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Article

One New Property of a Class of Linear Time-Optimal Control Problems

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Abstract: The following paper deals with a new property of linear time-optimal control problems with real eigenvalues of the system. This property unveils the possibility of synthesizing the time-optimal control without describing the switching hyper-surfaces. Furthermore, the novel technique offers an alternative solution to the classic example of the time-optimal control of a double integrator system.

Keywords: time-optimal control; minimum time control; Pontryagin's maximum principle; synthesis of optimal systems; linear systems; switching surface

MSC: 49N35; 93B50; 49N05; 93B52; 93C05

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1. Introduction

Since the first studies of Feldbaum [1,2], Pontryagin's Principle of Maximum [3], etc., the theory of linear time-optimal control problem has gained maturity—the main theoretical issues have been thoroughly studied and answered [4–9]. This historical evolution and facts provide a solid background of the progress in this field. The achieved state of knowledge in this field establishes the foundation for further exploration and advancement. Achieving a transition from one system state to another in a minimum time with maximum utilization of the available system resources—control within the constraints of both control inputs and state space variables—in a form of synthesis still presents an attractive topic for further research.

In synchrony with the above mentioned, the authors in the recently published book [10] state, "there has been tremendous progress in numerical methods in optimal control over the past fifteen years that has led to the solutions of some specific and very difficult problems" and, in particular, the introduction of geometrical methods, more specifically—"a first illustration of the power of geometric methods that go well beyond the conditions of the maximum principle and lead to deep results about the structure of optimal solutions". The geometric approach to the optimal control of a double integrator is also discussed in [11,12].

In a recent publication on the topic [13], the authors say that, "this paper has proposed a global time optimal control law for triple integrator with input saturation and full state constraints", and in terms of the results, "An analytical state feedback form control law has been synthesized based on the switching surfaces and curves".

The authors also mention "there are plenty of researches trying to solve the problem analytically, while there is still no complete time optimal analytical solution for systems higher than second order".

This is noteworthy considering Pontryagin's original sources. In reference [3] (Chapter 3, § 20, § 21, Example 3), the author and his colleagues describe the solution of the problem of a linear time-optimal control system fulfilling the condition of normality with real non-positive eigenvalues and one control input as follows.

The time-optimal control for such a type of linear system has maximum n (the order of the system) intervals of constancy, i.e., the number of switchings is maximum $(n-1)$; the state-space of the system is separated into manifold M_n, $M_{n-1}, \ldots, M_1$ of dimensions, respectively, $1, 2, \ldots, n$. The manifold M_n consists of all the points for which the time-optimal control has one interval of constancy. Supposing $|u| \leq 1$, the trajectory of the system under the control +1 ending at the state-space origin is defined as M_n^+, while the trajectory of the system ending at the state-space origin but under the control -1 is defined as M_n^-. Together, M_n^+ and M_n^- compose the switching curve M_n. The final stage of the time-optimal process represents a movement alongside M_n^+ or M_n^-. All the trajectories of the system ending at a point of the curve M_n^- under the control +1 fill the surface M_{n-1}^+. Analogically, all the trajectories ending at a point of the curve M_n^+ under the control -1 fill the surface M_{n-1}^-. Combining M_{n-1}^+ and M_{n-1}^-, we obtain the switching surface M_{n-1}, so the last two stages of each time-optimal process are in M_{n-1}. In the same manner, the rest of the manifolds are constructed. The manifold M_i is of dimension $(n-i+1)$; M_{i+1} is entirely in M_i and divides it into two areas M_i^+ and M_i^-; M_i^+ consists of all the trajectories under the control +1 ending at a point of M_{i+1}^-, while M_i^- consists of all the trajectories under the control -1 ending at a point of M_{i+1}^+. The last manifold M_1 coincides with the whole state-space of the system. The synthesizing function is depicted as:

$$u(x) = \begin{cases} +1 & \textit{in all areas} \quad M_i^+, \\ -1 & \textit{in all areas} \quad M_i^-. \end{cases}$$

So, in order to synthesize the time-optimal control for a given system fulfilling the above conditions, one needs to describe properly the switching surfaces M_i^+ and M_i^-.

Despite the progress in the field, finding a new solution for the problem discussed above by Pontryagin and others without the need of directly describing the respective manifolds M_i^+ and M_i^- renders it more appealing by conducting a deeper investigation of the state-space geometric properties of this time-optimal control problem.

A novel method for synthesizing the time-optimal control for a class of controllable linear systems of any order with real non-positive simple eigenvalues and one input is developed and further explored in the dissertation [14] and the following papers [15–17]. It is founded on some new state-space properties of the considered linear time-optimal control problem and the exclusion of switching surfaces description serves as its main advantage. The study [18] illustrates an example of a possible application of the method in practice.

Therefore, it is worthwhile trying to expand the thus developed solution of synthesizing the linear time-optimal control without the description of switching surfaces and curves to the more general case as the one described by Pontryagin and colleagues, in particular, a controllable linear system with one input and real non-positive eigenvalues, but not just non-positive simple eigenvalues.

The current paper is structured in the following way. In Section 2, a new property of the linear time-optimal control problem is theoretically represented. In Section 3, the author compares the classic solution of the time-optimal control problem of a double integrator to the alternatively suggested novel way by application of the new property. Section 4 represents a detailed discussion of the obtained results.

2. Formulation of the Problem and Solution

Let us consider the following linear time-optimal control problem of order n, $n \geq 2$. The system is described by the equations:

$$\begin{aligned} \dot{x}_i &= \sum_{j=1}^{n-1} a_{ij} x_j + b_i u, \qquad i = 1, 2, \ \ldots, \ n-1, \\ \dot{x}_n &= \sum_{j=1}^{n-1} a_{nj} x_j + \lambda_n x_n + b_n u. \end{aligned} \tag{1}$$

Let us suppose it is controllable as well as possessing real non-positive eigenvalues. It should be mentioned that every normal system with real eigenvalues could be transformed to such a type of presentation.

The initial state at the moment $t_0 = 0$ of the system (1) is

$$\boldsymbol{x}_0 = \begin{pmatrix} x_{10} & \cdots & x_{(n-1)0} & x_{n0} \end{pmatrix}^T \tag{2}$$

and the target state at the moment t_f represents the origin of the system's state-space where t_f is unspecified

$$\boldsymbol{x}\left(t_f\right) = \boldsymbol{x}_f = \left(\underbrace{0 \ \cdots \ 0 \ \ 0}_{n}\right)^T. \tag{3}$$

The admissible control $u(t)$ is a piecewise continuous function that takes its values in the range of

$$-u_0 \le u(t) \le u_0,\ u_0 = const > 0, \tag{4}$$

which is continuous on the boundaries of the set of allowed values (4) and with regard to the points of discontinuity τ we have

$$u(\tau) = u(\tau + 0). \tag{5}$$

The problem is to find an admissible control $u(\boldsymbol{x})$ which transfers the system (1) from its initial state (2) to the final state (3) in minimum time, i.e., minimizing the performance index

$$J = t_f \to min. \tag{6}$$

Let us refer to this problem as "Problem P(n)".

The form of the equations of the system (1) allows the introduction of the linear sub-system of order $(n-1)$

$$\begin{aligned} \dot{\boldsymbol{x}}_{n-1} &= A_{n-1}\boldsymbol{x}_{n-1} + B_{n-1}u, \\ y_{n-1} &= C_{n-1}\boldsymbol{x}_{n-1} \end{aligned} \tag{7}$$

with the state-space vector

$$\boldsymbol{x}_{n-1} = \begin{pmatrix} x_1 & \cdots & x_{n-1} \end{pmatrix}^T \tag{8}$$

and scalar output y_{n-1} where the matrices $A_{n-1},\ B_{n-1},\ C_{n-1}$ are, respectively,

$$\begin{gathered} A_{n-1} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1,n-1} \\ a_{21} & a_{22} & \dots & a_{2,n-1} \\ \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n-1} \end{pmatrix}, \\ B_{n-1} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{pmatrix},\quad C_{n-1} = \begin{pmatrix} a_{n1} & a_{n2} & \dots & a_{n,n-1} \end{pmatrix}. \end{gathered} \tag{9}$$

Thus, the system (1) could be represented by (7) in the following form which is also depicted in Figure 1.

$$\begin{aligned} \dot{\boldsymbol{x}}_{n-1} &= A_{n-1}\boldsymbol{x}_{n-1} + B_{n-1}u, \\ y_{n-1} &= C_{n-1}\boldsymbol{x}_{n-1}, \\ \dot{x}_{n-1} &= \lambda_n x_n + y_{n-1} + b_n u. \end{aligned} \tag{10}$$

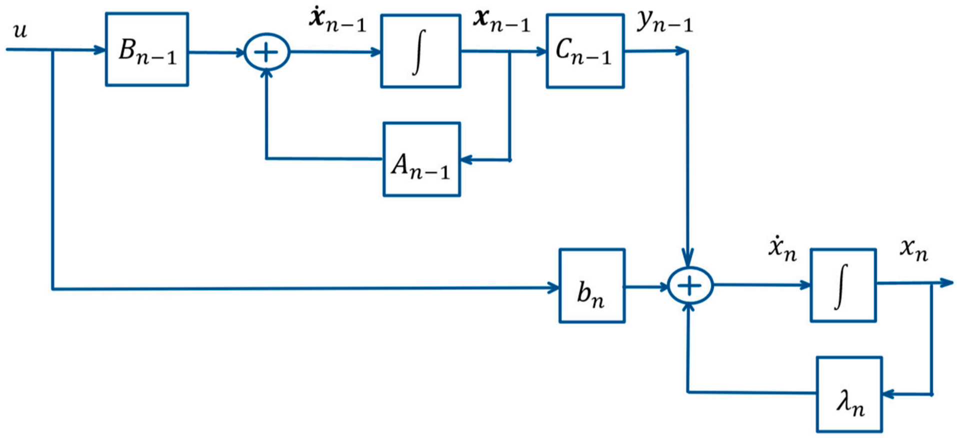


Figure 1. Schematic representation of the initial system (1) in form (10).

With regard to the sub-system (7), its initial state may be represented by $\boldsymbol{x}_{(n-1)0}$ (11) and the relationship between the initial states of both the system and the sub-system may be described as (12).

$$\boldsymbol{x}_{(n-1)0} = \left(\underbrace{x_{10} \quad \cdots \quad x_{(n-1)0}}_{n-1}\right)^T. \tag{11}$$

$$\boldsymbol{x}_0 = \begin{pmatrix} \boldsymbol{x}_{(n-1)0} \\ x_{n0} \end{pmatrix}. \tag{12}$$

Let us formulate the following linear time-optimal control problem of order $(n-1)$ which we shall call "Problem $\mathrm{P}(n-1)$". The system is defined by Equation (7). The initial state of the system (7) at the moment $t_0 = 0$ is (11) and the target state at the moment $t_{(n-1)f}$, which one should bear in mind is not initially specified, is the origin of the $(n-1)$-dimensional state-space of the system (7)

$$\boldsymbol{x}_{n-1}\left(t_{(n-1)f}\right) = \boldsymbol{x}_{(n-1)f} = \left(\underbrace{0 \quad \cdots \quad 0}_{n-1}\right)^T. \tag{13}$$

The admissible control $u(t)$ represents a piecewise continuous function that takes its values in the range of (4), which is continuous on the boundaries of the set of allowed values (4). Regarding the points of discontinuity τ we have (5). The Problem $\mathrm{P}(n-1)$ consists of synthesizing an admissible control $u(\boldsymbol{x}_{n-1})$ which on the one hand transfers the system (7) from its initial (11) to final state (13) and on the other hand, minimizes the performance index

$$J_{n-1} = t_{(n-1)f} \to min. \tag{14}$$

Let us assume we have found the solution of Problem $\mathrm{P}(n-1)$ and denote by $t^o_{(n-1)f}$ the optimal time defined as the minimum time of (14)

$$t^o_{(n-1)f} = min(J_{n-1}), \tag{15}$$

by $u^o_{n-1}(t),\ t \in \left[0,\ t^o_{(n-1)f}\right]$—the optimal control, and $\boldsymbol{x}^o_{n-1}(t),\ t \in \left[0, t^o_{(n-1)f}\right]$—the optimal trajectory in the $(n-1)$-dimensional state-space of the system (7), which is described by

$$\begin{gathered} \boldsymbol{x}^o_{n-1}(t) = e^{A_{n-1}t}\boldsymbol{x}_{(n-1)0} + \int\limits_0^t e^{A_{n-1}\tau}B_{n-1}u^o_{n-1}(t-\tau)d\tau \\ \text{for } t \in \left[0,\ t^o_{(n-1)f}\right]. \end{gathered} \tag{16}$$

$$x^o_{n-1}\left(t^o_{(n-1)f}\right) = \left(\underbrace{0 \quad \cdots \quad 0}_{n-1}\right)^T. \tag{17}$$

Let us denote the scalar output of the system (7) as a representation of the optimal vector-function $x^o_{n-1}(t),\ t \in \left[0,\ t^o_{(n-1)f}\right]$, resulting as $y^o_{n-1}(t),\ t \in \left[0,\ t^o_{(n-1)f}\right]$. In that case, $y^o_{n-1}(t)$ stands for

$$y^o_{n-1}(t) = C_{n-1}x^o_{n-1}(t) \text{ for } t \in \left[0,\ t^o_{(n-1)f}\right], \tag{18}$$

$$y^o_{n-1}\left(t^o_{(n-1)f}\right) = C_{n-1}x^o_{n-1}\left(t^o_{(n-1)f}\right) = 0. \tag{19}$$

Let us define x^1_{n0} (21) as an initial state of the n-th coordinate of the state-space vector x of the system (1) or (10) and consider the trajectory $x^1(t)$ in the n-dimensional state-space of Problem $\mathrm{P}(n)$ with initial state in the point x^1_0 and coordinates (20) and (21) under the optimal control $u^o_{n-1}(t),\ t \in \left[0,\ t^o_{(n-1)f}\right]$, of Problem $\mathrm{P}(n-1)$.

$$x^1_0 = \begin{pmatrix} x_{(n-1)0} \\ x^1_{n0} \end{pmatrix}, \tag{20}$$

$$x^1_{n0} = -\frac{\int_0^{t^o_{(n-1)f}} e^{\lambda_n(t-\tau)}\left(y^o_{n-1}(\tau) + b_n u^o_{n-1}(\tau)\right)d\tau}{e^{\lambda_n t^o_{(n-1)f}}}. \tag{21}$$

Given the characteristics of the system as defined in (1), the vector-function $x^1(t)$ presented as (10) specifies (22). According to (16), the first $(n-1)$ variables of the vector-function in (22) typify the optimal vector-function $x^o_{n-1}(t)$ of Problem $\mathrm{P}(n-1)$. Regarding the last n-th variable of $x^1(t)$ in (22), the function $y_{n-1}(\tau)$ depicts the scalar output of the system (7), which in this case is the result of the optimal vector-function $x^o_{n-1}(t)$, $t \in \left[0,\ t^o_{(n-1)f}\right]$. Then, in terms of the above mentioned and in consonance with (18), $y_{n-1}(\tau)$ equals $y^o_{n-1}(\tau)$. Thus, we obtain (23) for $x^1(t)$ (22).

$$x^1(t) = \begin{pmatrix} e^{A_{n-1}t}x_{(n-1)0} + \int\limits_0^t e^{A_{n-1}\tau}B_{n-1}u^o_{n-1}(t-\tau)d\tau \\ e^{\lambda_n t}x^1_{n0} + \int\limits_0^t e^{\lambda_n(t-\tau)}\left(y_{n-1}(\tau) + b_n u^o_{n-1}(\tau)\right)d\tau \end{pmatrix} \\ \text{for } t \in \left[0,\ t^o_{(n-1)f}\right]. \tag{22}$$

$$x^1(t) = \begin{pmatrix} x^o_{n-1}(t) \\ e^{\lambda_n t}x^1_{n0} + \int\limits_0^t e^{\lambda_n(t-\tau)}\left(y^o_{n-1}(\tau) + b_n u^o_{n-1}(\tau)\right)d\tau \end{pmatrix} \\ \text{for } t \in \left[0,\ t^o_{(n-1)f}\right]. \tag{23}$$

With regard to $x^1(t)$ (23) at the moment $t = t^o_{(n-1)f}$ we obtain

$$x^1\left(t^o_{(n-1)f}\right) = \begin{pmatrix} x^o_{n-1}\left(t^o_{(n-1)f}\right) \\ \begin{pmatrix} e^{\lambda_n t^o_{(n-1)f}}x^1_{n0} + \\ + \int_0^{t^o_{(n-1)f}} e^{\lambda_n(t-\tau)}\left(y^o_{n-1}(\tau) + b_n u^o_{n-1}(\tau)\right)d\tau \end{pmatrix} \end{pmatrix}. \tag{24}$$

Then, substituting $\boldsymbol{x}^o_{n-1}\left(t^o_{(n-1)f}\right)$ for (17) and x^1_{n0} for (21), the following result is achieved.

$$\boldsymbol{x}^1\left(t^o_{(n-1)f}\right) = \begin{pmatrix} \begin{pmatrix} \underbrace{0 \quad \cdots \quad 0}_{n-1} \end{pmatrix}^T \\ \begin{pmatrix} e^{\lambda_n t^o_{(n-1)f}}\left(-\frac{\int_0^{t^o_{(n-1)f}} e^{\lambda_n(t-\tau)}\left(y^o_{n-1}(\tau)+b_n u^o_{n-1}(\tau)\right)d\tau}{e^{\lambda_n t^o_{(n-1)f}}}\right)+ \\ +\int_0^{t^o_{(n-1)f}} e^{\lambda_n(t-\tau)}\left(y^o_{n-1}(\tau)+b_n u^o_{n-1}(\tau)\right)d\tau \end{pmatrix} \end{pmatrix}. \tag{25}$$

$$\boldsymbol{x}^1\left(t^o_{(n-1)f}\right) = \begin{pmatrix} \begin{pmatrix} \underbrace{0 \quad \cdots \quad 0}_{n-1} \end{pmatrix}^T \\ \begin{pmatrix} -\int_0^{t^o_{(n-1)f}} e^{\lambda_n(t-\tau)}\left(y^o_{n-1}(\tau)+b_n u^o_{n-1}(\tau)\right)d\tau \\ +\int_0^{t^o_{(n-1)f}} e^{\lambda_n(t-\tau)}\left(y^o_{n-1}(\tau)+b_n u^o_{n-1}(\tau)\right)d\tau \end{pmatrix} \end{pmatrix}. \tag{26}$$

$$\boldsymbol{x}^1\left(t^o_{(n-1)f}\right) = \begin{pmatrix} \begin{pmatrix} \underbrace{0 \quad \cdots \quad 0}_{n-1} \end{pmatrix}^T \\ 0 \end{pmatrix} = \begin{pmatrix} \underbrace{0 \cdots \quad 0 \quad 0}_{n} \end{pmatrix}^T. \tag{27}$$

Thus, we obtain that for Problem $\mathrm{P}(n)$ the trajectory $\boldsymbol{x}^1(t)$ in the n-dimensional state-space of the system (1) or (10) with initial point $\boldsymbol{x}^1_0$ (20) and (21) under the optimal control $u^o_{n-1}(t),\ t \in \left[0,\ t^o_{(n-1)f}\right]$, of Problem $\mathrm{P}(n-1)$ ends at the moment $t = t^o_{(n-1)f}$ at the origin of the n-dimensional state-space of Problem $\mathrm{P}(n)$. Taking into account that the function $u^o_{n-1}(t),\ t \in \left[0,\ t^o_{(n-1)f}\right]$, represents the optimal control of Problem $\mathrm{P}(n-1)$ and thereby it is a piecewise constant function with an amplitude u_0 and a number of switchings maximum $(n-2)$, i.e., the number of intervals of constancy maximum is $(n-1)$ [7] (Chapter 2, §6, Theorem 2.11, p. 116), one comes to the conclusion that the trajectory $\boldsymbol{x}^1(t)$ lies wholly on the switching hyper-surface of Problem $\mathrm{P}(n)$.

Let us now consider the trajectory $\boldsymbol{x}(t)$ (28) in the n-dimensional state-space of Problem $\mathrm{P}(n)$ with an initial point representing the initial state $\boldsymbol{x}_0$ (2) or (12) of Problem $\mathrm{P}(n)$ under the optimal control $u^o_{n-1}(t),\ t \in \left[0,\ t^o_{(n-1)f}\right]$, of Problem $\mathrm{P}(n-1)$. According to (16), the first $(n-1)$ variables of the vector-function in (28) account for the optimal vector-function $\boldsymbol{x}^o_{n-1}(t)$ of Problem $\mathrm{P}(n-1)$. With regard to the last variable of $\boldsymbol{x}(t)$ in (28), the function $y_{n-1}(\tau)$ is the scalar output of the system (7), which is actually the result of the optimal vector-function $\boldsymbol{x}^o_{n-1}(t),\ t \in \left[0, t^o_{(n-1)f}\right]$. In consonance with (18), the function $y_{n-1}(\tau)$ therefore denotes $y^o_{n-1}(\tau)$ in this case. Thus, we obtain (29) for $\boldsymbol{x}(t)$ (28).

$$\boldsymbol{x}(t) = \begin{pmatrix} e^{A_{n-1}t}\boldsymbol{x}_{(n-1)0} + \int\limits_0^t e^{A_{n-1}\tau}B_{n-1}u^o_{n-1}(t-\tau)d\tau \\ e^{\lambda_n t}x_{n0} + \int\limits_0^t e^{\lambda_n(t-\tau)}\left(y_{n-1}(\tau)+b_n u^o_{n-1}(\tau)\right)d\tau \end{pmatrix} \tag{28}$$

$$\text{for } t \in \left[0,\ t^o_{(n-1)f}\right].$$

$$\boldsymbol{x}(t) = \begin{pmatrix} \boldsymbol{x}^o_{n-1}(t) \\ e^{\lambda_n t} x_{n0} + \int\limits_0^t e^{\lambda_n (t-\tau)} \left(y^o_{n-1}(\tau) + b_n u^o_{n-1}(\tau) \right) d\tau \end{pmatrix} \tag{29}$$

$$\text{for } t \in \left[0,\ t^o_{(n-1)f}\right].$$

Let us consider now the difference between the two vector-functions $\boldsymbol{x}(t)$ (29) and $\boldsymbol{x}^1(t)$ (23). Thus, we obtain consecutively

$$\begin{aligned} &\boldsymbol{x}(t) - \boldsymbol{x}^1(t) \\ &= \begin{pmatrix} \boldsymbol{x}^o_{n-1}(t) \\ e^{\lambda_n t} x_{n0} + \int\limits_0^t e^{\lambda_n (t-\tau)} \left(y^o_{n-1}(\tau) + b_n u^o_{n-1}(\tau) \right) d\tau \end{pmatrix} \\ &- \begin{pmatrix} \boldsymbol{x}^o_{n-1}(t) \\ e^{\lambda_n t} x^1_{n0} + \int\limits_0^t e^{\lambda_n (t-\tau)} \left(y^o_{n-1}(\tau) + b_n u^o_{n-1}(\tau) \right) d\tau \end{pmatrix} \\ &\text{for } t \in \left[0,\ t^o_{(n-1)f}\right]. \end{aligned} \tag{30}$$

$$\boldsymbol{x}(t) - \boldsymbol{x}^1(t) = \begin{pmatrix} \Big(\underbrace{0 \quad \cdots \quad 0}_{n-1} \Big)^T \\ e^{\lambda_n t} \left(x_{n0} - x^1_{n0} \right) \end{pmatrix} \tag{31}$$

$$\text{for } t \in \left[0,\ t^o_{(n-1)f}\right].$$

As for the last n-th coordinate of (31) $e^{\lambda_n t}\left(x_{n0} - x^1_{n0}\right)$ for $t \in \left[0,\ t^o_{(n-1)f}\right]$ we could state that:

1. If $x_{n0} = x^1_{n0}$, then the initial state $\boldsymbol{x}_0$ (2) or (12) of Problem $\mathrm{P}(n)$ coincides with the point $\boldsymbol{x}^1_0$ with coordinates (20)–(21). As already illustrated, $\boldsymbol{x}^1_0$ represents a point of the switching hyper-surface of Problem $\mathrm{P}(n)$ and the trajectory with the initial point $\boldsymbol{x}^1_0$ under the optimal control $u^o_{n-1}(t),\ t \in \left[0, t^o_{(n-1)f}\right]$, of Problem $\mathrm{P}(n-1)$ lies wholly on the switching hyper-surface of Problem $\mathrm{P}(n)$ and ends at the moment $t = t^o_{(n-1)f}$ at the origin of the n-dimensional state-space of the system (1) or (10) of Problem $\mathrm{P}(n)$;
2. If $x_{n0} \neq x^1_{n0}$, then the initial state $\boldsymbol{x}_0$ (2) or (12) of Problem $\mathrm{P}(n)$ does not coincide with the point $\boldsymbol{x}^1_0$ with coordinates (20)–(21). The expression $e^{\lambda_n t}\left(x_{n0} - x^1_{n0}\right)$ for $t \in \left[0,\ t^o_{(n-1)f}\right]$ does not change its sign and is not equal to zero because $t^o_{(n-1)f}$ is a finite time. Thus, the trajectory with initial state $\boldsymbol{x}_0$ (2) or (12) of Problem $\mathrm{P}(n)$ under the optimal control $u^o_{n-1}(t),\ t \in \left[0, t^o_{(n-1)f}\right]$, of Problem $\mathrm{P}(n-1)$ lies entirely above or below the switching hyper-surface of Problem $\mathrm{P}(n)$ nowhere intersecting it and ends at the moment $t = t^o_{(n-1)f}$ at a point of the coordinate axis x_n different from zero.

Thus, the following theorem has been proven.

Theorem 1. *The trajectory of the system (1) or (10) with initial point in $\boldsymbol{x}_0$ (2) under the optimal control $u^o_{n-1}(t),\ t \in \left[0,\ t^o_{(n-1)f}\right]$ of Problem $P(n-1)$ lies wholly on the switching hyper-surface of Problem $P(n)$ and ends at the moment $t = t^o_{(n-1)f}$ at the origin of the n-dimensional state-space of the system (1) or (10) of Problem $P(n)$ or lies entirely above or below the switching hyper-surface of Problem $P(n)$ nowhere intersecting it and ends at the moment $t = t^o_{(n-1)f}$ at a point of the coordinate axis x_n different from zero.*

3. Example

Let us consider the following example of synthesizing the time-optimal control of a double integrator (§ 3. Example. The problem of synthesis, p. 38) [7]; (Chapter 7, Problem 7.1, p. 150) [11,12]. It is noteworthy to mention that the above problem of synthesis, as it is already an established example, has found a place in online optimal control courses on world platforms with video content [19–22]. It should be noted that these online resources are often volatile and unavailable after some time. In the first place, an illustration of this classical synthesis will be presented, and thereafter the synthesis as an expansion and update of the method [14] by the new property.

The system is described by the variables y (position) and v (velocity) and represents

$$\begin{aligned} \tfrac{dy}{dt} &= v, \\ \tfrac{dv}{dt} &= u. \end{aligned} \tag{32}$$

Let the constraints of the admissible control u (4), (5) be

$$-u_0 \leq u(t) \leq u_0,\ u_0 = 1. \tag{33}$$

3.1. Classical Synthesis

The switching curve S_2 in the phase plane yv is described by

$$\begin{aligned} S_2 &= \gamma^+ \cup \gamma^- \cup (0,0), \\ \gamma^+ &= \left\{ (y,v) : y = \tfrac{v^2}{2u_0},\ v < 0 \right\}, \\ \gamma^- &= \left\{ (y,v) : y = \tfrac{-v^2}{2u_0},\ v > 0 \right\}. \end{aligned} \tag{34}$$

The two pieces γ^+ and γ^- of the switching curve S_2 are the parts of the parabolas representing the phase trajectories going through the origin of the phase plane in case of constant control $u = u_0$ or $u = -u_0$, respectively.

The two areas R^+ and R^- in the phase plane,

$$\begin{aligned} R^+ &= \left\{ (y,v) : y + sign(v)\tfrac{v^2}{2u_0} < 0 \right\}, \\ R^- &= \left\{ (y,v) : y + sign(v)\tfrac{v^2}{2u_0} > 0 \right\}, \end{aligned} \tag{35}$$

below and above the switching curve S_2 (34), respectively, encompass the areas where the optimal control takes a value u_0 with regard to the points of R^+ and $(-u_0)$ with regard to the points of R^-. The areas R^+ and R^- as well as the parts γ^+ and γ^- of S_2 are shown in the following Figure 2.

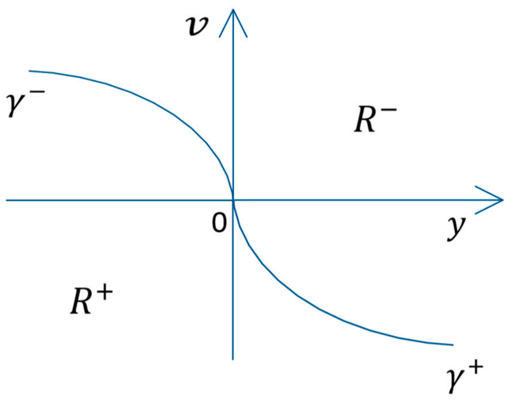


Figure 2. Representation of the areas R^+ and R^- as well as the two parts γ^+ and γ^- of the switching curve S_2 in the phase plane yv.

The time-optimal control is synthesized in the form

$$u(y,v) = \begin{cases} 0 & when \quad (y,v) \equiv (0,\ 0), \\ +u_0 & when \quad (y,v) \in R^+ \cup \gamma^+, \\ -u_0 & when \quad (y,v) \in R^- \cup \gamma^-. \end{cases} \tag{36}$$

After substitution of R^+ and R^- for (35) as well as γ^+ and γ^- for (34) in (36) the synthesized optimal control appears as

$$u(y,v) = \begin{cases} \qquad 0 \quad when \quad (y,v) \equiv (0,0), \\ +u_0 \quad \begin{pmatrix} when & \left(y + sign(v)\frac{v^2}{2u_0} < 0\right) \\ or\ when & \left(y - \frac{v^2}{2u_0} = 0,\ v < 0\right) \end{pmatrix}, \\ -u_0 \quad \begin{pmatrix} when & \left(y + sign(v)\frac{v^2}{2u_0} > 0\right) \\ or\ when & \left(y + \frac{v^2}{2u_0} = 0,\ v > 0\right) \end{pmatrix}. \end{cases} \tag{37}$$

3.2. Synthesis Based on the New Property and the Method [14]

Let us now consider the synthesis in terms of the method developed in [14] and expanded by the new property. One of the founding properties of the described method regards the trajectory in the state-space of a time-optimal control problem of higher order now being defined by the solution for the lower order, taking into consideration that all the time-optimal control problems of descending order are generated by the problem of the utmost order and form a class of problems. Thus, the method now allows a synthesis to be defined without the description of the switching hyper-surfaces. As we have shown here, the new property represents an expansion covering the general case of controllable linear systems with one input and real non-positive eigenvalues. Therefore, the simple non-positive system's eigenvalues of the method demonstrated in [14] is now omitted as an initial restriction. The example here considers a system of order two with double zero eigenvalue, so the synthesis is directly based on the solution of the problem of order one, which also allows the solution of the initial problem to be expressed analytically.

Step 1. First, we make a suitable change of variables through (38) and obtain a representation by $(x_1,\ x_2)$, which could also be performed by the matrix T (39) and (40) via (41).

$$y = x_2, \quad v = x_1. \tag{38}$$

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{39}$$

$$\begin{gathered} T^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ T^{-1}T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E. \end{gathered} \tag{40}$$

$$\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{41}$$

Thus, we obtain (43) and (44) from the initial system (32) through its matrix representation (42).

$$\begin{pmatrix} \dot{y} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u. \tag{42}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = T^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + T^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}u. \tag{43}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u. \tag{44}$$

The system (44) is now in the form (1). Then, (44) in form (10) is represented as (45) and (46) (as (1) in form (10)). The sub-system of (45) and (46) is (47) or (48).

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= A_1 \boldsymbol{x}_1 + B_1 u, \\ y_1 &= C_1 \boldsymbol{x}_1, \\ \dot{x}_2 &= \lambda_2 x_2 + y_1 + b_2 u, \end{aligned} \tag{45}$$

$$\begin{aligned} &\boldsymbol{x}_1 = (x_1), \\ &A_1 = (0), B_1 = (b_1) = (1),\ C_1 = (1), \\ &\lambda_2 = 0_1, b_2 = 0. \end{aligned} \tag{46}$$

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= A_1 \boldsymbol{x}_1 + B_1 u, \\ y_1 &= C_1 \boldsymbol{x}_1. \end{aligned} \tag{47}$$

$$\begin{aligned} \dot{x}_1 &= 0 x_1 + 1 u, \\ y_1 &= 1 x_1. \end{aligned} \tag{48}$$

Step 2. Solving Problem $\mathrm{P}(1)$. The eigenvalue of A_1 is 0. The optimal control of Problem $\mathrm{P}(1)$, $u_1^o(t)$ for $t \in \left[0,\ t_{1f}^o\right]$, is (49) and (50) [14] (pp. 50–52).

$$u_1^o(t) = \begin{cases} 0 & when \quad x_{10} = 0, \\ s_{11}^o u_0 \quad for \quad t \in \left[0, t_{11}^o\right] & when \quad x_{10} \neq 0. \end{cases} \tag{49}$$

$$\begin{aligned} s_{11}^o &= -sign(b_1 x_{10}), \\ t_{11}^o &= \tfrac{|x_{10}|}{|b_1| u_0}. \end{aligned} \tag{50}$$

$$min J_1 = t_{1f}^o = \begin{cases} 0 & when \quad x_{10} = 0, \\ t_{11}^o & when \quad x_{10} \neq 0. \end{cases} \tag{51}$$

Step 3. Calculating the value of the variable x_{2w}. The variable x_{kw} is defined in [14] (pp. 39–40), [15] (p. 320), [16] (p. 41) and in the case of expanding the class of time-optimal control problems here, it represents at $k = n$ the n-th coordinate of the vector $\boldsymbol{x}(t)$ (29) at the moment $t = t_{(n-1)f}^o$. In case $n = 2$, the variable x_{2w} represents (52). With regard to the system (47) or (48) of Problem $\mathrm{P}(1)$, the variable x_{2w} (52) becomes (53) and after simplifying—(55).

$$x_{2w} = e^{\lambda_2 t} x_{20} + \int_0^{t_{1f}^o} e^{\lambda_2 (t-\tau)} (y_1^o(\tau) + b_2 u_1^o(\tau)) d\tau. \tag{52}$$

$$x_{2w} = x_{20} + x_{10} t_{11}^o + \frac{b_1 s_{11}^o u_0}{2} {t_{11}^o}^2. \tag{53}$$

$$x_{2w} = x_{20} + \frac{x_{10}|x_{10}|}{|b_1| u_0} + \frac{(-sign(b_1 x_{10})) {x_{10}}^2}{2 b_1 u_0}. \tag{54}$$

$$x_{2w} = x_{20} + \frac{sign(x_{10}) {x_{10}}^2}{2|b_1| u_0}. \tag{55}$$

Step 4. Applying the theorem for synthesizing the optimal function in the initial state [14] (Theorem 3.2, pp. 40–43), [15] (Theorem 3, p. 320), and [16] (Theorem 3, p. 41).

According to this theorem and its corollaries, the time-optimal control in the initial state of Problem $\mathrm{P}(2)$ represents (56).

$$u^o(0) = u^o(x_{10}, x_{20}) = \begin{cases} u_0 & when \quad x_{2+}x_{2w} > 0, \\ u_1^o(0) & when \quad x_{2+}x_{2w} = 0, \\ -u_0 & when \quad x_{2+}x_{2w} < 0. \end{cases} \tag{56}$$

The variable x_{k+}, respectively, x_{2+} in (56), is a term introduced in [14] (p. 38) and [15] (pp. 319–320) and defines the relationship between the points on axis x_k of the state-space of the system of Problem $\mathrm{P}(k)$ from the considered class of problems and the switching hyper-surface of the same Problem $\mathrm{P}(k)$. The value of the variable x_{k+} is determined by a procedure called "axes initialization" (Chapter 3, Section 3.3, pp. 60–88) [14] and (pp. 41–45) [16].

With regard to the example

$$x_{2+} = -1. \tag{57}$$

Hence, this means that all the points of the negative semi-axis Ox_2 are above the switching curve of Problem P(2) and the optimal control value for them is $+u_0$ while all the points of the positive semi-axis Ox_2 are below the switching curve of Problem P(2) and the optimal control value for them is $-u_0$.

Thus, after substitution x_{2w} for (55) and x_{2+} for (57) taking into consideration the initial state (x_{10}, x_{20}) based on (56), we obtain

$$u^o(0) = u^o(x_{10}, x_{20}) = \begin{cases} u_0 & when \quad -\left(x_{20} + \frac{sign(x_{10})x_{10}{}^2}{2|b_1|u_0}\right) > 0, \\ u_1^o(0) & when \quad \left(x_{20} + \frac{sign(x_{10})x_{10}{}^2}{2|b_1|u_0}\right) = 0\ , \\ -u_0 & when \quad -\left(x_{20} + \frac{sign(x_{10})x_{10}{}^2}{2|b_1|u_0}\right) < 0. \end{cases} \tag{58}$$

So, the synthesized optimal function with regard to a state (x_1, x_2) is

$$u^o(x_1, x_2) = \begin{cases} u_0 & when \quad -\left(x_2 + \frac{sign(x_1)x_1{}^2}{2|b_1|u_0}\right) > 0, \\ -sign(b_1x_1)u_0 & when \quad \left(x_2 + \frac{sign(x_1)x_1{}^2}{2|b_1|u_0}\right) = 0, \\ -u_0 & when \quad -\left(x_2 + \frac{sign(x_1)x_1{}^2}{2|b_1|u_0}\right) < 0. \end{cases} \tag{59}$$

Taking into account $b_1 = 1$ according to (46), (59) becomes

$$u^o(x_1, x_2) = \begin{cases} u_0 & when \quad \left(x_2 + \frac{sign(x_1)x_1{}^2}{2u_0}\right) < 0, \\ -sign(x_1)u_0 & when \quad \left(x_2 + \frac{sign(x_1)x_1{}^2}{2u_0}\right) = 0, \\ -u_0 & when \quad \left(x_2 + \frac{sign(x_1)x_1{}^2}{2u_0}\right) > 0. \end{cases} \tag{60}$$

Bearing in mind the relation (38) or (41) between $(y,\ v)$ and (x_1, x_2), one can easily appreciate that the analytical expression of the synthesized here optimal control (60) is identical with the expression obtained by the classical synthesis (37).

3.3. Simulation Results

For instance, let us depict the following two initial states

$$(y_0, v_0) = (10,\ 0). \tag{61}$$

$$(y_0, v_0) = (-10,\ 0). \tag{62}$$

The corresponding initial states in the state-space (x_1, x_2) of the system (44) are, respectively,

$$(x_{10}, x_{20}) = (0,\ 10). \tag{63}$$

$$(x_{10}, x_{20}) = (0, -10). \tag{64}$$

In Step 2, according to (49) and (50) and with regard to (63), we obtain

$$\begin{gathered} s_{11}^o = 0,\ t_{11}^o = 0, \\ t_{1f}^o = 0, \\ u_1^o(t) = 0. \end{gathered} \tag{65}$$

In Step 3, with regard to x_{2w} according to (53), we obtain

$$x_{2w} = x_{20} = 10. \tag{66}$$

Thus, in Step 4 in reference to (56) and (57), the result for the time-optimal control in the initial state (63) is

$$u^o(0) = u^o(0,\ 10) = -u_0 = -1. \tag{67}$$

Analogically, in accord with the initial state (64) in Step 2, likewise the initial state (63) in Step 2, we again obtain (65), but in terms of x_{2w} the result is

$$x_{2w} = x_{20} = -10, \tag{68}$$

which leads to

$$u^o(0) = u^o(0, -10) = u_0 = 1. \tag{69}$$

Figure 3 shows the near time-optimal processes with an accuracy of $\varepsilon_r = 0.001$ with regard to the considered initial states while the trajectories in the phase plane yv of the system (32) are shown in Figure 4a. The blue and red phase trajectories outline the initial states $(y_0, v_0) = (10,\ 0)$ and $(y_0, v_0) = (-10,\ 0)$, respectively. The near time-optimal trajectories relating to the corresponding initial states in the state-space $(x_1,\ x_2)$ of (44) $(x_{10},\ x_{20}) = (0,\ 10)$ and $(x_{10},\ x_{20}) = (0, -10)$ are represented in the phase plane x_1x_2 of (44) in Figure 4b. The blue trajectory concerns the state $(x_{10}, x_{20}) = (0,\ 10)$, however, the red one—$(x_{10}, x_{20}) = (0, -10)$. The conversion of the trajectories shown in Figure 4b by the relation (41) returns the identical result shown in Figure 4a.

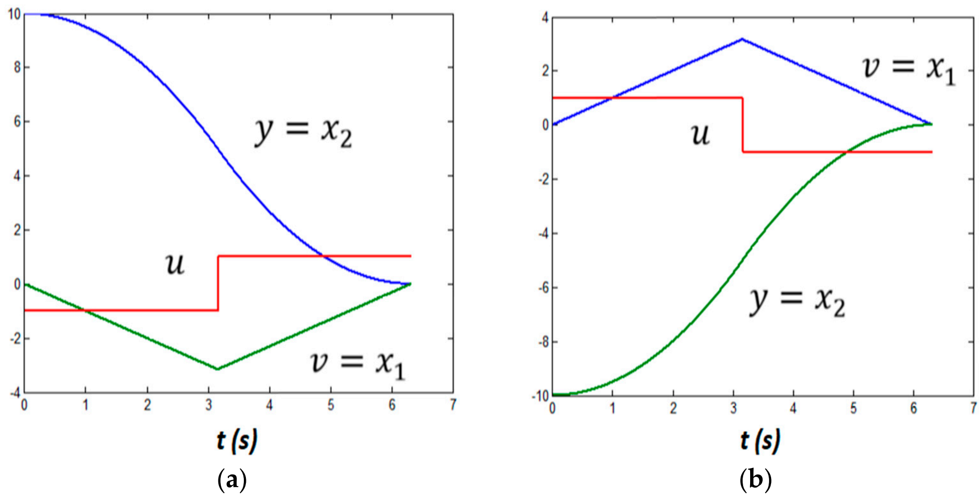
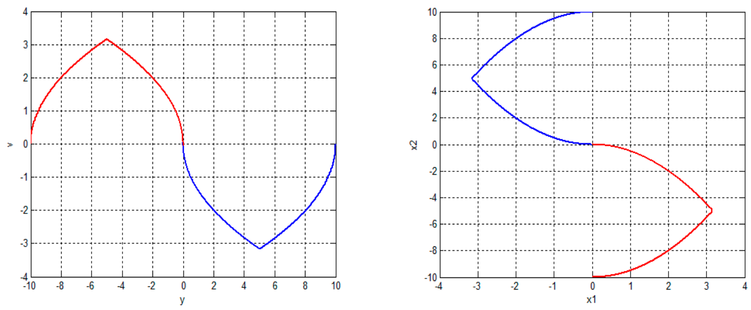


Figure 3. Near time-optimal process with an accuracy of $\varepsilon_r = 0.001$ referring to the initial state: (**a**) $(y_0, v_0) = (10,\ 0)$ with corresponding $(x_{10}, x_{20}) = (0,\ 10)$; (**b**) $(y_0, v_0) = (-10,\ 0)$ with corresponding $(x_{10}, x_{20}) = (0, -10)$.



(**a**) (**b**)

Figure 4. Phase trajectories of the near time-optimal processes with an accuracy of $\varepsilon_r = 0.001$ in the phase plane: (**a**) yv of the system (32); (**b**) x_1x_2 of the system (44).

4. Discussion

If the assumption of real non-positive eigenvalues of the system, in particular the constraints on the eigenvalues of subsystem (7), is omitted as they lack any specific characteristics, then in accordance with the idea and derivation technique it will simply turn out that the trajectory of the system with the initial state $\boldsymbol{x}_0$ (2), obtained under the action of the optimal control of Problem $\mathrm{P}(n-1)$, will coincide with the trajectory with the initial point $\boldsymbol{x}_0^1$ with coordinates (20) and (21) or that this trajectory will be completely below or above the last one in a vertical direction, determined by the axis x_n and will end at a point on the axis x_n different from the coordinate origin.

In order to serve the idea of synthesis this result is a matter of separate research. In case of not considering the spectral structure of the system matrix the number of switches or intervals of constancy, although finite, is not limited by the order of the system. At first glance, it might be appropriate to look at a certain area around the coordinate origin, if we do not deviate from the idea of the approach.

However, taking into account that the eigenvalues of the system are real non-positive, then on the basis of the theorem for the number of intervals of constancy [7] (Chapter 2, §6, Theorem 2.11, p. 116), we obtain that the initial point $\boldsymbol{x}_0$ of Problem $\mathrm{P}(n)$ and the end point of the obtained trajectory located on the axis x_n have the same relationship to the switching surface.

In [14–16], a novel property of the state space of the system has been defined, in particular that the positive and negative parts of the coordinate axes lie outside the switching hypersurface on opposite sides of the surface and the optimal control for the points of these axes is exactly with n number of intervals of constancy.

Then, finding the optimal control at the initial point $\boldsymbol{x}_0$ of Problem $\mathrm{P}(n)$ becomes a significantly easier task because it only requires solving the easier Problem $\mathrm{P}(n-1)$, which significantly reduces the computational load and the knowledge of the relation of the positive or negative parts of the axis x_n to the switching hyper-surface. The latter can again be obtained by solving the not so difficult problem of the lower order, but under specific initial conditions [14–16]. These data can be retrieved in advance and the process is called "axes initialization". Relying on a straightforward geometrical concept, this advantage of the approach is of significant benefit when solving high-order problems by immersing the initial problem in a class of problems Problem $\mathrm{P}(n)$, Problem $\mathrm{P}(n-1)$, … Problem $\mathrm{P}(1)$ and returning by reverse order to the initial Problem $\mathrm{P}(n)$.

Besides the property proved here, there is still a rigorous need to prove other properties of the problem in the case of its expansion. In [18], the author presents several interesting results of numerical experiments for near time-optimal control of a scanning lidar system based on the described method. Furthermore, the numerical aspects of the currently developed technique imply a close connection with linear programming.

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