

Article

Development of the Method of Averaging in Clifford Geometric Algebras

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Abstract: We develop the method of averaging in Clifford (geometric) algebras suggested by the author in previous papers. We consider operators constructed using two different sets of anticommuting elements of real or complexified Clifford algebras. These operators generalize Reynolds operators from the representation theory of finite groups. We prove a number of new properties of these operators. Using the generalized Reynolds operators, we give a complete proof of the generalization of Pauli's theorem to the case of Clifford algebras of arbitrary dimension. The results can be used in geometry, physics, engineering, computer science, and other applications.

Keywords: Clifford algebra; geometric algebra; method of averaging; Reynolds operator; Pauli's theorem

MSC: 15A66; 11E88



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1. Introduction

In the present paper, we develop the method of averaging in Clifford (geometric) algebras suggested by the author in [1–3]. Namely, we introduce generalized Reynolds operators in Clifford algebras and prove a number of new properties of these operators. Theorems 1–10 are new. We use generalized Reynolds operators to prove generalized Pauli's theorem (see Theorems 11 and 12), which has been formulated for the first time in a brief report [4] without a proof. The main idea of these theorems is to present an algorithm to compute the element T that connects two sets of Clifford algebra elements that satisfy the main anticommutative conditions. The proofs of Theorems 11 and 12 are presented in this paper for the first time. The results are used in geometry, physics, and engineering, in particular, for n -dimensional Weyl, Majorana, and Majorana–Weyl spinors, to study relations between spin and orthogonal groups for the Dirac and the Yangs–Mills equations in pseudo-Euclidean space and on non-trivial curved manifolds, etc.

This paper is organized as follows. In Section 1, we discuss Reynolds operators and generalized Reynolds operators in Clifford algebras and representation theory of finite groups. In Section 2, we prove some auxiliary statements about sets of anticommuting elements of the Clifford algebra. In Section 3, we prove a number of new properties of generalized Reynolds operators in Clifford algebras. In Section 4, we consider some other operators and prove their properties. In Section 5, we prove that these operators are also related to generalized Reynolds operators in the case of odd dimension. In Section 6, we use these operators to deduce an algorithm to compute an element that connects two sets of anticommuting elements of the Clifford algebra. The discussion follows in Section 7.

Let e_a be generators of the real Clifford algebra $\mathcal{C}_{p,q}$ (or geometric algebra, see, for example, [5–9]), and $e_A = e_{a_1 a_2 \dots a_k} = e_{a_1} e_{a_2} \dots e_{a_k}$ are basis elements enumerated by ordered multi-indices $A = a_1 a_2 \dots a_k$, $a_1 < a_2 < \dots < a_k$ with a length between 0 and n (the element $e := e_\emptyset$ with empty multi-index is the identity element). The indices

a, a_1, a_2, \dots take the values from 1 to n . The generators satisfy the main anticommutative conditions of Clifford algebra

$$e_a e_b + e_b e_a = 2\eta_{ab} e,$$

where $\eta = (\eta_{ab}) = \text{diag}(1, \dots, 1, -1, \dots, -1)$ is the diagonal matrix with its first p entries equal to 1 and the last q entries equal to -1 on the diagonal, $p + q = n$. We denote inverses of generators by

$$e^a := \eta^{ab} e_b = (e_a)^{-1}, \quad a = 1, \dots, n,$$

and inverses of basis elements by $e^A := (e_A)^{-1}$. We use the Einstein summation convention.

Let us consider the Reynolds operator acting on the Clifford algebra

$$\frac{1}{|G|} \sum_{g \in G} g^{-1} U g, \quad U \in \mathcal{C}_{p,q}$$

where $|G|$ is the number of elements in a finite subgroup G of the group $\mathcal{C}_{p,q}^\times$ of all invertible elements of $\mathcal{C}_{p,q}$. Reynolds operators are widely used in the representation theory of finite groups (see [10–13]).

In [1], we consider the operator

$$\frac{1}{2^n} \sum_{A \in I} e_A U e^A, \tag{1}$$

which is the Reynolds operator of the Salingaros vee group [14–18]

$$G_{p,q} = \{\pm e_A, A \in I\}, \quad \dim G_{p,q} = 2^{n+1}, \tag{2}$$

$$I = \{\emptyset, 1, \dots, n, 12, 13, \dots, 1 \dots n\}. \tag{3}$$

The operator (1) is a projection onto the center $\text{Cen}(\mathcal{C}_{p,q})$ of the Clifford algebra $\mathcal{C}_{p,q}$.

We can consider Reynolds operators acting on $\mathcal{C}_{p,q}$ of the Salingaros vee group $G_{p',q'}$ of another Clifford algebra $\mathcal{C}_{p',q'} \subset \mathcal{C}_{p,q}$. These operators “average” an action of the Salingaros vee group $G_{p',q'}$ on the Clifford algebra $\mathcal{C}_{p,q}$. For example, the operator

$$\frac{1}{2}(U + (e_A)^{-1} U e_A) \tag{4}$$

is the Reynolds operator of the group $\{\pm e, \pm e_A\}$ of order 4, where $(e_A)^2 = e$ or $(e_A)^2 = -e$, which is isomorphic to the Salingaros vee group $G_{1,0}$ (or $G_{0,1}$). This operator is a projection onto the subset of elements that commute with the generator e_A (see [1], Section 6).

More generally, we can consider the operators

$$\frac{1}{|S|} \sum_{A \in S} e_A U e^A, \quad U \in \mathcal{C}_{p,q}, \tag{5}$$

where $|S|$ is the number of elements of some subset S of the set I (3). In particular cases, we obtain (1) and (4). For some subsets S , the operators (5) are not Reynolds operators. In the present paper, we consider the following new operators, which generalize (5):

$$\frac{1}{|S|} \sum_{A \in S} \beta_A U \gamma^A, \quad U \in \mathcal{C}_{p,q}, \quad \gamma^A := (\gamma_A)^{-1}, \tag{6}$$

where the elements γ_A and β_A are generated by two different sets of Clifford algebra elements $\{\gamma_a, a = 1, \dots, n\}$ and $\{\beta_a, a = 1, \dots, n\}$ that satisfy the main anticommutative conditions of the Clifford algebras $\mathcal{C}_{p,q}$. If $S = I$, then the corresponding sets of elements $\{\pm \gamma_A, A \in I\}$ and $\{\pm \beta_A, A \in I\}$ form Salingaros vee groups of dimensions 2^{n+1} or 2^n (see Lemma 1). We call the operators (6) *generalized Reynolds operators* in the case $S = I$. Sometimes we consider operators (6) with subsets S such that the corresponding sets of

elements $\{\pm\gamma_A, A \in S\}$ and $\{\pm\beta_A, A \in S\}$ do not form groups (see Section 4) but are related to generalized Reynolds operators in the case of odd n (see Section 5).

We formulate statements of this paper not only for the case of the real Clifford algebras $\mathcal{C}l_{p,q}$, but also for the case of the complexified Clifford algebras $\mathbb{C} \otimes \mathcal{C}l_{p,q}$ (see, for example, [7] (p. 139) and [5] (p. 80)), which are important for applications. It is convenient for us to use the notation $\mathcal{C}l_{p,q}^{\mathbb{F}}$ ($\mathbb{F} = \mathbb{R}$ or \mathbb{C}) when results are true for both cases $\mathcal{C}l_{p,q}^{\mathbb{R}} := \mathcal{C}l_{p,q}$, $\mathcal{C}l_{p,q}^{\mathbb{C}} := \mathbb{C} \otimes \mathcal{C}l_{p,q}$.

2. On Sets of Anticommuting Clifford Algebra Elements

Let us consider a set of Clifford algebra elements

$$\{\gamma_a, a = 1, \dots, n\} \in \mathcal{C}l_{p,q}^{\mathbb{F}}, \quad \gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab}e \tag{7}$$

and the corresponding set

$$\mathfrak{B} = \{\gamma_A, A \in I\} = \{e, \gamma_a, \gamma_{ab}, \gamma_{abc}, \dots, \gamma_{1\dots n}\}, \quad a < b < c < \dots, \tag{8}$$

where $\gamma_{a_1} \cdots \gamma_{a_k}$ is denoted by $\gamma_{a_1\dots a_k}$ for $a_1 < \dots < a_k$. We have the following lemma (see also [19] (pp. 289–290) and [20] (pp. 127–128)). We present the proof for the convenience of the reader.

Lemma 1.

1. If $n = p + q$ is even, then the set (8) is linearly independent.
2. If $n = p + q$ is odd, then either
 - we have $\gamma_{1\dots n} = \pm e_{1\dots n}$ and (8) is linearly independent, or
 - we have $\gamma_{1\dots n} = \pm e$ and (8) is linearly dependent (this is possible only in the case $p - q \equiv 1 \pmod{4}$), or
 - we have $\gamma_{1\dots n} = \pm ie$ and (8) is linearly dependent (this is possible only in case $\mathbb{F} = \mathbb{C}$, $p - q \equiv 3 \pmod{4}$).

Note that in the cases $\mathbb{F} = \mathbb{R}$ when the set (8) is linearly independent, this set is another basis of $\mathcal{C}l_{p,q}$. In the cases when the set is linearly dependent, it is the set of two bases of the subalgebra of $\mathcal{C}l_{p,q}$, which is isomorphic to another Clifford algebra $\mathcal{C}l_{p',q'}$ of dimension 2^{n-1} , $p' + q' = n - 1$. For example, in $\mathcal{C}l_{2,1}$ with the set of generators e_1, e_2, e_3 , we can take $\gamma_1 = e_1, \gamma_2 = e_2, \gamma_3 = e_{12}$ and obtain the set $\mathfrak{B} = \{e, e_1, e_2, e_{12}, e_2, -e_1, -e\}$ of two bases of $\mathcal{C}l_{2,0}$.

Proof. Suppose that the set (8) is linearly dependent. Then there exist not all zero scalars $u, u_1, \dots, u_{1\dots n}$ such that

$$ue + u_1\gamma_1 + \dots + u_{1\dots n}\gamma_{1\dots n} = 0.$$

At least one scalar is non-zero $u_B \neq 0$ for some multi-index B . Multiplying both sides of the equation by $\frac{\gamma_B}{u_B}$, we obtain

$$e + v_1\gamma_1 + \dots + v_{1\dots n}\gamma_{1\dots n} = 0 \tag{9}$$

for some other scalars $v_1, \dots, v_{1\dots n}$. At least one of these scalars is nonzero because $e \neq 0$.

For any element γ_A (except $\gamma_{1\dots n}$ in the case of odd n and except e in the case of any n), there exists the element γ_a such that γ_A anticommutes with γ_a (if $|A|$ is even, then we can take $a \in A$; if $|A|$ is odd, then we can take $a \notin A$). Let us choose some γ_A and some γ_a that anticommutes with γ_A . Multiplying both sides of (9) on the right by γ_a and on the left by $(\gamma_a)^{-1}$, we obtain the equation like (9) but with another sign of the term $v_A\gamma_A$ (the signs of some other terms may also change). Adding both sides of this new equation and both sides of (9), we obtain again the equation like (9) but without the term $v_A\gamma_A$ (and some other terms) and with other scalars. Further, we repeat this process for other terms.

In the case of even n , after a finite number of steps, we obtain $e = 0$, i.e., a contradiction. Hence, the set (8) is linearly independent.

In the case of odd n , after a finite number of steps, we obtain $e + w_{1\dots n}\gamma_{1\dots n} = 0$ for some scalar $w_{1\dots n}$, i.e., $\gamma_{1\dots n} = \lambda e$ for some scalar λ . We have

$$(\gamma_{1\dots n})^2 = (-1)^{\frac{n(n-1)}{2}}(-1)^qe. \tag{10}$$

If $\frac{n(n-1)}{2} + q$ is even, which is equivalent to $p - q \equiv 1 \pmod{4}$, then $\gamma_{1\dots n} = \pm e$. If $\frac{n(n-1)}{2} + q$ is odd, which is equivalent to $p - q \equiv 3 \pmod{4}$, then $\gamma_{1\dots n} = \pm ie$, which is possible only in the complex case. In other cases, the set (8) is linearly independent. Since the conditions (7), it follows that $\gamma_{1\dots n}$ is in the center (note that the center $\text{Cen}(\mathcal{C}_{p,q}^{\mathbb{F}}) = \{U \in \mathcal{C}_{p,q}^{\mathbb{F}} \mid UV = VU, \forall V \in \mathcal{C}_{p,q}^{\mathbb{F}}\}$ of $\mathcal{C}_{p,q}^{\mathbb{F}}$ is the subspace $\mathcal{C}_{p,q}^{0\mathbb{F}}$ in the case of even n and the subspace $\mathcal{C}_{p,q}^{0\mathbb{F}} \oplus \mathcal{C}_{p,q}^{n\mathbb{F}}$ in the case of odd n) of the Clifford algebra. We have $\gamma_{1\dots n} = ae + be_{1\dots n}$ for some scalars a and b . We obtain

$$(\gamma_{1\dots n})^2 = (a^2 + b^2(-1)^{\frac{n(n-1)}{2}+q})e + 2abe_{1\dots n}. \tag{11}$$

From (10) and (11), we obtain $ab = 0$. If $b = 0$, then $\gamma_{1\dots n} = ae$, and we obtain a contradiction because the set (8) is linearly independent. Hence, $a = 0$ and $b^2 = 1$, i.e., $\gamma_{1\dots n} = \pm e_{1\dots n}$. \square

Let us consider the operator

$$F(U) = \frac{1}{2^n} \sum_{A \in I} \gamma_A U \gamma^A, \quad \gamma^A := (\gamma_A)^{-1}, \quad U \in \mathcal{C}_{p,q}^{\mathbb{F}}$$

where we have a sum over the multi-index $A \in I$ (3). Note that in the real case $\mathbb{F} = \mathbb{R}$, if $\{\gamma_A, A \in I\}$ is a basis of $\mathcal{C}_{p,q}$, then F is the Reynolds operator of the Salingaros vee group, see [1].

Let us denote a vector subspace spanned by the elements $e_{a_1\dots a_k}$ enumerated by the ordered multi-indices of length k by $\mathcal{C}_{p,q}^{k\mathbb{F}}$. Any Clifford algebra element $U \in \mathcal{C}_{p,q}^{\mathbb{F}}$ can be written in the form

$$U = ue + \sum_a u_a e_a + \sum_{a_1 < a_2} u_{a_1 a_2} e_{a_1 a_2} + \dots + u_{1\dots n} e_{1\dots n}, \quad u, u_a, \dots, u_{1\dots n} \in \mathbb{F}.$$

We use the projection operator (note that this operation coincides up to a constant with the trace of the corresponding matrix representation, see [21,22]) $\pi_0(U) = u$ onto the vector subspace $\mathcal{C}_{p,q}^{0\mathbb{F}}$ (where $e \equiv 1$). Additionally, we consider the following operator: $\pi_n(U) = u_{1\dots n}$ for $U \in \mathcal{C}_{p,q}^{\mathbb{F}}$.

If the set $\{\gamma_A, A \in I\}$ is linearly independent, then we have (this is proved for the real case $\mathbb{F} = \mathbb{R}$ in [1])

$$F(U) = \frac{1}{2^n} \sum_A \gamma_A U \gamma^A = \begin{cases} \pi_0(U)e, & \text{if } n \text{ is even;} \\ \pi_0(U)e + \pi_n(U)e_{1\dots n}, & \text{if } n \text{ is odd.} \end{cases} \tag{12}$$

Operator F is a projection $F^2 = F$ onto the center of $\mathcal{C}_{p,q}^{\mathbb{F}}$. Note that here and after we use notation $F^2(U) := F(F(U)) \neq (F(U))^2$ and a similar notation for the squares of all other operators in this paper. In the language of [11], $\text{Cen}(\mathcal{C}_{p,q})$ is the “ring of invariants” of the Salingaros vee group.

3. Generalized Reynolds Operators

Let us consider two different sets of Clifford algebra elements

$$\{\gamma_a, a = 1, \dots, n\} \in \mathcal{C}_{p,q}^{\mathbb{R}}, \quad \gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab}e, \tag{13}$$

$$\{\beta_a, a = 1, \dots, n\} \in \mathcal{C}_{p,q}^{\mathbb{R}}, \quad \beta_a \beta_b + \beta_b \beta_a = 2\eta_{ab}e, \tag{14}$$

and generalized Reynolds operators

$$H(U) = \frac{1}{2^n} \sum_{A \in I} \beta_A U \gamma^A, \quad P(V) = \frac{1}{2^n} \sum_{A \in I} \gamma_A V \beta^A, \quad U, V \in \mathcal{C}_{p,q}^{\mathbb{R}}, \tag{15}$$

where we have the sum over multi-index $A \in I = \{\emptyset, 1, \dots, n, 12, 13, \dots, 1 \dots n\}$ of 2^n terms, $\beta_A = \beta_{a_1 \dots a_k} := \beta_{a_1} \dots \beta_{a_k}$ for $a_1 < \dots < a_k$, $\gamma^A := (\gamma_A)^{-1}$, $\beta^A := (\beta_A)^{-1}$. We consider both operators (15) at the same time because we want to study how they are related (see Theorem 2).

Theorem 1. We have

$$\beta_B H(U) = H(U) \gamma_B, \quad \gamma_B P(V) = P(V) \beta_B, \quad \forall B \in I. \tag{16}$$

The operators H and P are projections:

$$H^2 = H, \quad P^2 = P,$$

where $H^2(U) := H(H(U))$ and $P^2(U) := P(P(U))$.

Proof. We have

$$\beta_B H(U) \gamma^B = \beta_B \frac{1}{2^n} \sum_A \beta_A U \gamma^A \gamma^B = \frac{1}{2^n} \sum_A (\beta_B \beta_A) U (\gamma_B \gamma^A) = \frac{1}{2^n} \sum_A \beta_A U \gamma^A = H(U).$$

Thus,

$$H^2(U) = \frac{1}{2^n} \sum_B \beta_B H(U) \gamma^B = \frac{1}{2^n} (2^n H(U)) = H(U).$$

Proof of the statement for the operator P is similar. \square

Theorem 2. Let us consider $\mathcal{C}_{p,q}^{\mathbb{R}}$, linearly independent sets $\{\gamma_A, A \in I\}$, $\{\beta_A, A \in I\}$, and the operators (15).

1. If $n = p + q$ is even, then

$$P(V)H(U) = H(U)P(V) = \pi_0(VH(U))e, \tag{17}$$

where

$$\pi_0(VH(U)) = \pi_0(H(U)V) = \pi_0(UP(V)) = \pi_0(P(V)U).$$

2. If $n = p + q$ is odd, then

$$P(V)H(U) = H(U)P(V) = \pi_0(VH(U))e + \pi_n(VH(U))e_{1 \dots n}, \tag{18}$$

where

$$\begin{aligned} \pi_0(VH(U)) &= \pi_0(H(U)V) = \pi_0(UP(V)) = \pi_0(P(V)U), \\ \pi_n(VH(U)) &= \pi_n(H(U)V) = \pi_n(UP(V)) = \pi_n(P(V)U). \end{aligned}$$

Proof. Using (12) and Theorem 1, we obtain

$$\begin{aligned}
 P(V)H(U) &= \frac{1}{2^n} \sum_A \gamma_A V \beta^A H(U) = \frac{1}{2^n} \sum_A \gamma_A V H(U) \gamma^A \\
 &= \begin{cases} \pi_0(VH(U))e, & \text{if } n \text{ is even;} \\ \pi_0(VH(U))e + \pi_n(VH(U))e_{1\dots n}, & \text{if } n \text{ is odd.} \end{cases}
 \end{aligned}$$

We can similarly obtain

$$\begin{aligned}
 P(V)H(U) &= \frac{1}{2^n} P(V) \sum_A \beta_A U \gamma^A = \frac{1}{2^n} \sum_A \gamma_A P(V) U \gamma^A \\
 &= \begin{cases} \pi_0(P(V)U)e, & \text{if } n \text{ is even;} \\ \pi_0(P(V)U)e + \pi_n(P(V)U)e_{1\dots n}, & \text{if } n \text{ is odd.} \end{cases}
 \end{aligned}$$

Further, we use the properties $\pi_0(AB) = \pi_0(BA)$ for arbitrary $A, B \in \mathcal{C}_{p,q}^{\mathbb{F}}$ (see [21,22]) and $\pi_n(AB) = \pi_n(BA)$ for arbitrary $A, B \in \mathcal{C}_{p,q}^{\mathbb{F}}$ with odd $n = p + q$ (see [23]). □

We use the following notation

$$I_{(0)} = \{A \in I, |A| \text{ is even}\}, \quad I_{(1)} = \{A \in I, |A| \text{ is odd}\}.$$

Note that the parity of an element γ_A is not the parity of length of the corresponding multi-index A . The same element γ_A may have multi-indices with lengths of different parity in different representations. In the case $\gamma_{1\dots n} = \pm e$, n is odd (see Lemma 1); we have 2^{n-1} pairs of coincident (up to sign) elements in the set $\mathfrak{B} = \{\gamma_A, A \in I\}$. In this case, the same element $\gamma_{1\dots n} = \pm e$ has a multi-index with even length in one representation ($\pm e$) and multi-index with odd length in another representation ($\gamma_{1\dots n}$).

To prove Theorem 3, we need the following statement.

Lemma 2. *Let us consider $\mathcal{C}_{p,q}^{\mathbb{F}}$, $p + q = n$ and the set of elements $\mathfrak{B} = \{\gamma_A, A \in I\}$ with the property (13). Then, each element of this set (if it is neither e nor $\gamma_{1\dots n}$) commutes with 2^{n-2} elements of the set \mathfrak{B} with a multi-index of even length, commutes with 2^{n-2} elements of the set \mathfrak{B} with a multi-index of odd length, anticommutes with 2^{n-2} elements of the set \mathfrak{B} with a multi-index of even length and anticommutes with 2^{n-2} elements of the set \mathfrak{B} with a multi-index of odd length. The element e commutes with all elements of the set \mathfrak{B} .*

1. *if n is even, then $\gamma_{1\dots n}$ commutes with all 2^{n-1} elements of the set \mathfrak{B} with a multi-index of even length and anticommutes with all 2^{n-1} elements of the set \mathfrak{B} with a multi-index of odd length;*
2. *if n is odd, then $\gamma_{1\dots n}$ commutes with all 2^n elements of the set \mathfrak{B} .*

Proof. In [1], we proved the particular case of this statement for the set $\{e_A, A \in I\}$. In this proof, we did not use the fact that this set is linearly independent (we use only anticommutative properties of the elements). Hence, the statement is valid for the set $\mathfrak{B} = \{\gamma_A, A \in I\}$, which is not always linearly independent (see Lemma 1). □

Theorem 3. *Let us consider $\mathcal{C}_{p,q}^{\mathbb{F}}$, $n = p + q$ and two sets (13) and (14). Then,*

$$\sum_{B \in I_{(0)}} H(\gamma_B) \gamma^B = \frac{1}{2^n} \sum_A \sum_{B \in I_{(0)}} \beta_A \gamma_B \gamma^A \gamma^B = \begin{cases} \frac{1}{2}(e + \beta_{1\dots n} \gamma^{1\dots n}), & \text{if } n \text{ is even;} \\ \frac{1}{2}(e + \beta_{1\dots n} \gamma^{1\dots n}), & \text{if } n \text{ is odd,} \end{cases} \tag{19}$$

$$\sum_{B \in I_{(1)}} H(\gamma_B) \gamma^B = \frac{1}{2^n} \sum_A \sum_{B \in I_{(1)}} \beta_A \gamma_B \gamma^A \gamma^B = \begin{cases} \frac{1}{2}(e - \beta_{1\dots n} \gamma^{1\dots n}), & \text{if } n \text{ is even;} \\ \frac{1}{2}(e + \beta_{1\dots n} \gamma^{1\dots n}), & \text{if } n \text{ is odd,} \end{cases} \tag{20}$$

$$\sum_B H(\gamma_B) \gamma^B = \frac{1}{2^n} \sum_A \sum_B \beta_A \gamma_B \gamma^A \gamma^B = \begin{cases} e, & \text{if } n \text{ is even;} \\ e + \beta_{1\dots n} \gamma^{1\dots n}, & \text{if } n \text{ is odd.} \end{cases} \tag{21}$$

Proof. Let us consider the following expressions in the case of even n (we swap all γ_B and γ^A and obtain plus or minus in each case):

$$\begin{aligned}
 \sum_A \beta_A e \gamma^A &= e + \beta_1 \gamma^1 + \dots + \beta_{1\dots n} \gamma^{1\dots n} = (e + \beta_1 \gamma^1 + \dots + \beta_{1\dots n} \gamma^{1\dots n})e, \\
 \sum_A \beta_A \gamma_1 \gamma^A &= \gamma_1 + \beta_1 \gamma_1 \gamma^1 + \dots + \beta_{1\dots n} \gamma_1 \gamma^{1\dots n} = (e + \beta_1 \gamma^1 - \dots - \beta_{1\dots n} \gamma^{1\dots n})\gamma_1, \\
 &\dots \\
 \sum_A \beta_A \gamma_{1\dots n} \gamma^A &= \gamma_{1\dots n} + \beta_1 \gamma_{1\dots n} \gamma^1 + \dots + \beta_{1\dots n} \gamma_{1\dots n} \gamma^{1\dots n} \\
 &= (e - \beta_1 \gamma^1 - \dots + \beta_{1\dots n} \gamma^{1\dots n})\gamma_{1\dots n}.
 \end{aligned}$$

Let us multiply (on the right) both sides of the first equation by e , the second equation by γ^1, \dots , the last equation by $\gamma^{1\dots n}$. Summing equalities with $B \in I_{(0)}$ (or $B \in I_{(1)}$) and using Lemma 2 (we must take into account the number of pluses and minuses), we obtain the statement of this theorem. In the case of odd n , the proof is similar. Summing (19) and (20), we obtain (21). \square

Theorem 4. Let us consider $\mathcal{C}_{p,q}^{\mathbb{R}}$, $n = p + q$, two sets (13) and (14), and the operator $H(U)$ (15).

1. If n is even, then there exists an element $U \in \{\gamma_A, A \in I\}$ such that $H(U)$ is nonzero. Moreover, we can take $U \in \{\gamma_A, A \in I_{(0)}\}$ if $\beta_{1\dots n} \neq -\gamma_{1\dots n}$ and we can take $U \in \{\gamma_A, A \in I_{(1)}\}$ if $\beta_{1\dots n} \neq \gamma_{1\dots n}$.
2. If n is odd and $\beta_{1\dots n} \neq -\gamma_{1\dots n}$, then there exists an element $U \in \{\gamma_A, A \in I\}$ such that $H(U)$ is nonzero. Moreover, we can take $U \in \{\gamma_A, A \in I_{(0)}\}$ and we can take $U \in \{\gamma_A, A \in I_{(1)}\}$ at the same time.

Proof. Let n be even. Suppose that for all elements U the operator $H(U) = \frac{1}{2^n} \sum_A \beta_A U \gamma^A$ equals zero. It follows from Theorem 3 (see (21)) that

$$2^n e = \sum_B \left(\sum_A \beta_A \gamma_B \gamma^A \right) \gamma^B = \sum_B 0 \gamma^B = 0$$

and we obtain a contradiction. Using (19) and (20), we can similarly prove the statement in the other cases. \square

We use Theorems 1, 2, and 4 to prove generalized Pauli’s theorem in Section 6.

4. Some Other Operators

Let us consider the following operators for the sets (13) and (14)

$$\begin{aligned}
 H'(U) &= \frac{1}{2^n} \left(\sum_{A \in I_{(0)}} \beta_A U \gamma^A - \sum_{A \in I_{(1)}} \beta_A U \gamma^A \right) = \sum_A \frac{(-1)^{|A|}}{2^n} \beta_A U \gamma^A, \quad U \in \mathcal{C}_{p,q}^{\mathbb{R}}, \tag{22} \\
 P'(V) &= \frac{1}{2^n} \left(\sum_{A \in I_{(0)}} \gamma_A V \beta^A - \sum_{A \in I_{(1)}} \gamma_A V \beta^A \right) = \sum_A \frac{(-1)^{|A|}}{2^n} \gamma_A V \beta^A, \quad V \in \mathcal{C}_{p,q}^{\mathbb{R}}.
 \end{aligned}$$

Theorem 5. We have

$$\begin{aligned}
 \beta_B H'(U) &= H'(U) \gamma_B, & \gamma_B P'(V) &= P'(V) \beta_B, & \forall B \in I_{(0)}, \\
 \beta_B H'(U) &= -H'(U) \gamma_B, & \gamma_B P'(V) &= -P'(V) \gamma_B, & \forall B \in I_{(1)},
 \end{aligned}$$

in particular,

$$\beta_b H'(U) = -H'(U) \gamma_b, \quad \gamma_b P'(V) = -P'(V) \beta_b, \quad \forall b = 1, \dots, n.$$

The operators H' and P' are projections:

$$H'(H'(U)) = H'(U), \quad P'(P'(V)) = P'(V).$$

Additionally, we have

$$H(H'(U)) = 0, \quad H'(H(U)) = 0, \quad P(P'(V)) = 0, \quad P'(P(V)) = 0.$$

Theorem 6. Let us consider $\mathcal{C}_{p,q}^{\mathbb{F}}$ and linearly independent sets $\{\gamma_A, A \in I\}$, $\{\beta_A, A \in I\}$.

1. If $n = p + q$ is even, then

$$H'(U)P'(V) = P'(V)H'(U) = \pi_0(H'(U)V)e,$$

where

$$\pi_0(H'(U)V) = \pi_0(VH'(U)) = \pi_0(P'(V)U) = \pi_0(UP'(V)).$$

2. If $n = p + q$ is odd, then

$$H'(U)P'(V) = \pi_0(H'(U)V)e + \pi_n(H'(U)V)e_{1\dots n},$$

where

$$\pi_0(H'(U)V) = \pi_0(VH'(U)) = \pi_0(P'(V)U) = \pi_0(UP'(V)),$$

$$\pi_n(H'(U)V) = \pi_n(VH'(U)) = \pi_n(P'(V)U) = \pi_n(UP'(V)).$$

Theorem 7. Let us consider $\mathcal{C}_{p,q}^{\mathbb{F}}$, $n = p + q$ and two sets (13) and (14). Then,

$$\sum_{B \in I_{(0)}} H'(\gamma_B)\gamma^B = \frac{1}{2^n} \sum_A \sum_{B \in I_{(0)}} (-1)^{|A|} \beta_A \gamma_B \gamma^A \gamma^B = \begin{cases} \frac{1}{2}(e + \beta_{1\dots n} \gamma^{1\dots n}), & \text{if } n \text{ is even;} \\ \frac{1}{2}(e - \beta_{1\dots n} \gamma^{1\dots n}), & \text{if } n \text{ is odd,} \end{cases}$$

$$\sum_{B \in I_{(1)}} H'(\gamma_B)\gamma^B = \frac{1}{2^n} \sum_A \sum_{B \in I_{(1)}} (-1)^{|A|} \beta_A \gamma_B \gamma^A \gamma^B = \begin{cases} \frac{1}{2}(e - \beta_{1\dots n} \gamma^{1\dots n}), & \text{if } n \text{ is even;} \\ \frac{1}{2}(e + \beta_{1\dots n} \gamma^{1\dots n}), & \text{if } n \text{ is odd,} \end{cases}$$

$$\sum_B H'(\gamma_B)\gamma^B = \sum_A \sum_B \frac{1}{2^n} (-1)^{|A|} \beta_A \gamma_B \gamma^A \gamma^B = \begin{cases} e, & \text{if } n \text{ is even;} \\ e - \beta_{1\dots n} \gamma^{1\dots n}, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 8. Let us consider $\mathcal{C}_{p,q}^{\mathbb{F}}$, $n = p + q$, two sets (13) and (14), and the operator $H'(U)$ (22).

1. If n is even, then there exists an element $U \in \{\gamma_A, A \in I\}$ such that $H'(U)$ is nonzero. Moreover, we can take $U \in \{\gamma_A, A \in I_{(0)}\}$ if $\beta_{1\dots n} \neq -\gamma_{1\dots n}$ and we can take $U \in \{\gamma_A, A \in I_{(1)}\}$ if $\beta_{1\dots n} \neq \gamma_{1\dots n}$.
2. If n is odd and $\beta_{1\dots n} \neq \gamma_{1\dots n}$, then there exists an element $U \in \{\gamma_A, A \in I\}$ such that $H'(U)$ is nonzero. Moreover, we can take $U \in \{\gamma_A, A \in I_{(0)}\}$ and we can take $U \in \{\gamma_A, A \in I_{(1)}\}$ at the same time.

Note that we have the condition $\beta_{1\dots n} \neq \gamma_{1\dots n}$ in Theorem 8 and the condition $\beta_{1\dots n} \neq -\gamma_{1\dots n}$ in Theorem 4 in the case of odd n .

Proof. We omit the proof of Theorems 5–8 because it is similar to the Proof of Theorems 1–4. \square

Let us consider the following operators for the sets (13) and (14)

$$H_{(0)}(U) = \frac{1}{2^{n-1}} \sum_{A \in I_{(0)}} \beta_A U \gamma^A, \quad H_{(1)}(U) = \frac{1}{2^{n-1}} \sum_{A \in I_{(1)}} \beta_A U \gamma^A, \quad U \in \mathcal{C}_{p,q}^{\mathbb{F}}$$

$$P_{(0)}(V) = \frac{1}{2^{n-1}} \sum_{A \in I_{(0)}} \gamma_A V \beta^A, \quad P_{(1)}(V) = \frac{1}{2^{n-1}} \sum_{A \in I_{(1)}} \gamma_A V \beta^A, \quad V \in \mathcal{C}_{p,q}^{\mathbb{F}}.$$

The corresponding operators in the particular case $\gamma_A = \beta_A = e_A, \mathbb{F} = \mathbb{R}$, are considered in [2]. We have

$$\begin{aligned}
 H(U) &= \frac{1}{2}(H_{(0)}(U) + H_{(1)}(U)), & H'(U) &= \frac{1}{2}(H_{(0)}(U) - H_{(1)}(U)), \\
 P(V) &= \frac{1}{2}(P_{(0)}(V) + P_{(1)}(V)), & P'(V) &= \frac{1}{2}(P_{(0)}(V) - P_{(1)}(V)).
 \end{aligned}$$

Theorem 9. *We have*

$$\begin{aligned}
 \beta_B H_{(0)}(U) &= H_{(0)}(U)\gamma_B, & \gamma_B P_{(0)}(V) &= P_{(0)}(V)\beta_B, & \forall B \in I_{(0)}, \\
 \beta_B H_{(0)}(U) &= H_{(1)}(U)\gamma_B, & \gamma_B P_{(0)}(V) &= P_{(1)}(V)\beta_B, & \forall B \in I_{(1)}, \\
 \beta_B H_{(1)}(U) &= H_{(1)}(U)\gamma_B, & \gamma_B P_{(1)}(V) &= P_{(1)}(V)\beta_B, & \forall B \in I_{(0)}, \\
 \beta_B H_{(1)}(U) &= H_{(0)}(U)\gamma_B, & \gamma_B P_{(1)}(V) &= P_{(0)}(V)\beta_B, & \forall B \in I_{(1)},
 \end{aligned}$$

and

$$\begin{aligned}
 H_{(0)}^2(U) &= H_{(1)}^2(U) = H_{(0)}(U), & P_{(0)}^2(V) &= P_{(1)}^2(V) = P_{(0)}(V), \\
 H_{(1)}(H_{(0)}(U)) &= H_{(0)}(H_{(1)}(U)) = H_{(1)}(U), \\
 P_{(1)}(P_{(0)}(V)) &= P_{(0)}(P_{(1)}(V)) = P_{(1)}(V), \\
 H_{(0)}(H(U)) &= H_{(1)}(H(U)) = H(H_{(0)}(U)) = H(H_{(1)}(U)) = H(U), \\
 P_{(0)}(P(V)) &= P_{(1)}(P(V)) = P(P_{(0)}(V)) = P(P_{(1)}(V)) = P(V).
 \end{aligned}$$

Proof. The proof is similar to the proof of Theorem 1. \square

Using the previous theorems, we obtain the following multiplication tables (see Tables 1 and 2) for the operators $H, H', H_{(0)}, H_{(1)}$ and the operators $P, P', P_{(0)}, P_{(1)}$.

Table 1. Multiplication table for the operators $H, H', H_{(0)}, H_{(1)}$.

	H	$H_{(0)}$	$H_{(1)}$	H'
H	H	H	H	0
$H_{(0)}$	H	$H_{(0)}$	$H_{(1)}$	H'
$H_{(1)}$	H	$H_{(1)}$	$H_{(0)}$	$-H'$
H'	0	H'	$-H'$	H'

Table 2. Multiplication table for the operators $P, P', P_{(0)}, P_{(1)}$.

	P	$P_{(0)}$	$P_{(1)}$	P'
P	P	P	P	0
$P_{(0)}$	P	$P_{(0)}$	$P_{(1)}$	P'
$P_{(1)}$	P	$P_{(1)}$	$P_{(0)}$	$-P'$
P'	0	P'	$-P'$	P'

5. Relation between Operators in the Case of Odd n

In this section, we prove that the operators considered in the previous section are related to the generalized Reynolds operators in the case of odd n .

We need the concept of adjoint multi-indices introduced in the previous work [1] of the author. We call ordered multi-indices $a_1 \dots a_k$ and $b_1 \dots b_l$ *adjoint multi-indices* if they have no common indices and they form a multi-index $1 \dots n$ of the length n . We write $b_1 \dots b_l = \widetilde{a_1 \dots a_k}$ and $a_1 \dots a_k = \widetilde{b_1 \dots b_l}$. We denote the sets of corresponding 2^{n-1} multi-indices by I_{Adj} and $\widetilde{I}_{Adj} = I \setminus I_{Adj}$. Therefore, for each multi-index in I_{Adj} , there exists an

adjoint multi-index in $\widetilde{I_{Adj}}$. We have $\mathfrak{B} = \{e_A \mid A \in I\} = \{e_A \mid A \in I_{Adj}\} \cup \{e_A \mid A \in \widetilde{I_{Adj}}\}$. For example, $I_{Adj} = I_{First}$, $\widetilde{I_{Adj}} = I \setminus I_{First} = I_{Last}$, where I_{First} consists of the first (in the order) 2^{n-1} multi-indices of the set I . In the case of odd n , we can write $I_{First} = \{A \in I, |A| \leq \frac{n-1}{2}\}$, $I_{Last} = \{A \in I, |A| \geq \frac{n+1}{2}\}$. In the case of even n , we can consider the following adjoint sets $I_{Adj} = I_{(0)}$, $\widetilde{I_{Adj}} = I_{(1)}$.

Let us consider the following operators in $C_{p,q}^{\mathbb{F}}$ with odd $n = p + q$ for the sets (13) and (14), and some $I_{Adj}, \widetilde{I_{Adj}}$:

$$H_{Adj}(U) = \frac{1}{2^{n-1}} \sum_{A \in I_{Adj}} \beta_A U \gamma^A, \quad H_{\widetilde{Adj}}(U) = \frac{1}{2^{n-1}} \sum_{A \in \widetilde{I_{Adj}}} \beta_A U \gamma^A,$$

$$H(U) = \frac{1}{2} (H_{Adj}(U) + H_{\widetilde{Adj}}(U)), \quad U \in C_{p,q}^{\mathbb{F}}.$$

Using Lemma 1, we conclude that the elements $\gamma_{1\dots n}$ and $\beta_{1\dots n}$ are equal to the elements $\pm e$ and $\pm e_{1\dots n}$ in the case of odd n and $\mathbb{F} = \mathbb{R}$, they are equal to the elements $\pm e, \pm ie$, and $\pm e_{1\dots n}$ in the case of even n and $\mathbb{F} = \mathbb{C}$. Thus, we have four (six in the complex case) different cases

$$\gamma_{1\dots n} = \pm \beta_{1\dots n}, \quad \gamma_{1\dots n} = \pm e_{1\dots n} \beta_{1\dots n}, \quad \gamma_{1\dots n} = \pm ie_{1\dots n} \beta_{1\dots n}.$$

We use notation

$$H_{First}(U) = \frac{1}{2^{n-1}} \sum_{A \in I_{First}} \beta_A U \gamma^A, \quad H_{Last}(U) = \frac{1}{2^{n-1}} \sum_{A \in I_{Last}} \beta_A U \gamma^A$$

in the following theorem.

Theorem 10. Let us consider $C_{p,q}^{\mathbb{F}}$ with odd $n = p + q$ and the sets (13) and (14).

1. If $\gamma_{1\dots n} = \beta_{1\dots n}$, then

$$H_{Adj}(U) = H(U).$$

In particular, we have

$$H(U) = H_{(0)}(U) = H_{(1)}(U) = H_{First}(U) = H_{Last}(U), \quad H'(U) = 0.$$

2. If $\gamma_{1\dots n} = -\beta_{1\dots n}$, then

$$H(U) = 0, \quad H_{Adj}(U) = -H_{\widetilde{Adj}}(U).$$

In particular, we have

$$H_{(0)}(U) = -H_{(1)}(U) = H'(U), \quad H_{First}(U) = -H_{Last}(U).$$

3. If $\gamma_{1\dots n} = e_{1\dots n} \beta_{1\dots n}$ (in the case $p - q \equiv 1 \pmod{4}$), then

$$H_{Adj}(U) = e_{1\dots n} H_{\widetilde{Adj}}(U), \quad H(U) = \frac{e + e_{1\dots n}}{2} H_{Adj}(U).$$

4. If $\gamma_{1\dots n} = -e_{1\dots n} \beta_{1\dots n}$ (in the case $p - q \equiv 1 \pmod{4}$), then

$$H_{Adj}(U) = -e_{1\dots n} H_{\widetilde{Adj}}(U), \quad H(U) = \frac{e - e_{1\dots n}}{2} H_{Adj}(U).$$

5. If $\gamma_{1\dots n} = ie_{1\dots n}\beta_{1\dots n}$ (in the case $\mathbb{F} = \mathbb{C}$ and $p - q \equiv 3 \pmod{4}$), then

$$H_{\text{Adj}}(U) = ie_{1\dots n}H_{\widetilde{\text{Adj}}}(U), \quad H(U) = \frac{e + ie_{1\dots n}}{2}H_{\text{Adj}}(U).$$

6. If $\gamma_{1\dots n} = -ie_{1\dots n}\beta_{1\dots n}$ (in the case $\mathbb{F} = \mathbb{C}$ and $p - q \equiv 3 \pmod{4}$), then

$$H_{\text{Adj}}(U) = -ie_{1\dots n}H_{\widetilde{\text{Adj}}}(U), \quad H(U) = \frac{e - ie_{1\dots n}}{2}H_{\text{Adj}}(U).$$

In particular, we have in Cases 3–6:

$$H_{\text{Adj}}(U) = H_{(0)}(U) = H_{(1)}(U) = H_{\text{First}}(U) = H_{\text{Last}}(U).$$

Proof. Using Lemma 1, we conclude that the elements $\beta_{1\dots n}$ and $\gamma_{1\dots n}$ are in the center of $\mathcal{C}_{p,q}^{\mathbb{F}}$ with odd $n = p + q$. Thus, these elements commute with all elements of $\mathcal{C}_{p,q}^{\mathbb{F}}$.

If $\beta_{1\dots n} = \gamma_{1\dots n}$, then

$$\beta_{a_1\dots a_m}U\gamma^{a_1\dots a_m} = \beta_{1\dots n}\gamma^{1\dots n}\beta_{a_1\dots a_m}U\gamma^{a_1\dots a_m} = \beta_{\widetilde{a_1\dots a_m}}U\gamma^{\widetilde{a_1\dots a_m}}.$$

Thus, we obtain $\sum_A \beta_A U\gamma^A = 2\sum_{A \in \text{I}_{\text{Adj}}} \beta_A U\gamma^A$ and the first statement of the theorem.

If $\beta_{1\dots n} = -\gamma_{1\dots n}$, then analogously $\beta_{a_1\dots a_m}U\gamma^{a_1\dots a_m} = -\beta_{\widetilde{a_1\dots a_m}}U\gamma^{\widetilde{a_1\dots a_m}}$ and

$$\sum_A \beta_A U\gamma^A = \sum_{A \in \text{I}_{\text{Adj}}} \beta_A U\gamma^A + \sum_{A \in \widetilde{\text{I}_{\text{Adj}}}} \beta_A U\gamma^A = 0.$$

If $\gamma_{1\dots n} = e_{1\dots n}\beta_{1\dots n}$, then

$$\beta_{a_1\dots a_m}U\gamma^{a_1\dots a_m} = \beta_{1\dots n}e_{1\dots n}\gamma^{1\dots n}\beta_{a_1\dots a_m}U\gamma^{a_1\dots a_m} = e_{1\dots n}\beta_{\widetilde{a_1\dots a_m}}U\gamma^{\widetilde{a_1\dots a_m}}.$$

Thus, we obtain $\sum_A \beta_A U\gamma^A = (e + e_{1\dots n})\sum_{A \in \text{I}_{\text{Adj}}} \beta_A U\gamma^A$ and the third statement of the theorem.

If $\gamma_{1\dots n} = -e_{1\dots n}\beta_{1\dots n}$, then $\beta_{a_1\dots a_m}U\gamma^{a_1\dots a_m} = -e_{1\dots n}\beta_{\widetilde{a_1\dots a_m}}U\gamma^{\widetilde{a_1\dots a_m}}$ and

$$\sum_A \beta_A U\gamma^A = (e - e_{1\dots n})\sum_{A \in \text{I}_{\text{Adj}}} \beta_A U\gamma^A.$$

If $\gamma_{1\dots n} = ie_{1\dots n}\beta_{1\dots n}$, then

$$\beta_{a_1\dots a_m}U\gamma^{a_1\dots a_m} = \beta_{1\dots n}ie_{1\dots n}\gamma^{1\dots n}\beta_{a_1\dots a_m}U\gamma^{a_1\dots a_m} = ie_{1\dots n}\beta_{\widetilde{a_1\dots a_m}}U\gamma^{\widetilde{a_1\dots a_m}}.$$

Thus, we obtain $\sum_A \beta_A U\gamma^A = (e + ie_{1\dots n})\sum_{A \in \text{I}_{\text{Adj}}} \beta_A U\gamma^A$.

If $\gamma_{1\dots n} = -ie_{1\dots n}\beta_{1\dots n}$, then $\beta_{a_1\dots a_m}U\gamma^{a_1\dots a_m} = -ie_{1\dots n}\beta_{\widetilde{a_1\dots a_m}}U\gamma^{\widetilde{a_1\dots a_m}}$ and

$$\sum_A \beta_A U\gamma^A = (e - ie_{1\dots n})\sum_{A \in \text{I}_{\text{Adj}}} \beta_A U\gamma^A.$$

This completes the proof. \square

We use Theorem 10 to prove generalized Pauli’s theorem in Section 6.

6. Using Generalized Reynolds Operators to Prove Pauli’s Theorem in Clifford Algebras

In this section, we show the application of generalized Reynolds operators to prove generalization of Pauli’s theorem [24] in the case of Clifford algebras of arbitrary dimension. We use the theorems from the previous sections of this paper.

Note the following well-known fact. The Clifford algebras $\mathcal{C}\ell_{p,q}^{\mathbb{F}}$ with even $n = p + q$ and $\mathcal{C}\ell_{p,q}$, $p - q \equiv 3 \pmod{4}$ are simple. The Clifford algebras $\mathcal{C}\ell_{p,q}$, $p - q \equiv 1 \pmod{4}$ and $\mathbb{C} \otimes \mathcal{C}\ell_{p,q}$ with odd $n = p + q$ are not simple; they are direct sums of two simple algebras. The existence (or non-existence) of an element T that connect two different sets of Clifford algebra elements that satisfy the main anticommutative conditions can be proved using representation theory. The main idea of the following statements is to present an algorithm to compute this element T . These statements have been formulated in a brief report [4] of the author without proof. In this paper, we demonstrate how we can prove them using generalized Reynolds operators.

We repeat the formulation of the theorems here (Theorems 11 and 12) for consistency of the presentation. The proof of the theorems is new and shows us the application of the method of averaging and generalized Reynolds operators.

Theorem 11 ([4]). *Let us consider $\mathcal{C}\ell_{p,q}^{\mathbb{F}}$ with even $n = p + q$. Let two sets of Clifford algebra elements $\gamma_a, \beta_a, a = 1, 2, \dots, n$ satisfy conditions*

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab}e, \quad \beta_a \beta_b + \beta_b \beta_a = 2\eta_{ab}e.$$

Then, the sets $\{\gamma_A, A \in I\}$ and $\{\beta_A, A \in I\}$ are linearly independent (in the real case $\mathbb{F} = \mathbb{R}$, they are new bases of $\mathcal{C}\ell_{p,q}$), and there exists a unique (up to multiplication by a real (respectively, complex) number) element $T \in \mathcal{C}\ell_{p,q}^{\mathbb{F}}$ such that

$$\gamma_a = T^{-1} \beta_a T, \quad \forall a = 1, \dots, n.$$

Additionally, we can obtain this element T in the following way:

$$T = H(U) = \frac{1}{2^n} \sum_A \beta_A U \gamma^A, \tag{23}$$

where U is an element

- of the set $\{\gamma_A, A \in I_{(0)}\}$ if $\beta_{1\dots n} \neq -\gamma_{1\dots n}$,
- of the set $\{\gamma_A, A \in I_{(1)}\}$ if $\beta_{1\dots n} \neq \gamma_{1\dots n}$,

such that $H(U) \neq 0$.

Proof. To obtain this statement, we use the properties of generalized Reynolds operators, namely, Theorems 1, 2, and 4. Linearly independence of the sets follows from Lemma 1. For two arbitrary elements $U, V \in \mathcal{C}\ell_{p,q}^{\mathbb{F}}$ and elements (15), we have (17) by Theorem 2. There exists U such that $H(U)$ is nonzero by Theorem 4. Further, we take element V such that $\pi_0(VH(U)) \neq 0$ (we can take V from the set of basis elements $\{e_A, A \in I\}$). Therefore, from (17), we see that $T = H(U)$ is invertible. Using Theorem 1, we obtain $\gamma_a = T^{-1} \beta_a T$, $\forall a = 1, \dots, n$. We obtain an algorithm to compute the element T (23) from Theorem 4.

Let us prove that T is unique up to multiplication by a constant. Suppose that we have two elements T_1, T_2 that satisfy $\gamma_a = T^{-1} \beta_a T$, $\forall a = 1, \dots, n$. Then, for any $a = 1, \dots, n$, we have $T_1^{-1} \beta_a T_1 = T_2^{-1} \beta_a T_2$. Let us multiply both sides of this equation on the left by T_1 and on the right by $(T_2)^{-1}$. We obtain $[T_1 T_2^{-1}, \beta_a] = 0$ for $a = 1, \dots, n$. Using $\text{Cen}(\mathcal{C}\ell_{p,q}^{\mathbb{F}}) = \mathcal{C}\ell_{p,q}^{0\mathbb{F}}$ (n is even), we obtain $T_1 = \mu T_2$, where $\mu \neq 0, \mu \in \mathbb{F}$. \square

To prove Theorem 12, we need Lemmas 3 and 4.

Lemma 3. *Let us consider $\mathcal{C}\ell_{p,q}^{\mathbb{F}}$, $n = p + q$ and the set (7).*

1. *If $n = p + q$ is even, then $\pi_0(\gamma_{a_1 \dots a_k}) = 0$, $k = 1, \dots, n$.*
2. *If $n = p + q$ is odd, then $\pi_0(\gamma_{a_1 \dots a_k}) = 0$, $k = 1, \dots, n - 1$ and*

$$\pi_0(\gamma_{1\dots n}) = \begin{cases} 0, & \text{if (8) is linearly independent;} \\ \pm 1, \pm i, & \text{if (8) is linearly dependent.} \end{cases}$$

Values $\pm i$ are possible only in the case of the complexified Clifford algebra.
 If n is odd, then $\pi_n(\gamma_{a_1 \dots a_k}) = 0$, $k = 1, \dots, n - 1$ and

$$\pi_n(\gamma_{1 \dots n}) = \begin{cases} \pm 1, & \text{if (8) is linearly independent;} \\ 0, & \text{if (8) is linearly dependent.} \end{cases}$$

Proof. For any element γ_A (except $\gamma_{1 \dots n}$ in the case of odd n and except e in the case of any n), there exists an element γ_a such that γ_A anticommutes with γ_a (if $|A|$ is even, then we can take $a \in A$; if $|A|$ is odd, then we can take $a \notin A$). We obtain

$$\pi_0(\gamma_A) = \pi_0(-\gamma_a \gamma_A (\gamma_a)^{-1}) = -\pi_0(\gamma_A)$$

and $\pi_0(\gamma_A) = 0$. Further, we use Lemma 1.

The statement for the operator π_n can be proved similarly using the following property: for any two elements U, V of $\mathcal{C}\ell_{p,q}^{\mathbb{F}}$ with odd $n = p + q$, we have $\pi_n(UV) = \pi_n(VU)$. It follows from the fact $\pi_n([U, V]) = 0$ (see [23]). \square

Lemma 4.

1. Let us consider $\mathcal{C}\ell_{p,q}^{\mathbb{F}}$ with odd $n = p + q$ such that $p - q \equiv 1 \pmod{4}$, the set (7), and elements

$$\sigma_a = e_{1 \dots n} \gamma_a, \quad a = 1, \dots, n$$

2. Let us consider $\mathbb{C} \otimes \mathcal{C}\ell_{p,q}$ with odd $n = p + q$ such that $p - q \equiv 3 \pmod{4}$, the set (7), and elements

$$\sigma_a = ie_{1 \dots n} \gamma_a, \quad a = 1, \dots, n$$

In both cases, the elements $\sigma_a, a = 1, \dots, n$ satisfy the conditions $\sigma_a \sigma_b + \sigma_b \sigma_a = 2\eta_{ab}e$. If $\{\gamma_A, A \in I\}$ is linearly independent, then $\{\sigma_A, A \in I\}$ is linearly dependent. If $\{\gamma_A, A \in I\}$ is linearly dependent, then $\{\sigma_A, A \in I\}$ is linearly independent.

Proof.

(1) Using (7), we obtain

$$\sigma_a \sigma_b + \sigma_b \sigma_a = e_{1 \dots n} \gamma_a e_{1 \dots n} \gamma_b + e_{1 \dots n} \gamma_b e_{1 \dots n} \gamma_a = 2\eta_{ab}e$$

because $e_{1 \dots n} \in \text{Cen}(\mathcal{C}\ell_{p,q}^{\mathbb{F}})$ and $(e_{1 \dots n})^2 = (-1)^{\frac{n(n-1)}{2}} (-1)^q e = e$ because $p - q \equiv 1 \pmod{4}$.

If $\gamma_{1 \dots n} = \pm e_{1 \dots n}$, then $\sigma_{1 \dots n} = \pm (e_{1 \dots n})^n \gamma_{1 \dots n} = \pm e_{1 \dots n} e_{1 \dots n} = \pm e$. If $\gamma_{1 \dots n} = \pm e$, then $\sigma_{1 \dots n} = \pm (e_{1 \dots n})^n \gamma_{1 \dots n} = \pm e_{1 \dots n} e = \pm e_{1 \dots n}$. Further, we use Lemma 1.

(2) The proof of the second statement of the lemma is similar to the proof of the first statement. We have

$$\sigma_a \sigma_b + \sigma_b \sigma_a = ie_{1 \dots n} \gamma_a ie_{1 \dots n} \gamma_b + ie_{1 \dots n} \gamma_b ie_{1 \dots n} \gamma_a = 2\eta_{ab}e,$$

because $e_{1 \dots n} \in \text{Cen}(\mathbb{C} \otimes \mathcal{C}\ell_{p,q})$ and $(e_{1 \dots n})^2 = (-1)^{\frac{n(n-1)}{2}} (-1)^q e = -e$ because $p - q \equiv 3 \pmod{4}$.

If $\gamma_{1 \dots n} = \pm e_{1 \dots n}$, then $\sigma_{1 \dots n} = \pm (ie_{1 \dots n})^n \gamma_{1 \dots n} = \pm ie_{1 \dots n} e_{1 \dots n} = \pm ie$. If $\gamma_{1 \dots n} = \pm ie$, then $\sigma_{1 \dots n} = \pm (ie_{1 \dots n})^n \gamma_{1 \dots n} = \pm ie_{1 \dots n} ie = \pm e_{1 \dots n}$. \square

Theorem 12 ([4]). Let us consider $\mathcal{C}\ell_{p,q}^{\mathbb{F}}$ with odd $n = p + q$. Suppose that two sets of Clifford algebra elements $\gamma_a, \beta_a, a = 1, 2, \dots, n$ satisfy the conditions

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab}e, \quad \beta_a \beta_b + \beta_b \beta_a = 2\eta_{ab}e.$$

Then, for $\mathcal{C}_{p,q}^{\mathbb{F}}$ of signature $p - q \equiv 1 \pmod{4}$, the element $\gamma_{1\dots n}$ either takes the values $\pm e_{1\dots n}$ and the set $\{\gamma_A, A \in \mathbb{I}\}$ is linearly independent or takes the values $\pm e$ and then the set is linearly dependent. The same is for the set $\{\beta_A, A \in \mathbb{I}\}$. We have cases 1–4 below.

For $\mathcal{C}_{p,q}^{\mathbb{F}}$ of signature $p - q \equiv 3 \pmod{4}$, the element $\gamma_{1\dots n}$ either takes the values $\pm e_{1\dots n}$ and the set $\{\gamma_A, A \in \mathbb{I}\}$ is linearly independent or takes the values $\pm ie$ (this is possible only in the case $\mathbb{F} = \mathbb{C}$) and then the set is linearly dependent. This is the same for the set $\{\beta_A, A \in \mathbb{I}\}$. We have cases 1–2, 5–6 below.

There exists a unique element T of $\mathcal{C}_{p,q}^{\mathbb{F}}$ (up to multiplication by an invertible element of the center of $\mathcal{C}_{p,q}^{\mathbb{F}}$) such that

1. $\gamma_a = T^{-1}\beta_a T, \quad \forall a = 1, \dots, n \quad \Leftrightarrow \beta_{1\dots n} = \gamma_{1\dots n};$
2. $\gamma_a = -T^{-1}\beta_a T, \quad \forall a = 1, \dots, n \quad \Leftrightarrow \beta_{1\dots n} = -\gamma_{1\dots n};$
3. $\gamma_a = e_{1\dots n}T^{-1}\beta_a T, \quad \forall a = 1, \dots, n \quad \Leftrightarrow \beta_{1\dots n} = e_{1\dots n}\gamma_{1\dots n};$
4. $\gamma_a = -e_{1\dots n}T^{-1}\beta_a T, \quad \forall a = 1, \dots, n \quad \Leftrightarrow \beta_{1\dots n} = -e_{1\dots n}\gamma_{1\dots n};$
5. $\gamma_a = ie_{1\dots n}T^{-1}\beta_a T, \quad \forall a = 1, \dots, n \quad \Leftrightarrow \beta_{1\dots n} = ie_{1\dots n}\gamma_{1\dots n};$
6. $\gamma_a = -ie_{1\dots n}T^{-1}\beta_a T, \quad \forall a = 1, \dots, n \quad \Leftrightarrow \beta_{1\dots n} = -ie_{1\dots n}\gamma_{1\dots n}$

For all cases, we have $\gamma_a = (\beta_{1\dots n}\gamma^{1\dots n})T^{-1}\gamma_a T, a = 1, \dots, n.$

Additionally, in the case of the real Clifford algebra $\mathcal{C}_{p,q}$ of signature $p - q \equiv 1 \pmod{4}$ and the complexified Clifford algebra $\mathbb{C} \otimes \mathcal{C}_{p,q}$ of arbitrary signature, the element T , whose existence is stated in cases 1–6 of the theorem, equals

$$T = H_{(0)}(U) = \frac{1}{2^{n-1}} \sum_{A \in \mathbb{I}_{(0)}} \beta_A U \gamma_A^{-1} \tag{24}$$

where U is an element of the set $\{\gamma_A + \gamma_B, A, B \in \mathbb{I}_{(0)}\}.$

In the case of the real Clifford algebra $\mathcal{C}_{p,q}$ of signature $p - q \equiv 3 \pmod{4}$, the element T , whose existence is stated in cases 1 and 2 of the theorem, equals $T = H_{(0)}(U)$, where U is element of the set $\{\gamma_A, A \in \mathbb{I}_{(0)}\}$ such that $H_{(0)}(U) \neq 0.$

Proof. Linearly independence (or linearly dependence) of the sets follows from Lemma 1. This implies that we have four cases in $\mathcal{C}_{p,q}$ and six cases in $\mathbb{C} \otimes \mathcal{C}_{p,q}.$

Cases 3–6 of the theorem are reduced to cases 1 and 2 by Lemma 4. We must change one of the given sets by the set σ_a in these cases. Case 2 of the theorem (when we have $\beta_{1\dots n} = -\gamma_{1\dots n}$) is reduced to case 1. We must consider the set $\sigma_a = -\beta_a$ for $a = 1, \dots, n.$ For this set, we have $\sigma_{1\dots n} = (-1)^n \beta_{1\dots n} = -\beta_{1\dots n} = \gamma_{1\dots n}$ and obtain $\gamma_a = T^{-1}\sigma_a T = -T^{-1}\beta_a T.$

Thus, we will consider and prove only case 1 of the theorem (when $\beta_{1\dots n} = \gamma_{1\dots n}).$ We will consider only the case $\beta_{1\dots n} = \gamma_{1\dots n} = \pm e_{1\dots n}$ (other cases are reduced to this case by Lemma 4).

Let us consider arbitrary elements $U, V \in \mathcal{C}_{p,q}^{\mathbb{F}}$ and expressions (15). Then, we have (16) and (18) by Theorems 1 and 2. We must prove that there exist elements $H(U)$ and V such that $\pi_0(VH(U))e + \pi_n(VH(U))e_{1\dots n}$ is invertible. Then, from (18), we will see that $T = H(U)$ is invertible, and from (16), we will obtain $\gamma_a = T^{-1}\beta_a T.$ We have

$$\begin{aligned} & (\pi_0(VH(U))e + \pi_n(VH(U))e_{1\dots n})(\pi_0(VH(U))e - \pi_n(VH(U))e_{1\dots n}) \\ &= (\pi_0^2(VH(U)) - \pi_n^2(VH(U))(-1)^{\frac{n(n-1)}{2}}(-1)^q)e = (\pi_0^2(VH(U)) \pm \pi_n^2(VH(U)))e, \end{aligned}$$

where sign “+” is in the case $p - q \equiv 3 \pmod{4}$ and sign “−” is in the case $p - q \equiv 1 \pmod{4}.$

- (I) Let us consider the case $p - q \equiv 1 \pmod{4}.$ We must choose elements $H(U)$ and V such that $\pi_0(H(U)V) \neq \pm \pi_n(H(U)V).$ By Theorem 4, there exists the element $U \in \{\gamma_B, B \in \mathbb{I}_{(0)}\}$ such that $H(U) \neq 0.$ Since $\{\gamma_A, A \in \mathbb{I}\}$ is a basis, $H(U)$ can be written in the form $H(U) = \sum_A h_A \gamma_A, h_A \in \mathbb{F}.$

If there exists a multi-index C such that $h_C \neq 0$ and $h_{\tilde{C}} \neq \pm h_C$ (where \tilde{C} and C are adjoint multi-indices, see the previous section and [1]), then we can take $V = \gamma_C$. Using Lemma 3, we obtain

$$\pi_0(H(U)V) = h_C \neq \pm h_{\tilde{C}} = \pm \pi_n(H(U)V).$$

If there is no such index C , then $H(U)$ can be represented in the form

$$H(U) = \sum_{j=1}^k \lambda_j \gamma_{A_j} (e + e_{1\dots n}) + \sum_{j=1}^l \mu_j \gamma_{B_j} (e - e_{1\dots n}), \quad \lambda_j, \mu_j \neq 0, \tag{25}$$

where all multi-indices A_j and B_j are different and any two of them do not constitute the full multi-index $1 \dots n$.

Using Theorem 3, we obtain $\sum_{B \in I_{(0)}} H(\gamma_B) \gamma^B = e$. We have at least one $U \in \{\gamma_B, B \in I_{(0)}\}$ such that $k \neq 0$ and at least one $U \in \{\gamma_B, B \in I_{(0)}\}$ such that $l \neq 0$. Therefore, there exists the element $U \in \{\gamma_A + \gamma_B, A \neq B, A, B \in I_{(0)}\}$ such that $k, l \neq 0$ in (25). Taking (25) and the element

$$V = \sum_{j=1}^k \frac{1}{\lambda_j} \gamma^{A_j} + \sum_{j=1}^l \frac{1}{\mu_j} \gamma^{B_j},$$

we obtain

$$\begin{aligned} H(U)V &= k(e + e_{1\dots n}) + \left(\sum_{i,j=1, i \neq j}^k \frac{\lambda_j}{\lambda_i} \gamma_{A_j} \gamma^{A_i} + \sum_{j=1}^k \sum_{i=1}^l \frac{\lambda_j}{\mu_i} \gamma_{A_j} \gamma^{B_i} \right) (e + e_{1\dots n}) \\ &+ l(e - e_{1\dots n}) + \left(\sum_{j=1}^l \sum_{i=1}^k \frac{\mu_j}{\lambda_i} \gamma_{B_j} \gamma^{A_i} + \sum_{i,j=1, i \neq j}^l \frac{\mu_j}{\mu_i} \gamma_{B_j} \gamma^{B_i} \right) (e - e_{1\dots n}), \end{aligned}$$

and $k + l = \pi_0(H(U)V) \neq \pm \pi_n(H(U)V) = \pm(k - l)$.

(II) Let us consider the case $p - q \equiv 3 \pmod{4}$. We must choose elements $H(U)$ and V such that $\pi_0^2(VH(U)) + \pi_n^2(VH(U)) \neq 0$. By Theorem 4, we always have element $U \in \{\gamma_A, A \in I_{(0)}\}$ such that $H(U) \neq 0$. We can always take element $V \in \{e_A, A \in I\}$ such that $\pi_0(VH(U)) \neq 0$ or $\pi_n(VH(U)) \neq 0$. In the case of the real Clifford algebra, the theorem is proved.

In the case of the complexified Clifford algebra, we must choose elements $H(U)$ and V such that $\pi_0(H(U)V) \neq \pm i \pi_n(H(U)V)$. Further proof is similar to the proof of the case $p - q \equiv 1 \pmod{4}$, but we consider the elements

$$H(U) = \sum_{j=1}^k \lambda_j \gamma_{A_j} (e + ie_{1\dots n}) + \sum_{j=1}^l \mu_j \gamma_{B_j} (e - ie_{1\dots n}), \quad \lambda_j, \mu_j \neq 0 \tag{26}$$

instead of the elements (25).

Proof of uniqueness of element $T = H(U)$ up to multiplication by an invertible Clifford algebra element is similar to the proof of uniqueness in the case of even n .

According to the proof above, we can find element T in different cases in the following form (up to multiplication by a nonzero constant):

1. $\beta_{1\dots n} = \gamma_{1\dots n} \Rightarrow T = \sum_A \beta_A U \gamma^A,$
2. $\beta_{1\dots n} = -\gamma_{1\dots n} \Rightarrow T = \sum_A (-1)^{|A|} \beta_A U \gamma^A,$
3. $\beta_{1\dots n} = e_{1\dots n} \gamma_{1\dots n} \Rightarrow T = \sum_{A \in I_{(0)}} \beta_A U \gamma^A + e_{1\dots n} \sum_{A \in I_{(1)}} \beta_A U \gamma^A,$
4. $\beta_{1\dots n} = -e_{1\dots n} \gamma_{1\dots n} \Rightarrow T = \sum_{A \in I_{(0)}} \beta_A U \gamma^A - e_{1\dots n} \sum_{A \in I_{(1)}} \beta_A U \gamma^A,$
5. $\beta_{1\dots n} = ie_{1\dots n} \gamma_{1\dots n} \Rightarrow T = \sum_{A \in I_{(0)}} \beta_A U \gamma^A + ie_{1\dots n} \sum_{A \in I_{(1)}} \beta_A U \gamma^A,$
6. $\beta_{1\dots n} = -ie_{1\dots n} \gamma_{1\dots n} \Rightarrow T = \sum_{A \in I_{(0)}} \beta_A U \gamma^A - ie_{1\dots n} \sum_{A \in I_{(1)}} \beta_A U \gamma^A.$

Using Theorem 10, we conclude that all these elements T equal (24) up to multiplication by a nonzero constant. \square

7. Discussion

In this paper, we develop the method of averaging in Clifford algebra suggested by the author in [1,2]. We consider specific operators (generalized Reynolds operators) in Clifford algebras and study their properties (see Sections 3–5). These operators allowed us to deduce an algorithm to compute elements that connect two different sets of Clifford algebra elements that satisfy the main anticommutative condition of Clifford algebra (see Section 6).

The results obtained in this paper are used in geometry, physics, engineering, and other applications. We use the results in the study of n -dimensional Weyl, Majorana, and Majorana–Weyl spinors [25] and in the theory of spin groups [26,27]. Using the algorithm from this paper, the method of calculating of elements of spin groups is presented. Some modification of this algorithm using zero divisors is discussed in [28,29] by other authors. The results of this paper are also used in problems related to the Dirac equation and spinors [30–32], the Higgs model [33], and applications of Riemannian geometry [34]. Note the other possible applications of the results in geometry, engineering, physics, and analysis [35–38]. We use generalized Reynolds operators in the proof of the local Pauli’s theorem [39] when both sets of Clifford algebra elements depend on the point of Euclidean space. These operators can be used to study the same question and other problems for the Dirac and the Yang–Mills equations on non-trivial curved manifolds. The technique developed in this paper can be generalized to the case of other algebraic (for example, to matrix algebras using the Cartan–Bott 8-periodicity) and geometric structures (the Atiyah–Kähler algebras [40–45] and the algebra of h-forms [46–48], which are geometric generalizations of the Clifford algebras).

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