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Innovating and Pricing Carbon-Offset Options of Asian Styles on the Basis of Jump Diffusions and Fractal Brownian Motions

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Abstract: Due to CO₂ emissions, humans are encountering grave environmental crises (e.g., rising sea levels and the grim future of submerged cities). Governments have begun to offset emissions by constructing emission-trading schemes (carbon-offset markets). Investors naturally crave carbon-offset options to effectively control risk. However, the research and practice for these options are relatively limited. This paper contributes to the literature in this area. Specifically, according to carbon-emission allowances' empirical distributions, we implement fractal Brownian motions and jump diffusions instead of traditional geometric Brownian motions. We contribute to extending the theoretical model based on carbon-offset option-pricing methods. We innovate the carbon-offset options of Asian styles. We authenticate the options' stochastic differential equations and analytically price the options in the form of theorems. We verify the parameter sensitivity of pricing formulas by illustrations. We also elucidate the practical implications of an emission-trading scheme.

Keywords: carbon offset; emission-trading scheme; carbon-offset investments; carbon-offset options; jump diffusions; fractal Brownian motions; stochastic differential equations

MSC: 60H15; 91G20



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1. Introduction

1.1. Alarming Global Warming and Carbon-Offset Options

We humans must accept the unquestionable fact that CO₂ does cause global warming. Even climate scientists were astonished at observing record temperatures at the poles (e.g., 40 degrees above normal in Antarctica) on 20 March 2022. (Data source: The Irish Times, <https://www.irishtimes.com/news/environment/scientists-astonished-by-heatwaves-at-north-and-south-poles-1.4831673>, 20 March 2022).

The Intergovernmental Panel on Climate Change (IPCC) projects that the global sea level will rise 0.6 m to 1.1 m by 2100. (Data source: IPCC 2019 report, <https://www.ipcc.ch/2019>, 22 December 2021). The rising seas will then engulf numerous major cities (e.g., New York City by the year 2080 (Data source: JSTOR Daily, <https://daily.jstor.org/new-york-city-underwater/>, 6 October 2021) and Shanghai and Guangzhou by 2100 (Data source: JY International Cultural Communications, <https://www.thatsmags.com/guangzhou/post/29864/shanghai-and-prd-at-risk-of-disappearing-under-rising-sea-levels>, 1 November 2019)). More alarmingly, even such pessimistic projections have become too optimistic under accelerating global warming day by day.

Humans are eagerly trying to combat global warming by restraining CO₂ consumption. Ref. [1] empirically confirms that economic growth increases CO₂ emissions, and urges the reduction. Governments have composed carbon-offset securities to curb CO₂ increments while preserving necessary economic growth. Promisingly, the European Union Emission-Trading System

(EU ETS) was pioneered in 2005 as the first international emission-trading system and has successfully launched three phases of action. (Data source: EU ETS, https://ec.europa.eu/clima/eu-action/eu-emissions-trading-system-eu-ets/development-eu-ets-2005-2020_en#ecl-inpage-1020, 3 January 2022). Ref. [2] methodically concentrated on the EU ETS and contended that the public does care about climate.

In particular, investors have applauded European Union Allowances futures and European Union Allowances options as ground-breaking carbon-offset derivatives. However, the trading volume of such options is still scant, in the context of 33.31 billion option contracts being traded worldwide in 2021. (Data source: Statista Incorporation, <https://www.statista.com/statistics/377025/global-futures-and-options-volume/>, 9 March 2022).

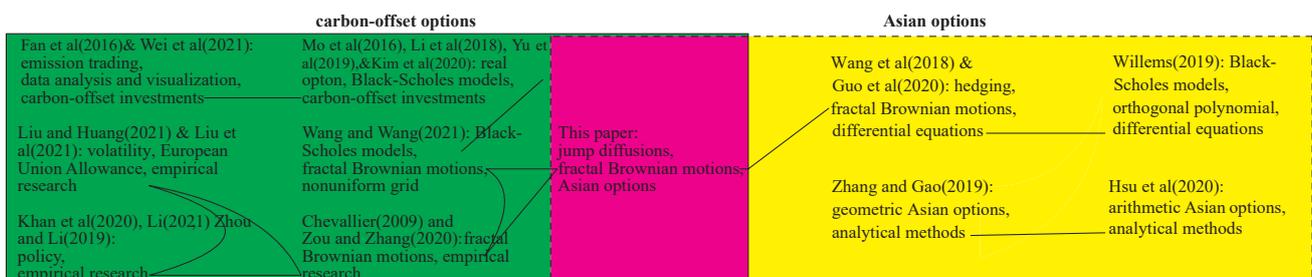
1.2. Carbon-Offset-Option Literature

The literature on carbon-offset options is also relatively scant. We depict several key pieces of literature in the green rectangle under the name “carbon-offset options” in the top part of Figure 1. We categorize similar research into a group and register the main research methods. For instance, Refs. [2,3] have investigated emission trading, data analysis and visualization, and carbon-offset investments. We additionally connect different groups’ analogous methods using lines. For example, the group of [2,3] and the group of [4–7] have both contemplated carbon-offset investments. The group of [8,9] and the group of [1,10,11] have both conducted empirical research. We will further review the references later (especially in Section 2).

Under a similar format, we also depict several key pieces of literature for Asian options in the yellow rectangle. For instance, the group of [12] and the group of [13,14] have both implemented differential equations. Refs. [15,16] applied analytical methods to derive pricing formulas.

As the intersection (in the pink shaded area) of the green rectangle and yellow rectangle, we depict this paper for Asian carbon-offset options based on jump diffusions and fractal Brownian motions in the top part of Figure 1. The group of [17], the group of [18,19], and the group of [13,14] have all harnessed fractal Brownian motions.

Moreover, we briefly present the originality in the bottom part of Figure 1.



Originality 1: We innovate carbon-offset options of Asian styles by jump diffusions and fractal Brownian motions and verify the following stochastic differential equation:

$$\frac{\partial V_t}{\partial t} + H\sigma_s^2 S_t^2 t^{2H-1} \frac{\partial^2 V_t}{\partial S_t^2} + \frac{A_t(\ln S_t - \ln A_t)}{t} \frac{\partial V_t}{\partial A_t} + \lambda E(V_{t+\delta t} - V_t) + (r - \lambda\theta_{j(t)})S_t \frac{\partial V_t}{\partial S_t} - rV_t = 0$$

We next resolve the call option by the equation above and boundary condition $V(S_T, A_T, T) = \max(A_T - K, 0)$ and resolve the put option by the equation above and boundary condition $V(S_T, A_T, T) = \max(K - A_T, 0)$

Originality 2: We analytically reckon the call-option value as $V(S_t, A_t, t) = e^{\tau+\xi} - r^{(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$ and reckon the put-option value as $V(S_t, A_t, t) = Ke^{-r(T-t)}N(-d_2) - e^{\tau+\xi} - r^{(T-t)}N(-d_1)$

Figure 1. The originality and several key pieces of literature [1–19].

1.3. Originality: Innovating and Pricing Carbon-Offset Options of Asian Styles by Jump Diffusions and Fractal Brownian Motions

Specifically in the area of carbon-offset options, this paper carries the following originality:

First, we innovate the options by justifying the jump-diffusion condition and fractal-Brownian-motion condition. Refs. [18,19] empirically discovered heavy tails, high peaks, and thus non-normal distributions for European carbon-offset futures. The discovery contradicts geometric Brownian motion properties (especially normal distributions). Therefore,

we extend geometric Brownian motions into fractal Brownian motions. The COVID-19 pandemic, astronomical amounts of money worldwide and the liquidity effect, and regional conflicts can bring uncommon volatility and uncertainty to financial markets. For instance, Refs. [20,21] discerned uncommon volatility in stock markets and bond markets, respectively. The exceptional volatility can hardly be prescribed by geometric Brownian motion. Therefore, we append jump-diffusion models. Moreover, we envisage the options as Asian styles, because the existing carbon-offset options typically belong to European styles or American styles (see [17,22]). Asia (especially China and Japan) shoulders important roles in carbon offset. Meanwhile, the path dependence of Asian options avoids the risk of exceptional volatility for underlying asset prices on the maturity date, which is not considered in the existing research on carbon-offset options.

Second, we analytically price the options in the form of theorems. The approximate analytical pricing formula of options is more concise than pricing formulas containing series terms (as displayed by [23]).

1.4. Paper Structure

The rest of this paper is organized as follows: We review the theoretical background in Section 2. We instigate the options and gauge the stochastic differential equations in Section 3. We analytically price the options in Section 4. We elucidate the pricing by an example in Section 5. We conclude this paper in Section 6.

2. Literature Review: Asian Options and Carbon-Offset Options

2.1. Classic Option Pricing

Ref. [24] seminaly lay the foundation of option pricing by assuming that the underlying asset follows a geometric Brownian motion. However, researchers later discovered the assumption's weakness in practice. Ref. [25] compared and related three major option-pricing methods. Ref. [26] stressed the discrepancy between the model of [24] and reality and modify the model.

To better delineate financial markets, researchers instigated the extensions. For instance, Ref. [27] (pp. 125–130) proposed jump-diffusion models, emphasizing uncommonly sizable stock-price jumps and prescribing that both continuous and jump processes command stock prices. Ref. [28] further enlightened the double exponential jump-diffusion models to overcome leptokurtic features (e.g., heavy tails) and volatility-smile features.

Specifically for jump diffusions, Ref. [29] (pp. 642–643) illuminated jump diffusions as follows:

$$dS_t = (\mu - \lambda\theta_{j(t)})S_t dt + \sigma_s S_t dz_t + dp_t \quad (1)$$

where

- $(\mu - \lambda\theta_{j(t)})S_t dt + \sigma_s S_t dz_t$ basically inscribes geometric Brownian motions (as delineated by [29] (pp. 323–324)).
- dp_t inscribes the jumps.
- Stochastic processes $\{z_t\}$ and $\{p_t\}$ are independent.
- For other terms of (1), we will extend (1) into (2) and then explain them in Section 3.
- For notations, we principally track the classic symbols of [29]. For stochastic processes (e.g., $\{X_t\}$), we interchangeably employ X_t and $X(t)$ by trailing the notation tradition (e.g., X_t of [30] (p. 184) and $X(t)$ of [31] (p. 199)).

Ref. [32] deployed (1) for Asian options.

The B-S model, based on the geometric Brownian motion (as displayed by [24]), assumes the log-normal property of the underlying asset price, and hardly describes random walks with skewness. For instance, Refs. [33,34] recognized the underlying asset peculiarity (e.g., heavy tails) and extended geometric Brownian motions into fractal Brownian motions. Scholars have further developed other diffusion processes for the fractal Brownian motion variant, for example, mixed fractal Brownian motion (see [35])

and sub-mixed fractal Brownian motion (see [36]). Consequently, scholars have gradually studied derivative prices jointly driven by jump diffusions and fractal Brownian motions (as displayed by [23]).

Specifically for fractal Brownian motions, Ref. [37] proposed a stochastic integral of fractal Brownian motions based on the Wick product. However, Ref. [38] first questioned the use of the Wick product and focused on the basic economic explanation beyond pure mathematical theories. Ref. [34] implemented the Wick product into the definition of portfolio value and the attribute of self-financing. Ref. [39] utilized properties of the fractal Taylor formula to develop the fractal Itô's formula with the Hurst exponent $H \in [\frac{1}{2}, 1)$. The method is different from the classic method of Wick product (as described by [37]). Ref. [13] presumed self-similar and long-term dependence characters for the assets, manipulated stochastic differential equations, and analytically assessed options. Ref. [40] utilized Itô's lemma and analytically estimated the options by Malliavin calculus. Ref. [41] added approximative fractal stochastic volatility to the double Heston jump-diffusion model and deduced the option-pricing formula.

Moreover, researchers have exploited partial differential equations and analytically valued options. Ref. [42] applied partial differential equations, adopted Mellin transformation, and valued options using Volterra integral equations. Ref. [43] established monotonous transaction costs, fashioned partial differential equations, and analytically assessed options.

2.2. Pricing Asian Options

Ref. [29] (p. 626) defined that an Asian option is a kind of option whose value is determined by the average price of the underlying asset during the option life. Ref. [29] (p. 626) prescribed the call-option value and put-option value on maturity T as follows:

$$\max(A_T - K, 0) \quad \max(K - A_T, 0)$$

where

- K is the exercise price;
- A_T is the asset's geometric average price $A_T = e^{\frac{1}{T} \int_0^T \ln S_t dt}$.

The analytical pricing formula of Asian options has been widely studied by scholars due to its concise form. Based on [24], Ref. [12] cogitated orthogonal polynomials, calculated series expansions for option prices, and established the series convergence. Ref. [13] explored the pricing formula of dual-asset Asian rainbow options under fractal Brownian motions based on differential equations. Ref. [14] targeted subdiffusive Brownian motions, executed hedging strategies, and analytically resolved option prices using partial differential equations. Ref. [15] experimented with double Heston models and evaluated first-order asymptotic expansions for the geometric Asian option price. Ref. [16] integrated Taylor-series expansions for deriving approximate analytical solutions for arithmetic Asian options.

2.3. Pricing Carbon-Offset Options

2.3.1. Underlying Assets' Empirical Distributions and Limitations of Geometric Brownian

A great deal of empirical research analyzes the time series and empirical distribution characteristics of carbon underlying asset prices.

Carbon financial asset prices have random fluctuations and present fractal features such as non-normal distribution, peak, and thick tail. Ref. [18] focused on European carbon-offset futures and portrayed the returns using leptokurtic features (e.g., heavy tails) and nonzero-skewness features. Ref. [19] also characterized European carbon-offset futures using heavy tails, high peaks, and thus non-normal distributions, and deduced the characterization as the cause for multiple-fractal situations. Ref. [44] used the fractal market hypothesis and evolutionary computing to analyze carbon futures trading and short-term price prediction.

Carbon financial asset prices occasionally have exceptional fluctuations and jumping features. Ref. [45] empirically discovered that the time series of carbon-emission-allowance price presents jumps and is non-stationary. Ref. [46] confirmed the Markov property of the carbon allowance price under a non-linear market fundamental model. Ref. [47] applied the mechanism transformation jump-diffusion model with hidden Markov chains to capture the jump and fluctuation clustering characteristics of EUA price returns.

Because the leptokurtic features contradict geometric Brownian motion properties (especially normal distributions) from the existing empirical research, scholars have gradually channeled fractal Brownian motions for carbon-offset derivatives. Jumping and non-stationary features from carbon-trading markets also contradict geometric Brownian motion properties (especially continuous). Therefore, scholars have harnessed jump-diffusion processes for carbon-offset derivatives.

2.3.2. Carbon-Offset Option-Pricing Models

The main research on carbon-offset option pricing usually adopts the classic Black–Scholes model as the basis for construction. Ref. [22] attested that the carbon-emission permit price driven by geometric Brownian motion is a martingale process in stochastic, continuous, and infinite time models. Ref. [17] modulated the Black–Scholes model into a mixed fractal Black–Scholes model and direct power-penalty approaches and nonuniform grid-based modifications in the problem of American carbon-emission derivative pricing. Ref. [48] proposed the application of carbon-offset options evaluated through a geometric Brownian motion model with regime-switching for carbon management, together with the high volatility of the carbon price dynamic.

2.3.3. Exploring Real Options

Researchers have made promising progress in carbon-offset options. In addition, the research concentrates on real options. For instance, Ref. [4] constructed real options for wind power, tuned the main parameters, and dissected the relationship between the emission-trading scheme and low-carbon-energy investments in China. Ref. [6] aimed for defer-type real options and abandon-type real options, and executed two-dimensional binomial lattices in the pricing. Ref. [5] heeded carbon-price fluctuations and, accordingly, fashioned real options, and enhanced investment performance. Ref. [7] was aware of fossil-energy prices and carbon-emission-allowance prices, and conceived real options for R&D investments.

2.3.4. Innovating Portfolio Selection

Considering the investment demand for carbon offsetting, scholars have proposed a multi-objective portfolio selection that takes into account both risks and returns. Investors can build constraints using the carbon-offset measure and operate portfolio selection and optimization (as experimented by [49]). Ref. [50] took the green innovation index as the third objective dimension of the portfolio-selection model.

2.3.5. Calibrating GARCH Models for the Volatility

The volatility prediction of underlying assets in the carbon-trading market has been another concern of researchers. Ref. [8] sampled European Union Allowances option prices on the European Energy Exchange, measured the volatility using GARCH models, and forecasted the future prices. Ref. [9] consumed the GARCH-MIDAS models for the volatility of European Union Allowances futures, contrasted the GARCH-MIDAS models and other GARCH models, and uncovered the outperformance of the GARCH-MIDAS models.

2.4. Appraising Key Literature

2.4.1. Urgency to Advance Carbon-Offset Options

Ref. [1] confirmed that economic growth increases CO₂ emissions, and thus urged reduction. Ref. [10] reviewed Chinese research and the practice of carbon offset, underscor-

ing the urgency for carbon-offset markets, and appraising the options. Ref. [11] empirically verified that options trading increases corporate investments and that the effect is stronger for corporations with higher information asymmetry difficulties.

2.4.2. Suitability For Jump Diffusions

In addition to global warming, the COVID-19 pandemic has unleashed crises of humanity, economy, and finance. With industry shutdown, loss of employment, and an unimaginable death toll of almost a million in the US alone by March 2022, the global economy is falling into recession. Moreover, the astronomical amount of money worldwide and the liquidity effect, and regional conflicts (e.g., the Russia–Ukraine war) can bring uncommon volatility and uncertainty to financial markets and put financial stability at great risk. For instance, Refs. [20,21] discerned uncommon volatility in stock markets and bond markets during the COVID-19 pandemic. In the carbon-emission trading market, Refs. [45,47] empirically discovered the jumping characteristics of carbon-emission allowance prices. Further research has claimed that the jump-diffusion model (JDM) proposed by [27] is the most suitable dynamic model for EUAs. Therefore, jump-diffusion models have surfaced as appropriate candidates.

2.4.3. Suitability for Fractal Brownian Motions

Refs. [18,19] empirically unearthed heavy tails, high peaks, and thus non-normal distributions for European carbon-offset futures and inferred the application of fractal Brownian motions. Ref. [17] unveiled fractal Black–Scholes models and affirmed the models’ effectiveness. Ref. [8] reasonably predicted carbon option prices with fractal Brownian motion considering the fractal characteristics of carbon-offset option prices.

3. Originating the Options and Verifying the Stochastic Differential Equation

3.1. Initiating Carbon-Offset Options of Asian Styles on the Basis of Jump Diffusions and Fractal Brownian Motions

We extend the geometric-Brownian-motion assumption of [24] (pp. 640–641), impose jumps for the underlying asset, and formulate the following fractal Brownian motions with jump diffusions for the asset (especially carbon-emission allowances):

$$dS_t = (\mu - \lambda\theta_{j(t)})S_t dt + \sigma_s S_t dB_s^H(t) + j(t)S_t dN_t \tag{2}$$

where

1. μ is the expectation (as documented by [29] (p. 323)).
2. σ_s is the volatility (as documented by [29] (pp. 323–324)).
3. On a probability space (Ω, \mathcal{F}, P) , we define $\{B_s^H(t)\}$ as a fractal Brownian motion (as documented by [29] (pp. 329–330)) with the Hurst exponent $H \in [\frac{1}{2}, 1)$ with the following property:

$$\text{Cov}(B_s^H(t), B_s^H(s)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}), \text{ for } s \in \mathbb{R} \text{ and } t \in \mathbb{R}$$

For $H = \frac{1}{2}$, $B_s^H(t)$ retreats to Brownian motions.

4. We inherit the formulation of [27] (pp. 128–129) and establish $\{N_t\}$ as a Poisson process. λ is the expected jump numbers per unit time.
5. We enrich the jump diffusions by erecting $j(t)$ as the jump multitude. $1 + j(t)$ comes from the following log-normal distribution:

$$\ln(1 + j(t)) \sim N(\mu_{j(t)}, \sigma_{j(t)}^2) \tag{3}$$

The distribution is fixed by the variance $\sigma_{j(t)}^2$ and expectation $\mu_{j(t)}$ as follows:

$$\mu_{j(t)} \equiv \ln(1 + \theta_{j(t)}) - \frac{1}{2}\sigma_{j(t)}^2 \tag{4}$$

We denote the expectation of $j(t)$ as follows:

$$\theta_{j(t)} \equiv E(j(t)) = E(e^{\ln(1+j(t))}) - 1 = e^{\mu_{j(t)} + \frac{1}{2}\sigma_{j(t)}^2} - 1 \tag{5}$$

6. At time t , if N_t jumps with probability $P = \lambda\delta t$, dN_t assumes 1 (i.e., $dN_t = 1$) in time interval $[t, t + \delta t]$ for sufficiently small δ . For term $j(t)S_t dN_t$ of (2), the underlying asset's price changes by $S_{t+\delta t} - S_t = j(t)S_t$. Otherwise (i.e., N_t does not jump at time t with probability $1 - \lambda\delta t$), dN_t assumes 0 (i.e., $dN_t = 0$) in time interval $[t, t + \delta t]$. For term $j(t)S_t dN_t$ of (2), the underlying asset's price does not change. In summary, we reiterate dN_t in time interval $[t, t + \delta t]$ as follows:

$$dN_t = \begin{cases} 1, \text{ jumps, probability } \lambda\delta t \\ 0, \text{ no-jump, probability } 1 - \lambda\delta t \end{cases} \tag{6}$$

7. Stochastic processes $\{B_s^H(t)\}$, $\{N_t\}$, and $\{j(t)\}$ are mutually independent at any time t .
8. At last, we pursue [24] (pp. 640–641) and assume the following conditions:
 - (a) continuous-time option trading,
 - (b) no-arbitrage opportunity,
 - (c) identical borrowing and lending interest rate r ,
 - (d) short-sales feasibility,
 - (e) no dividend during the option life, and
 - (f) frictionless markets in the form of no transaction cost or tax.

3.2. Verifying the Stochastic Differential Equation and Boundary Conditions

Traditionally for options based on geometric Brownian motions, Ref. [29] (p. 349) instructs the following stochastic differential equation for the option value V_t :

$$\frac{\partial V_t}{\partial t} + rS_t \frac{\partial V_t}{\partial S_t} + \frac{1}{2}\sigma_s^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} - rV_t = 0 \tag{7}$$

Investors resolve a specific option by the boundary conditions for (7). For instance, the condition for European call options on maturity T with exercise price K is as follows:

$$V_T = \max(S_T - K, 0) \tag{8}$$

Because we have already extended geometric Brownian motions into fractal Brownian motions and appended jump diffusions in (2), we correspondingly reckon the counterpart of (7) in this subsection. Of course, the counterpart is much more complicated.

For S_t of (2) with time from 0 to t , we institute the following path-dependent stochastic process $\{A_t\}$:

$$A_t = e^{\frac{1}{t} \int_0^t \ln S_\tau d\tau} \tag{9}$$

The option value V_t depends on S_t , A_t , and t as follows:

$$V_t = V(S_t, A_t, t) \tag{10}$$

We then augment (7) in the following theorem:

Theorem 1. For carbon-offset options of Asian styles based on jump diffusions and fractal Brownian motions (2), the options' value V_t is imposed by the following stochastic differential equation:

$$\begin{aligned} \frac{\partial V_t}{\partial t} + H\sigma_s^2 S_t^2 t^{2H-1} \frac{\partial^2 V_t}{\partial S_t^2} + \frac{A_t(\ln S_t - \ln A_t)}{t} \frac{\partial V_t}{\partial A_t} \\ + \lambda E(V_{t+\delta t} - V_t) + (r - \lambda\theta_{j(t)})S_t \frac{\partial V_t}{\partial S_t} - rV_t = 0 \end{aligned} \tag{11}$$

where S_t and the associated terms are introduced in (2); A_t is introduced in (9).

Proof of Theorem 1.

1. We exploit dynamic-hedging strategies (as outlined by [29] (pp. 422–423)) and build the following portfolio Π_t :

$$\Pi_t = V_t - \chi S_t \tag{12}$$

where χ is the weight for S_t . We trail [29] (pp. 422–423) and assign χ as follows:

$$\chi = \frac{\partial V_t}{\partial S_t} \tag{13}$$

The value of Π_t changes from time t to time $t + \delta t$ as follows:

$$\Pi_{t+\delta t} - \Pi_t = (V_{t+\delta t} - \chi S_{t+\delta t}) - (V_t - \chi S_t) = (V_{t+\delta t} - V_t) - \chi(S_{t+\delta t} - S_t)$$

We consult [29] (p. 349) and rewrite the model above as follows:

$$\Delta \Pi_t = \Delta V_t - \chi \Delta S_t \tag{14}$$

Because [27] (p. 129) tends the no-jump case and jump case at time t for (1), we also tend the cases for (2) below.

2. For the no-jump case as the simpler situation, we define the event ζ_1 as follows:

$$\zeta_1 \equiv \{N_t \text{ does not jump at time } t\}$$

We recognize $dN_t = 0$ by the formulation of (2) and notice the following probability by (6):

$$P(\zeta_1) = P(dN_t = 0) = 1 - \lambda \delta t \tag{15}$$

With $dN_t = 0$, we simplify (2) as follows:

$$dS_t = \mu S_t dt + \sigma_s S_t dB_s^H(t) \tag{16}$$

On the basis of (9) and (16), we operate fractal Itô's lemma (as described by [39] (pp. 4814–4816)) to V_t of (10) as follows:

$$\begin{aligned} \Delta V_t = & \left(\frac{\partial V_t}{\partial t} + \mu S_t \frac{\partial V_t}{\partial S_t} + H\sigma_s^2 S_t^2 t^{2H-1} \frac{\partial^2 V_t}{\partial S_t^2} + \frac{A_t(\ln S_t - \ln A_t)}{t} \frac{\partial V_t}{\partial A_t} \right) \delta t \\ & + \sigma_s S_t \frac{\partial V_t}{\partial S_t} \delta B_s^H(t) \end{aligned}$$

We substitute ΔV_t above into (14) as follows:

$$\begin{aligned} \Delta \Pi_t = & \left(\frac{\partial V_t}{\partial t} + \mu S_t \left(\frac{\partial V_t}{\partial S_t} - \chi \right) + H\sigma_s^2 S_t^2 t^{2H-1} \frac{\partial^2 V_t}{\partial S_t^2} + \frac{A_t(\ln S_t - \ln A_t)}{t} \frac{\partial V_t}{\partial A_t} \right) \delta t \\ & + \sigma_s S_t \left(\frac{\partial V_t}{\partial S_t} - \chi \right) \delta B_s^H(t) \end{aligned} \tag{17}$$

We have already assigned $\chi = \frac{\partial V_t}{\partial S_t}$ of (13), so the term $\sigma_s S_t (\frac{\partial V_t}{\partial S_t} - \chi) \delta B_s^H(t)$ becomes 0 for (17). By (13)–(17), we develop $\Delta \Pi_t$ as follows:

$$\Delta \Pi_t = \left(\frac{\partial V_t}{\partial t} + H\sigma_s^2 S_t^2 t^{2H-1} \frac{\partial^2 V_t}{\partial S_t^2} + \frac{A_t(\ln S_t - \ln A_t)}{t} \frac{\partial V_t}{\partial A_t} \right) \delta t$$

For $\Delta \Pi_t$ above, we then take the expectation or precisely conditional expectation on ζ_1 as follows:

$$E(\Delta \Pi_t | \zeta_1) = \left(\frac{\partial V_t}{\partial t} + H\sigma_s^2 S_t^2 t^{2H-1} \frac{\partial^2 V_t}{\partial S_t^2} + \frac{A_t(\ln S_t - \ln A_t)}{t} \frac{\partial V_t}{\partial A_t} \right) \delta t \tag{18}$$

3. For the jump case, we define the event ζ_2 as follows:

$$\zeta_2 \equiv \{N_t \text{ jumps at time } t\}$$

We recognize $dN_t = 1$ by the formulation of (2) and notice the following probability by (6):

$$P(\zeta_2) = P(dN_t = 1) = \lambda \delta t \tag{19}$$

Due to the jump and jump multitude $j(t)$ (as designated for (2)), we update V_t of (10) as follows:

$$V_{t+\delta t} = V((1 + j(t))S_t, A_{t+\delta t}, t + \delta t) \tag{20}$$

We substitute (20) into (14) as follows:

$$\Delta \Pi_t = (V((1 + j(t))S_t, A_{t+\delta t}, t + \delta t) - V(S_t, A_t, t)) - \chi j(t) S_t$$

For $\Delta \Pi_t$ above, we postulate (5), still postulate $\chi = \frac{\partial V_t}{\partial S_t}$ of (13), and take the expectation or precisely conditional expectation on ζ_2 as follows:

$$E(\Delta \Pi_t | \zeta_2) = E(V_{t+\delta t} - V_t) - \theta_{j(t)} S_t \frac{\partial V_t}{\partial S_t} \tag{21}$$

4. By the no-jump case and jump case, we manipulate total-expectation law (as described by [51] (p. 299)) for $\Delta \Pi_t$ as follows:

$$\begin{aligned} E(\Delta \Pi_t) &= E(E(\Delta \Pi_t | \zeta_1 \cup \zeta_2)) \\ &= E(\Delta \Pi_t | \zeta_1) P(\zeta_1) + E(\Delta \Pi_t | \zeta_2) P(\zeta_2), \text{ by (15) and (19)} \\ &= E(\Delta \Pi_t | \zeta_1) (1 - \lambda \delta t) + E(\Delta \Pi_t | \zeta_2) (\lambda \delta t), \text{ by (18) and (21)} \\ &= \left(\frac{\partial V_t}{\partial t} + H\sigma_s^2 S_t^2 t^{2H-1} \frac{\partial^2 V_t}{\partial S_t^2} + \frac{A_t(\ln S_t - \ln A_t)}{t} \frac{\partial V_t}{\partial A_t} \right) \delta t (1 - \lambda \delta t) \\ &\quad + (E(V_{t+\delta t} - V_t) - \theta_{j(t)} S_t \frac{\partial V_t}{\partial S_t}) (\lambda \delta t), \text{ by rearrangement} \\ &= \left(\frac{\partial V_t}{\partial t} + H\sigma_s^2 S_t^2 t^{2H-1} \frac{\partial^2 V_t}{\partial S_t^2} + \frac{A_t(\ln S_t - \ln A_t)}{t} \frac{\partial V_t}{\partial A_t} \right) \delta t \\ &\quad - \lambda \theta_{j(t)} S_t \frac{\partial V_t}{\partial S_t} \delta t + \lambda E(V_{t+\delta t} - V_t) \delta t \\ &= \left(\frac{\partial V_t}{\partial t} + H\sigma_s^2 S_t^2 t^{2H-1} \frac{\partial^2 V_t}{\partial S_t^2} + \frac{A_t(\ln S_t - \ln A_t)}{t} \frac{\partial V_t}{\partial A_t} \right) (\delta t)^2 \tag{22} \end{aligned}$$

Because [51] (p. 44) posits $(\delta t)^2 = 0$ for sufficiently small δ , so do we and thus infer as follows:

$$\lambda \left(\frac{\partial V_t}{\partial t} + H\sigma_s^2 S_t^2 t^{2H-1} \frac{\partial^2 V_t}{\partial S_t^2} + \frac{A_t(\ln S_t - \ln A_t)}{t} \frac{\partial V_t}{\partial A_t} \right) (\delta t)^2 = 0$$

By the zero item above, we advance and rearrange (22) as follows:

$$E(\Delta \Pi_t) = \left(\frac{\partial V_t}{\partial t} + H\sigma_s^2 S_t^2 t^{2H-1} \frac{\partial^2 V_t}{\partial S_t^2} + \frac{A_t(\ln S_t - \ln A_t)}{t} \frac{\partial V_t}{\partial A_t} - \lambda \theta_{j(t)} S_t \frac{\partial V_t}{\partial S_t} + \lambda E(V_{t+\delta t} - V_t) \right) \delta t \tag{23}$$

Because [29] (p. 349 and 422–423) sketches the no-arbitrage opportunity and thus risk-free conditions for Π_t of (12) in time interval $[t, t + \delta t]$, Π_t earns interest-rate r as follows:

$$E(\Delta \Pi_t) = r \Pi_t \delta t \tag{24}$$

We equate (23) and (24) and finally obtain (11).

□

We then decipher the call option and put option by indicating the boundary conditions in the following theorems:

Theorem 2. For carbon-offset options of Asian styles based on jump diffusions and fractal Brownian motions (2), the call option is decoded by the following stochastic differential Equation (11) and boundary condition on maturity T :

$$\begin{cases} \frac{\partial V_t}{\partial t} + H\sigma_s^2 S_t^2 t^{2H-1} \frac{\partial^2 V_t}{\partial S_t^2} + \frac{A_t(\ln S_t - \ln A_t)}{t} \frac{\partial V_t}{\partial A_t} \\ + \lambda E(V_{t+\delta t} - V_t) + (r - \lambda \theta_{j(t)}) S_t \frac{\partial V_t}{\partial S_t} - r V_t = 0 \\ V(S_T, A_T, T) = \max(A_T - K, 0) \end{cases}$$

where $V(S_T, A_T, T)$ is based on (10), A_T is based on (9), and K is the exercise price.

Proof of Theorem 2. By Theorem 1, the call option satisfies (11). The boundary condition is that investors harness the option value $V(S_T, A_T, T) = \max(A_T - K, 0)$ on maturity T . □

Theorem 3. For carbon-offset options of Asian styles based on jump diffusions and fractal Brownian motions (2), the put option is decoded by the following stochastic differential Equation (11) and boundary condition on maturity T :

$$\begin{cases} \frac{\partial V_t}{\partial t} + H\sigma_s^2 S_t^2 t^{2H-1} \frac{\partial^2 V_t}{\partial S_t^2} + \frac{A_t(\ln S_t - \ln A_t)}{t} \frac{\partial V_t}{\partial A_t} \\ + \lambda E(V_{t+\delta t} - V_t) + (r - \lambda \theta_{j(t)}) S_t \frac{\partial V_t}{\partial S_t} - r V_t = 0 \\ V(S_T, A_T, T) = \max(K - A_T, 0) \end{cases}$$

Proof of Theorem 3. By Theorem 1, the put option satisfies (11). The boundary condition is that investors harness the option value $V(S_T, A_T, T) = \max(K - A_T, 0)$ on maturity T . □

4. Pricing the Options

In the following theorems, we strive to analytically reveal the call-option value and put-option value.

Theorem 4. For the carbon-offset options $V(S_t, A_t, t)$ of (10) of Asian styles based on jump diffusions and fractal Brownian motions, the call-option value of Theorem 2 is approximately calculated as follows:

$$V(S_t, A_t, t) = e^{\tau + \xi_\tau - r(T-t)} N(d_1) - Ke^{-r(T-t)} N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{2\tau + \xi_\tau - \ln K}{\sqrt{2\tau}} & d_2 &= \frac{\xi_\tau - \ln K}{\sqrt{2\tau}} & N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt \\ \tau &= \frac{1}{6} \lambda (\mu_{j(t)}^2 + \sigma_{j(t)}^2) \frac{(T-t)^3}{T^2} + H\sigma_s^2 \left(\frac{T^{2H} - t^{2H}}{2H} - \frac{T^{2H+1} - t^{2H+1}}{(2H+1)T} + \frac{T^{2H+2} - t^{2H+2}}{2(H+1)T^2} \right) \\ \xi_\tau &= \frac{t}{T} \ln \frac{A_t}{S_t} + \ln S_t + \frac{(T-t)^2}{2T} (r - \lambda\theta_{j(t)} + \lambda\mu_{j(t)}) \\ &+ \frac{\sigma_s^2(T^{2H} - t^{2H})}{2} - \frac{H\sigma_s^2(T^{2H+1} - t^{2H+1})}{(2H+1)T} \end{aligned}$$

Proof of Theorem 4. Overall, directly computing the term $E(V_{t+\delta t} - V_t)$ of Theorem 2 is difficult, so we approximate the term using the Taylor series, substitute variables three times, and reckon the option value in the following steps:

1. For the first variable substitution, we substitute S_t by x and substitute $\ln(1 + j(t))$ by η as follows:

$$x = \ln S_t \quad \eta = \ln(1 + j(t)) \tag{25}$$

Of course, x and η depend on t . Due to the complex computation below, we suppress t for expression clarity. We total the partial derivatives by the substitution as follows:

$$\frac{\partial V_t}{\partial S_t} = \frac{1}{S_t} \frac{\partial V_t}{\partial x} \quad \frac{\partial^2 V_t}{\partial S_t^2} = \frac{1}{S_t^2} \left(\frac{\partial^2 V_t}{\partial x^2} - \frac{\partial V_t}{\partial x} \right)$$

We transport $\frac{\partial V_t}{\partial S_t}$ and $\frac{\partial^2 V_t}{\partial S_t^2}$ into (11) as follows:

$$\begin{aligned} &\frac{\partial V_t}{\partial t} + H\sigma_s^2 t^{2H-1} \frac{\partial^2 V_t}{\partial x^2} + (r - \lambda\theta_{j(t)} - H\sigma_s^2 t^{2H-1}) \frac{\partial V_t}{\partial x} \\ &+ \frac{A_t(x - \ln A_t)}{t} \frac{\partial V_t}{\partial A_t} + \lambda E(V(\eta + x, A_t, t) - V(x, A_t, t)) - rV_t = 0 \end{aligned} \tag{26}$$

We focus on η of $V(\eta + x, A_t, t)$ for fixed t and introduce $V(\eta + x, A_t, t) = V(\eta + x)$. We operate the following Taylor series with respect to η and drop the cubic or higher-moment terms (as traditionally established by [52] (p. 10)):

$$\begin{aligned} V(\eta + x) &= V(x) + \eta \frac{\partial V_t}{\partial x} + \frac{1}{2} \eta^2 \frac{\partial^2 V_t}{\partial x^2} + o(\eta^2) \\ &\approx V(x) + \eta \frac{\partial V_t}{\partial x} + \frac{1}{2} \eta^2 \frac{\partial^2 V_t}{\partial x^2} \end{aligned} \tag{27}$$

We then take the expectation of (27) as follows:

$$E(V(\eta + x) - V(x)) \approx \frac{\partial V_t}{\partial x} E_\eta(\eta) + \frac{1}{2} \frac{\partial^2 V_t}{\partial x^2} E_\eta(\eta^2) \tag{28}$$

By (3), (4), and (25), we recognize the expectation of η as $E_\eta(\eta) = \mu_{j(t)}$ and recognize the variance of η as $D_\eta(\eta) = \sigma_{j(t)}^2$. We gauge $E_\eta(\eta^2)$ as follows:

$$E_\eta(\eta^2) = (E_\eta(\eta))^2 + D_\eta(\eta) = \mu_{j(t)}^2 + \sigma_{j(t)}^2$$

We substitute $E_\eta(\eta)$ and $E_\eta(\eta^2)$ into (28) as follows:

$$E(V(\eta + x, A_t, t) - V(x, A_t, t)) = \mu_{j(t)} \frac{\partial V_t}{\partial x} + \frac{1}{2}(\mu_{j(t)}^2 + \sigma_{j(t)}^2) \frac{\partial^2 V_t}{\partial x^2} \tag{29}$$

We further substitute (29) into (26) as follows:

$$\frac{\partial V_t}{\partial t} + \varphi_1(t) \frac{\partial^2 V_t}{\partial x^2} + \varphi_2(t) \frac{\partial V_t}{\partial x} + \frac{A_t(x - \ln A_t)}{t} \frac{\partial V_t}{\partial A_t} - rV_t = 0 \tag{30}$$

where

$$\varphi_1(t) = \frac{1}{2} \lambda (\mu_{j(t)}^2 + \sigma_{j(t)}^2) + H\sigma_s^2 t^{2H-1} \tag{31}$$

$$\varphi_2(t) = r - \lambda \theta_{j(t)} + \lambda \mu_{j(t)} - H\sigma_s^2 t^{2H-1} \tag{32}$$

By (25), we reexpress the boundary condition $V(S_T, A_T, T) = \max(A_T - K, 0)$ of Theorem 2 as follows:

$$V(x, A_T, T) = \max(A_T - K, 0) \tag{33}$$

- For the second variable substitution, we perform the following substitution in order to simplify (30):

$$G_t = \frac{1}{T}(t \ln A_t + (T - t)x) \quad U(G_t, t) = V(x, A_t, t) \tag{34}$$

By (34), we rewrite (33) as follows:

$$U(G_T, T) = \max(e^{G_T} - K, 0) \tag{35}$$

We calculate the following partial derivatives:

$$\begin{aligned} \frac{\partial V_t}{\partial x} &= \frac{T-t}{T} \frac{\partial U_t}{\partial G_t} & \frac{\partial^2 V_t}{\partial x^2} &= \left(\frac{T-t}{T}\right)^2 \frac{\partial^2 U_t}{\partial G_t^2} \\ \frac{\partial V_t}{\partial A_t} &= \frac{t}{TA_t} \frac{\partial U_t}{\partial G_t} & \frac{\partial V_t}{\partial t} &= \frac{\partial U_t}{\partial t} + \frac{\ln A_t - x}{T} \frac{\partial U_t}{\partial G_t} \end{aligned}$$

We bring the partial derivatives above into (30) as follows:

$$\frac{\partial U_t}{\partial t} + \varphi_3(t) \frac{\partial^2 U_t}{\partial G_t^2} + \varphi_4(t) \frac{\partial U_t}{\partial G_t} - rU_t = 0 \tag{36}$$

where

$$\varphi_3(t) = \left(\frac{T-t}{T}\right)^2 \varphi_1(t) \quad \varphi_4(t) = \frac{T-t}{T} \varphi_2(t) \tag{37}$$

- For the third variable substitution, we follow [14] and introduce the following substitution:

$$\tau = \alpha(t) \quad \xi_\tau = G_t + \beta(t) \quad \omega(\tau, \xi_\tau) = U(G_t, t)e^{\gamma(t)} \tag{38}$$

We will configure $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ later in this step. We presume $\alpha(T) = \beta(T) = \gamma(T) = 0$. By (38), we reexpress the boundary condition (35) as follows:

$$\omega(0, \xi_0) = \max(e^{\xi_0} - K, 0) \tag{39}$$

Moreover, by (38), we compute the following partial derivatives:

$$\begin{aligned} \frac{\partial U_t}{\partial t} &= e^{-\gamma(t)} \left(\frac{\partial \omega}{\partial \tau} \alpha'(t) + \frac{\partial \omega}{\partial \xi_\tau} \beta'(t) - \omega \gamma'(t) \right) \\ \frac{\partial U_t}{\partial G_t} &= e^{-\gamma(t)} \frac{\partial \omega}{\partial \xi_\tau} \quad \frac{\partial^2 U_t}{\partial G_t^2} = e^{-\gamma(t)} \frac{\partial^2 \omega}{\partial \xi_\tau^2} \end{aligned}$$

We bring the partial derivatives above into (36) as follows:

$$\frac{\partial \omega}{\partial \tau} \alpha'(t) + \varphi_3(t) \frac{\partial^2 \omega}{\partial \xi_\tau^2} + (\varphi_4(t) + \beta'(t)) \frac{\partial \omega}{\partial \xi_\tau} - (r + \gamma'(t)) \omega = 0 \tag{40}$$

By (40), we demand the following equations:

$$\alpha'(t) + \varphi_3(t) = 0 \quad \varphi_4(t) + \beta'(t) = 0 \quad r + \gamma'(t) = 0 \tag{41}$$

By (31)–(32) and (37), we solve (41) as follows:

$$\begin{aligned} \alpha(t) &= \int_t^T \varphi_3(v) dv = \int_t^T \left(\frac{T-v}{T} \right)^2 \left(\frac{1}{2} \lambda (\mu_{j(t)}^2 + \sigma_{j(t)}^2) + H \sigma_s^2 v^{2H-1} \right) dv \\ &= \frac{1}{6} \lambda (\mu_{j(t)}^2 + \sigma_{j(t)}^2) \frac{(T-t)^3}{T^2} \\ &\quad + H \sigma_s^2 \left(\frac{T^{2H} - t^{2H}}{2H} - \frac{T^{2H+1} - t^{2H+1}}{(2H+1)T} + \frac{T^{2H+2} - t^{2H+2}}{2(H+1)T^2} \right) \end{aligned} \tag{42}$$

$$\begin{aligned} \beta(t) &= \int_t^T \varphi_4(v) dv \\ &= \int_t^T \left(\frac{T-v}{T} \right) (r - \lambda \theta_{j(t)} + \lambda \mu_{j(t)} - H \sigma_s^2 v^{2H-1}) dv \\ &= \frac{(T-t)^2}{2T} (r - \lambda \theta_{j(t)} + \lambda \mu_{j(t)}) \\ &\quad + \frac{\sigma_s^2 (T^{2H} - t^{2H})}{2} - \frac{H \sigma_s^2 (T^{2H+1} - t^{2H+1})}{(2H+1)T} \end{aligned} \tag{43}$$

$$\gamma(t) = \int_t^T r dv = r(T-t) \tag{44}$$

We convey (42)–(44) into (40) and consider (39) as follows:

$$\begin{cases} \frac{\partial \omega}{\partial \tau} = \frac{\partial^2 \omega}{\partial \xi_\tau^2} \\ \omega(0, \xi_0) = \max(e^{\xi_0} - K, 0) \end{cases} \tag{45}$$

Equation (45) is a heat equation (as described by [53] (p. 254)). By (34) and (38), we usher in τ and ξ_τ (as dictated in Theorem 4) as follows:

$$\begin{aligned} \tau = \alpha(t) &= \frac{1}{6} \lambda (\mu_{j(t)}^2 + \sigma_{j(t)}^2) \frac{(T-t)^3}{T^2} \\ &\quad + H \sigma_s^2 \left(\frac{T^{2H} - t^{2H}}{2H} - \frac{T^{2H+1} - t^{2H+1}}{(2H+1)T} + \frac{T^{2H+2} - t^{2H+2}}{2(H+1)T^2} \right) \end{aligned} \tag{46}$$

$$\begin{aligned} \xi_\tau = G_t + \beta(t) &= \frac{1}{T} (t \ln A_t + (T-t) \ln S_t) + \beta(t) \\ &= \frac{t}{T} \ln \frac{A_t}{S_t} + \ln S_t + \frac{(T-t)^2}{2T} (r - \lambda \theta_{j(t)} + \lambda \mu_{j(t)}) \\ &\quad + \frac{\sigma_s^2 (T^{2H} - t^{2H})}{2} - \frac{H \sigma_s^2 (T^{2H+1} - t^{2H+1})}{(2H+1)T} \end{aligned} \tag{47}$$

Ref. [53] (p. 254) crack the heat equation and we borrow their solution as follows:

$$\begin{aligned} \omega(\tau, \xi_\tau) &= \frac{1}{2\sqrt{\pi\tau}} \int_{\ln K}^{+\infty} (e^y - K)e^{-\frac{(y-\xi_\tau)^2}{4\tau}} dy \\ &= \frac{1}{2\sqrt{\pi\tau}} \int_{\ln K}^{+\infty} e^{y-\frac{(y-\xi_\tau)^2}{4\tau}} dy - \frac{K}{2\sqrt{\pi\tau}} \int_{\ln K}^{+\infty} e^{-\frac{(y-\xi_\tau)^2}{4\tau}} dy \\ &= \phi_1 - \phi_2 \end{aligned} \tag{48}$$

We take $\frac{y-\xi_\tau-2\tau}{\sqrt{2\tau}} = t$ and calculate ϕ_1 as follows:

$$\begin{aligned} \phi_1 &= \frac{1}{2\sqrt{\pi\tau}} \int_{\ln K}^{+\infty} e^{y-\frac{(y-\xi_\tau)^2}{4\tau}} dy = e^{\tau+\xi_\tau} \frac{1}{\sqrt{2\pi}} \int_{-\frac{2\tau+\xi_\tau-\ln K}{\sqrt{2\tau}}}^{+\infty} e^{-\frac{t^2}{2}} dt \\ &= e^{\tau+\xi_\tau} N\left(\frac{2\tau + \xi_\tau - \ln K}{\sqrt{2\tau}}\right) \end{aligned} \tag{49}$$

We take $\frac{y-\xi_\tau}{\sqrt{2\tau}} = t$ and calculate ϕ_2 as follows:

$$\begin{aligned} \phi_2 &= \frac{K}{2\sqrt{\pi\tau}} \int_{\ln K}^{+\infty} e^{-\frac{(y-\xi_\tau)^2}{4\tau}} dy = \frac{K}{\sqrt{2\pi}} \int_{-\frac{\xi_\tau-\ln K}{\sqrt{2\tau}}}^{+\infty} e^{-\frac{t^2}{2}} dt \\ &= KN\left(\frac{\xi_\tau - \ln K}{\sqrt{2\tau}}\right) \end{aligned} \tag{50}$$

We transfer (49)–(50) into (48) and compute the following solution of (45):

$$\omega(\tau, \xi_\tau) = e^{\tau+\xi_\tau} N\left(\frac{2\tau + \xi_\tau - \ln K}{\sqrt{2\tau}}\right) - KN\left(\frac{\xi_\tau - \ln K}{\sqrt{2\tau}}\right) = e^{\tau+\xi_\tau} N(d_1) - KN(d_2) \tag{51}$$

where

$$d_1 = \frac{2\tau + \xi_\tau - \ln K}{\sqrt{2\tau}} \quad d_2 = \frac{\xi_\tau - \ln K}{\sqrt{2\tau}} \quad N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt$$

- At last, we revert the three-round variable substitutions (25), (34), and (38) back to (10) as follows:

$$V(S_t, A_t, t) = V(x, A_t, t) = U(G_t, t) = e^{-\gamma(t)} \omega(\tau, \xi_\tau) \tag{52}$$

We relocate (44) and (51) to (52) and obtain the pricing formula in Theorem 4 as follows:

$$\begin{aligned} V(S_t, A_t, t) &= e^{-r(T-t)} (e^{\tau+\xi_\tau} N(d_1) - KN(d_2)) \\ &= e^{\tau+\xi_\tau-r(T-t)} N(d_1) - Ke^{-r(T-t)} N(d_2) \end{aligned}$$

□

Theorem 5. For the carbon-offset options $V(S_t, A_t, t)$ of (10) of Asian styles based on jump diffusions and fractal Brownian motions, the put-option value of Theorem 3 is approximately calculated as follows:

$$V(S_t, A_t, t) = Ke^{-r(T-t)} N(-d_2) - e^{\tau+\xi_\tau-r(T-t)} N(-d_1)$$

Proof of Theorem 5. Overall, we follow the computations and steps of the proof of Theorem 4 as follows:

- For the first variable substitution, we exactly follow step 1 of the proof of Theorem 4.

- For the second variable substitution, we still perform (34) and rewrite the boundary condition $V(S_T, A_T, T) = \max(K - A_T, 0)$ of Theorem 3 as follows:

$$U(G_T, T) = \max(K - e^{G_T}, 0)$$

We still designate (37).

- For the third variable substitution, we still perform (38), obtain (42)–(44), and configure the following heat equation

$$\begin{cases} \frac{\partial \omega}{\partial \tau} = \frac{\partial^2 \omega}{\partial \xi_\tau^2} \\ \omega(0, \xi_0) = \max(K - e^{\xi_0}, 0) \end{cases} \tag{53}$$

We still assign (46)–(47) and obtain the following solution for (53):

$$\begin{aligned} \omega(\tau, \xi_\tau) &= \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\ln K} (K - e^y) e^{-\frac{(y-\xi_\tau)^2}{4\tau}} dy \\ &= \frac{K}{2\sqrt{\pi\tau}} \int_{-\infty}^{\ln K} e^{-\frac{(y-\xi_\tau)^2}{4\tau}} dy - \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\ln K} e^{y-\frac{(y-\xi_\tau)^2}{4\tau}} dy \\ &\equiv \phi_3 - \phi_4 \end{aligned} \tag{54}$$

We define $\frac{y-\xi_\tau}{\sqrt{2\tau}} = t$ and calculate ϕ_3 as follows:

$$\begin{aligned} \phi_3 &= \frac{K}{2\sqrt{\pi\tau}} \int_{-\infty}^{\ln K} e^{-\frac{(y-\xi_\tau)^2}{4\tau}} dy = \frac{K}{\sqrt{2\pi}} \int_{\frac{\xi_\tau - \ln K}{\sqrt{2\tau}}}^{+\infty} e^{-\frac{t^2}{2}} dt \\ &= KN\left(-\frac{\xi_\tau - \ln K}{\sqrt{2\tau}}\right) \end{aligned}$$

We define $\frac{y-\xi_\tau-2\tau}{\sqrt{2\tau}} = t$ and calculate ϕ_4 as follows:

$$\begin{aligned} \phi_4 &= \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\ln K} e^{y-\frac{(y-\xi_\tau)^2}{4\tau}} dy = e^{\tau+\xi_\tau} \frac{1}{\sqrt{2\pi}} \int_{\frac{2\tau+\xi_\tau-\ln K}{\sqrt{2\tau}}}^{+\infty} e^{-\frac{t^2}{2}} dt \\ &= e^{\tau+\xi_\tau} N\left(-\frac{2\tau + \xi_\tau - \ln K}{\sqrt{2\tau}}\right) \end{aligned}$$

We restore ϕ_3 and ϕ_4 above back to (54) and obtain the following solution for (53):

$$\begin{aligned} \omega(\tau, \xi_\tau) &= \phi_3 - \phi_4 = KN\left(-\frac{\xi_\tau - \ln K}{\sqrt{2\tau}}\right) - e^{\tau+\xi_\tau} N\left(-\frac{2\tau + \xi_\tau - \ln K}{\sqrt{2\tau}}\right) \\ &= KN(-d_2) - e^{\tau+\xi_\tau} N(-d_1) \end{aligned} \tag{55}$$

- At last, we revert the three-round variable substitutions (25), (34), and (38) back to (10) and obtain (52). We relocate (44) and (55) to (52) and obtain the pricing formula in Theorem 5 as follows:

$$\begin{aligned} V(S_t, A_t, t) &= e^{-r(T-t)} \omega(\tau, \xi_\tau) = e^{-r(T-t)} (KN(-d_2) - e^{\tau+\xi_\tau} N(-d_1)) \\ &= Ke^{-r(T-t)} N(-d_2) - e^{\tau+\xi_\tau-r(T-t)} N(-d_1) \end{aligned}$$

□

5. Illustrations

In this section, we postulate the following option parameters and dissect the parameter sensitivity by individually attuning one parameter:

$$\begin{aligned}
 T = 1.5 \quad K = 100 \quad S_0 = 100 \quad \sigma_s = 0.20 \quad r = 0.020 \\
 \lambda = 1 \quad H = 0.7 \quad \mu_{j(t)} = 0.13 \quad \sigma_{j(t)} = 0.15
 \end{aligned}
 \tag{56}$$

where the Hurst exponent H is set in the parameter range $[\frac{1}{2}, 1)$ of [39]. For default parameter assumptions, see [17] (p. 5) for setting the exercise price K and the interest-rate r . See ref. [22] (p. 455) for the setting range of the expectation and volatility of underlying asset prices.

At the beginning with time $t = 0$, we calculate the call-option value of Theorem 4 as follows:

$$V(S_0, A_0, 0) = e^{\tau + \zeta_\tau - rT} N(d_1) - Ke^{-rT} N(d_2)
 \tag{57}$$

We also calculate the put-option value of Theorem 5 as follows:

$$V(S_0, A_0, 0) = Ke^{-rT} N(-d_2) - e^{\tau + \zeta_\tau - rT} N(-d_1)
 \tag{58}$$

By Theorem 4 and (56)–(58), we reckon the following parameters:

$$\begin{aligned}
 \theta &= e^{0.13 + 0.5 \times 0.15^2} - 1 = 0.1517 \\
 \tau &= \frac{1 \times (0.13^2 + 0.15^2) \times 1.5}{6} \\
 &+ \left(\frac{1}{2 \times 0.7} - \frac{1}{2 \times 0.7 + 1} + \frac{1}{2 \times 0.7 + 2} \right) \times 0.7 \times 0.2^2 \times 1.5^{2 \times 0.7} = 0.0391
 \end{aligned}$$

$$\begin{aligned}
 \zeta_\tau &= \ln 100 + \frac{(0.02 - 1 \times 0.1517 + 1 \times 0.13) \times 1.5}{2} \\
 &+ \left(\frac{1}{2} - \frac{0.7}{2 \times 0.7 + 1} \right) \times 0.2^2 \times 1.5^{2 \times 0.7} = 4.6186 \\
 d_1 &= \frac{2 \times 0.0391 + 4.6186 - \ln 100}{\sqrt{2 \times 0.0391}} = 0.3276 \\
 d_2 &= \frac{4.6186 - \ln 100}{\sqrt{2 \times 0.0391}} = 0.0480
 \end{aligned}$$

By (57) and the parameters above, we calculate the call-option value at time $t = 0$ as follows:

$$\begin{aligned}
 V(S_0, A_0, 0) &= e^{0.0391 + 4.6186 - 0.02 \times 1.5} \times N(0.3276) - 100 \times e^{-0.02 \times 1.5} \times N(0.0480) \\
 &= 13.8878
 \end{aligned}$$

By (58) and the parameters above, we calculate the put-option value at time $t = 0$ as follows:

$$\begin{aligned}
 V(S_0, A_0, 0) &= 100 \times e^{-0.02 \times 1.5} \times N(-0.0480) - e^{0.0391 + 4.6186 - 0.02 \times 1.5} \times N(-0.3276) \\
 &= 8.6573
 \end{aligned}$$

We then modify interest rate r by $r = 0.020, r = 0.025, r = 0.030, r = 0.035$, and $r = 0.040$ but hold other parameters of (56) unchanged. We depict the effect on the call-option value in Figure 2. We illustrate the underlying asset price (2) at time $t = 0$ in the horizon axis and the option value (57) at time $t = 0$ in the vertical axis. We portray the value for $r = 0.020$ by a solid red curve and the value for $r = 0.040$ by a solid green curve. Under the same figure format, we depict the effect on the put-option value (58) at time $t = 0$ in Figure 3.

Under the same figure format, we also modify Hurst exponent H and depict the effect on the call-option value and on the put-option value in Figures 4 and 5. Furthermore, we

modify expected jump numbers λ and depict the effect on the call-option value and on the put-option value in Figures 6 and 7.

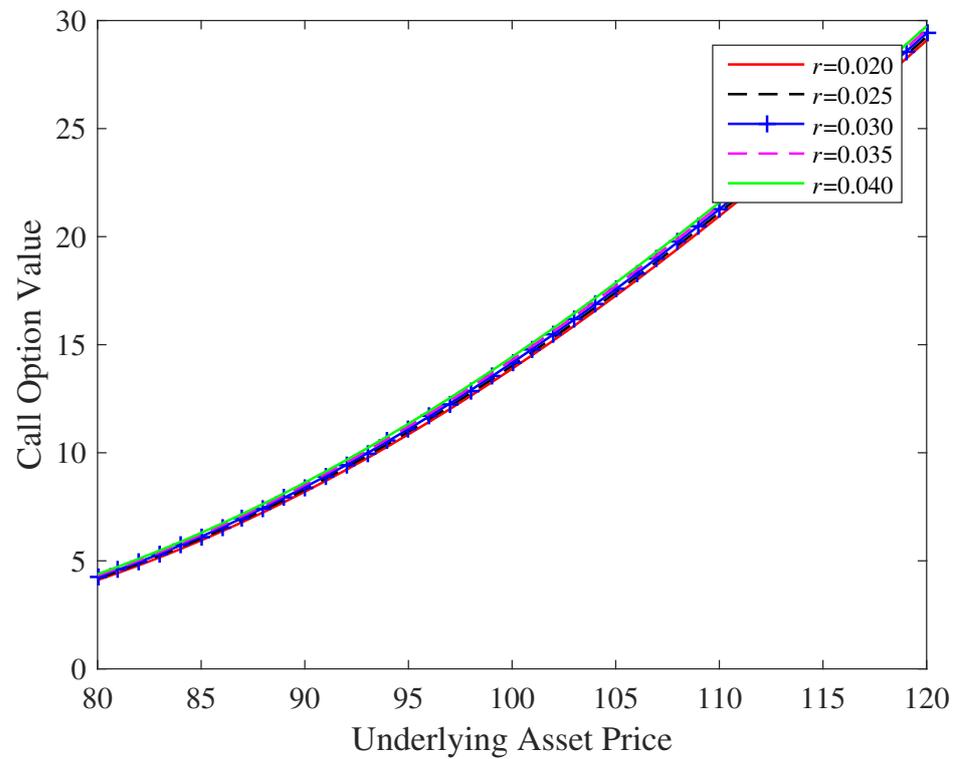


Figure 2. Modifying interest rate r and observing the effect on the call-option value.

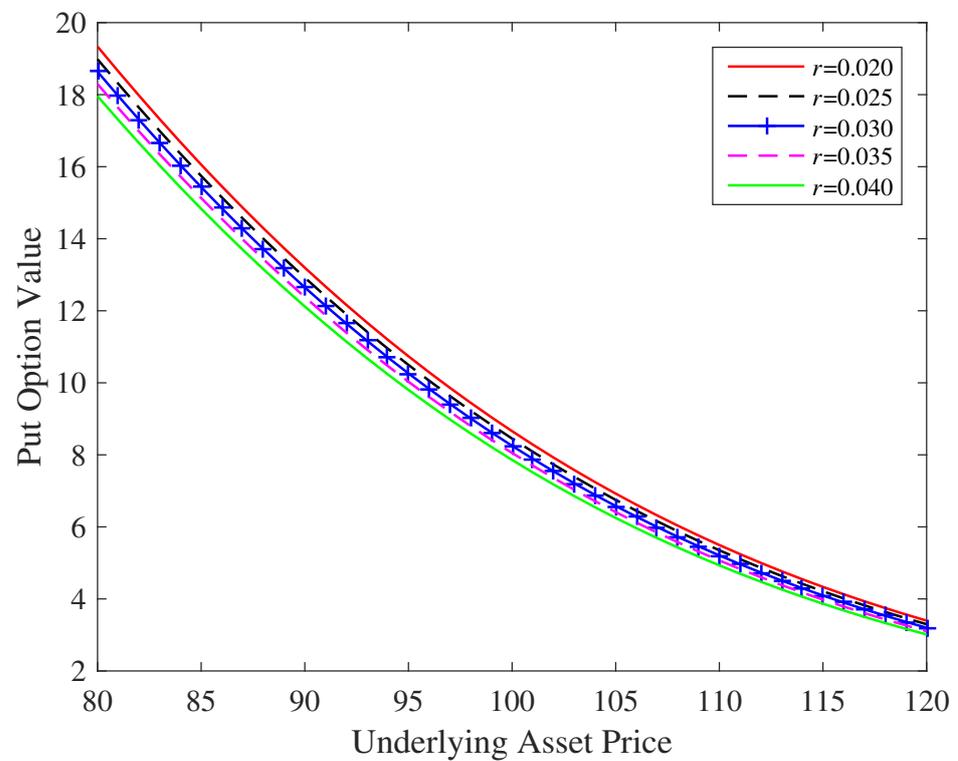


Figure 3. Modifying interest rate r and observing the effect on the put-option value.

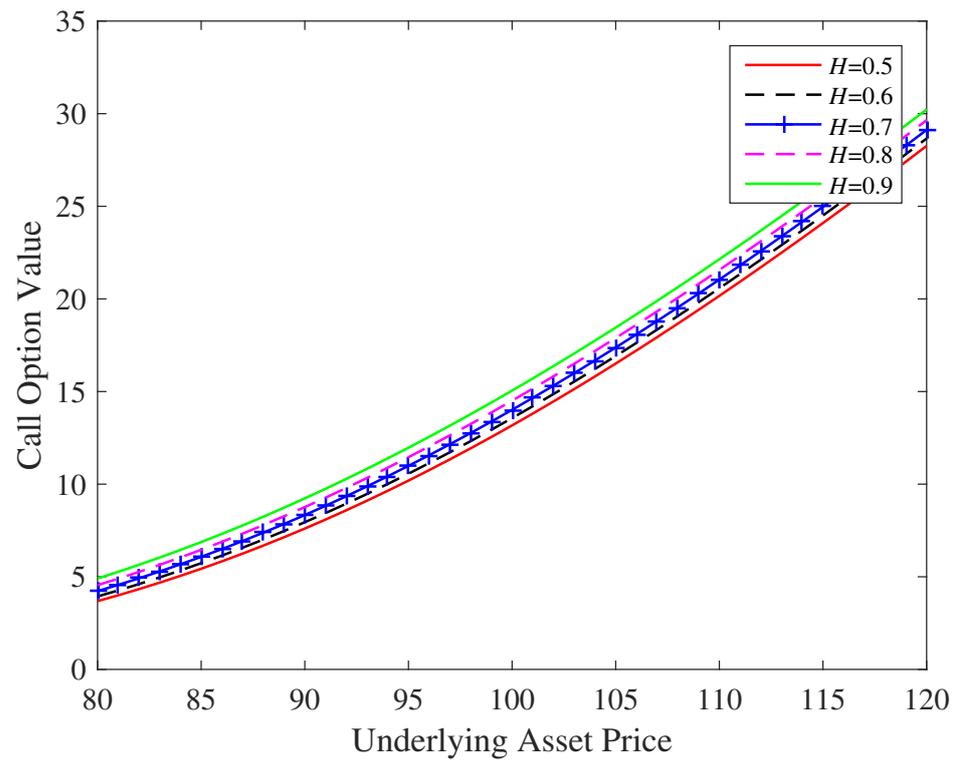


Figure 4. Modifying Hurst exponent H and observing the effect on the call-option value.

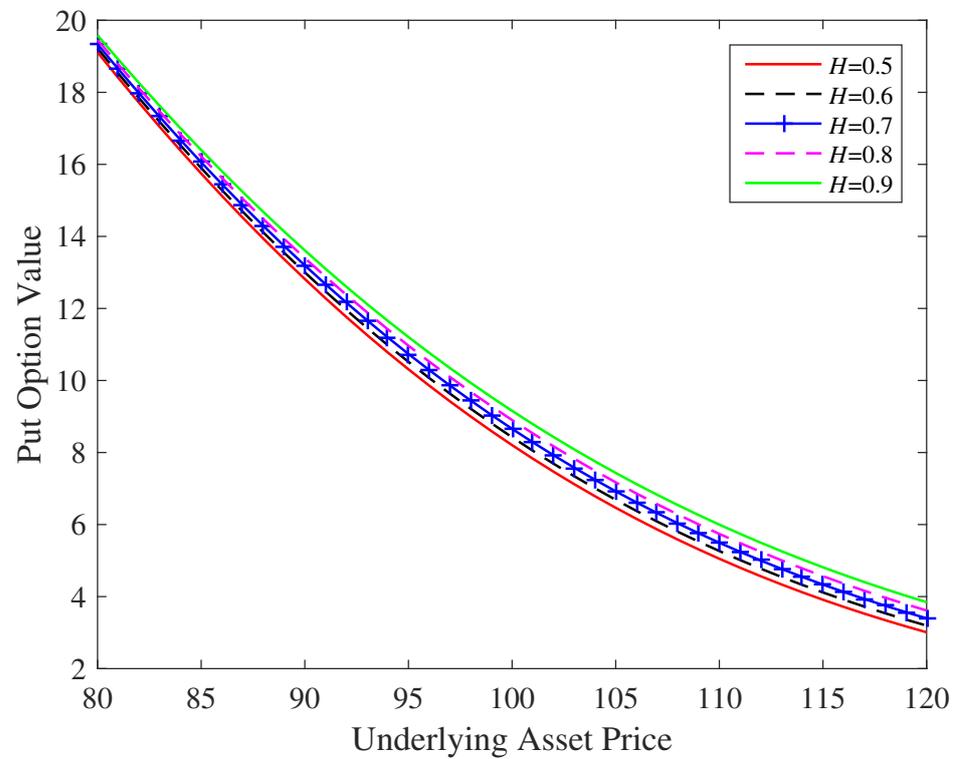


Figure 5. Modifying Hurst exponent H and observing the effect on the put-option value.

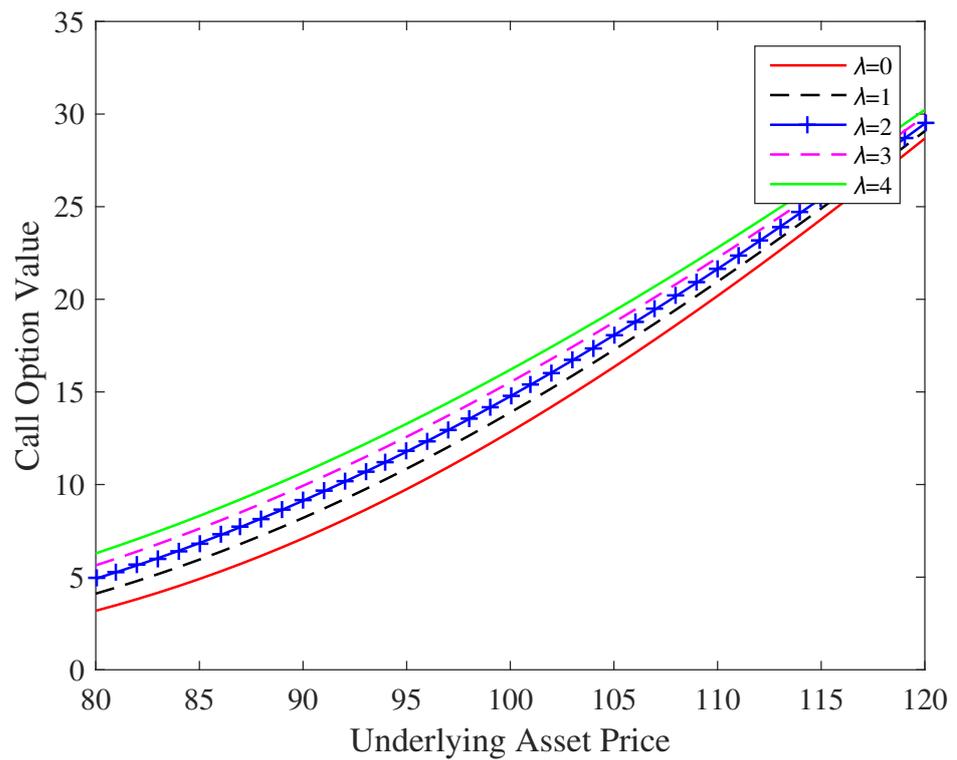


Figure 6. Modifying expected jump numbers λ and observing the effect on the call-option value.

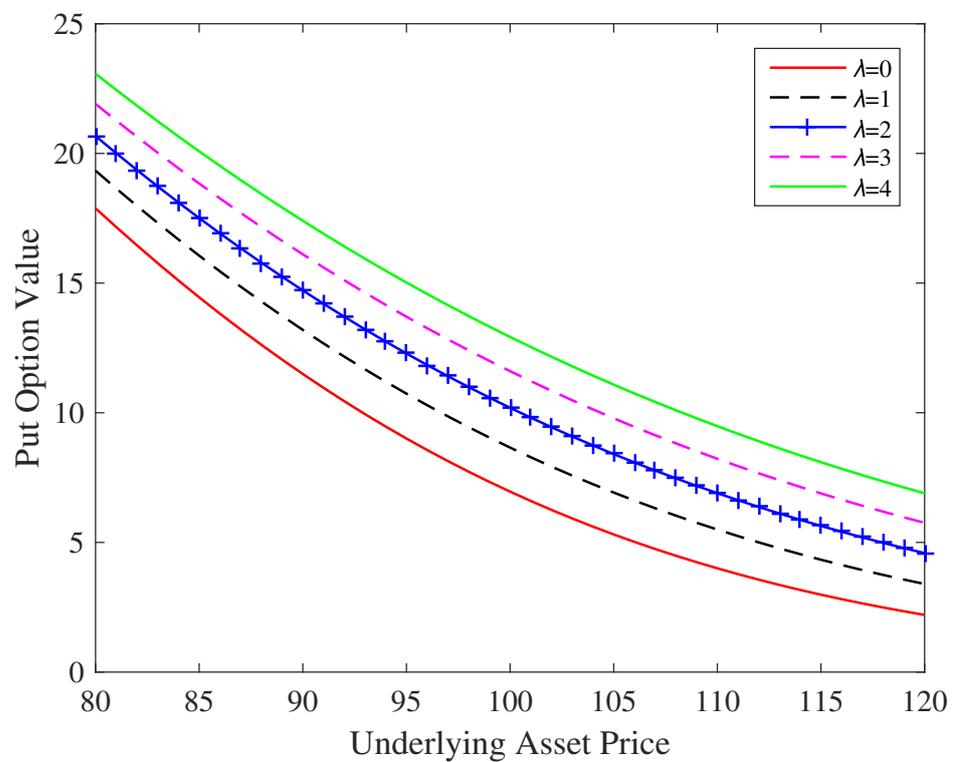


Figure 7. Modifying expected jump numbers λ and observing the effect on the put-option value.

We have deposited the data and codes for this paper at Harvard Dataverse <https://doi.org/10.7910/DVN/O4VXBD>.

6. Conclusions

6.1. Future Directions

In future studies, we can design the terms of carbon-offset option contracts based on underlying assets such as the carbon-emission allowance or the carbon-neutral index, including the contract type, expiration date, strike price, and other terms of the carbon-offset option, to provide a reference for launching carbon options in the carbon-emission trading market. In addition, we can also analyze the time series of the underlying asset price of carbon-offset options based on the historical data from the carbon-emission trading market, estimate the volatility of carbon-offset options through the GARCH model, and then use our pricing model to calculate the initial price of carbon-offset option contracts.

6.2. Concluding Remarks

Our Asian-style carbon-offset option-pricing model considers the fractal and jump characteristics of carbon financial underlying assets and provides a theoretical reference for pricing. With the implementation of the double carbon policy, the carbon-emission trading market can explore carbon-offset option financial derivatives, improve market activity, and enrich the trading variety system.

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