

## Article

# The Logic of Homophily Dynamics in Heterogeneous Networks: Axiomatization, Model Checking and Validity Checking

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**Abstract:** Social networks have received considerable attention from the modal logic community. In this article, we study and characterize one of the most important principles in the field of social networks. Homophily, which means similarity breeds association, reveals the nature of social organization. In order to be able to express similarity and association together, we generalize the basic network and then define the heterogeneous network. The heterogeneous network is also defined to provide a good foundation for the use of logical approaches. The Logic of Homophily  $LH_{G,M}$  that we propose in this article is based on Computation Tree Logic and Formal Concept Analysis.  $LH_{G,M}$  describes the homophily dynamics of the heterogeneous networks at a specified similarity coefficient. Furthermore, we not only axiomatize the  $LH_{G,M}$  and prove that the axiom system  $\mathbf{LH}_{G,M}^n$  is sound and complete, but we also prove that the model checking and the validity checking for  $LH_{G,M}$  are both PSPACE-complete.

**Keywords:** homophily; computation tree logic; implication between attributes; axiom system; model checking; validity checking

MSC: 03B44



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## 1. Introduction

With the rapid development in recent years, social network analysis (SNA) has been considered as an effective tool for solving various problems in sociology, economics, and management. As a model that can intuitively represent knowledge, beliefs, and social problems, social networks also received a lot of attention from modal logicians (e.g., [1–7]). In this article, we will characterize one of the most influential principles of social networks, *homophily*, from the perspective of logic.

Homophily describes the basic social organization rule that similarity breeds association [8]. In other words, those who have more of the same attributes will be more likely to tend to associate with each other. For example, people that have the same interests, have the same academic background, and have the same hometown are more likely to become friends.

In short, a general network is a structure composed of an object set and a relation between objects, which is usually depicted by a graph, in which objects and relationships are represented by nodes and edges. For example, in Figure 1, objects 1, 3 and 4 are, respectively, associated with 2. Similarly, the affiliation network is a structure composed of an object set, an attribute set and a relation between object and attribute. For example, in Figure 2 (for all figures in this article, unshaded circles represent objects, shaded circles represent attributes, straight lines represent relationships between objects and attributes, and curved lines represent relationships between objects), the object 1 has the attribute *a* and *b*, both the object 3 and 4 have the attribute *c*. However, neither of the two most common types of networks can be directly used to represent homophily since they can only represent association and similarity, respectively. In order to study homophily on the basis of networks, sociologists often use additional data tables and some statistical methods,

which are not suitable for logic. Therefore, we will use the heterogeneous network (*HetNet* for short, e.g., Figure 3), which is a direct combination of general network and affiliation network and can represent both association and similarity.

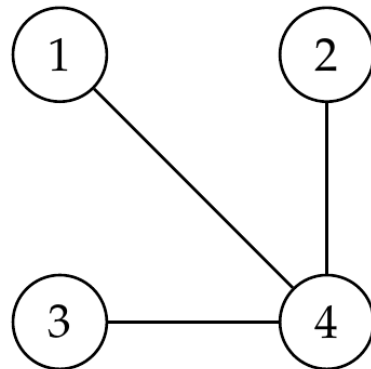


Figure 1. General Network.

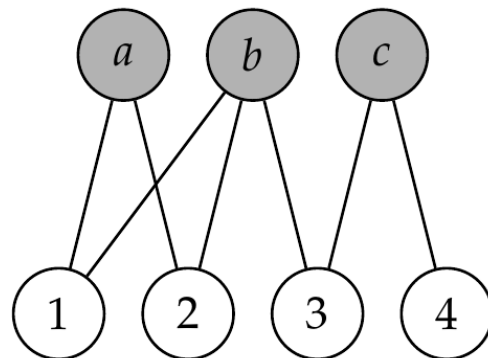


Figure 2. Affiliation Network.

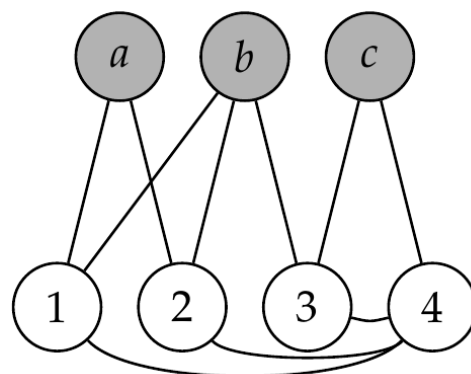


Figure 3. Heterogeneous Network.

In this article, the logic of homophily will be based on the HetNets to study and describe the dynamics of networks under the principle of homophily. In a HetNet, if a pair of objects has a certain number of the same attributes, it is considered that the two are similar in a certain degree, and thus they will be associated with each other. If in the HetNet, all similar objects have already been associated with each other, it is considered that the network is *homogenized*.

The most related work to this article is *logic of allies and enemies* (LAE) [5–7,9]. LAE is based on *computation tree logic* (CTL), and mainly characterizes another important structural theory in social networks—structure balance theory. Similarly, the logic of homophily in this paper will be also based on CTL to characterize the dynamics of the HetNet under

homophily. In addition, logic of homophily will introduce the methods in *formal concept analysis* (FCA) to represent and extract the implicit information of the HetNet.

Previous logic study on social network structure is only about the structure balance theory and stays on the general object networks. In this article, we generalize the logic study of the structure of social networks to HetNets and the principle of homophily. We first combine general object networks with affiliation networks to propose HetNets that can be used to characterize the principle of homophily; secondly, we define similarity and homophily in HetNets and construct the homophily-evolution models; thirdly, we establish the logic of homophily, which is based not only on CTL but also on FCA, and is, therefore, more expressive, to describe and analyze the HetNet and its homophily dynamic; fourthly, we axiomatize the logic and prove that its axiom system is sound and complete; finally, we propose the model checking and validity checking problem and prove that they are both PSPACE-complete by improving the algorithms.

The structure of this article is as follows: In Section 2, we first introduce the definition of HetNet and define the similarity and homophily on it, and then establish the homophily-evolution model to describe the dynamics. In Section 3, we first introduce the implication between attributes on HetNet, then define the language and semantics of logic of homophily  $LH_{G,M}$ , and give some valid formulas. In Section 4, we first introduce the description formula of the HetNet, and then define and explain the axiom system  $LH_{G,M}^n$ , and finally prove the soundness and the completeness of  $LH_{G,M}^n$ . In Section 5, we express what the model checking for  $LH_{G,M}$  is and then prove it is PSPACE-complete. In Section 6, we express what the validity checking for  $LH_{G,M}$  is and then prove it is PSPACE-complete. In Section 7, we summarize our work in this article and illustrate the directions for future work.

## 2. HetNets and Homophily

**Definition 1.** HetNet is a quadruple  $N = (G, M, I, R)$ , where  $G$  and  $M$  are non-empty finite sets,  $I \subseteq G \times M$ ,  $R \subseteq G \times G$ , and  $R$  is reflexive and symmetric.

In the above definition, requiring the relation  $R$  to be reflexive is to reflect that “the object is always associated with itself”. In order to avoid ambiguity, we specifically require that  $G \cap M = \emptyset$ . Then, in order to simplify the problem, we will only consider the case that object relations always appear in pairs, that is,  $R$  is required to be symmetric.  $G$  and  $M$  denote the object set and attribute set, respectively,  $I$  and  $R$  denote the relationship between objects and attributes and the relationship between objects, respectively.

As mentioned in Section 1, the HetNet given in the Definition 1 is one of the simplest and most intuitive heterogeneous networks and is a direct composition of a general network and an affiliation network. In other words, any HetNet  $N = (G, M, I, R)$  can be regarded as composed of the affiliation network  $(G, M, I)$  and the general network  $(G, R)$ . For example, the network in Figure 3 is combined by the network in Figure 1 and the network in Figure 2.

### 2.1. Similarity and Homophily

Consider sets  $X, Y$ .  $\uparrow: 2^X \times 2^{X \times Y} \rightarrow 2^Y$  and  $\downarrow: 2^Y \times 2^{X \times Y} \rightarrow 2^X$ , respectively, denote the following operators (where,  $A \subseteq X$ ,  $B \subseteq Y$ ,  $E \subseteq X \times Y$ ):

$$A^{\uparrow E} = \{y \in Y \mid \forall a \in A, aEy\},$$

$$B^{\downarrow E} = \{x \in X \mid \forall b \in B, xEb\}.$$

For the sets  $X, Y$ , if the relation  $E$  is fixed, this pair of operators can be regarded as a pair of concept-forming operators in FCA.

FCA, first proposed by Rudolf Wille in [10], is a formal tool that has a solid theoretical foundation and is widely used in various areas, e.g., knowledge representation, information management, and data mining. Not only is the basic structure in FCA, called *formal context*, isomorphic to the affiliation network, but the methods of FCA can also be used to represent and extract implicit information in networks [11,12]. Although our work is

heavily inspired by FCA, in this article, we will only introduce FCA where necessary since this work has not yet covered the FCA methods in depth (See [13] for more details about FCA).

**Definition 2.** Consider  $N = (G, M, I, R)$  and  $O \subseteq G$ , the similarity of the object set  $O$  is defined as  $\text{Sim}(O) = |O^{\uparrow I}|$ . Let  $0 < n \leq |M|$  be a similarity coefficient. If  $\text{Sim}(O) \geq n$ , the objects in set  $O$  are considered to be  $n$ -similar.

Intuitively,  $O^{\uparrow I}$  represents the common attributes in  $M$  of objects in  $O$ , and the more they have the same attributes, the more similar they are and the higher the similarity coefficient the set has. For example, in Figure 3,  $\text{Sim}(\{1, 2\}) = |\{a, b\}| = 2$ ,  $\text{Sim}(\{3, 4\}) = |\{d\}| = 1$ . Therefore, objects 1 and 2 are 2-similar and also 1-similar, but objects 3 and 4 are only 1-similar. In other words, objects 1 and 2 are more similar than 3 and 4. Since  $\emptyset^{\uparrow I} = M$ , the empty set always has similarities and has the largest similarity. For any  $g \in G$ ,  $\{g\}^{\uparrow I} = \{m \in M \mid gIm\}$ , so if one object has attributes, it is always the most similar to itself.

From the perspective of FCA,  $\cdot^{\uparrow I}$  is one of concept-forming operators of formal context  $K = (G, M, I)$  that is isomorphic to the component affiliation network  $(G, M, I)$  of HetNet  $(G, M, I, R)$ , and any set of objects with similarity is a subset of the *extension* of one certain maximal non-empty formal concept in  $K$ . Therefore, as long as all the maximal non-empty formal concepts are obtained, all object sets with similarity in a HetNet can be also obtained. Existing polynomial-delay algorithms for computing formal concepts can do this efficiently (see [14] for details).

**Definition 3.** Consider  $N = (G, M, I, R)$ ,  $O \subseteq G$  and a similarity coefficient  $0 < n \leq |M|$ . If the object set  $O$  is  $n$ -similar and  $O \subseteq O^{\uparrow R}$ , the set  $O$  is homogenized at  $n$ . If all  $n$ -similar object sets in  $N$  are homogenized, then the HetNet  $N$  is called homogenized at the given similarity coefficient  $n$ . Otherwise, it is called not homogenized at  $n$ .

In the above definition,  $\cdot^{\uparrow R}$  can be understood as one of the concept-forming operators of formal context  $(G, G, R)$  that is isomorphic to the component general network  $(G, R)$  of HetNet  $(G, M, I, R)$ , and  $O^{\uparrow R}$  represents the objects that are associated with all objects in  $O$ . Similarly, the empty set is always homogenized since  $\emptyset^{\uparrow R} = G$ . And because  $R$  is reflexive, any single-point set with similarity is always homogenized. In particular, if a HetNet does not have any  $n$ -similar set, then it is vacuously homogenized at  $n$ . Take Figure 3 as an example. Since  $\{3, 4\}^{\uparrow R} = \{3, 4\}$  and  $\{1, 2\}^{\uparrow R} = \emptyset$ ,  $\{3, 4\}$  is homogenized at 2 but  $\{3, 4\}$  is not even homogenized at 1.

The principle of homophily describes the process of individual aggregation and organization caused by similarity. In other words, homophily shows that the network has a tendency to be homogenized.

## 2.2. Homophily-Evolution Models

**Definition 4.** Consider HetNet  $N = (G, M, I, R)$  and a similarity coefficient  $0 < n \leq |M|$ . HetNet  $N' = (G, M, I, R')$ , where  $R \subseteq R'$ , is called the  $n$ -immediate successor of  $N$  (denoted  $N \rightsquigarrow_n N'$ ) if and only if  $N'$  satisfies the following conditions:

- (1) If  $N$  is homogenized, then  $N' = N$ ;
- (2) If  $N$  is not homogenized, there is only one  $n$ -similar object set  $\{i, j\} \subseteq G$ , where  $i, j$  are different objects, satisfying that the object set is not homogenized on  $N$ , but homogenized on  $N'$ , i.e.,  $\text{Sim}(\{i, j\}) \geq n$ ,  $(i, j) \notin R$ ,  $(j, i) \notin R$ ,  $(i, j) \in R'$  and  $(j, i) \in R'$ .

The Definition 4 means that each update of the non-homogenized HetNet can and can only make a pair of objects with  $n$ -similarity associated. Although it is more intuitive to associate objects with higher similarity first, the updates are “equal” to each pair of objects that will be homogenized since we do not intend to discuss this issue in depth in

this article. Take Figure 3 as an example. The HetNet in Figures 3–5 are its 1-immediate successor, 2-immediate successor and 3-immediate successor, respectively. The update made in Figure 4 is to associate object 2 with 3 since they have one common attribute  $b$ . The update made in Figure 5 is to associate object 1 with 2 since they have two common attributes  $a$  and  $b$ . In other words, the HetNet in Figure 5 is also a 1-immediate successor. Because there are no 3-similar sets in Figure 3's HetNet, it is homogenized at 3 and thus its only 3-immediate successor is itself.

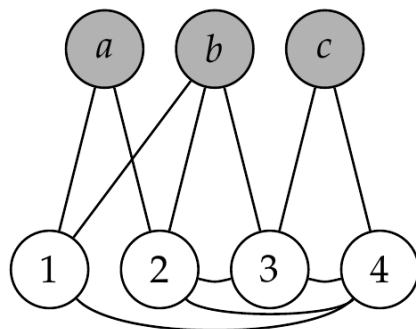


Figure 4. 1-immediate Successor.

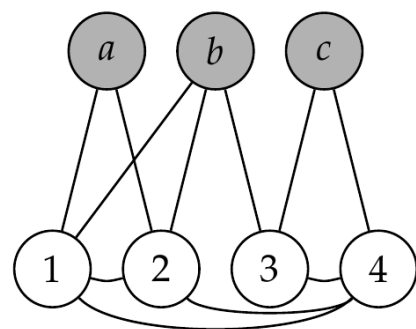


Figure 5. 2-immediate Successor.

**Definition 5.** Consider HetNet  $N = (G, M, I, R)$  and  $0 < n \leq |M|$ . If there is a sequence  $N_0 \rightsquigarrow_n N_1 \rightsquigarrow_n \dots \rightsquigarrow_n N_m$  ( $m$  is a positive integer) such that  $N_0 = N$ , and either  $m = 0$ , or for any  $0 \leq i < m$ ,  $N_i$  is not homogenized, then if and only if  $N_m$  is homogenized, we call this sequence the  $n$ -homophily-evolution line of  $N$  ( $n$ -timeline for short, denoted  $l^n(N)$ ) and call  $N_i$  the  $n$ -successor of  $N$ . The  $n$ -homophily-evolution tree  $t^n N = (\mathcal{N}, \mathcal{S})$  of  $N$  ( $n$ -timetree, for short) is a tuple, where  $\mathcal{N} = \{N' \mid N' \text{ is the } n\text{-successor of } N\}$ ,  $\mathcal{S} = \{(U, V) \mid U, V \in \mathcal{N} \text{ and } U \rightsquigarrow_n V\}$ .

By the immediate successor relation given in Definition 4, a HetNet will gradually form an  $n$ -timetree, which is non-transitive, antisymmetric, and has self-loops only at the terminal nodes. Since the network will eventually reach homogenized, the depth of this timetree is finite. Let  $len(l^n)$  and  $dep(t^n)$  be the length of  $n$ -timeline and the depth of  $n$ -timetree, respectively.  $len(l^n)$  is the number of HetNets in an  $n$ -timeline and  $dep(t^n)$  is the length of the longest  $n$ -timeline in it.

**Proposition 1.** The depth of  $n$ -timetree is limited by  $\frac{|G|(|G|-1)}{2} + 1$ .

For a HetNet  $N = (G, M, I, R)$ , only when any two of the objects are not associated but are all  $n$ -similar, the number of updates of  $N$  can reach the maximum.

### 3. Logic of Homophily $LH_{G,M}$

In this section, we will introduce the *implication between attributes* (IbA) and generalize it on HetNet, and then establish the *Logic of Homophily*  $LH_{G,M}$ .

### 3.1. Implication between Attributes and Index

IbA, or the IbA system is an important method in FCA, which is used to represent and extract information from formal contexts. As the “language” of formal contexts, the IbA system describes and characterizes a formal context by expressing the “implicative relationship” between attributes. In other words, “the attribute set  $A$  “implicates” the attribute set  $B$ ” actually means “any object that has all the attributes in  $A$  also has all the attributes in  $B$ ”. Consider an affiliation network or a formal context  $(G, M, I)$  and attribute sets  $A, B \subseteq M$ . An IbA of the formal context is denoted by  $A \Rightarrow B$ .  $A \Rightarrow B$  is valid on the formal context  $(G, M, I)$  if and only if  $A^{\downarrow I} \subseteq B^{\downarrow I}$ .

Since the HetNet and the homophily principle involve not only the relationship between objects and attributes, but also the association between objects, IbA in FCA cannot be directly used to describe the HetNet and its homophily. However, because the formal context is isomorphic to the affiliation network, and the general network  $(G, R)$  can also be expressed in the form of the affiliation network  $(G, G, R)$ , i.e., the HetNet can also be similarly expressed, the IbA method can be naturally generalized to the HetNets.

Consider a HetNet  $N = (G, M, I, R)$ . Let  $\hat{G} = \{\hat{g} \mid g \in G\}$  be its *index set*,  $\hat{R} = R \cup \{(g, \hat{g}) \mid g \in G\}$  be its *index expansion relation*.

**Definition 6.** An IbA on HetNet  $N = (G, M, I, R)$  is a tuple  $(A, B)$  (denoted as  $A \Rightarrow B$ ), where  $A, B \subseteq G \cup \hat{G} \cup M$ . IbA  $A \Rightarrow B$  is valid on a HetNet  $N$  if and only if  $A^{\downarrow I\hat{R}} \subseteq B^{\downarrow I\hat{R}}$ .

In definition 6,  $I\hat{R}$  is shorthand for  $I \cup \hat{R}$  and This type of abbreviation in the following also means the same. In addition, for the convenience of writing and avoiding ambiguity, brackets around sets in IbA are usually omitted and the string will be arranged in the order of numbers, indexes, and letters, and the numbers and letters will be arranged in the order of natural numbers and alphabets, respectively. For example, we write  $1\hat{3}a \Rightarrow bd$  as an IbA instead of  $\{1, a, \hat{3}\} \Rightarrow \{b, d\}$ .

The generalization made by Definition 6 is essentially to use the symbols in the object set and index set to represent the attributes of being associated with the “object represented by this symbol” and attributes of only being associated with the “object represented by this symbol”, respectively. For example, IbA  $1 \Rightarrow a$  means “the objects associated with object 1 all have the attribute  $a$ ”, and “1” in this IbA means the attribute “has a relationship with the object 1”, instead of “the object 1”.  $\hat{1} \Rightarrow b$  means “the object only associated with 1 has the attribute  $b$ ”, and since all objects in the HetNet are associated with themselves,  $\hat{1} \Rightarrow b$  actually means “object 1 has attribute  $b$ ”.

### 3.2. Syntax and Semantics

The language  $\mathcal{L}_{G,M}$  of logic of homophily  $LH_{G,M}$  is built on the given object set  $G$  and attribute set  $M$  (where,  $G$  and  $M$  is finite, non-empty and satisfies  $G \cap M = \emptyset$ ). The formula  $\phi$  of  $\mathcal{L}_{G,M}$  is defined as follows (where  $A, B \subseteq G \cup M \cup \hat{G}$ ):

$$\phi ::= A \Rightarrow B \mid \top \mid \text{Hom} \mid \neg\phi \mid \phi \vee \phi \mid \text{AX}\phi \mid \text{E}(\phi \cup \phi) \mid \text{A}(\phi \cup \phi).$$

The formula  $A \Rightarrow B$  represents the IbA  $A \Rightarrow B$ , Hom represents the HetNet is homogenized, and the propositional connectives and temporal operators are consistent with the conventional definitions (more information about CTL temporal operators, please refer to [15,16]). Other propositional connectives and temporal operators are also defined as usual. For example,  $\phi \rightarrow \psi = \neg\phi \vee \psi$ ,  $\text{EX}\phi = \neg\text{AX}\neg\phi$ ,  $\text{AF}\phi = \text{A}(\top \cup \phi)$ . In addition, also for the convenience of writing, in the following, the expression of “atomic formula”  $A \Rightarrow B$  is consistent with the specification of IbA in the Section 3.1.

Consider  $LH_{G,M}$ . The formula  $\phi \in \mathcal{L}_{G,M}$  is *satisfied* on a HetNet  $N = (G, M, I, R)$  and at a similarity coefficient  $n$  (denoted as  $N \models_n \phi$ ) defined as follows (where  $A, B \subseteq G \cup M \cup \hat{G}$ ,  $\psi \in \mathcal{L}_{G,M}$ ,  $0 < n \leq |M|$ ):

|                                        |        |                                                                                                                                                                                     |
|----------------------------------------|--------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $N \models_n A \Rightarrow B$          | $\iff$ | $\text{IbA } A \Rightarrow B \text{ is valid on } N.$                                                                                                                               |
| $N \models_n \top$                     |        | always.                                                                                                                                                                             |
| $N \models_n \text{Hom}$               | $\iff$ | $N \rightsquigarrow_n N.$                                                                                                                                                           |
| $N \models_n \neg\phi$                 | $\iff$ | $N \not\models_n \phi.$                                                                                                                                                             |
| $N \models_n \phi \vee \psi$           | $\iff$ | $N \models_n \phi \text{ or } N \models_n \psi.$                                                                                                                                    |
| $N \models_n \text{AX}\phi$            | $\iff$ | $\forall N', \text{ if } N \rightsquigarrow_n N', \text{ then } N' \models_n \phi.$                                                                                                 |
| $N \models_n \text{E}(\phi \cup \psi)$ | $\iff$ | There is a timeline $N_0 \rightsquigarrow_n N_1 \rightsquigarrow_n \dots$ with $N_0 = N$ ,<br>$\exists i \in \mathbb{N}, N_i \models_n \psi$ , for any $j < i, N_j \models_n \phi.$ |
| $N \models_n \text{A}(\phi \cup \psi)$ | $\iff$ | For any timeline $N_0 \rightsquigarrow_n N_1 \rightsquigarrow_n \dots$ with $N_0 = N$ ,<br>$\exists i \in \mathbb{N}, N_i \models_n \psi$ , for any $j < i, N_j \models_n \phi.$    |

Take Figure 3 as an example to briefly explain the semantics. Let  $N$  denote the network in it. First,  $N \models_n 2a \Rightarrow b, N \models_n \neg 4 \Rightarrow b, N \models_n \neg c \Rightarrow b$  represent “all objects associated with object 2 that have attribute  $a$  have attribute  $b$ ”, “object 4 does not have attribute  $b$ ”, and “there is an object that has attribute  $c$  and does not have attribute  $b$ ”. Then,  $N \models_1 \text{EX}(\dot{1} \Rightarrow 2)$  and  $N \models_1 \neg \text{Hom}$  represent, respectively, “there is a 1-immediate successor of  $N$  such that object 1 is associated with object 2” and “the HetNet  $N$  is not homogenized at 1”. Next,  $N \models_1 \text{A}(\neg \dot{1} \Rightarrow 3 \vee \neg \dot{1} \Rightarrow 2\text{UHom})$  means “On any timeline starting from  $N$  with the similarity coefficient 1 until the network homogenized, there is always that ‘object 1 has no association with 3 or object 1 is no associated with 2’”. Finally,  $N \models_2 \neg \text{EX}(\dot{1} \Rightarrow 2)$  and  $N \models_3 \text{Hom}$  shows that “there is not a 2-immediate successor of  $N$  such that object 1 is associated with object 2” and “the HetNet  $N$  is homogenized at 3”.

Given a formula  $\phi \in LH_{G,M}$  and a similarity coefficient  $n$ .  $\phi$  is called *satisfied* if and only if there is an  $N = (G, M, I, R)$  s.t.  $N \models_n \phi$ .  $\phi$  is called *valid* (denoted  $\models_n \phi$ ) if and only if for any  $N = (G, M, I, R)$ ,  $N \models_n \phi$ . Consider a formula set  $\Phi$  and a HetNet  $N$ .  $N \models_n \Phi$  if and only if for any  $\psi \in \Phi$ ,  $N \models_n \psi$ . In addition,  $\phi$  is called a *semantic consequence* of  $\Phi$  (denoted  $\Phi \models_n \phi$ ) if and only if for any HetNet  $N$ , if  $N \models_n \Phi$ , then  $N \models_n \phi$ .

The Proposition 2 gives some basic valid formulas of  $LH_{G,M}$ .

**Proposition 2.** For  $LH_{G,M}$ , the following formulas are valid (where  $A \subseteq G \cup M \cup \dot{G}$ ,  $a \in M$ ,  $\dot{g}, \dot{g}' \in \dot{G}$ ,  $0 < n \leq |M|$ ):

- (1)  $\models_n \text{AFHom}$ ;
- (2)  $\models_n \dot{g} \Rightarrow A \rightarrow \text{AX} \dot{g} \Rightarrow A$ ;
- (3)  $\models_n \neg \dot{g} \Rightarrow a \rightarrow \text{AX} \neg \dot{g} \Rightarrow a$ ;

The three valid formulas in the Proposition 2 state the following: (1) The HetNet will eventually be homogenized; (2) The update will not change the existing relationship; (3) The update will not generate new relationships between objects and attributes.

#### 4. Axiom System $LH_{G,M}^n$

The axiomatization of the logic of homophily  $LH_{G,M}$  relies on the description method first introduced by [9]. In short, the description is a formula used to characterize the HetNet. The description formula in this article is based on the regular atomic formula.

Consider  $LH_{G,M}$ . If the atomic formula  $A \Rightarrow B$  satisfies  $B \subseteq G \cup M$ , then  $A \Rightarrow B$  is called *regular*. Otherwise,  $A \Rightarrow B$  is *irregular*. For any regular atomic formula  $\phi = A \Rightarrow B$ , the *complement* of  $\phi$  (denoted as  $\sim \phi$ ) is a conjunction defined as follows:

$$\sim (A \Rightarrow B) = \bigwedge_{a \in (G \cup M) \setminus B} (\neg A \Rightarrow a).$$

For any  $g \in G$ ,  $A_g$  represents the subset of  $G \cup M$  only determined and selected by  $g$ . The *description*, denoted as  $\delta$ , is composed of the following series of regular formulas and their complements:

$$\delta = \bigwedge_{g \in G} (g \Rightarrow A_g \wedge \sim (g \Rightarrow A_g)).$$

For the conjunction  $\psi$  in which each branch is a regular atomic formula, let  $\sim \psi$  represent the conjunction of the complement of all branch formulas in  $\psi$ . Let  $\delta^+$  be the conjunction of all regular atomic formulas in  $\delta$ , and  $\sim \delta^+$  be the conjunction of complements of all its regular atomic formulas, and obviously  $\delta \leftrightarrow \delta^+ \wedge \sim \delta^+$ .

For the HetNet  $N = (G, M, I, R)$ , the *description* of  $N$ ,  $\delta(N)$  is defined as follows (the order of branches of conjunction is determined and  $IR$  is shorthand for  $I \cup R$ ):

$$\delta(N) = \bigwedge_{g \in G} (g \Rightarrow \{g\}^{\uparrow IR} \wedge \sim (g \Rightarrow \{g\}^{\uparrow IR})).$$

Since in a HetNet, objects are always associated with themselves and the associations between objects are always symmetric, not all description formulas describe a HetNet. In this article, we specifically stipulate the order of branches of conjunction in the description formula, i.e., the atomic formula is always on the left side of its complement, arrange the atomic formulas in the order of natural numbers according to the object indexes in them, and arrange the branches of complements of atomic formulas in the order of natural numbers and alphabetical order according to the symbols on the right of  $\Rightarrow$ . Therefore, it is not difficult to prove that if  $\delta(N)$  is the description of HetNet  $N$ ,  $\delta(N)$  is the unique description of  $N$ . Take Figure 3 as an example. Let  $N$  denote the network in it. Its description formula  $\delta(N) = (\dot{1} \Rightarrow 14ab \wedge \dot{2} \Rightarrow 24ab \wedge \dot{3} \Rightarrow 34bc \wedge \dot{4} \Rightarrow 1234c) \wedge ((\neg \dot{1} \Rightarrow 2 \wedge \neg \dot{1} \Rightarrow 3 \wedge \neg \dot{1} \Rightarrow c) \wedge (\neg \dot{2} \Rightarrow 1 \wedge \neg \dot{2} \Rightarrow 3 \wedge \neg \dot{2} \Rightarrow c) \wedge (\neg \dot{3} \Rightarrow 1 \wedge \neg \dot{3} \Rightarrow 2 \wedge \neg \dot{3} \Rightarrow a) \wedge (\neg \dot{4} \Rightarrow a \wedge \neg \dot{4} \Rightarrow b))$ .

Table 1 represents the axiom system  $\mathbf{LH}_{G,M}^n$  of the logic of homophily  $LH_{G,M}$  with the similarity coefficient  $n$ . In Table 1,  $\phi, \psi, \chi$  are arbitrary formulas,  $A, B, C, D \subseteq G \cup M \cup \dot{G}$ ,  $a \in M$ ,  $E^n \subseteq M$  and  $|E^n| \geq n$ ,  $g, g' \in G$ ,  $\dot{g}, \dot{g}' \in \dot{G}$ ,  $\dot{O} \Rightarrow A$  and  $\sim \dot{O} \Rightarrow A$ , respectively, represent  $\bigwedge_{g \in O} \dot{g} \Rightarrow A$  and  $\bigwedge_{g \in O} \neg \dot{g} \Rightarrow A$  ( $O$  is not empty and  $O \subseteq G$ ),  $\sigma_{(g,g')}$  means  $\bigwedge_{a \in M} (\dot{g} \Rightarrow a \leftrightarrow \neg \dot{g}' \Rightarrow a)$ ,  $\delta$  represents the description of a HetNet, and *homophily axiom* is the following formula (where  $\delta_1, \delta_2, \dots, \delta_m$  enumerate the descriptions of all HetNets that are homogenized at  $n$ ):

$$(HF) \quad \text{Hom} \leftrightarrow (\delta_1 \vee \delta_2 \vee \dots \vee \delta_m).$$

The axiom system  $\mathbf{LH}_{G,M}^n$  consists of three parts: CTL axioms and rules, axioms about IbAs and axioms and rules about HetNet and update.

CTL axioms and rules are consistent with those in [16].

In the axioms about IbAs, (Ref), (Frag), (Comp), and (Trans) are the axioms of Armstrong's system (see [17] for more details). (Frag) $_{\neg}$ , (Comp) $_{\neg}$  are the negative extensions of the (Frag) and (Comp). (Incl) indicates the feature of the inclusion relationship. (Abs) explain how the attribute set  $A$  implicates the attribute set  $B$ . (Emp) means that the set of attributes not owned by any object implicates any set of attributes.

In the axioms and rules about HetNet and updates, (Loop) indicates that the object is always associated with itself. (Loop) $_{\neg}$  indicates that the index symbol only indicates the relationship between the object and itself. (Symm) indicates that the relationship between objects is always symmetric. (A-gen) and (A-gen) $_{\neg}$  are valid formulas introduced in Section 3.2. (E-gen) represents the relationship generation in an update. (E-gen) $_{\neg}$  represents the relationship that will not be generated in an update. (Exp) represents the immediate successor. (Hom) and (Hom) $_{\neg}$  indicate that the immediate successor of a homogenized HetNet is always itself and the immediate successor of a HetNet which is not homogenized cannot be itself. In fact, (Hom) and (Hom) $_{\neg}$  can be derived from (HF), (A-gen), (A-gen) $_{\neg}$ , (E-gen) and (Exp). However, for the convenience of the follow-

ing proofs, these two formulas are still retained as axioms. (HF), (Tm<sub>1</sub>) and (Tm<sub>2</sub>) are consistent with the similar axioms and rules in [9].

Consider a formula set  $\Phi$  and a formula  $\phi$ . If  $\phi$  can be derived by axioms and rules of  $\text{LH}_{G,M}^n$ ,  $\phi$  is a *theorem* of  $\text{LH}_{G,M}^n$  (denoted  $\vdash_n \phi$ ). If  $\phi$  can be derived from  $\Phi$  by axioms and rules of  $\text{LH}_{G,M}^n$ ,  $\Phi \vdash_n \phi$ .

**Table 1.** Axiom system  $\text{LH}_{G,M}^n$ .

| CTL axioms and rules:                     |                                                                                                                                                                                                                                      |
|-------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (PC)                                      | Substitutions of all propositional tautologies                                                                                                                                                                                       |
| (K <sub>x</sub> )                         | $AX(\phi \rightarrow \psi) \rightarrow (AX\phi \rightarrow AX\psi)$                                                                                                                                                                  |
| (D <sub>x</sub> )                         | $EX\top$                                                                                                                                                                                                                             |
| (EU)                                      | $E(\phi U \psi) \leftrightarrow \psi \vee (\phi \wedge EXE(\phi U \psi))$                                                                                                                                                            |
| (AU)                                      | $A(\phi U \psi) \leftrightarrow \psi \vee (\phi \wedge AXA(\phi U \psi))$                                                                                                                                                            |
| (MP)                                      | From $\phi$ and $\phi \rightarrow \psi$ to infer $\psi$                                                                                                                                                                              |
| (Nec)                                     | From $\phi$ to infer $AX\phi$                                                                                                                                                                                                        |
| (A-ind)                                   | From $\psi \vee (\phi \wedge AX\chi) \rightarrow \chi$ to infer $A(\phi U \psi) \rightarrow \chi$                                                                                                                                    |
| (E-ind)                                   | From $\psi \vee (\phi \wedge EX\chi) \rightarrow \chi$ to infer $E(\phi U \psi) \rightarrow \chi$                                                                                                                                    |
| Axioms about IbAs:                        |                                                                                                                                                                                                                                      |
| (Ref)                                     | $A \Rightarrow A$                                                                                                                                                                                                                    |
| (Frag)                                    | $A \Rightarrow B \cup C \rightarrow A \Rightarrow B$                                                                                                                                                                                 |
| (Frag <sub>¬</sub> )                      | $\neg A \cup B \Rightarrow C \rightarrow \neg A \Rightarrow C$                                                                                                                                                                       |
| (Comp)                                    | $A \Rightarrow B \wedge C \Rightarrow D \rightarrow (A \cup C \Rightarrow B \cup D)$                                                                                                                                                 |
| (Comp <sub>¬</sub> )                      | $\neg A \Rightarrow B \wedge \neg A \Rightarrow C \rightarrow \neg A \Rightarrow B \cup C$                                                                                                                                           |
| (Trans)                                   | $A \Rightarrow B \wedge B \Rightarrow C \rightarrow A \Rightarrow C$                                                                                                                                                                 |
| (Incl)                                    | $(A \Rightarrow B \wedge \neg A \Rightarrow C) \rightarrow \neg B \Rightarrow C$                                                                                                                                                     |
| (Abs)                                     | $(\dot{O} \Rightarrow A \wedge \dot{O} \Rightarrow B \wedge \sim (G \setminus O) \Rightarrow A) \rightarrow A \Rightarrow B$                                                                                                         |
| (Emp)                                     | $\sim \dot{G} \Rightarrow A \rightarrow A \Rightarrow B$                                                                                                                                                                             |
| Axioms and rules about HetNet and update: |                                                                                                                                                                                                                                      |
| (Loop)                                    | $\dot{g} \Rightarrow g$                                                                                                                                                                                                              |
| (Loop <sub>¬</sub> )                      | $\neg g \Rightarrow \dot{g}' \quad (g \neq g')$                                                                                                                                                                                      |
| (Symm)                                    | $\dot{g} \Rightarrow g' \leftrightarrow \dot{g}' \Rightarrow g$                                                                                                                                                                      |
| (A-gen)                                   | $\dot{g} \Rightarrow A \rightarrow AX\dot{g} \Rightarrow A$                                                                                                                                                                          |
| (A-gen <sub>¬</sub> )                     | $\neg \dot{g} \Rightarrow a \rightarrow AX\neg \dot{g} \Rightarrow a$                                                                                                                                                                |
| (E-gen)                                   | $(\dot{g} \Rightarrow E^n \wedge \dot{g}' \Rightarrow E^n) \rightarrow EX\dot{g} \Rightarrow g'$                                                                                                                                     |
| (E-gen <sub>¬</sub> )                     | From $\delta \rightarrow \neg \dot{g} \Rightarrow g' \wedge \sigma_{(g,g')}$ to infer $\delta \rightarrow \neg EX\dot{g} \Rightarrow g'$                                                                                             |
| (Exp)                                     | From $\delta \rightarrow \neg \dot{g} \Rightarrow g' \wedge EX\dot{g} \Rightarrow g'$<br>to infer $\delta \rightarrow EX(\dot{g} \Rightarrow g' \wedge \sim (\delta^+ \wedge \dot{g} \Rightarrow g' \wedge \dot{g}' \Rightarrow g))$ |
| (HF)                                      | Homophily axiom                                                                                                                                                                                                                      |
| (Hom)                                     | $(\delta \rightarrow \text{Hom}) \leftrightarrow (\delta \rightarrow AX\delta)$                                                                                                                                                      |
| (Hom <sub>¬</sub> )                       | $(\delta \rightarrow \neg \text{Hom}) \leftrightarrow (\delta \rightarrow \neg AX\delta)$                                                                                                                                            |
| (Tm <sub>1</sub> )                        | From $\delta \rightarrow \text{Hom}$ to infer $\delta \rightarrow (A(\phi U \psi) \rightarrow \psi)$                                                                                                                                 |
| (Tm <sub>2</sub> )                        | From $\delta \rightarrow \text{Hom}$ to infer $\delta \rightarrow (E(\phi U \psi) \rightarrow \psi)$                                                                                                                                 |

#### 4.1. Soundness

We will prove the validity of all other axioms and rules except the CTL axioms and rules and Armstrong's axioms which have been proven to be valid. Next, a lemma used to assist the following proofs will be first proved.

**Lemma 1.** Consider  $\text{LH}_{G,M}$ . For any description  $\delta$ , HetNet  $N$  and similarity coefficient  $n$ ,  $\delta$  is the description of  $N$  iff  $N \models_n \delta$ .

**Proof.** By definition, left to right obviously holds. Only from right to left is shown here. Consider the description  $\delta = \bigwedge_{g \in G} (\dot{g} \Rightarrow A_g \wedge \sim (\dot{g} \Rightarrow A_g))$ , the HetNet  $N = (G, M, I, R)$  and the similarity coefficient  $n$ . If  $\delta$  does not describe  $N$ , then there exists  $g' \in G$  such that  $A_{g'} \neq \{g'\}^{\downarrow IR}$ , i.e., either  $A_{g'} \subset \{g'\}^{\uparrow IR}$ , or  $\{g'\}^{\uparrow IR} \subset A_{g'}$ . For the former, according to the definition of  $\sim (\dot{g} \Rightarrow A_g)$ , there must exist  $a \in \{g'\}^{\uparrow IR}$  so that  $\neg \dot{g}' \Rightarrow a$  is a branch of  $\delta$ . Therefore,  $N \not\models_n \delta$ . Similarly, for the latter, there must exist  $a \in A_{g'}$  such that  $N \not\models_n \dot{g}' \Rightarrow a$  (otherwise,  $a \in \{g'\}^{\uparrow IR}$ ). Thus,  $N \not\models_n \delta$ .  $\square$

Here, we only show the axioms and rules that should be proved, and the proof will be given in the Appendix A.

**Theorem 1.** For  $LH_{G,M}$ , the following axioms and rules are valid (where  $\phi, \psi$  are arbitrary formulas,  $A, B, C \subseteq G \cup M \cup \dot{G}$ ,  $a \in M$ ,  $g, g' \in G$ ,  $\dot{g}, \dot{g}' \in \dot{G}$ ,  $E^n \subseteq M$  and  $|E^n| \geq n$  ( $0 < n \leq |M|$ ),  $O$  is not empty and  $O \subseteq G$ ,  $\delta$  is a description of HetNet):

- (Frag $_{\neg}$ )  $\neg A \cup B \Rightarrow C \rightarrow \neg A \Rightarrow C$ ;
- (Comp $_{\neg}$ )  $\neg A \Rightarrow B \wedge \neg A \Rightarrow C \rightarrow \neg A \Rightarrow B \cup C$ ;
- (Incl)  $(A \Rightarrow B \wedge \neg A \Rightarrow C) \rightarrow \neg B \Rightarrow C$ ;
- (Abs)  $(\dot{O} \Rightarrow A \wedge \dot{O} \Rightarrow B \wedge \sim (G \setminus O) \Rightarrow A) \rightarrow A \Rightarrow B$ ;
- (Loop)  $\dot{g} \Rightarrow g$ ;
- (Loop $_{\neg}$ )  $\neg g \Rightarrow \dot{g}' \quad (g \neq g')$ ;
- (Symm)  $\dot{g} \Rightarrow g' \leftrightarrow \dot{g}' \Rightarrow g$ ;
- (A-gen)  $\dot{g} \Rightarrow A \rightarrow AX\dot{g} \Rightarrow A$ ;
- (A-gen $_{\neg}$ )  $\neg \dot{g} \Rightarrow a \rightarrow AX\neg \dot{g} \Rightarrow a$ ;
- (E-gen)  $(\dot{g} \Rightarrow E^n \wedge \dot{g}' \Rightarrow E^n) \rightarrow EX\dot{g} \Rightarrow g'$ ;
- (E-gen $_{\neg}$ ) From  $\delta \rightarrow \neg \dot{g} \Rightarrow g' \wedge \sigma_{(g,g')}$  to infer  $\delta \rightarrow \neg EX\dot{g} \Rightarrow g'$ ;
- (Exp) From  $\delta \rightarrow \neg \dot{g} \Rightarrow g' \wedge EX\dot{g} \Rightarrow g'$  to infer  $\delta \rightarrow EX(\dot{g} \Rightarrow g' \wedge \sim (\delta^+ \wedge \dot{g} \Rightarrow g' \wedge \dot{g}' \Rightarrow g))$ ;
- (HF) Homophily axiom;
- (Hom)  $(\delta \rightarrow \text{Hom}) \leftrightarrow (\delta \rightarrow AX\delta)$ ;
- (Hom $_{\neg}$ )  $(\delta \rightarrow \neg \text{Hom}) \leftrightarrow (\delta \rightarrow \neg AX\delta)$ ;
- (Tm $_1$ ) From  $\delta \rightarrow \text{Hom}$  to infer  $\delta \rightarrow (A(\phi \cup \psi) \rightarrow \psi)$ ;
- (Tm $_2$ ) From  $\delta \rightarrow \text{Hom}$  to infer  $\delta \rightarrow (E(\phi \cup \psi) \rightarrow \psi)$ .

**Theorem 2 (Soundness).** For any formula set  $\Phi$  and formula  $\phi$ , if  $\Phi \vdash_n \phi$ , then  $\Phi \models_n \phi$ .

#### 4.2. Completeness

In this section, we will first prove that the tomorrow formula is the theorem of  $LH_{G,M}^n$ . Tomorrow formula, denoted as  $(t(\delta))$ , is the following formula:

$$(t(\delta)) \quad \delta \rightarrow (AX(\delta_1 \vee \dots \vee \delta_n) \wedge EX\delta_1 \wedge \dots \wedge EX\delta_n).$$

$\delta_1, \dots, \delta_n$  is the description of the  $n$ -immediate successor of the HetNet described by  $\delta$ .  $(t(\delta))$  can be abbreviated as  $\delta \rightarrow \nabla\{\delta_1, \dots, \delta_n\}$ ,  $\nabla$  is the so-called *cover modality*.

**Lemma 2.**  $\vdash_n \delta \rightarrow \nabla\{\delta_1, \dots, \delta_n\}$ .

**Proof.** Consider a description  $\delta$  which describes a HetNet and a similarity coefficient  $n$ .

If  $\delta$  describes a HetNet that is homogenized at  $n$ , according to (HF),  $\vdash_n \delta \rightarrow \text{Hom}$ . By (HOM),  $\vdash_n \delta \rightarrow AX\delta$ , i.e.,  $\vdash_n \delta \rightarrow \nabla\{\delta\}$ .

If  $\delta$  does not describe a homogenized HetNet, then  $\vdash_n \delta \rightarrow \neg(\delta_1 \vee \delta_2 \vee \dots \vee \delta_n)$ , i.e.,  $\vdash_n \delta \rightarrow \neg \text{Hom}$ . According to (Hom $_{\neg}$ ) and  $\vdash_n \delta \leftrightarrow \delta^+ \wedge \sim \delta^+$ ,  $\vdash_n \delta \rightarrow \neg AX(\delta^+ \wedge \sim \delta^+)$ , i.e.,  $\vdash_n \delta \rightarrow EX(\neg \delta^+ \vee \neg(\sim \delta^+))$ . And by (A-gen),  $\vdash_n \delta \rightarrow AX\delta^+$ . Therefore,  $\vdash_n \delta \rightarrow EX(\neg(\sim \delta^+))$ . According to (A-gen $_{\neg}$ ), for all formulas in the form of  $\neg \dot{g} \Rightarrow a$  in  $\sim \delta^+$  (where  $a \in M$ ),  $\vdash_n \delta \rightarrow AX\neg \dot{g} \Rightarrow a$ . Similarly, according to (E-gen $_{\neg}$ ), for all

formulas in the form of  $\neg \dot{g} \Rightarrow g'$  (where  $g, g' \in G$ ) in  $\sim \delta^+$ , if  $\vdash_n \delta \rightarrow \sigma_{(g, g')}$ , then  $\vdash_n \delta \rightarrow \text{AX} \neg \dot{g} \Rightarrow g'$ . Therefore, there exists  $g, g' \in G$  such that  $\neg \dot{g} \Rightarrow g'$  is the branch of  $\sim \delta^+$  and  $\vdash_n \delta \rightarrow \neg(\sigma_{(g, g')})$ , then according to (E-gen),  $\vdash_n \delta \rightarrow \text{EX} \dot{g} \Rightarrow g'$ , i.e.,  $\vdash_n \delta \rightarrow \neg \dot{g} \Rightarrow g' \wedge \text{EX} \dot{g} \Rightarrow g'$ . Thus, according to (Exp), there exists  $g, g' \in G$  such that  $\vdash_n \delta \rightarrow \text{EX}(\dot{g} \Rightarrow g' \wedge \sim(\delta^+ \wedge \dot{g} \Rightarrow g' \wedge \dot{g}' \Rightarrow g))$ , and since (A-gen) and (Symm), there exists  $g, g' \in G$  such that  $\vdash_n \delta \rightarrow \text{EX}((\delta^+ \wedge \dot{g} \Rightarrow g' \wedge \dot{g}' \Rightarrow g) \wedge \sim(\delta^+ \wedge \dot{g} \Rightarrow g' \wedge \dot{g}' \Rightarrow g))$ , obviously  $(\delta^+ \wedge \dot{g} \Rightarrow g' \wedge \dot{g}' \Rightarrow g) \wedge \sim(\delta^+ \wedge \dot{g} \Rightarrow g' \wedge \dot{g}' \Rightarrow g)$  is equivalent to the description of one HetNet. Suppose for  $\delta$ , there are  $m$  pairs of  $g, g' \in G$  that satisfy the above conditions. Let  $\delta_1, \dots, \delta_m$  represent the above equivalent HetNet descriptions, respectively.  $\vdash_n \delta \rightarrow \text{EX} \delta_1 \wedge \dots \wedge \text{EX} \delta_m$ . According to the definition of  $n$ -immediate successor, it is easy to know that  $\delta_1, \dots, \delta_m$  are the descriptions of  $n$ -immediate successor of the HetNet described by  $\delta$ . Finally, since  $G$  is finite, it is easy to verify that the above  $g, g' \in G$  has exhausted all the cases, thus,  $\vdash_n \delta \rightarrow \text{AX}(\delta_1 \vee \dots \vee \delta_m) \wedge \text{EX} \delta_1 \wedge \dots \wedge \text{EX} \delta_m$ , i.e.,  $\vdash_n \delta \rightarrow \nabla\{\delta_1, \dots, \delta_m\}$ .  $\square$

**Lemma 3 ([9]).** Suppose  $\vdash_n \delta \rightarrow \nabla\{\delta_1, \dots, \delta_m\}$ . The following situations hold:

- (1) If for all  $i = 1, \dots, m$ ,  $\vdash_n \delta_i \rightarrow \phi$ , then  $\vdash_n \delta \rightarrow \text{AX}\phi$ ;
- (2) If there exists  $i \in \{1, \dots, m\}$  such that  $\vdash_n \delta_i \rightarrow \neg\phi$ , then  $\vdash_n \delta \rightarrow \neg\text{AX}\phi$ .

**Proof.** Suppose  $\vdash_n \delta \rightarrow \nabla\{\delta_1, \dots, \delta_m\}$ .

- (1) Suppose that for all  $i = 1, \dots, m$ ,  $\vdash_n \delta_i \rightarrow \phi$ .  $\vdash_n \delta_1 \vee \dots \vee \delta_m \rightarrow \phi$ . By (Nec),  $\vdash_n \text{AX}(\delta_1 \vee \dots \vee \delta_m \rightarrow \phi)$ . By ( $K_x$ ),  $\vdash_n \text{AX}(\delta_1 \vee \dots \vee \delta_m) \rightarrow \text{AX}\phi$ . Since  $\vdash_n \delta \rightarrow \nabla\{\delta_1, \dots, \delta_m\}$ ,  $\vdash_n \delta \rightarrow \text{AX}\phi$ .
- (2) Suppose that there exists  $i \in \{1, \dots, m\}$  s.t.  $\vdash_n \delta_i \rightarrow \neg\phi$ . In other words, there exists  $i \in \{1, \dots, m\}$  s.t.  $\vdash_n \phi \rightarrow \neg\delta_i$ . By (Nec),  $\vdash_n \text{AX}(\phi \rightarrow \neg\delta_i)$ . By ( $K_x$ ),  $\vdash_n \text{AX}\phi \rightarrow \text{AX}\neg\delta_i$ , i.e.,  $\vdash_n \text{EX}\delta_i \rightarrow \neg\text{AX}\phi$ . Since  $\vdash_n \delta \rightarrow \nabla\{\delta_1, \dots, \delta_m\}$ ,  $\vdash_n \delta \rightarrow \neg\text{AX}\phi$ .

$\square$

**Lemma 4.** For any description  $\delta$  and formula  $\phi$ , one and only one of  $\vdash_n \delta \rightarrow \phi$  and  $\vdash_n \delta \rightarrow \neg\phi$  must hold.

**Proof.** According to the definition of description, the description is obviously consistent, so  $\vdash_n \delta \rightarrow \phi$  and  $\vdash_n \delta \rightarrow \neg\phi$  cannot be held at the same time.

Applying induction on  $\phi$  to prove that for any description  $\delta$  and formula  $\phi$ , either  $\vdash_n \delta \rightarrow \phi$  or  $\vdash_n \delta \rightarrow \neg\phi$  must hold.

$\phi = A \Rightarrow B$ : (1) For any  $\delta$  and an atomic formula in the form of  $\dot{g} \Rightarrow A$  (where  $g \in G$ ,  $A \subseteq G \cup M \cup \dot{G}$ ), either  $\vdash_n \delta \rightarrow \dot{g} \Rightarrow A$  or  $\vdash_n \delta \rightarrow \neg\dot{g} \Rightarrow A$  must hold.

Let  $\psi$  be an atomic formula in the form of  $\dot{g} \Rightarrow A$ . If  $\psi$  is the conjunction branch of  $\delta^+$ , or can be derived from one of the conjunction branches by (Frag), then  $\vdash_n \delta \rightarrow \psi$ . According to (Frag) and (Comp),  $\vdash_n \psi \leftrightarrow \bigwedge_{a \in A} (\dot{g} \Rightarrow a)$ . If  $\psi$  is not a conjunction of  $\delta^+$  and cannot be obtained by (Frag), then there is at least one conjunction branch in  $\bigwedge_{a \in A} (\dot{g} \Rightarrow a)$  that also satisfy this condition. Let  $\dot{g} \Rightarrow a'$  be the formula that satisfies the condition. Suppose that  $\dot{g} \Rightarrow a'$  is  $\dot{g} \Rightarrow \dot{g}$ . If there is no other formula that satisfies the condition, according to (Ref),  $\vdash_n \delta \rightarrow \psi$ . If there are other formulas, it is obviously impossible to be  $\dot{g} \Rightarrow \dot{g}$ . If  $\dot{g} \Rightarrow a'$  is of the form  $\dot{g} \Rightarrow g'$  (where  $g' \in G$  and  $g \neq g'$ ), then according to (Frag), (Loop) and (Loop<sub>-</sub>),  $\vdash_n \delta \rightarrow \neg\psi$ . If it is not the above two cases, then according to the definition of  $\delta$ , it is easy to know that  $\neg\dot{g} \Rightarrow a'$  must be the conjunction branch of  $\sim \delta^+$ . Therefore,  $\vdash_n \delta \rightarrow \neg\dot{g} \Rightarrow a'$ , i.e.,  $\vdash_n \delta \rightarrow \neg\psi$ . In summary, (1) holds.

(2) Consider  $\delta$  and non-empty sets  $O \subseteq G$  and  $A \subseteq G \cup M \cup \dot{G}$ .

(2.1)  $\vdash_n \delta \rightarrow \dot{O} \Rightarrow A$  or  $\vdash_n \delta \rightarrow \neg\dot{O} \Rightarrow A$  must hold.

(2.2)  $\vdash_n \delta \rightarrow \sim \dot{O} \Rightarrow A$  or  $\vdash_n \delta \rightarrow \neg(\sim \dot{O} \Rightarrow A)$  must hold.

(2.3) If  $\vdash_n \delta \rightarrow \hat{O} \Rightarrow A$ , then there exists  $O' \subseteq G$  satisfying  $O \subseteq O'$  such that  $\vdash_n \delta \rightarrow \hat{O}' \Rightarrow A$  and  $\vdash_n \delta \rightarrow \sim (G \setminus O') \Rightarrow A$ .

According to (1), (2.1) and (2.2) obviously hold. Therefore, according to (1), (2.1), (2.2) and the definition of description, it is easy to know that (2.3) holds.

(3) Consider a description  $\delta$  and  $A, B \subseteq G \cup M \cup \hat{G}$ . If there exists  $\hat{g} \in \hat{G}$  such that  $\vdash_n \delta \rightarrow \hat{g} \Rightarrow A$  and  $\vdash_n \delta \rightarrow \neg \hat{g} \Rightarrow B$ , then according to (Incl),  $\vdash_n \delta \rightarrow \neg A \Rightarrow B$ . If for any  $\hat{g} \in \hat{G}$ ,  $\nvdash_n \delta \rightarrow \hat{g} \Rightarrow A$  or  $\nvdash_n \delta \rightarrow \neg \hat{g} \Rightarrow B$ , then according to (1), for any  $\hat{g} \in \hat{G}$ , either  $\vdash_n \delta \rightarrow \neg \hat{g} \Rightarrow A$ , or  $\vdash_n \delta \rightarrow \hat{g} \Rightarrow B$  (3.1). If there is a non-empty set  $O \subseteq G$  such that  $\vdash_n \delta \rightarrow \hat{O} \Rightarrow A$ , then according to (2.3), there exists  $O' \subseteq G$  satisfying  $O \subseteq O'$  to make  $\vdash_n \delta \rightarrow \hat{O}' \Rightarrow A$  and  $\vdash_n \delta \rightarrow \sim (G \setminus O') \Rightarrow A$  hold. Thus, according to (3.1),  $\vdash_n \delta \rightarrow \hat{O}' \Rightarrow B$ . Therefore, according to (Abs),  $\vdash_n \delta \rightarrow A \Rightarrow B$ . If there is no such non-empty set  $O \subseteq G$  that  $\vdash_n \delta \rightarrow \hat{O} \Rightarrow A$ , i.e., for any  $g \in G$ ,  $\vdash_n \delta \rightarrow \neg \hat{g} \Rightarrow A$ . According to (Emp),  $\vdash_n \delta \rightarrow A \Rightarrow B$ .

Therefore, when  $\phi$  is the atomic formula  $A \Rightarrow B$ , the conclusion holds.

$\phi = \neg\psi$ : Suppose  $\nvdash_n \delta \rightarrow \neg\psi$ . According to the induction hypothesis,  $\vdash_n \delta \rightarrow \psi$ , i.e.,  $\vdash_n \delta \rightarrow \neg\neg\psi$ .

$\phi = \psi \vee \chi$ : Suppose  $\nvdash_n \delta \rightarrow \psi \vee \chi$ .  $\nvdash_n \delta \rightarrow \psi$  and  $\nvdash_n \delta \rightarrow \chi$ . According to the induction hypothesis,  $\vdash_n \delta \rightarrow \neg\psi$  and  $\vdash_n \delta \rightarrow \neg\chi$ , i.e.,  $\vdash_n \delta \rightarrow \neg(\psi \vee \chi)$ .

$\phi = AX\psi$ : Suppose  $\nvdash_n \delta \rightarrow AX\psi$ . According to Lemma 2 and Lemma 3 (1), there exists  $i = 1, \dots, m$  such that  $\nvdash_n \delta_i \rightarrow \psi$  (where for  $i = 1, \dots, m$ ,  $\delta_1, \dots, \delta_m$  are the descriptions of the  $n$ -immediate successor of  $\delta$ ). According to the induction hypothesis,  $\vdash_n \delta_i \rightarrow \neg\psi$ . Therefore, by Lemma 3 (2),  $\vdash_n \delta \rightarrow \neg AX\psi$ .

$\phi = \text{Hom}$ : Suppose  $\nvdash_n \delta \rightarrow \text{Hom}$ . According to (Hom),  $\nvdash_n \delta \rightarrow AX\delta$ . According to the induction hypothesis,  $\vdash_n \delta \rightarrow \neg AX\delta$ . Then according to (Hom $_{\neg}$ ),  $\vdash_n \delta \rightarrow \neg \text{Hom}$ .

$\phi = E(\psi U \chi)$ : Suppose  $\nvdash_n \delta \rightarrow E(\psi U \chi)$ . According to (EU),  $\nvdash_n \delta \rightarrow \chi \vee (\psi \wedge \text{EXE}(\psi U \chi))$ . Assume that  $\nvdash_n \delta \rightarrow \chi$  and  $\nvdash_n \delta \rightarrow \psi$ , according to the induction hypothesis,  $\vdash_n \delta \rightarrow \neg\chi$  and  $\vdash_n \delta \rightarrow \neg\psi$ . Therefore,  $\vdash_n \delta \rightarrow \neg(\chi \vee (\psi \wedge \text{EXE}(\psi U \chi)))$  and according to (EU),  $\vdash_n \delta \rightarrow \neg E(\psi U \chi)$ . Assume that  $\nvdash_n \delta \rightarrow \chi$  and  $\nvdash_n \delta \rightarrow \text{EXE}(\psi U \chi)$ . For  $i = 1, \dots, m$ , let  $\delta_1, \dots, \delta_m$  be the descriptions of the  $n$ -immediate successor of  $\delta$ . According to the Lemma 3 (2), for  $i = 1, \dots, m$ ,  $\nvdash_n \delta_i \rightarrow E(\psi U \chi)$ . Thus, it is back to the condition at the beginning of the proof. Repeat finitely many times until the description  $\delta'$  describes the homogenized HetNet, i.e.,  $\vdash_n \delta' \rightarrow \text{Hom}$ . Since  $\nvdash_n \delta' \rightarrow E(\psi U \chi)$ ,  $\nvdash_n \delta' \rightarrow \chi \vee (\psi \wedge \text{EXE}(\psi U \chi))$ . If  $\nvdash_n \delta' \rightarrow \chi$  and  $\nvdash_n \delta' \rightarrow \psi$ , the conclusion also holds. Assume that  $\nvdash_n \delta' \rightarrow \chi$  and  $\nvdash_n \delta' \rightarrow \text{EXE}(\psi U \chi)$ . According to the Lemma 3 (2) and  $\delta' \rightarrow \nabla \delta'$ ,  $\nvdash_n \delta' \rightarrow E(\psi U \chi)$  and thus  $\vdash_n \delta' \rightarrow \neg E(\psi U \chi)$  by (Tm $_2$ ). Using the Lemma 3 (1) and (EU) repeatedly, we can finally obtain  $\vdash_n \delta \rightarrow \neg E(\psi U \chi)$ .

$\phi = A(\psi U \chi)$ : The case of  $\phi = A(\psi U \chi)$  is similar to  $\phi = E(\psi U \chi)$ , as long as (Tm $_1$ ) is used instead of (Tm $_2$ ), it can be proved.

□

**Lemma 5.** Consider a description  $\delta$ , a maximal consistent formula set  $\Phi$  and a formula  $\phi$ . If  $\delta, \phi \in \Phi$ , then  $\vdash_n \delta \rightarrow \phi$ .

**Proof.** Suppose  $\delta, \phi \in \Phi$  and  $\nvdash_n \delta \rightarrow \phi$ . By the Lemma 3,  $\vdash_n \delta \rightarrow \neg\phi$ . And according to the definition of the maximal consistent set,  $\neg\phi \in \Phi$ . Therefore,  $\Phi$  is inconsistent, contradictory, and the assumption does not hold. □

Consider  $LH_{G,M}$ . For any maximal consistent set  $\Phi$ , the relation in the HetNet  $N^\Phi = (G, M, I, R)$  of  $\Phi$  is defined as follows (where  $g, g' \in G$ ,  $m \in M$  and  $\hat{g} \in \hat{G}$ ):

$$\begin{aligned} (g, m) \in I & \iff \hat{g} \Rightarrow m \in \Phi, \\ (g, g') \in R & \iff \hat{g} \Rightarrow g' \in \Phi. \end{aligned}$$

**Lemma 6.** Consider any description  $\delta$  of the HetNet and a set of maximal consistent  $\Phi$ , the following propositions are equivalent:

- (1)  $N^\Phi \models_n \delta$ , (2)  $\delta$  describes  $N^\Phi$ , (3)  $\delta \in \Phi$ .

**Proof.** According to the definition, it is easy to know that (1) is equivalent to (2). Suppose (2) holds. Let  $\dot{g} \Rightarrow A$  represent any conjunction branch of  $\delta^+$ . According to the definition of  $N^\Phi$ , for any  $a \in A$ ,  $\dot{g} \Rightarrow a \in \Phi$ . Therefore, by (Comp),  $\dot{g} \Rightarrow A \in \Phi$ , i.e.,  $\delta^+ \in \Phi$ . In the same way,  $\sim \delta^+ \in \Phi$  can also be proved. Therefore,  $\delta \in \Phi$ . Suppose (3) holds. Consider  $\dot{g} \Rightarrow a$  (where  $g \in G, a \in G \cup M$ ). According to the Lemma 5, if  $\dot{g} \Rightarrow a \in \Phi$ , then  $\vdash_n \delta \rightarrow \dot{g} \Rightarrow a$ , and if  $\neg \dot{g} \Rightarrow a \in \Phi$ , then  $\vdash_n \delta \rightarrow \neg \dot{g} \Rightarrow a$ . Therefore, one and only one of  $\dot{g} \Rightarrow a \in \Phi$  and  $\neg \dot{g} \Rightarrow a \in \Phi$  must hold. Thus, by definition,  $N^\Phi$  is completely determined by  $\delta$ , i.e.,  $\delta$  describes  $N^\Phi$ .  $\square$

**Lemma 7.** Consider any maximal consistent formula set  $\Phi$ .  $N^\Phi \models_n \Phi$ .

**Proof.** Suppose  $\delta$  describes  $N^\Phi$ . According to the Lemma 6,  $N^\Phi \models_n \delta$  and  $\delta \in \Phi$ . And according to the Lemma 5, for any  $\phi \in \Phi$ ,  $\vdash_n \delta \rightarrow \phi$ . According to the Theorem 2, for any  $\phi \in \Phi$ ,  $\models_n \delta \rightarrow \phi$ . Therefore, for any  $\phi \in \Phi$ ,  $N^\Phi \models_n \phi$ , i.e.,  $N^\Phi \models_n \Phi$ .  $\square$

**Theorem 3 (Completeness).** For any formula set  $\Phi$  and formula  $\phi$ , if  $\Phi \models_n \phi$ , then  $\Phi \vdash_n \phi$ .

Since the timetrees of the logic of homophily  $LH_{G,M}$  are all finite, i.e., they must be able to be homogenized in the end, the axiom system  $\mathbf{LH}_{G,M}^n$  has strong completeness.

## 5. Model Checking

In this section, we consider the model checking problem of  $LH_{G,M}$ . Consider  $\mathcal{L}_{G,M}$ ,  $N = (G, M, I, R)$  and a similarity coefficient  $n$ . For any  $\phi \in \mathcal{L}_{G,M}$ , the model checking problem is to check whether  $N \models_n \phi$ .

**Lemma 8.** The model checking for  $LH_{G,M}$  is PSPACE-hard.

**Proof.** First, since the model checking problem (TQBF) and the validity checking problem (QBFSAT) for quantified boolean formulas (QBF) are both PSPACE-complete [18], we use the QBFSAT problem for reduction to prove the model checking problem for  $LH_{G,M}$  is PSPACE-hard.

Let  $\psi_m = Q_1 x_1 Q_2 x_2 \cdots Q_m x_m \phi(x_1, x_2, \dots, x_m)$  be the QBF, where  $Q_i = \exists$  or  $\forall$ ,  $\phi$  is a propositional formula, and the value range of  $x_1, x_2, \dots, x_m$  is  $\{0, 1\}$ .

Consider a positive integer  $n$ . Let  $N_m^n = (G, M, I, R)$  satisfy the following conditions:

1.  $G = \{0_i, 1_i, 2_i \mid 1 \leq i \leq m\}$ ;
2.  $M = \{a_i^1, \dots, a_i^n, b_i^1, \dots, b_i^n \mid 1 \leq i \leq m\}$ ;
3.  $I = \{(0_i, a_i^1), (1_i, a_i^1), \dots, (0_i, a_i^n), (1_i, a_i^n), (1_i, b_i^1), (2_i, b_i^1), \dots, (1_i, b_i^n), (2_i, b_i^n) \mid 1 \leq i \leq m\}$ ;
4.  $R = \{(0_i, 0_i), (1_i, 1_i), (2_i, 2_i) \mid 1 \leq i \leq m\}$ .

In the HetNet  $N$ , the object sets that are  $n$ -similar but not homogenized must be in the form of  $\{0_i, 1_i\}$  or  $\{1_i, 2_i\}$ . In other words, an update of the HetNet  $N$  can only add at most one association between  $0_i$  and  $1_i$  or  $1_i$  and  $2_i$  for a given  $i$ . Therefore, the selection of the object sets can be described by the value of  $x_i$ . If  $\{0_i, 1_i\}$  becomes homogenized in an update,  $x_i$  takes 1. And if  $\{1_i, 2_i\}$  becomes homogenized,  $x_i$  takes 0.

Let  $\alpha_i = \neg 0_i \rightarrow 1_i \wedge \neg 1_i \rightarrow 2_i$ ,  $\alpha_i^1 = 0_i \rightarrow 1_i$ , and  $\alpha_i^0 = 1_i \rightarrow 2_i$ .  $\alpha_i^1$  means  $x_i$  takes 1,  $\alpha_i^0$  means  $x_i$  takes 0, and  $\alpha_i$  represents the initial situation, i.e., the situation where the value of  $x_i$  is undecided. The inductive definition of  $\beta_m \in \mathcal{L}_{G,M}$  is as follows:

1.  $\beta_0 = \phi(\alpha_1^1, \alpha_2^1, \dots, \alpha_m^1)$ ;

$$2. \quad \beta_{i+1} = \begin{cases} \alpha_{i+1} \wedge \text{AX}(\neg\alpha_{i+1} \rightarrow \beta_i) & \text{if } Q_i = \forall \\ \alpha_{i+1} \wedge \text{EX}(\neg\alpha_{i+1} \wedge \beta_i) & \text{if } Q_i = \exists \end{cases}$$

For any  $\beta_m$ , we should only consider the  $m$ -step  $n$ -successors of the root HetNet and since  $\neg\alpha_{i+1}$  should be satisfied first in every update, for any  $1 \leq j \leq m$ , it is impossible for  $\alpha_j^1$  and  $\alpha_j^0$  to be satisfied at the same time in the above  $m$ -step  $n$ -successors. Therefore,  $N_m^n \models_n \beta_m$  if and only if  $\psi_m$  is satisfied. For the HetNet  $N$  defined above,  $\text{dep}(t^n(N_m^n)) = 2m + 1$ . And the length of  $\beta_m$  is also linear-related to  $m$ . The above reduction is thus a linear-time reduction. The model checking problem for  $LH_{G,M}$  is PSPACE-hard.  $\square$

**Lemma 9.** *The model checking problem for  $LH_{G,M}$  is in PSPACE.*

**Proof.** First, determining whether an atomic formula is satisfied on a HetNet or not requires only a few basic calculations on sets. In addition, the model checking algorithms for basic propositional connectives are polynomial-time. Therefore, what we should consider carefully here are Hom and temporal operators.

Consider a HetNet  $N = (G, M, I, R)$  and a similarity coefficient  $n$ . The Algorithm 1 can determine whether  $N \models_n \text{Hom}$  or not. Since Algorithm 1 performs at most  $\frac{|G|(|G|-1)}{2} + 1$  calculations, it is in PSPACE.

---

**Algorithm 1:** Model Checking Algorithm for Hom

---

```

initialize i = 0;
for i < | G | do
    j = i + 1;
    for j < | G | do
        if (i, j)  $\notin$  R and  $\text{Sim}(\{i, j\}) \geq n$  then
            return FALSE
        else
            j++;
        end
    end
    i++;
end
return TRUE

```

---

The algorithms for temporal operators are based on the algorithms for CTL given in [19] since the temporal operators in  $LH_{G,M}$  come from CTL. Here, we only give the algorithm of  $E(\phi U \psi)$  (Algorithm 2) as an example and other cases are shown in Appendix B.

According to Algorithm 2, it is not difficult to find that it is a recursive algorithm. Each run of the algorithm either shortens the input formula or makes input HetNet “jump to” one of its immediate successors. Therefore, since the formula is finite, and the HetNet will finally reach homogenized, the algorithm will eventually halt.

The model checking algorithms for CTL temporal operators are all in PSPACE with respect to the size of the input computation tree and the length of the input formula. However, the size of the computation tree of an input HetNet  $(G, M, I, R)$  is at most  $(\frac{|G|(|G|-1)}{2} + 1)!$ , i.e., the growth of the size of the computation tree with that of the HetNet can even be greater than exponential. To reduce storage space usage, we use the depth-first algorithm like Algorithm 2, and release the storage space of the non-overlapping part of the old branch before starting to calculate a new branch. In other words, we do not need the storage space that can store the whole time tree, but only the space that can store the longest timeline, i.e., the size of the storage space is linear to the depth of the tree.

**Algorithm 2:** Model Checking Algorithm for  $E(\phi U \psi)$ 


---

```

if  $N \models_n \psi$  then
  | return TRUE
else if  $N \not\models_n \phi$  then
  | return FALSE
else if  $N \models_n \phi$  then
  | if  $N \models_n \text{Hom}$  then
  | | return FALSE
  | else
  | |  $\mathcal{N} = \{N' \mid N \rightsquigarrow_n N'\};$ 
  | | for  $N' \in \mathcal{N}$  do
  | | | if  $N' \models_n E(\phi U \psi)$  then
  | | | | return TRUE
  | | | end
  | | end
  | | return FALSE
  | end
end

```

---

Therefore, the model checking for  $LH_{G,M}$  is in PSPACE.  $\square$

**Theorem 4.** The model checking for  $LH_{G,M}$  is PSPACE-complete.

## 6. Validity Checking

Consider non-empty finite sets  $G$  and  $M$  s.t.  $G \cap M = \emptyset$ , and a similarity coefficient  $n$ . For any  $\phi \in \mathcal{L}_{G,M}$ , the validity checking problem is to check whether  $\models_n \phi$ .

**Lemma 10.** The validity checking for  $LH_{G,M}$  is PSPACE-hard.

**Proof.** Consider a QBF  $\psi_m = Q_1 x_1 Q_2 x_2 \cdots Q_m x_m \phi(x_1, x_2, \dots, x_m)$ , where  $m$  is any positive integer,  $Q_i = \exists$  or  $\forall$ ,  $\phi$  is a propositional formula, and the value range of  $x_1, x_2, \dots, x_m$  is  $\{0, 1\}$ , and a positive integer  $n$ .

Let  $G = \{0_i, 1_i, 2_i \mid 1 \leq i \leq m\}$ ,  $M = \{a_i^1, \dots, a_i^n, b_i^1, \dots, b_i^n \mid 1 \leq i \leq m\}$ , and  $\delta_m^n = \bigwedge_{i=1}^m ((0_i \Rightarrow 0_i a_i^1 a_i^2 \dots a_i^n \wedge \sim (0_i \Rightarrow 0_i a_i^1 a_i^2 \dots a_i^n)) \wedge (1_i \Rightarrow 1_i a_i^1 a_i^2 \dots a_i^n b_i^1 b_i^2 \dots b_i^n \wedge \sim (1_i \Rightarrow 1_i a_i^1 a_i^2 \dots a_i^n b_i^1 b_i^2 \dots b_i^n)) \wedge (2_i \Rightarrow 2_i b_i^1 b_i^2 \dots b_i^n \wedge \sim (2_i \Rightarrow 2_i b_i^1 b_i^2 \dots b_i^n)))$ . It is obvious that  $\delta_m^n$  is the description of the HetNet  $N_m^n$  in the Proof of Lemma 8. Here, we take the value of the similarity coefficient as  $n$ . Since  $|M| = 2mn$  and hence  $1 \leq n \leq |M|$ , if  $N \models_n \delta_m$ ,  $N \models_n \beta_m$  if and only if  $\psi_m$  is satisfied ( $\beta_m$  is the same as that defined in the Proof of Lemma 8). Therefore,  $\models_n \delta_m^n \rightarrow \beta_m$  if and only if  $\psi_m$  is satisfied. The validity checking for  $LH_{G,M}$  is PSPACE-hard.  $\square$

**Lemma 11.** The validity checking for  $LH_{G,M}$  is in PSPACE.

**Proof.** When non-empty finite sets  $G$  and  $M$  and a similarity coefficient  $n$  are given, the number of HetNets composed of  $G$  and  $M$  is determined and apparently finite. Let  $\mathcal{N}$  be the set of all those HetNets. We can check whether  $\models_n \phi$  by enumeration, i.e., we can use model checking algorithms to check whether  $N \models_n \phi$  for all  $N \in \mathcal{N}$ . Since the model checking algorithm is in PSPACE and the storage only needs to keep one HetNet during each model checking process, the validity checking for  $LH_{G,M}$  is in PSPACE.  $\square$

**Theorem 5.** The validity checking for  $LH_{G,M}$  is PSPACE-complete.

## 7. Conclusions

In this article, we explore the homophily dynamics of HetNets at a determined similarity coefficient from the perspective of logic. More precisely, we establish the Logic of Homophily  $LH_{G,M}$ , which describes the homophily dynamics, construct a sound and complete axiom system  $\mathbf{LH}_{G,M}^n$ , and propose the PSPACE-complete model checking and validity checking algorithms for  $LH_{G,M}$ . IbAs in FCA are actually some first-order formulas constrained by universal quantifiers. Index expansion enables these first-order formulas to represent “exists and only exists”, and to represent the edges in a HetNet. The method of description formulas based on index expanded IbAs is the key to proving the completeness of axiom systems and solving the validity checking problem. The axiom system  $\mathbf{LH}_{G,M}^n$  not only provides the axioms and rules that directly describe the update, but can also be able to prove the tomorrow theorems, which can only be axioms in previous work [9]. Finally, the model checking algorithms provide the basis for implementation.

There are two main directions for future work:

1. The Logic of Homophily  $LH_{G,M}$  is based on CTL and FCA. However, in this article, we only discussed the temporal logic aspect of  $LH_{G,M}$ , but did not conduct an in-depth study of the aspect of FCA. Therefore, we will explore the properties of  $LH_{G,M}$  from the perspective of IbA systems (e.g., Attribute Exploration) in the following work.
2. Cognitive balance theory, as another important dynamic theory in SNA, is not only closely related to the theory of homophily, but we have also conducted some work on this theory (e.g., [9]). Thus, we will consider further exploring the principle of sentiment homophily by introducing cognitive balance theory. In other words, for sentiment homophily, only when two objects like or dislike a certain attribute at the same time, they can be associated with. Since sentiment homophily can better and more accurately describe the formation of object relationships, it is very important and meaningful to generalize the logic of homophily to the logic of sentiment homophily.

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## Abbreviations

The following abbreviations are used in this manuscript:

|        |                                               |
|--------|-----------------------------------------------|
| SNA    | Social network analysis                       |
| HetNet | Heterogeneous network                         |
| LAE    | Logic of allies and enemies                   |
| CTL    | Computation tree logic                        |
| FCA    | Formal concept analysis                       |
| IbA    | Implication between attributes                |
| QBF    | Quantified Boolean Formula                    |
| TQBF   | True Quantified Boolean Formula               |
| QBFSAT | Satisfiability of Quantified Boolean Formulas |

## Appendix A. Proof of Theorem 1

**Proof.** Consider a HetNet  $N = (G, M, I, R)$  and a similarity coefficient  $n$ .

- (Frag<sub>¬</sub>) Suppose  $N \models_n \neg A \cup B \Rightarrow C$ . There is a  $g \in G$  s.t.  $g \in A \cup B^{\downarrow IR}$  and  $g \notin C^{\downarrow IR}$ , i.e.,  $g \in A^{\downarrow IR}$  and  $g \notin C^{\downarrow IR}$ . Therefore,  $N \models_n \neg A \Rightarrow C$ .
- (Comp<sub>¬</sub>) Suppose  $N \models_n \neg A \Rightarrow B \wedge \neg A \Rightarrow C$ . There is a  $g \in G$  s.t.  $g \in A^{\downarrow IR}$ ,  $g \notin B^{\downarrow IR}$  and  $g \notin C^{\downarrow IR}$  i.e.,  $g \in A^{\downarrow IR}$  and  $g \notin B \cup C^{\downarrow IR}$ . Therefore,  $N \models_n \neg A \Rightarrow B \cup C$ .
- (Incl) Suppose  $N \models_n A \Rightarrow B \wedge \neg A \Rightarrow C$ . There is a  $g \in G$  s.t.  $g \in A^{\downarrow IR}$ ,  $g \in B^{\downarrow IR}$  and  $g \notin C^{\downarrow IR}$ . Therefore,  $N \models_n \neg B \Rightarrow C$ .
- (Abs) Suppose  $N \models_n \dot{O} \Rightarrow A \wedge \dot{O} \Rightarrow B \wedge \sim (G \setminus O) \Rightarrow A$ . By the definition of  $\dot{O}$ , for any  $o \in O$ ,  $N \models_n \dot{o} \Rightarrow A \wedge \dot{o} \Rightarrow B$  and for any  $o' \in G \setminus O$ ,  $N \models_n \neg \dot{o}' \Rightarrow A$ . Therefore,  $A^{\downarrow IR} = O$  and  $O \subseteq B^{\downarrow IR}$ , i.e.,  $A^{\downarrow IR} \subseteq B^{\downarrow IR}$ . Thus,  $N \models_n A \Rightarrow B$ .
- (Loop) Since  $R$  is reflexive,  $N \models_n \dot{g} \Rightarrow g$ .
- (Loop<sub>¬</sub>) Consider  $g, g' \in G$  s.t.  $g' \neq g$ . According to (Loop), for any  $o \in G$ ,  $\dot{o} \Rightarrow o$ , i.e.,  $o \in \dot{o}^{\downarrow IR}$ . Therefore,  $g \in \dot{g}^{\downarrow IR}$ . Since  $\dot{g}^{\downarrow IR} = \{g'\}$ ,  $g \notin \dot{g}^{\downarrow IR}$ , i.e.,  $N \models_n \neg g \Rightarrow \dot{g}'$ .
- (Symm) Since  $R$  is symmetric,  $N \models_n \dot{g} \Rightarrow g' \leftrightarrow g' \Rightarrow g$ .
- (A-gen) Suppose  $N \models_n \neg AX\dot{g} \Rightarrow A$ .  $N \models_n EX\neg\dot{g} \Rightarrow A$ . There is an  $n$ -immediate successor of  $N$ ,  $N' = (G, M, I, R')$ , s.t.  $N' \models_n \neg\dot{g} \Rightarrow A$ . Therefore,  $g \notin A^{\downarrow IR'}$ . Since  $R \subseteq R'$ , i.e.,  $A^{\downarrow IR} \subseteq A^{\downarrow IR'}$ ,  $g \notin A^{\downarrow IR}$ . Thus,  $N \models_n \neg\dot{g} \Rightarrow A$ .
- (A-gen<sub>¬</sub>) Suppose  $N \models_n \neg\dot{g} \Rightarrow a$ . Since for any  $n$ -immediate successor of  $N$ ,  $I$  keeps the same. Therefore,  $N \models_n AX\neg\dot{g} \Rightarrow a$ .
- (E-gen) Suppose  $N \models_n \dot{g} \Rightarrow E^n \wedge \dot{g}' \Rightarrow E^n$ . By the definition of satisfaction,  $\{g, g'\}$  is  $n$ -similar on  $N$ . If  $(g, g') \in R$ ,  $N \models_n EX\dot{g} \Rightarrow g'$  by (A-gen). If  $(g, g') \notin R$ , there is an  $n$ -immediate successor  $N' = (G, M, I, R')$  of  $N$  satisfying  $(g, g') \in R'$ , i.e.,  $N' \models_n \dot{g} \Rightarrow g'$  and  $N \models_n EX\dot{g} \Rightarrow g'$ .
- (E-gen<sub>¬</sub>) If  $\delta$  describes  $N$  and  $N \models_n \delta \rightarrow \neg\dot{g} \Rightarrow g' \wedge \sigma_{(g, g')}$ , then according to the lemma 1,  $N \models_n \neg\dot{g} \Rightarrow g' \wedge \sigma_{(g, g')}$ . Therefore,  $\{g, g'\}$  is not  $n$ -similar on  $N$ , and  $(g, g') \notin R$ , i.e., for any  $n$ -immediate successor  $N' = (G, M, I, R')$  of  $N$ ,  $(g, g') \notin R'$ . Thus,  $N \models_n \neg EX\dot{g} \Rightarrow g'$ ,  $N \models_n \delta \rightarrow \neg EX\dot{g} \Rightarrow g'$ .
- (Exp) If  $\delta$  describes  $N$  and  $N \models_n \delta \rightarrow \neg\dot{g} \Rightarrow g' \wedge EX\dot{g} \Rightarrow g'$ , then according to the lemma 1,  $N \models_n \neg\dot{g} \Rightarrow g' \wedge EX\dot{g} \Rightarrow g'$ . Thus,  $\{g, g'\}$  is  $n$ -similar on  $N$ ,  $(g, g') \in R$ , and there is an  $n$ -immediate successor of  $N$ ,  $N' = (G, M, I, R')$  satisfying that  $(g, g') \in R'$ . For this  $n$ -immediate successor  $N'$ , obviously  $N' \models_n \dot{g} \Rightarrow g' \wedge \sim (\delta^+ \wedge \dot{g} \Rightarrow g' \wedge \dot{g}' \Rightarrow g)$ . Therefore,  $N \models_n EX(\dot{g} \Rightarrow g' \wedge \sim (\delta^+ \wedge \dot{g} \Rightarrow g' \wedge \dot{g}' \Rightarrow g))$ , i.e.,  $N \models_n \delta \rightarrow EX(\dot{g} \Rightarrow g' \wedge \sim (\delta^+ \wedge \dot{g} \Rightarrow g' \wedge \dot{g}' \Rightarrow g))$ .
- (HF) Since the  $n$ -immediate successor of a HetNet that is homogenized at  $n$  is itself, the homophily axiom is valid.
- (Hom) Suppose  $N \models_n \delta \rightarrow \text{Hom}$ . If  $\delta$  describes  $N$ ,  $N$  is homogenized at  $n$ . By the definition,  $N \models_n AX\delta$ . Therefore,  $N \models_n (\delta \rightarrow \text{Hom}) \rightarrow (\delta \rightarrow AX\delta)$ . Suppose  $N \models_n \delta \rightarrow AX\delta$ . If  $\delta$  describes  $N$ ,  $N \models_n AX\delta$ , i.e.,  $N \models_n \text{Hom}$ . Therefore,  $N \models_n (\delta \rightarrow \text{Hom}) \leftrightarrow (\delta \rightarrow AX\delta)$ .
- (Hom<sub>¬</sub>) Suppose  $N \models_n \delta \rightarrow \neg\text{Hom}$ . If  $\delta$  describes  $N$ , there is a HetNet  $N'$  s.t.  $N \rightsquigarrow_n N'$  and  $N' \neq N$ . Therefore,  $N' \models_n \delta$  and hence  $N \models_n (\delta \rightarrow \neg\text{Hom}) \rightarrow (\delta \rightarrow \neg AX\delta)$ . Suppose  $N \models_n \delta \rightarrow \neg AX\delta$ . If  $\delta$  describes  $N$ ,  $N \models_n \neg AX\delta$ , i.e.,  $N \models_n EX\neg\delta$ . There is a HetNet  $N'$  s.t.  $N \rightsquigarrow_n N'$  and  $N' \neq N$ . Therefore,  $N \models_n \neg\text{Hom}$  and hence  $N \models_n (\delta \rightarrow \neg\text{Hom}) \leftrightarrow (\delta \rightarrow \neg AX\delta)$ .
- (Tm<sub>1</sub>) Suppose  $N \models_n \delta \rightarrow \text{Hom}$  and  $\delta$  describes  $N$ .  $N$  is homogenized at  $n$ . If  $N \models_n \psi$ , the conclusion holds directly. If not, i.e.,  $N \models_n \neg\psi$ , by (Hom),  $N \models_n AX\neg\psi$ , i.e.,  $N \models_n \neg EX\psi$ . According to (A-ind),  $N \models_n A(\phi \cup \psi) \rightarrow \psi$ . Therefore,  $N \models_n \delta \rightarrow (A(\phi \cup \psi) \rightarrow \psi)$ .
- (Tm<sub>2</sub>) By (Tm<sub>1</sub>) and (E-ind), the conclusion holds directly.

□

## Appendix B. Algorithms of Model Checking

In this appendix, we show model checking algorithms for  $AX\phi$  and  $A(\phi \cup \psi)$ .

**Algorithm A1:** Model Checking Algorithm for  $AX\phi$ 


---

```

 $\mathcal{N} = \{N' \mid N \rightsquigarrow_n N'\};$ 
for  $N' \in \mathcal{N}$  do
  if  $N' \not\models_n \phi$  then
    return FALSE
  end
end
return TRUE

```

---

**Algorithm A2:** Model Checking Algorithm for  $A(\phi U \psi)$ 


---

```

if  $N \models_n \psi$  then
  return TRUE
else if  $N \not\models_n \phi$  then
  return FALSE
else if  $N \models_n \phi$  then
  if  $N \models_n \text{Hom}$  then
    return FALSE
  else
     $\mathcal{N} = \{N' \mid N \rightsquigarrow_n N'\};$ 
    for  $N' \in \mathcal{N}$  do
      if  $N' \not\models_n A(\phi U \psi)$  then
        return FALSE
      end
    end
    return TRUE
  end
end

```

---

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