of schools as shown in Table 12.

Table 12. Score of alternatives.

Alternatives	Score
S_1	3.7
S_2	0.2
S_3	−1.8
S_4	−0.9
S_5	−1.3

Step 6. From the Table 12, we can see that alternative S_1 is the best selection.

Scores of alternatives for the selection of best school for implanted child are shown in Figure 3.

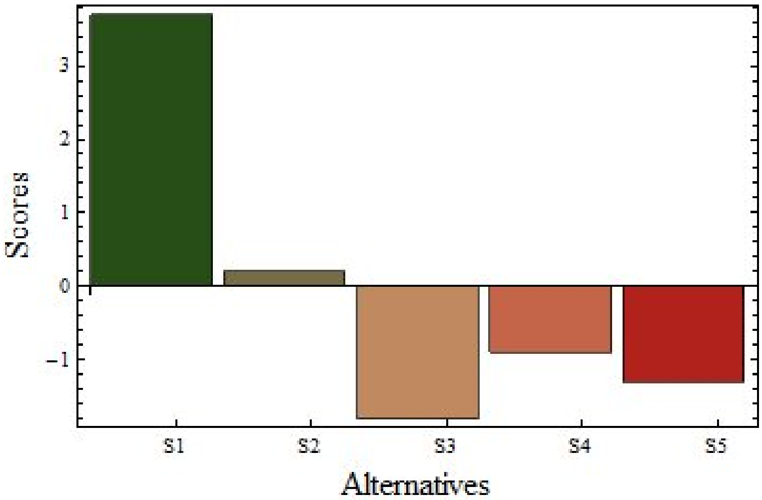

Figure 3. Scores of alternatives for the selection of best school for implanted child.

In Figure 3, the bar chart serves as a visual representation of the alternative values derived from the evaluation criteria outlined in Table 12. The chart allows for a clear and concise comparison of the different schools (S_1, S_2, S_3, S_4, and S_5) based on their respective scores.

At the top of the bar chart, we can observe that Beacon House School (S_1) stands out with the highest score of 3.7 among all the schools. This score reflects its excellent performance in the evaluation criteria, indicating that it outperformed the other schools in the assessment.

The City School (S_2) follows closely behind Beacon House School, securing the second-highest score. This suggests that The City School is also a strong competitor and performed admirably in the comparison.

However, the alternatives at the bottom of the bar chart, namely The Pak-American School (S_3), The Educators School (S_4), and The Superior Montessori School (S_5), obtained relatively lower scores. These scores indicate that these schools had comparatively lesser acceptability or performance based on the assessment criteria.

The bar chart, as an essential part of the decision-making process, provides a visual tool to discern the varying levels of performance among the alternatives. It simplifies the comparison, allowing decision-makers to identify the top-performing school (Beacon House School) and observe the relative positions of the other schools in terms of their scores.

By incorporating a bar chart into the decision-making process, the evaluation becomes more intuitive and accessible. Decision-makers can make well-informed choices by considering the graphical representation of the schools' performance, facilitating a clearer understanding of their respective strengths and weaknesses based on the evaluation criteria.

The visual comparison aids in selecting the most appropriate school that aligns with the decision-maker's preferences and requirements.

As compared to Asmat et al. [40,41] our proposed method has assessed the hybrid structure in hemirings and investigated several properties of hybrid h-bi-ideals. The proposed prime hybrid h-bi-ideals are an extension of hybrid h-bi-ideals and we have investigated various aspects of hemirings. Also, this work has conducted a characterization of certain classes of hemirings based on these prime hybrid h-bi-ideals. Furthermore, we have utilized the hybrid structure in the decision-making process with the help of examples from real-world situations to explore new directions in algebraic development and tackle practical problems with improved uncertainty-handling abilities by utilizing hybrid structures in hemirings. Furthermore, our proposed method distinguishes itself from the existing literature by adopting a hybrid structure-based model rather than focusing solely on algebraic structures like BCK/BCI algebras and semigroups, as seen in previous studies [35–39].

6. Conclusions

In summary, the paper highlights the significance of hybrid structures in mathematical and decision-making domains. This study focuses on investigating the properties of hybrid h-bi-ideals within the context of hemirings. These hybrid h-bi-ideals include prime, strongly prime, semiprime, irreducible, and strongly irreducible. By employing the hybrid h-bi-ideals, the paper provides insightful characterizations of h-hemiregular and h-intra-hemiregular hemirings. This analysis contributes to a deeper understanding of the algebraic structures associated with these types of hemirings. To this end, we present a decision-making algorithm based on hybrid structures that has been successfully applied to solve a significant real-world problem. This showcases the effectiveness of the proposed approach in providing robust solutions in situations involving imprecise or uncertain data. The practical utility of the findings is demonstrated through the successful application of the proposed decision-making algorithm to solve real-world problems under imprecise conditions.

The proposed hybrid structure-based model empowers decision-makers in complex situations with uncertainty. It helps them understand uncertainties comprehensively and make effective choices in diverse scenarios. By using both crisp and fuzzy information, it improves decision outcomes in various domains. Also, the proposed algorithm considers both quantitative and qualitative information, enhancing the decision-making process and reducing risks.

On the contrary to this, defining membership functions for fuzzy and soft sets requires extensive domain knowledge, and interpreting dual representation demands specialized expertise, making implementation and maintenance of the approach more challenging. Additionally, gathering precise and accurate data, especially for subjective or qualitative information, can be difficult. The effectiveness of the proposed hybrid structures relies on sufficient and reliable data; in situations with scarce or unreliable data, their accuracy and effectiveness may be compromised.

However, in the future, the applicability of hybrid structures may be assessed in different algebraic structures including rings, semirings, and lattices, acquiring insightful knowledge into their adaptability and efficiency. To handle complex situational decisions with multiple objectives and criteria more successfully, one could combine these hybrid structures with multi-criteria decision-making methodologies. Furthermore, comparative studies between the existing models and hybrid structures could be considered to understand their respective strengths and limitations in various decision-making scenarios.

Author Contributions: Conceptualization, A.H. and A.K.; Software, A.H., A.K., N.F. and D.M.K.; Validation, N.F., D.M.K., R.A.R.B. and M.E.; Formal analysis, A.H., A.K. and R.A.R.B.; Investigation, A.K., N.F., D.M.K. and M.E.; Resources, A.H., N.F., D.M.K., R.A.R.B. and M.E.; Data curation, A.H., N.F., R.A.R.B. and M.E.; Writing—original draft, A.H.; Writing—review editing, A.K.; Visualization, N.F., D.M.K., R.A.R.B. and M.E.; Supervision, A.K.; Project administration, D.M.K. and M.E.; Funding acquisition, R.A.R.B. All authors have read and agreed to the published version of the manuscript.

Funding: The technical and financial support—The Ministry of Education and King Abdulaziz University, DSR, Jeddah, Saudi Arabia.

Data Availability Statement: Not applicable.

Acknowledgments: This research work was funded by Institutional Fund Projects under grant no. (IFPIP: 550-150-1443). The authors gratefully acknowledge the technical and financial support provided by the Ministry of Education and King Abdulaziz University, DSR, Jeddah, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Example A1. *Suppose that there are six houses in the initial universe set $\mathbb{C}$ given by $\mathbb{C} = \{H_1, H_2, H_3, H_4, H_5, H_6\}$. Let a set of parameters $\mathbb{N} = \{\varphi_A, \varphi_B, \varphi_C, \varphi_D\}$ be a set of status of houses in which φ_A stands for the parameter "beautiful", φ_B stands for the parameter "cheap", φ_C stands for the parameter "in good location", φ_D stands for the parameter "in green surrounding". We define the binary operation $\diamond$ and $\dotplus$ on $\mathbb{N}$ by the Cayley table in Table A1.*

Table A1. Cayley table for the binary operations $\dotplus$ and $\diamond$.

$\dotplus$	φ_A	φ_B	φ_C	φ_D
φ_A	φ_A	φ_B	φ_C	φ_D
φ_B	φ_B	φ_A	φ_B	φ_C
φ_C	φ_C	φ_B	φ_B	φ_C
φ_D	φ_D	φ_C	φ_C	φ_B

$\diamond$	φ_A	φ_B	φ_C	φ_D
φ_A	φ_A	φ_A	φ_A	φ_A
φ_B	φ_A	φ_B	φ_B	φ_B
φ_C	φ_A	φ_B	φ_B	φ_B
φ_D	φ_A	φ_B	φ_B	φ_B

Then $(\mathbb{N},\dotplus,\diamond)$ is a hemiring. Let $\mathbb{Z}_{\tilde{1}} = (\xi^S_{1z}, \eta^F_{1z}), \mathbb{Z}_{\tilde{2}} = (\xi^S_{2z}, \eta^F_{2z})$ and $\mathbb{Z}_{\tilde{3}} = (\xi^S_{3z}, \eta^F_{3z})$ be a any ḦyH-BI in $\mathbb{N}$ over $\mathbb{C}$ which is given by Tables A2–A4.

Table A2. Tabular representation of ḦyH-BI $\mathbb{Z}_{\tilde{1}} = (\xi^S_{1z}, \eta^F_{1z})$.

$\mathbb{N}$	ξ^S_{1z}	η^F_{1z}
φ_A	$\{H_1, H_2, H_3, H_4, H_5, H_6\}$	0.2
φ_B	$\{H_1, H_2, H_3, H_4, H_6\}$	0.4
φ_C	$\{H_1, H_2, H_3\}$	0.7
φ_D	$\{H_1\}$	0.8

Table A3. Tabular representation of ḦyH-BI $\mathbb{Z}_{\tilde{2}} = (\xi^S_{2z}, \eta^F_{2z})$.

$\mathbb{N}$	ξ^S_{2z}	η^F_{2z}
φ_A	$\{H_1, H_2, H_3, H_4\}$	0.2
φ_B	$\{H_1, H_2, H_3\}$	0.3
φ_C	$\{H_1, H_2\}$	0.6
φ_D	$\{H_1\}$	0.8

Table A4. Tabular representation of ḦyH-BI $\mathbb{Z}_{\tilde{3}} = (\xi^S_{3z}, \eta^F_{3z})$.

$\mathbb{N}$	$\boldsymbol{\xi^S_{3z}}$	$\boldsymbol{\eta^F_{3z}}$
φ_A	$\{H_1, H_3, H_4, H_6\}$	0.3
φ_B	$\{H_3, H_4, H_5\}$	0.5
φ_C	$\{H_4, H_5\}$	0.7
φ_D	$\{H_5\}$	0.9

It is a routine calculation to verify that if $\left(\xi^S_{2z} \circledast_H \xi^S_{3z}\right) \sqcap \left(\xi^S_{3z} \circledast_H \xi^S_{2z}\right) \sqsubseteq \xi^S_{1z}$ and $\left(\eta^F_{2z} \odot_H \eta^F_{3z}\right) \veebar \left(\eta^F_{3z} \odot_H \eta^F_{2z}\right) \succcurlyeq \eta^F_{1z}$ implies $\xi^S_{2z} \sqsubseteq \xi^S_{1z}$ or $\xi^S_{3z} \sqsubseteq \xi^S_{1z}$ and $\eta^F_{2z} \succcurlyeq \eta^F_{1z}$ or $\eta^F_{3z} \succcurlyeq \eta^F_{1z}$ for all $\varphi_A,\varphi_B,\varphi_C,\varphi_D$ in $\mathbb{N}$. This implies that $(\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{3}})\widetilde{\cap}(\mathbb{Z}_{\tilde{3}} \boxtimes \mathbb{Z}_{\tilde{2}})\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$ gives $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$ or $\mathbb{Z}_{\tilde{3}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$. Thus, $\mathbb{Z}_{\tilde{1}}$ is a strongly prime ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$.

Example A2. *Suppose that there are six houses in the initial universe set $\mathbb{C}$ given by $\mathbb{C} = \{H_1, H_2, H_3, H_4, H_5, H_6\}$. Let a set of parameters $\mathbb{N} = \{\varphi_A,\varphi_B,\varphi_C,\varphi_D\}$ be a set of status of houses in which φ_A stands for the parameter "beautiful", φ_B stands for the parameter "cheap", φ_C stands for the parameter "in good location", φ_D stands for the parameter "in green surrounding". We define the binary operation $\diamond$ and $\dotplus$ on $\mathbb{N}$ by the Cayley table in Table A5.*

Table A5. Cayley table for the binary operations $\dotplus$ and $\diamond$.

$\dotplus$	$\boldsymbol{\varphi_A}$	$\boldsymbol{\varphi_B}$	$\boldsymbol{\varphi_C}$	$\boldsymbol{\varphi_D}$
φ_A	φ_A	φ_B	φ_C	φ_D
φ_B	φ_B	φ_A	φ_B	φ_C
φ_C	φ_C	φ_B	φ_B	φ_C
φ_D	φ_D	φ_C	φ_C	φ_B

$\diamond$	$\boldsymbol{\varphi_A}$	$\boldsymbol{\varphi_B}$	$\boldsymbol{\varphi_C}$	$\boldsymbol{\varphi_D}$
φ_A	φ_A	φ_A	φ_A	φ_A
φ_B	φ_A	φ_B	φ_B	φ_B
φ_C	φ_A	φ_B	φ_B	φ_B
φ_D	φ_A	φ_B	φ_B	φ_B

Then $(\mathbb{N},\dotplus,\diamond)$ is a hemiring. Let $\mathbb{Z}_{\tilde{1}} = (\xi^S_{1z}, \eta^F_{1z})$ and $\mathbb{Z}_{\tilde{2}} = (\xi^S_{2z}, \eta^F_{2z})$ be a any ḦyH-BI in $\mathbb{N}$ over $\mathbb{C}$ which is given by Tables A6 and A7.

Table A6. Tabular representation of ḦyH-BI $\mathbb{Z}_{\tilde{1}} = (\xi^S_{1z}, \eta^F_{1z})$.

$\mathbb{N}$	$\boldsymbol{\xi^S_{1z}}$	$\boldsymbol{\eta^F_{1z}}$
φ_A	$\{H_1, H_2, H_3, H_4, H_5, H_6\}$	0.2
φ_B	$\{H_1, H_2, H_3, H_4\}$	0.4
φ_C	$\{H_1, H_2\}$	0.7
φ_D	$\{H_1\}$	0.8

Table A7. Tabular representation of ḦyH-BI $\mathbb{Z}_{\tilde{2}} = (\xi^S_{2z}, \eta^F_{2z})$.

$\mathbb{N}$	$\boldsymbol{\xi^S_{2z}}$	$\boldsymbol{\eta^F_{2z}}$
φ_A	$\{H_2, H_3, H_4, H_6\}$	0.3
φ_B	$\{H_1, H_2, H_3\}$	0.5
φ_C	$\{H_1\}$	0.7
φ_D	$\{H_1\}$	0.8

It is a routine calculation to verify that if $\left(\xi^S_{2z} \circledast_H \xi^S_{2z}\right) \sqsubseteq \xi^S_{1z}$ and $\left(\eta^F_{2z} \odot_H \eta^F_{2z}\right) \succcurlyeq \eta^F_{1z}$ implies $\xi^S_{2z} \sqsubseteq \xi^S_{1z}$ and $\eta^F_{2z} \succcurlyeq \eta^F_{1z}$ for all $\varphi_A,\varphi_B,\varphi_C,\varphi_D$ in $\mathbb{N}$. This implies that $(\mathbb{Z}_{\tilde{2}} \boxtimes \mathbb{Z}_{\tilde{2}})\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$ gives $\mathbb{Z}_{\tilde{2}}\widetilde{\subseteq}\mathbb{Z}_{\tilde{1}}$. Thus, $\mathbb{Z}_{\tilde{1}}$ is a semiprime ḦyH-BI of $\mathbb{N}$ over $\mathbb{C}$.

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