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Fuzzy Metrics in Terms of Fuzzy Relations

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Abstract: In this paper, we study the concept of fuzzy metrics from the perspective of fuzzy relations. Specifically, we analyze the commonly used definitions of fuzzy metrics. We begin by noting that crisp metrics can be uniquely characterized by linear order relations. Further, we explore the criteria that crisp relations must satisfy in order to determine a crisp metric. Subsequently, we extend these conditions to obtain a fuzzy metric and investigate the additional axioms involved. Additionally, we introduce the definition of an extensional fuzzy metric or E-d-metric, which is a fuzzification of the expression $d(x,y) = t$. Thus, we examine fuzzy metrics from both the linear order and from the equivalence relation perspectives, where one argument is a value $d(x,y)$ and the other is a number within the range $[0,+\infty)$.

Keywords: metric; order relation; fuzzy metric; fuzzy relation; fuzzy order; fuzzy equivalence; extensional fuzzy metric

MSC: 03E72; 54E35

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1. Introduction

Since L.A. Zadeh introduced fuzzy sets in 1965 [1], researchers have been actively exploring ways to integrate traditional mathematical concepts and theories into the fuzzy sets context. Among the pioneering successes in this endeavor were the development of fuzzy topologies by C.L. Chang [2], the introduction of fuzzy algebraic structures by A. Rosenfeld [3], fuzzy category theory by A. Šostak [4], etc. Fuzzy sets have also been widely used for practical applications that involve uncertainty, vagueness, and imprecision. They have already proven their efficiency in natural language processing, decision-making, pattern recognition, and optimization problems. As the potential applications of fuzzy metrics in real-world problem solving became evident, the idea of establishing a fuzzy counterpart to a metric space gained traction. Several researchers took on this challenge, and notable contributions to the field of fuzzy metrics were made by I. Kramosil and J. Michalek [5], A. George and P. Veeramani [6], Z. Deng [7], and O. Kaleva and S. Seikkala [8]. It is worth mentioning that each of these researchers used different initial prerequisites in their approach, which adds diversity to the developments in the field. These advancements in fuzzy metrics open up new avenues for studying and addressing complex real-world issues through the flexible and adaptable nature of fuzzy sets and metrics. As research in this area continues to progress, we can anticipate even more valuable applications and insights into a wide range of problems. Currently, there is growing interest in exploring the topological properties of fuzzy metrics, as this line of study holds promise not only for theoretical constructions but also for fixed-point theorems and various practical applications. Regarding the investigation of the topological properties of classical fuzzy metrics, extensive references can be found in [9–21]. Although fuzzy metrics have demonstrated successful applications in image processing problems [22–24], their full potential remains untapped. These metrics hold significant promise, particularly in addressing segmentation, spectralization, and compression problems. Furthermore, fuzzy metrics have showcased their capability in tackling optimization problems [25]. As research in this area continues to progress, it is likely that we will witness even more innovative applications and fruitful outcomes from the study of fuzzy metrics and their properties. Continuing the exploration of fuzzy metrics from the perspective of fuzzy relations opens up exciting avenues for enhancing our theoretical understanding of their properties. By delving into the fuzzy relations aspect, we can establish insightful connections between various structures and gain deeper insights into the fundamental nature of fuzzy metrics. Fuzzy relations provide a powerful framework to analyze the relationships and interactions within fuzzy metrics, shedding light on the underlying mathematical intricacies. This approach not only offers a fresh lens to examine existing fuzzy metrics but also allows us to uncover hidden patterns and unveil novel properties. Moreover, investigating fuzzy metrics from the fuzzy relations point of view paves the way for constructing new examples. By leveraging the inherent flexibility of fuzzy relations, we can create innovative fuzzy metric spaces that possess specific properties tailored to tackle real-world problems. The synergy between fuzzy metrics and fuzzy relations also holds promise for cross-disciplinary applications. It enables us to leverage a wealth of knowledge from different fields and merge their insights to address complex challenges more comprehensively. In conclusion, venturing into the study of fuzzy metrics through the lens of fuzzy relations not only enriches our theoretical understanding of these structures but also opens up a vast landscape of possibilities for practical applications.

In the current literature, the concept of a fuzzy metric is predominantly based on the axioms introduced in [6,9], which are essentially a reformulation of the original axioms defined in [5]. In [5], the idea of defining a fuzzy metric stems from the assumption that the evaluated value $d(x,y)$ of a crisp metric d to be fuzzified or approximated is smaller than a predetermined real number t. In other words, the statement $d(x,y) < t$ is fuzzified. This paper aims to justify this fact. It is crucial to take into account this fact when working with applications.

Thus, the main idea of the paper is to show how the notion of a fuzzy metric arises from the crisp order relation R, by demonstrating that every metric d can be determined by an order relation R. Subsequently, we investigate the criteria that crisp relations must satisfy in order to establish a crisp metric. Furthermore, we fuzzify the axioms of R to obtain a fuzzy metric and examine the conditions that a fuzzy relation R must satisfy in order to be considered a fuzzy metric. Finally, we introduce a different approach to the fuzzy metric concept, where we extend a crisp metric d on a set X by means of a fuzzy equivalence relation E on the set $\mathbb{R}^+$. We call it an E-d metric or an extensional fuzzy metric.

The paper is structured as follows. In Section 2, we provide a recap of the key results and concepts that are used in the paper. Specifically, we introduce and discuss triangular norms and fuzzy relations. Section 3 is devoted to the examination of classical metrics and their representation by means of linear order relations. The primary objective of the paper is addressed in Section 4. Here, we analyze the existing definition of fuzzy metrics and propose a method for its construction, employing fuzzy order relations. Extensional fuzzy metrics are explored in Section 5. Finally, in Section 6, we conclude the paper.

2. Preliminaries

2.1. Triangular Norms

We start with the definition of a t-norm, which plays a crucial role in the definition of transitivity for fuzzy relations:

Definition 1 ([26])**.** *A triangular norm (t-norm for short) is a binary operation T on the unit interval $[0,1]$, i.e., a function $T : [0,1]^2 \to [0,1]$, such that for all $a,b,c \in [0,1]$, the following four axioms are satisfied:*

- $T(a,b) = T(b,a)$ *(commutativity);*
- $T(a,T(b,c)) = T(T(a,b),c)$ *(associativity);*
- $T(a,b) \leq T(a,c)$ *whenever* $b \leq c$ *(monotonicity);*

- $T(a,1) = a$ *(a boundary condition).*

Some of the commonly used t-norms are mentioned below:

- $T_M(a,b) = \min(a,b)$, the minimum t-norm;
- $T_P(a,b) = a \cdot b$, the product t-norm;
- $T_L(a,b) = \max(a+b-1,0)$, the Łukasiewicz t-norm;
- $T_H(a,b) = \begin{cases} \frac{a \cdot b}{a+b-a\cdot b} & \text{if } a^2+b^2 \neq 0 \\ 0 & \text{otherwise} \end{cases}$, the Hamacher t-norm.

A t-norm T is called Archimedean if and only if, for all pairs $(a,b) \in (0,1)^2$, there is $n \in \mathbb{N}$ such that $T^n(a) < b$, where $T^n(a)$ is defined by induction: $T^1(a) = T(a,a), T^2(a) = T(a, T^1(a)), \ldots, T^n(a) = T(a, T^{n-1}(a))$.

Product, Łukasiewicz, and Hamacher t-norms are Archimedean, while minimum t-norm is not.

2.2. Fuzzy Relations

We continue with an overview of basic definitions and results of fuzzy relations. Definitions of a fuzzy order relation and a fuzzy equivalence relation were first introduced by L.A. Zadeh in 1971 [27] under the names of a fuzzy ordering and a fuzzy similarity relation. In our paper, we use results of a fuzzy order defined with respect to a fuzzy equivalence relation studied in [28,29].

Definition 2 ([27])**.** *A fuzzy binary relation R on a set S is a mapping $R: S \times S \to [0,1]$.*

Definition 3 (see, e.g., [28])**.** *A fuzzy binary relation E on a set S is called a fuzzy equivalence relation with respect to a t-norm T (or a T-equivalence) if and only if the following three axioms are fulfilled for all $a,b,c \in S$:*

1. $E(a,a) = 1$ *reflexivity;*
2. $E(a,b) = E(b,a)$ *symmetry;*
3. $T(E(a,b), E(b,c)) \leq E(a,c)$ *T-transitivity.*

Definition 4 ([29])**.** *A fuzzy binary relation L on a set S is called a fuzzy order relation with respect to a t-norm T and a T-equivalence E (or T-E-order) if and only if the following three axioms are fulfilled for all $a,b,c \in S$:*

1. $L(a,b) \geq E(a,b)$ *E-reflexivity;*
2. $T(L(a,b), L(b,c)) \leq L(a,c)$ *T-transitivity;*
3. $T(L(a,b), L(b,a)) \leq E(a,b)$ *T-E-antisymmetry.*

A fuzzy order relation L is called strongly linear if and only if for all $a,b \in S$:

$$\max(L(a,b), L(b,a)) = 1.$$

The following theorem states that strongly linear fuzzy order relations are uniquely characterized as fuzzifications of crisp linear orders. Preliminary, let us recall the definition of compactability:

Definition 5 ([29])**.** *Let $\preceq$ be a crisp order on a set S, and let E be a fuzzy equivalence relation on S. E is called compatible with $\preceq$ if and only if the following implication holds for all $a,b,c \in S$: $a \preceq b \preceq c \Rightarrow E(a,c) \leq E(b,c)$ and $E(a,c) \leq E(a,b)$.*

Theorem 1 ([29])**.** *Let L be a binary fuzzy relation on S, and let E be a T-equivalence on S. Then, the following two statements are equivalent:*

1. *L is a strongly linear T-E-order on S;*

2. *There is a linear order $\preceq$ with which the relation E is compatible, such that L can be represented as follows:*
$$L(a,b) = \begin{cases} 1, & \text{if } a \preceq b \\ E(a,b), & \text{otherwise.} \end{cases}$$

3. Crisp Metrics

The concepts of a metric and a metric space, first introduced by M. Fréchet in 1906 [30], now belong to the most fundamental concepts of modern mathematics. For convenience of presentation, we recall them in the next definition:

Definition 6. *Metric space is an ordered pair (X,d), where X is a set and d is a metric on X, i.e., a function $d : X \times X \to [0,\infty)$, satisfying the following axioms for all points $x,y,z \in X$:*

1. $d(x,y) = 0$ *if and only if* $x = y$;
2. $d(x,y) = d(y,x)$;
3. $d(x,y) + d(y,z) \geq d(x,z)$.

As the next theorem shows, metric spaces (X,d) are fully characterized by pairs (X,R_d), where $R_d : X \times X \times [0,\infty) \to \{0,1\}$ and $R_d(x,y,t) = 1$ if and only if $d(x,y) < t$.

Theorem 2. *A metric d on a set X is uniquely determined by the following function: $R_d : X \times X \times [0,\infty) \to \{0,1\}$, where for all $x,y \in X$ and $t \in [0,\infty)$, $R_d(x,y,t) = 1$ if and only if $d(x,y) < t$.*

Proof. Let us prove that two metrics d_1 and d_2 differ if and only if $R_{d_1} \neq R_{d_2}$, where $R_{d_1}(x,y,t) = 1$ if and only if $d_1(x,y) < t$, and $R_{d_2}(x,y,t) = 1$ if and only if $d_2(x,y) < t$. Thus, $d_1 \neq d_2$ if and only if there exist $x,y \in X$, such that $d_1(x,y) \neq d_2(x,y)$. The last one is fulfilled if and only if there exist $x,y \in X$, such that $d_1(x,y) < d_2(x,y)$ or $d_2(x,y) < d_1(x,y)$. If $d_1(x,y) < d_2(x,y)$ than $R_{d_1}(x,y,d_2(x,y)) = 1$ and $R_{d_2}(x,y,d_2(x,y)) = 0$. If $d_2(x,y) < d_1(x,y)$, then $R_{d_2}(x,y,d_1(x,y)) = 1$ and $R_{d_1}(x,y,d_1(x,y)) = 0$; thus, $R_{d_1} \neq R_{d_2}$.

Further, if $R_{d_1} \neq R_{d_2}$, then there exist $x,y \in X$ and $t \in [0,\infty)$ such that $R_{d_1}(x,y,t) = 1$ and $R_{d_2}(x,y,t) = 0$ or $R_{d_1}(x,y,t) = 0$ and $R_{d_2}(x,y,t) = 1$. If $R_{d_1}(x,y,t) = 1$ and $R_{d_2}(x,y,t) = 0$, then $d_1(x,y) < t$ and $d_2(x,y) \geq t$; thus, $d_1 \neq d_2$. □

Now we investigate how to define $R_d : X \times X \times [0,\infty) \to \{0,1\}$ in order to reflect axioms from Definition 6:

1. If $R_d(x,y,t) = 1$ if and only if $d(x,y) < t$, then, if $t = 0$, $d(x,y) < 0$ cannot be fulfilled for any $x,y \in X$ and $R(x,y,0) = 0$. However, we still want to invent an axiom for R_d that is equivalent to the axiom $d(x,y) = 0$ if and only if $x = y$. The axiom is:
$$R_d(x,y,t) = 1 \ \forall t > 0 \iff x = y. \tag{1}$$
Let us prove that this axiom is equivalent to the axiom $d(x,y) = 0 \iff x = y$:
If $d(x,y) = 0$, then obviously $R_d(x,y,t) = 1$ for all $t > 0$, and from (1), it follows that $x = y$. If $x = y$, then from (1), $R_d(x,y,t) = 1$ for all $t > 0$, which means $d(x,y) < t$ for all $t > 0$, and then $d(x,y) = 0$.
Let us prove the opposite. If $R_d(x,y,t) = 1$ for all $t > 0$, then $d(x,y) < t$ for all $t > 0$, which implies $d(x,y) = 0$ and, finally, $x = y$. The opposite direction is also fulfilled.
2. $$R_d(x,y,t) = R_d(y,x,t) \tag{2}$$
It is obvious that condition (2) is equivalent to axiom (2) from Definition 6:
$$R_d(x,y,t) = 1 \iff d(x,y) < t \iff d(y,x) < t \iff R_d(y,x,t) = 1.$$
3. $$R_d(x,y,t) \wedge R_d(y,z,s) \leq R_d(x,z,t+s). \tag{3}$$

Inequality (3) comes from the assertion:

$$d(x,y) < t \;\&\; d(y,z) < s \Longrightarrow d(x,z) \le d(x,y) + d(y,z) < t+s.$$

Now, we prove that (3) is equivalent to axiom (3) from Definition 6. If $R_d(x,y,t) = 0$ or $R_d(y,z,s) = 0$, then obviously (3) is fulfilled. If $R_d(x,y,t) = 1$ and $R_d(y,z,s) = 1$, then $d(x,y) < t$ and $d(y,z) < s$, which means $d(x,z) \le d(x,y) + d(y,z) < t+s$ and, finally, $R_d(x,z,t+s) = 1$. Now we prove that $d(x,y) + d(y,z) \ge d(x,z)$ if (3) is fulfilled. Assuming that $d(x,y) + d(y,z) < d(x,z)$, then there exist $t,s \in [0,\infty)$ such that $d(x,y) + d(y,z) < t+s < d(x,z)$ and $d(x,y) < t \;\&\; d(y,z) < s$. Thus, $R_d(x,y,t) = 1$ and $R_d(y,z,s) = 1$, but $R_d(x,z,t+s) = 0$.

Thus, axioms (1)–(3) from Definition 6 are equivalent to the following axioms for function $R_d : X \times X \times [0,\infty) \to \{0,1\}$, such that $R_d(x,y,t) = 1$ if and only if $d(x,y) < t$ for all $x,y \in X$ and $t,s \in [0,\infty)$:

1. $R_d(x,y,t) = 1 \; \forall t > 0 \iff x = y$;
2. $R_d(x,y,t) = R_d(y,x,t)$;
3. $R_d(x,y,t) \wedge R_d(y,z,s) \le R_d(x,z,t+s)$.

The question is whether a metric d on a set X is uniquely determined by a function $R : X \times X \times (0,\infty) \to \{0,1\}$, satisfying for all $x,y \in X$ and $t \in [0,\infty)$ the three above-mentioned conditions.

It is clear that function R, which satisfies the three above-mentioned conditions, is non-decreasing with respect to the third argument:

$$R(x,y,t) \wedge R(y,y,s) = R(x,y,t) \wedge 1 = R(x,y,t) \le R(x,y,t+s)$$

for all $t,s > 0$. That means that, for the fixed x,y, the value $R(x,y,t) = 0$ when t is less than or equal to /less than some λ and $R(x,y,t) = 1$ otherwise. Then, we can define a metric $d : X \times X \to [0,\infty)$ as $d(x,y) = \inf\{t : R(x,y,t) = 1\}$. The only thing to take into account is that two functions could define the same metric (if R differs for fixed x,y only in one point); thus, we ask function R to be left-semicontinuous to be in accordance with the condition $R(x,y,t) = 1 \Longrightarrow d(x,y) < t$. Note that the metric $d : X \times X \to [0,\infty)$ can also be defined as $d(x,y) = \sup\{t : R(x,y,t) = 0\}$, which is equal to $d(x,y) = \max\{t : R(x,y,t) = 0\}$, since R is left-semicontinuous.

Thus, we have the following theorem:

Theorem 3. *A metric d on a set X is uniquely determined by a function $R : X \times X \times (0,\infty) \to \{0,1\}$, which is left-semicontinuous with respect to the third argument and for which the following conditions are fulfilled for all $x,y \in X$ and $t,s \in (0,\infty)$:*

1. $R(x,y,t) = 1 \; \forall t > 0 \iff x = y$;
2. $R(x,y,t) = R(y,x,t)$;
3. $R(x,y,t) \wedge R(y,z,s) \le R(x,z,t+s)$.

Proof. Let $R : X \times X \times (0,\infty) \to \{0,1\}$ be a function satisfying conditions (1)–(3). Taking into account condition (3), the function R is non-decreasing by the third argument, and taking into account the left-semicontinuity of R, R can be illustrated by Figure 1:

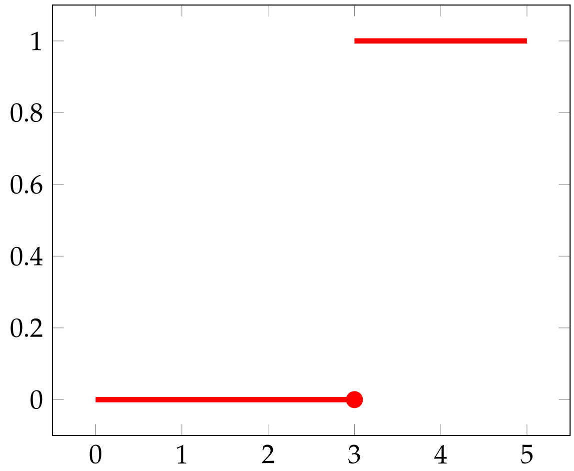

Figure 1. A figure demonstrating the function values $R(x,y,t)$, where x,y are fixed and t is changing from 0 to infinity.

Then we can build a function $d : X \times X \to [0,\infty)$ as $d(x,y) = \inf\{t : R(x,y,t) = 1\}$. It is obvious that, if $R_1 \neq R_2$, then $d_1 \neq d_2$, where $d_1(x,y) = \inf\{t : R_1(x,y,t) = 1\}$ and $d_2(x,y) = \inf\{t : R_2(x,y,t) = 1\}$. Now, we prove that d defined as $d(x,y) = \inf\{t : R(x,y,t) = 1\}$ satisfies the axioms from Definition 6:

1. If $x = y$, then $R(x,y,t) = 1$ for all $t > 0$ and $d(x,y) = \inf\{t : R(x,y,t) = 1\} = 0$. On the other hand, if $d(x,y) = 0$, then $\inf\{t : R(x,y,t) = 1\} = 0$. Thus, $R(x,y,t) = 1$ for all $t > 0$, which means $x = y$.
2. $d(x,y) = \inf\{t : R(x,y,t) = 1\} = \inf\{t : R(y,x,t) = 1\} = d(y,x)$, since $R(x,y,t) = R(y,x,t)$.
3. Now we prove that $d(x,y) + d(y,z) \geq d(x,z)$. Indeed, if $d(x,y) + d(y,z) < d(x,z)$, then there exist $t,s \in (0,\infty)$, such that $d(x,y) + d(y,z) < t + s < d(x,z)$ and $d(x,y) < t$ & $d(y,z) < s$, where $d(x,y) = \inf\{t_1 : R(x,y,t_1) = 1\} < t$ and $d(y,z) = \inf\{t_2 : R(y,z,t_2) = 1\} < s$. Thus, $R(x,y,t) = 1$ and $R(y,z,s) = 1$, but $R(x,y,t) \wedge R(y,z,s) \leq R(x,z,t+s)$, and thus $R(x,z,t+s) = 1$. However, this leads to a contradiction with $R(x,z,t+s) = 0$, which is fulfilled since $t + s < d(x,z) = \inf\{t_3 : R(x,z,t_3) = 1\}$.

Now let $d : X \times X \to [0,\infty)$ be a metric; we define $R : X \times X \times (0,\infty) \to \{0,1\}$, where for all $x,y \in X$ and $t \in (0,\infty)$, $R(x,y,t) = 1$ if and only if $d(x,y) < t$. It has already been shown that, if $d_1 \neq d_2$, then $R_1 \neq R_2$. Further, let us prove that, for the defined function R, conditions (1)–(3) are fulfilled:

1. If $R(x,y,t) = 1$ for all $t > 0$, then $d(x,y) < t$ for all $t > 0$, which implies $d(x,y) = 0$ and, finally, $x = y$. If $x = y$, then $d(x,y) = 0$ and $d(x,y) < t$ for all $t > 0$, which is $R(x,y,t) = 1$ for all $t > 0$.
2. $R(x,y,t) = 1 \iff d(x,y) < t \iff d(y,x) < t \iff R(y,x,t) = 1$. Taking into account that R can take only values 0 and 1, we conclude:
$$R(x,y,t) = R(y,x,t).$$
3. If $R(x,y,t) = 0$ or $R(y,z,s) = 0$, then obviously (3) is fulfilled. If $R(x,y,t) = 1$ and $R_d(y,z,s) = 1$, then $d(x,y) < t$ and $d(y,z) < s$, which means $d(x,z) \leq d(x,y) + d(y,z) < t + s$ and, finally, $R(x,z,t+s) = 1$.

□

Remark 1. *In the previous theorem, it was sufficient to define the domain of R as $X \times X \times (0,\infty)$ (not including 0 in the interval $(0,\infty)$). Intuitively, it could be explained by the fact that $d(x,y)$ cannot be less than 0. On the other hand, this does not prevent us from defining $d(x,x)$. If we still* *want to work with domain $X \times X \times [0, \infty)$ for R, we should define $R(x, x, 0)$ for all $x \in X$, since otherwise it could be both 0 and 1.*

If, in the previous proof, we want to define $d(x, y)$ as $\sup\{t : R(x, y, t) = 0\}$, we should add the following condition for R:

$$R(x, x, 0) = 0.$$

Remark 2. *The function $R : X \times X \times [0, \infty) \to \{0, 1\}$ can also be determined as a crisp relation $R : Y \times [0, \infty) \to \{0, 1\}$, where $Y = X \times X$. Based on this fact, we will call a function $R : X \times X \times [0, \infty) \to \{0, 1\}$ as a parametric relation.*

From the above theorems, we obtain the following principal result:

Corollary 1. *Given a metric $d : X \times X \to [0, \infty)$, by setting $R_d(x, y, t) = 1 \iff d(t) < t$, we obtain a parametric relation satisfying properties (1)–(3). Conversely, having a parametric relation $R : X \times X \times (0, \infty) \to \{0, 1\}$ satisfying properties (1)–(3) by setting $d(x, y) = \inf\{t : R(x, y, t) = 1\}$, we obtain a metric. Additionally, $d_{R_d} = d$ for every metric d and, if the parametric relation R satisfies properties (1)–(3) and is left-continuous, then $R_{d_R} = R$.*

According to the definition of nonexpansive, continuous, and uniformly continuous functions in terms of metric spaces (X, d), it is possible to define these functions in terms of spaces (X, R), where R satisfies properties (1)–(3) of Theorem 3 and is left-semicontinuous. In the next propositions, we suppose that (X_1, R_{d_1}) and (X_2, R_{d_2}) are spaces isomorphic to (X_1, d_1) and (X_2, d_2) in the sense of Corollary 1, ($d_1(x, y) = \inf\{t : R_{d_1}(x, y, t) = 1\}$ and $d_2(x, y) = \inf\{t : R_{d_1}(x, y, t) = 1\}$):

Proposition 1. *A function $f : (X_1, R_{d_1}) \to (X_2, R_{d_2})$ is nonexpansive if and only if, for every pair of points x and y in X_1, it holds that:*

$$R_2(f(x), f(y), t) \leq R_1(x, y, t).$$

Proposition 2. *A function $f : (X_1, R_{d_1}) \to (X_2, R_{d_2})$ is continuous if and only if, for every $x \in X_1$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every point y in X_1, it holds that:*

$$R_1(x, y, \delta) \leq R_2(f(x), f(y), \varepsilon).$$

Proposition 3. *A function $f : (X_1, R_{d_1}) \to (X_2, R_{d_2})$ is uniformly continuous if and only if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every pair of points x and y in X_1, it holds that:*

$$R_1(x, y, \delta) \leq R_2(f(x), f(y), \varepsilon).$$

The proof of the previous three propositions relies on the direct application of nonexpansive, continuous, and uniformly continuous functions in terms of metric spaces (X, d) and Theorem 3. It is possible to study categorical aspects of metric spaces in terms of metrics defined by relations, but we left the study of this topic for the future.

We continue in this paper to explain the definition of commonly used fuzzy metrics by extending the definition of a metric space in terms of relation R taking values in unit interval $[0, 1]$.

4. Fuzzy Metrics

Now we can use the last theorem from the previous section to define a fuzzy metric expanding the set $\{0, 1\}$ to the interval $[0, 1]$ and using arbitrary t-norm T instead of the minimum t-norm that was used in the previous section:

Definition 7. *A fuzzy metric on a set X is a function $M : X \times X \times [0, \infty) \to [0, 1]$ satisfying the following axioms for all $x, y, z \in X$ and $t, s \in [0, \infty)$:*

0. $M(x, x, 0) = 0$;
1. $M(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$;
2. $M(x, y, t) = M(y, x, t)$;
3. $T(M(x, y, t), M(y, z, s)) \leq M(x, z, t + s)$;
4. $M(x, y, -) : [0, \infty) \to [0, 1]$ *is left-semicontinuous.*

The above definition, with the more strict axiom (0):

0. $M(x, y, 0) = 0 \; \forall x, y \in X$,

is a definition of a fuzzy metric introduced by Kramosil and Michalek [5] for a measurable real function $T : [0, 1] \times [0, 1] \to [0, 1]$ such that $T(1, 1) = 1$ and revised by Grabiec [9] for a t-norm T. To be precise, Kramosil and Michalek used the additional axiom:

5. $M(x, y, -) : (0, \infty) \to [0, 1]$ is nondecreasing, and $\lim_{t\to\infty} M(x, y, t) = 1$.

In the case of Definition 7, the nondecreasing condition is fulfilled in the case of any t-norm T, and the condition $\lim_{t\to\infty} M(x, y, t) = 1$ is skipped by other authors since it comes from the statistical metric spaces and does not play any role in the context of fuzzy sets.

Example 1. *These examples fulfill fuzzy metrics axioms (0)–(2) and (4) and axiom (3) for the corresponding t-norm and for any crisp metric d that is used for the construction:*

1.
$$M_1(x, y, t) = \begin{cases} e^{-\frac{d(x,y)}{t}}, & \text{if } t \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Axiom (3) is fulfilled for product t-norm T.

2.
$$M_2(x, y, t) = \begin{cases} \frac{t}{t+d(x,y)}, & \text{if } t \neq 0 \\ 0, & \text{otherwise;} \end{cases}$$

Axiom (3) is fulfilled for any t-norm T.

In the same way as mentioned in the previous section, it is possible to define the fuzzy metric $M : X \times X \times (0, \infty) \to [0, 1]$ (not including 0 in the interval $(0, \infty)$), requesting M to fulfill axioms (1)–(4) and skipping axiom (0).

Defining fuzzy metrics in this way, we should clearly understand that the value $M(x, y, t)$ shows the degree to which $d(x, y) < t$ for a metric d, which is explained by the roots of this definition proposed in the previous section.

The conditions (0) and (1) are quite strong especially when they are used together. Condition (0) shows that $M(x, y, 0) = 0$ since, for any metric and for all $x, y \in X$, condition $d(x, y) < 0$ is not fulfilled, i.e., $d(x, y) \geq 0$ for any $x, y \in X$. In the fuzzy sense, this leads to the assumption that if $d(x, y) \geq t$, then $M(x, y, t)$ should be always 0. The condition (1) leads to the assumption that, if $d(x, y) < t$, then $M(x, y, t) = 1$, but it is not clear why it is fulfilled only in the case $x = y$. Both assumptions together lead us to the crisp case explained in the previous section.

In [6], the authors slightly modified axioms (1)–(4) and defined a fuzzy metric as a function M with domain $X \times X \times (0, \infty)$:

Definition 8 ([6])**.** *A fuzzy metric on a set* X *is a function* $M : X \times X \times (0, \infty) \to [0, 1]$ *satisfying the following axioms for all* $x, y, z \in X$ *and* $t, s \in (0, \infty)$:

0. $M(x, y, t) > 0$;
1. $M(x, y, t) = 1$ *if and only if* $x = y$;
2. $M(x, y, t) = M(y, x, t)$;
3. $T(M(x, y, t), M(y, z, s)) \leq M(x, z, t + s)$;
4. $M(x, y, -) : [0, \infty) \to [0, 1]$ *is continuous.*

In this definition, the authors do not allow function M to take the value 0 and allow it to take the value 1 only when $x = y$:

$$M(x, x, t) = 1.$$

These requirements are quite strong.

Additionally, in using this definition, it is impossible to construct a crisp metric d from the function M even if we use the definition of fuzzy linear order R, where $R(d(x, y), t) = M(x, y, t)$. This means that it is not clear which metric d the fuzzy metric M fuzzifies.

To overcome the problem of revealing the metric d that is fuzzified by M, we propose two approaches. The first approach is to define the fuzzy metric as the function $M : X \times X \times [0, \infty) \to [0, 1]$:

$$M(x, y, t) = \begin{cases} 0, & \text{if } t < t_{x,y}. \\ R(x, y, t), & \text{otherwise,} \end{cases} \tag{4}$$

where $R(x, y, t)$ satisfies conditions (2) and (3) from Definition 7.

The second idea is to define a fuzzy metric as a function $M : X \times X \times [0, \infty) \to [0, 1]$ by

$$M(x, y, t) = \begin{cases} 1, & \text{if } t > t_{x,y}. \\ R(x, y, t), & \text{otherwise,} \end{cases} \tag{5}$$

where $R(x, y, t)$ satisfies conditions (2) and (3) from Definition 7. In this case, we also require M to be continuous. In both cases, we can construct a crisp metric $d(x, y) = t_{x,y}$ that is fuzzified by M. The second idea is more natural, and a similar approach was investigated in [31].

Example 2. *These examples fulfill condition (5) and axioms (2) and (3) from Definition 7 for the corresponding t-norm and for any crisp metric d:*

1.
$$M_3(x, y, t) = \begin{cases} 1, & \text{if } d(x, y) < t \\ \max(1 - |d(x, y) - t|, 0), & \text{otherwise.} \end{cases}$$

Axiom (3) is fulfilled for the Łukasiewicz t-norm.

2.
$$M_4(x, y, t) = \begin{cases} 1, & \text{if } d(x, y) < t \\ e^{-|d(x,y)-t|}, & \text{otherwise.} \end{cases}$$

Axiom (3) is fulfilled for the product t-norm.

3.
$$M_5(x, y, t) = \begin{cases} 1, & \text{if } d(x, y) < t \\ \frac{1}{1+|d(x,y)-t|}, & \text{otherwise.} \end{cases}$$

Axiom (3) is fulfilled for the Hamacher t-norm.

Here, we propose axioms sufficient for the fuzzy metric to generate a crisp metric:

Theorem 4. *Let function $M : X \times X \times [0, \infty) \to [0, 1]$ satisfy the following axioms for all $x, y, z \in X$ and $t, s \in [0, \infty)$:*

0. $M(x, y, 0) = 0$;
1. $M(x, y, t) = 1$ *for all* $t > 0$, *if and only if* $x = y$;
2. $M(x, y, t) = M(y, x, t)$;
3. $T(M(x, y, t), M(y, z, s)) \leq M(x, z, t + s)$;
4. $M(x, y, -) : [0, \infty) \to [0, 1]$ *is continuous for* $x \neq y$.

Then, the function $d : X \times X \to [0, \infty)$ defined as $d(x, y) = \inf\{t : R(x, y, t) > \lambda\}$, where λ is a fixed real number from interval $(0, 1)$, is a metric if λ is an idempotent element for t-norm T.

Proof. We prove that d, defined as $d(x, y) = \inf\{t : R(x, y, t) > \lambda\}$, satisfies the axioms from Definition 6:

1. If $x = y$, then $R(x, y, t) = 1$ for all $t > 0$, and $d(x, y) = \inf\{t : R(x, y, t) > \lambda\} = 0$. On the other hand, if $d(x, y) = 0$, then $\inf\{t : R(x, y, t) > \lambda\} = 0$. Thus, since M is continuous by the third argument and $M(x, y, 0) = 0$, we conclude $x = y$. Actually, to prove this theorem, it is enough to request that function M is continuous only at 0 and is left-semicontinuous for other points.
2. $d(x, y) = \inf\{t : R(x, y, t) > \lambda\} = \inf\{t : R(y, x, t) > \lambda\} = d(y, x)$, since $R(x, y, t) = R(y, x, t)$.
3. Now we prove that $d(x, y) + d(y, z) \geq d(x, z)$. Indeed, if $d(x, y) + d(y, z) < d(x, z)$, then there exist $t, s \in (0, \infty)$, such that $d(x, y) + d(y, z) < t + s < d(x, z)$ and $d(x, y) < t$ & $d(y, z) < s$, where $d(x, y) = \inf\{t_1 : R(x, y, t_1) > \lambda\} < t$ and $d(y, z) = \inf\{t_2 : R(y, z, t_2) > \lambda\} < s$. Thus, $R(x, y, t) > \lambda$ and $R(y, z, s) > \lambda$, but $\lambda = T(\lambda, \lambda) < T(R(x, y, t), R(y, z, s)) \leq R(x, z, t + s)$, which means $R(x, z, t + s) > \lambda$. However, this leads to a contradiction with $R(x, z, t + s) < \lambda$, which is fulfilled since $t + s < d(x, z) = \inf\{t_3 : R(x, z, t_3) > \lambda\}$.

□

Example 3. *Function M_2 from Example 1 fulfill axioms (0)–(4) of Theorem 4. Thus, it is possible to apply the result of Theorem 4 and build a metric $d(x, y) = \inf\{t : M_2(x, y, t) > \lambda\}$, where λ is a fixed real number from interval $(0, 1)$.*

Let us come back to our initial idea of defining a metric through an order, but this time in a fuzzy sense. We first introduce a definition of a compatible fuzzy relation with an order $\leq$.

Definition 9. *Let $\leq$ be a linear order on a set S. Fuzzy relation $R : S \times S \to [0, 1]$ is called compatible with $\leq$ if and only if $R(a, b) \leq R(a, c)$ whenever $b \leq c$ and $R(b, c) \leq R(a, c)$ whenever $a \leq b$.*

This property can be interpreted as follows: if we have a three-element chain $a < b < c$, then the degree that $a < c$ is greater then the degree of $a < b$ and of $b < c$.

The next theorem shows that it is enough for a fuzzy relation $R : X \times X \to [0, 1]$, defined as $R(d(x, y), t)$ for a metric d, to be compatible with $\leq$ (where $\leq$ is a linear order on $S = [0, \infty)$) to fulfill the axioms from Definition 7. Thus, we do not need to require T-transitivity of the fuzzy relation R.

Theorem 5. *Let $d : X \times X \to [0, \infty)$ be a metric. A function $M : X \times X \times [0, \infty) \to [0, 1]$, defined as $M(x, y, t) = R(d(x, y), t)$, where $R : [0, \infty) \times [0, \infty) \to [0, 1]$, which is compatible with $\leq$ on $[0, \infty)$, is left-semicontinuous with respect to the second argument, and satisfies conditions*

$$R(a, t) = 1 \ \forall t > 0 \iff a = 0$$

$$R(0, 0) = 0,$$

is a fuzzy metric.

Proof. Let us prove that function $M : X \times X \times [0, \infty) \to [0, 1]$ satisfies the following axioms for all $x, y, z \in X$ and $t, s \in [0, \infty)$:

0. $M(x, x, 0) = 0$, since $R(0, 0) = 0$;
1. $M(x, y, t) = R(d(x, y), t) = 1$ for all $t > 0$ if and only if $d(x, y) = 0$, but that it is fulfilled if and only if $x = y$;

2. $M(x, y, t) = M(y, x, t)\ \forall x, y \in X,\ \forall t \in [0, \infty)$ since $d(x, y) = d(y, x)$;
3.
$$T(M(x, y, t), M(y, z, s)) = T(R(d(x, y), t), R(d(y, z), s)) \leq$$
$$R(d(x, y) + d(y, z), t + s) \leq R(d(x, z), t + s).$$
Thus, $T(M(x, y, t), M(y, z, s)) \leq M(x, z, t + s)\ \forall x, y, z \in X,\ \forall s, t \in [0, \infty)$.

□

Example 4. *Functions M_3, M_4, and M_5 from Example 2 can be constructed as $R_i(d(x, y), t)$, where $R_i : [0, \infty) \times [0, \infty) \to [0, 1]$ are represented as in Theorem 1:*

$$R_i(a, b) = \begin{cases} 1, & \text{if } a \leq b \\ E_i, & \text{otherwise,} \end{cases}$$

where

1. $E_1(a, b) = \max(1 - |a - b|, 0)$;
2. $E_2(a, b) = e^{-|a-b|}$;
3. $E_3(a, b) = \frac{1}{1+|a-b|}$.

5. Extensional Fuzzy Metrics

In this section, we invite the reader to trace the development of the ideas of the previous sections. Whereas in the previous section we fuzzified the statement $d(x, y) < t$, where d is a metric and $t \in [0, \infty)$, here we explain the idea of fuzzification of the statement $d(x, y) = t$.

Consider a metric space (X, d) and a T-equivalence relation E. We define a fuzzy metric as an extension of the given metric d with respect to a T-equivalence relation E on the set $[0, \infty)$ (codomain of the metric d). In the definition of an extensional fuzzy metric, we use a strongly linear T-E-order on $[0, \infty)$, defined as:

$$R_E(a, b) = \begin{cases} 1, & \text{if } a \leq b \\ E(a, b), & \text{otherwise} \end{cases}. \tag{6}$$

Thus, whereas in the previous section we used a fuzzy order relation, here we rely on a fuzzy equivalence relation. This approach has been developed in [32]; we outline here the main ideas to illustrate the approach and the logical development of the ideas of the previous section.

We propose to define a fuzzy metric as the degree to which the observed distance $d(x, y)$ between points x and y is equal to the real number t, or equal in a certain fuzzy sense determined by fuzzy equivalence E. That is, we define a fuzzy metric (called the E-d-metric) as a mapping $M_{Ed} : X \times X \times [0, \infty) \to [0, 1]$ as follows:

Definition 10. *Let d be a crisp metric on a set X, $t \in [0, \infty)$ and E be a fuzzy T-equivalence. Let a mapping $M_{Ed} : X \times X \times [0, \infty) \to [0, 1]$ be defined as:*

$$M_{Ed}(x, y, t) = E(d(x, y), t). \tag{7}$$

The fuzzy set M_{Ed} is called an extensional fuzzy metric determined by metric d and fuzzy equivalence E or E-d-metric if the following condition is satisfied:

$$T(E(d(x, y), t), E(d(y, z), s)) \leq R_E(d(x, z), t + s). \tag{8}$$

Condition (8) shows that $d(x, y) = t$ and that $d(y, z) = s$ implies $d(x, z) \leq t + s$ in a certain fuzzy sense. In other words, it is a fuzzy version of the triangular inequality.

If we have a crisp fuzzy equivalence relation:

$$E(a,b) = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{otherwise} \end{cases},$$

and corresponding T-E-order, then condition (8) holds for any t-norm T and metric d; it actually follows from the triangular inequality of the metric d.

We finish this section by noting the fact that the inequality (8) is quite natural and is fulfilled for Archimedean t-norms automatically.

Theorem 6. *Let T be a continuous Archimedean t-norm, and let T-equivalence be defined by:*

$$E(a,b) = g^{(-1)}(|a-b|),$$

where g is an additive generator of t-norm T. Then, the condition

$$T(E(d(x,y),t), E(d(y,z),s)) \leq R_E(d(x,z), t+s)$$

is fulfilled for any metric d.

Example 5. *Let d be a crisp metric, $t \in [0,\infty)$. Then, we have the following examples of the E-d-metric:*

- $M_{E_L d}(x,y,t) = E_L(d(x,y),t) = \max(1 - |d(x,y) - t|, 0)$ *in the case of T, which is the Łukasiewicz t-norm;*
- $M_{E_P d}(x,y,t) = E_P(d(x,y),t) = e^{-|d(x,y)-t|}$ *in the case of T, which is the product t-norm;*
- $M_{E_H d}(x,y,t) = E_H(d(x,y),t) = \frac{1}{1+|d(x,y)-t|}$ *in the case of T, which is the Hamacher t-norm.*

6. Conclusions

In this paper, we explained the definitions of fuzzy metrics used in the literature and analyzed which notions they fuzzified. This explanation is important for finding possible applications, since in applying fuzzy constructions, we should clearly understand the essence of the construction. Thus, we draw the reader's attention to the fact that the classical fuzzy metric definition directly arises from the fuzzification of the expression $d(x,y) < t$. To explain this, first we studied which properties fulfill the crisp relation $R : X \times X \times [0,\infty) \to \{0,1\}$ to uniquely define a crisp metric $d : X \times X \to [0,\infty)$. Then, we fuzzified these conditions in order to obtain a fuzzy metric. Namely, we allowed relation R to be fuzzy or to take values in the interval $[0,1]$, and we used the t-norm as a generalized conjunction instead of a crisp conjunction. Then, we analyzed the obtained conditions. Since relation R in the crisp case determines a metric d and is defined as $R(x,y,t) = 1$ if and only if $d(x,y) < t$, we invite the reader to understand that, in Kramosil–Michalek, Grabisch, and George–Veeramani fuzzy metric cases, the expression $d(x,y) < t$ is fuzzified. We also studied which conditions a fuzzy relation should fulfill in order to determine a fuzzy metric. For completeness, we recalled and revised the definition of a fuzzy metric that fuzzifies the expression $d(x,y) = t$, as introduced in [32].

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