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Bounded Solutions of Semi-Linear Parabolic Differential Equations with Unbounded Delay Terms

Allaberen Ashyralyev [1,2,3] and Sa'adu Bello Mu'azu [4,5,*]

[1] Department of Mathematics, Bahcesehir University, Istanbul 34353, Turkey; allaberen.ashyralyev@eng.bau.edu.tr
[2] Department of Mathematics, Peoples' Friendship University of Russia, 117198 Moscow, Russia
[3] Institute of Mathematics and Mathematical Modeling, Almaty 050010, Kazakhstan
[4] Department of Mathematics, Faculty of Arts and Sciences, Near East University, TRNC, Mersin 10, Nicosia 99138, Turkey
[5] Department of Mathematics, Faculty of Physical Sciences, Kebbi State University of Science and Technology, Aliero P.O. Box 1144, Nigeria
* Correspondence: 20194087@std.neu.edu.tr or saadbm13.sbm@gmail.com

Abstract: In the present work, an initial boundary value problem (IBVP) for the semi-linear delay differential equation in a Banach space with unbounded positive operators is studied. The main theorem on the uniqueness and existence of a bounded solution (BS) of this problem is established. The application of the main theorem to four different semi-linear delay parabolic differential equations is presented. The first- and second-order accuracy difference schemes (FSADSs) for the solution of a one-dimensional semi-linear time-delay parabolic equation are considered. The new desired numerical results of this paper and their discussion are presented.

Keywords: bounded solution; delay parabolic equations; uniqueness and existence; unbounded delay term

MSC: 35K60; 65M06

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1. Introduction

The modeling of biological, physical and sociological processes is carried out by using differential equations (DEs) with delay terms. They are employed to simulate oscillation systems that occur in nature. A sampled data control theory provides a classic illustration of the occurrence of time delay (for instance, see [1–4]). It is well known that the unbounded delay term present in delay differential equations makes it challenging to analyze these equations. Additionally, for a few studies, analytical solutions are provided. As a result of this, studies on numerical methods make up for the dearth of theoretical research. Particularly, the finite difference method is one of the primary techniques employed in this field.

Lu [5], investigated monotone iterative methods for finite-difference solutions of reaction–diffusion systems with time delays and provided improved iterative schemes by using the upper–lower solution approach of the Gauss–Seidel method or the Jacobi method.

The initial value problem (IVP) for linear DPPDEs was studied by Ashyralyev and Sobolevskii [6]; they provided a sufficient condition for the stability of the solution to this problem and obtained the stability estimates of solutions in Hölder norms. Different types of delay parabolic problems were investigated by Ashyralyev and Ağırseven in [7,8]. They provided convergence and stability theorems.

Finally, theorems on the uniqueness and existence of a BS of nonlinear delay parabolic equations were established by Ashyralyev, Agirseven and Ceylanin in [9]. They provided necessary conditions for the existence of a unique BS of nonlinear delay parabolic equations.

We consider the IVP

$$\begin{cases} \frac{dv}{dt} + Av(t) = f(t, B(t)v(t), B(t)v(t-d)), t \in [0, \infty), \\ v(t) = \varphi(t), t \in [-d, 0] \end{cases} \quad (1)$$

for the semi-linear differential equation in a Banach space E with linear unbounded operators A and $B(t)$ with dense domains $D(A) \subset D(B(t))$ in any arbitrary Banach space E. Assume that A is a very positive operator in E. That means $-A$ is the generator of the analytic semigroup $\exp\{-tA\}t \in [0, \infty)$ of the linear bounded operators with exponentially decreasing norm when $t \to \infty$. The following estimates are valid:

$$\|\exp\{-tA\}\|_{E\to E} \le Pe^{-\delta t}, \quad \|tA\exp\{-tA\}\|_{E\to E} \le P, t \in (0, \infty) \quad (2)$$

for some $P > 0$, $\delta > 0$. Let $B(t)$ be closed operators. The operator function $B(t)$ is strongly continuous on $D(A)$ and $\left\|B(t)A^{-1/2}\right\|_{E\to E} \le H$.

A function $v(t)$ is called a solution to problem (1) if it satisfied the following conditions:

1. $v(t)$ is a continuously differentiable function on $[-d, \infty)$.
2. The element $v(t) \in D(A)$ $\forall t \in [-d, \infty)$, and the function $Av(t)$ is continuous on $[-d, \infty)$.
3. $v(t)$ satisfies the equation and the initial condition (1).

In the present work, we aim to provide necessary conditions for the existence of a unique BS to problem (1). A semi-linear parabolic differential equation with an unbounded delay term is used to establish a theorem on the uniqueness and existence of a BS for problem (1). Four different semi-linear DPPDEs are used to illustrate the main theorem's application. Overall, it is precisely difficult to obtain the solution of semi-linear problems. Consequently, the FSADSs for the solution of semi-linear one-dimensional DPPDE are shown. Numerical results are provided. It should be noted that past publications [10–14] have looked into the BS of nonlinear one-dimensional parabolic and hyperbolic differential equations with time delay. However, due to the generality of the strategy used in this research, a larger class of multidimensional delay semi-linear DEs can be treated.

2. Theorem on Existence and Uniqueness

We reduced problem (1) into an integral equation of the form

$$v(t) = e^{-A(t-(m-1)\theta)}v((m-1)d) + \int_{(m-1)d}^{t} e^{-A(t-s)} f(s, B(s)v(s), B(s)v(s-d))ds,$$

$$t \in [(m-1)d, md], m \in N, v(t) = \varphi(t), t \in [-d, 0]$$

in $[0, \infty) \times E$, and the recursive formula for the solution of problem (1) by using successive approximations is

$$\begin{aligned} v_i(t) &= e^{-A(t-(m-1)d)}v_i((m-1)d) + \int_{(m-1)d}^{t} e^{-A(t-s)} f(s, B(s)v_{i-1}(s), B(s)v_i(s-d))ds, \\ v_0(t) &= e^{-A(t-(m-1)d)}v((m-1)d), t \in [(m-1)d, md], m \in N, i \in N, \end{aligned}$$

$$v(t) = \varphi(t), t \in [-d, 0]. \quad (3)$$

Here, N is the set of natural numbers.

Theorem 1. *Assume that the hypotheses below are fulfilled:*

1. $\varphi : [-d, 0] \times D\left(A^{\frac{1}{2}}\right) \longrightarrow E$ *be continuous function and*

$$\|\varphi(t)\|_{D\left(A^{\frac{1}{2}}\right)} \leq H. \tag{4}$$

2. $f : [0, \infty) \times D\left(A^{\frac{1}{2}}\right) \times D\left(A^{\frac{1}{2}}\right) \longrightarrow E$ *is a bounded and continuous function, i.e.,*

$$\|f(A^{\frac{1}{2}}v, A^{\frac{1}{2}}u)\|_E \leq \bar{H} \tag{5}$$

and with respect to z, the Lipschitz condition holds:

$$\|f(A^{\frac{1}{2}}v, A^{\frac{1}{2}}z) - f(A^{\frac{1}{2}}u, A^{\frac{1}{2}}z)\|_E \leq L\|A^{\frac{1}{2}}v - A^{\frac{1}{2}}u\|_E. \tag{6}$$

Here, $H, \bar{H}, L$ are positive constants and $L < \frac{1}{2Pd^{\frac{1}{2}}}$. Then, the problem (1) has a unique BS in $[0, \infty) \times E$.

Proof of Theorem 1. Using the interval $t \in [0, d]$, we can written problem (1) as

$$\frac{dv}{dt} + Av(t) = f(t, B(t)v(t), B(t)\varphi(t-d)), v(0) = \varphi(0)$$

which in an equivalent integral form, becomes

$$v(t) = e^{-At}\varphi(0) + \int_0^t e^{-A(t-s)} f(s, B(s)v(s), B(s)\varphi(s-d))ds. \tag{7}$$

In accordance with the recursive approximation approach (3), we obtain

$$v_i(t) = e^{-At}\varphi(0) + \int_0^t e^{-A(t-s)} f(s, B(s)v_{i-1}(s), B(s)\varphi(s-d))ds, i = 1, 2, \ldots \tag{8}$$

Consequently,

$$v(t) = v_0(t) + \sum_{i=0}^{\infty}(v_{i+1}(t) - v_i(t)), \tag{9}$$

where

$$v_0(t) = e^{-At}\varphi(0).$$

From (2) and (4), it follows that

$$\|A^{\frac{1}{2}}v_0(t)\|_E = \|e^{-At}\|_{E\to E}\|A^{1/2}\varphi(0)\|_E \leq HP.$$

Using Equation (8) along with estimates (2) and (5), we obtain

$$\|A^{\frac{1}{2}}v_1(t) - A^{\frac{1}{2}}v_0(t)\|_E$$

$$\leq \int_0^t \|A^{\frac{1}{2}}e^{-A(t-s)}\|\|f(s, B(s)A^{-\frac{1}{2}}A^{\frac{1}{2}}u_0, B(s)A^{-\frac{1}{2}}A^{\frac{1}{2}}\varphi(s-d))\|_E ds \leq 2\bar{H}Pt^{\frac{1}{2}}.$$

By triangle inequality, we have

$$\|A^{\frac{1}{2}}v_1(t)\|_E \leq HP + 2\bar{H}Pt^{\frac{1}{2}}.$$

Using Formula (8) along with estimates (2), (5), and (6), we obtain

$$
\begin{aligned}
&\|A^{\frac{1}{2}}v_2(t) - A^{\frac{1}{2}}v_1(t)\|_E \\
&\le \int_0^t \|A^{\frac{1}{2}}e^{-A(t-s)}\|\|f(s, B(s)v_1, B(s)\varphi(s-d)) - f(s, B(s)v_0, B(s)\varphi(s-d))\|_E ds \\
&\le LP\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}}\|B(s)v_1(s) - B(s)v_0(s)\|_E ds \le 2LP^2\bar{H}\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}}s^{\frac{1}{2}}ds \\
&\le 4LP^2\bar{H}t.
\end{aligned}
$$

Then,

$$\|A^{\frac{1}{2}}v_2(t)\|_E \le HP + 2\bar{H}Pt^{\frac{1}{2}} + 4LP^2\bar{H}t.$$

Let

$$\|A^{\frac{1}{2}}v_n(t) - A^{\frac{1}{2}}v_{n-1}(t)\|_E \le \frac{\bar{H}}{L}(2LPt^{\frac{1}{2}})^n.$$

Therefore, we obtain

$$
\begin{aligned}
&\|A^{\frac{1}{2}}v_{n+1}(t) - A^{\frac{1}{2}}v_n(t)\|_E \\
&\le \int_0^t \|A^{\frac{1}{2}}e^{-A(t-s)}\|\|f(B(s)v_n, B(s)\varphi(s-d)) - f(B(s)v_{n-1}, B(s)\varphi(s-d))\|_E ds \\
&\le P\int_0^t L\|B(s)v_n(s) - B(s)v_{n-1}(s)\|_E ds \le P\int_0^t L\frac{\bar{H}}{L}(2LPs^{\frac{1}{2}})^n ds \\
&\le \frac{\bar{H}}{L}(2LPt^{\frac{1}{2}})^{n+1}.
\end{aligned}
$$

Henceforth, for any $n, n \ge 1$, we obtain

$$\|A^{\frac{1}{2}}v_{n+1}(t) - A^{\frac{1}{2}}v_n(t)\|_E \le \frac{\bar{M}}{L}(2LPt^{\frac{1}{2}})^{n+1}$$

and

$$\|A^{\frac{1}{2}}v_{n+1}(t)\|_E \le MP + 2\bar{M}Pt^{\frac{1}{2}} + \ldots + \frac{\bar{M}}{L}(2LPt^{\frac{1}{2}})^{n+1}$$

by mathematical induction. It is implied by that equation and Equation (9) that

$$
\begin{aligned}
&\|A^{\frac{1}{2}}v(t)\|_E \le \|A^{\frac{1}{2}}v_0(t)\|_E \\
&\quad + \sum_{i=0}^{\infty}\|A^{\frac{1}{2}}v_{i+1}(t) - A^{\frac{1}{2}}v_i(t)\|_E \le HP + \sum_{i=0}^{\infty}\frac{\bar{H}}{L}(2LPt^{\frac{1}{2}})^{i+1} < \infty, t \in [0,d].
\end{aligned}
$$

This shows that problem (1) solution exists and is bounded in $[0,d] \times E$.

From $t \in [d, 2d]$, it follows that $0 \le t - d \le d$. We denote that

$$\varphi_1(t) = v(t-d), t \in [d, 2d],$$

and suppose that problem (1) has a BS in $[d, 2d] \times E$. Replacing t and $t-d$, we can write

$$\|A^{\frac{1}{2}}\varphi_1(t)\| \le H_1$$

and

$$\|f(A^{\frac{1}{2}}v_0(t), A^{\frac{1}{2}}\varphi_1(t))\|_E \le \bar{H}_1.$$

According to successive approximation of Formula (3), we can write

$$v_0(t) = e^{-A(t-d)}\varphi_1(d)$$
$$v_i(t) = e^{-A(t-d)}\varphi_1(d) + \int_d^t e^{-A(t-s)} f(B(s)v_{i-1}(s), B(s)\varphi_1(s))ds, i = 1, 2, \ldots$$

In the same way, for any $r, r \geq 1$, we obtain

$$\|A^{\frac{1}{2}}v_{r+1}(t) - A^{\frac{1}{2}}v_r(t)\|_E \leq \frac{\bar{H}_1}{L}(2LPt^{\frac{1}{2}})^{r+1},$$

and

$$\|A^{\frac{1}{2}}v_{r+1}(t)\|_E \leq H_1P + 2\bar{H}_1Pt^{\frac{1}{2}} + \ldots + \frac{\bar{H}_1}{L}(2LPt^{\frac{1}{2}})^{r+1}.$$

From that, it implies that

$$\|A^{\frac{1}{2}}v(t)\|_E \leq \|A^{\frac{1}{2}}v_0(t)\|_E$$
$$+ \sum_{i=0}^{\infty} \|A^{\frac{1}{2}}v_{i+1}(t) - A^{\frac{1}{2}}v_i(t)\|_E \leq H_1P + \sum_{i=0}^{\infty} \frac{\bar{H}_1}{L}(2LPt^{\frac{1}{2}})^{i+1} < \infty, t \in [d, 2d].$$

This proves that problem (1)'s solution exists, and it is bounded in $[d, 2d] \times E$.

In the same procedure one, can establish that

$$\|A^{\frac{1}{2}}v(t)\|_E \leq H_1P + \sum_{i=0}^{\infty} \frac{\bar{H}_1}{L}(2LPt^{\frac{1}{2}})^{i+1}, t \in [nd, (n+1)d],$$

where H_n and $\bar{H}_n$ are bounded. This shows that problem (1)'s solution exists and is bounded in $[nd, (n+1)d] \times E$. Overall, the constructed function $v(t)$ of problem (1) is a BS in $[0, \infty) \times E$.

We shall now show that this solution to problem (1) is unique. Suppose that problem (1) has a BS solution $u(t)$ and that $u(t) \neq v(t)$. We write down $z(t) = u(t) - v(t)$. Hence, for $z(t)$, we obtain that

$$\begin{cases} \frac{dz}{dt} + Az(t) = f(B(s)u(t), B(s)u(t-d)) - f(B(s)v(t), B(s)v(t-d)), t \in (0, \infty), \\ z(t) = 0, t \in [-d, 0]. \end{cases}$$

We consider $t \in [0, d]$. As $u(t-d) = u(t-d) = \varphi(t-d)$, we obtain

$$\begin{cases} \frac{dz}{dt} + Az(t) = f(B(s)u(t), B(s)\varphi(t-d)) - f(B(s)v(t), B(s)\varphi(t-d)), t \in (0, \infty), \\ z(t) = 0, t \in [-d, 0]. \end{cases}$$

Henceforth,

$$z(t) = e^{-At}z(0) + \int_0^t e^{-A(t-s)}[f(B(s)v(s), B(s)\varphi(s-d)) - f(B(s)u(s), B(s)\varphi(s-d))]ds.$$

Using (2) and (5), we obtain

$$\|A^{\frac{1}{2}}z(t)\|_E \leq \int_0^t \|A^{\frac{1}{2}}e^{-A(t-s)}\|\|f(B(s)v(s), B(s)\varphi(s-d)) - f(B(s)u(s), \varphi(s-d))\|_E ds$$
$$\leq PL\int_0^t \|B(s)v(s) - B(s)u(s)\|_E ds \leq PL\int_0^t \|A^{\frac{1}{2}}z(s)\|_E ds.$$

By means of integral inequality, we can write

$$\|A^{\frac{1}{2}}z(t)\|_E \leq 0.$$

This implies that $A^{\frac{1}{2}}z(t) = 0$, which proves that problem (1)'s solution is unique and bounded in $[0, d] \times E$.

Using a similar procedure and mathematical induction, we can show that problem (1)'s solution is unique and bounded in $[0, \infty) \times E$. $\square$

Remark 1. *The approach used in the current study also makes it possible to demonstrate, under certain presumptions, that there exists a unique BS to the IVP for semi-linear parabolic equations*

$$\begin{cases} \frac{dv}{dt} + A(t)v(t) = f(t, B(t)v(t), B(t)v([t])), 0 < t < \infty, \\ v(0) = \varphi \end{cases} \tag{10}$$

in a Banach space E with unbounded operators $A(t)$ and $B(t)$.

Remark 2. *It is known that various problems in fluid mechanics dynamics, elasticity and other areas of physics lead to fractional parabolic-type differential equations. Methods of solutions of problems for linear fractional differential equations have been studied extensively by many researchers (see, e.g., [15–22] and the references given therein). The approach used in the current study also makes it possible to demonstrate, under certain presumptions, that there exists a unique BS to the IVP for semi-linear fractional parabolic equations*

$$\begin{cases} \frac{dv}{dt} + Av(t) + D^{\alpha}v(t) = f(t, B(t)v(t), B(t)v(t-d)), t \in [0, \infty), \\ v(t) = \varphi(t), t \in [-d, 0] \end{cases} \tag{11}$$

in a Banach space E with unbounded operators A and $B(t)$. Here, $\alpha \in [0, 1)$.

3. Applications

We begin by considering an IBVP for semi-linear one-dimensional DPPDEs with the Dirichlet condition:

$$\begin{cases} v_t(t, x) - a(x)v_{xx}(t, x) + \delta v(t, x) = f(t, x, v_x(t, x), v_x(t-d, x)), \\ t \in (0, \infty), x \in (0, l) \\ v(t, x) = \varphi(t, x), \varphi(t, 0) = \varphi(t, l) = 0, t \in [-d, 0], x \in [0, l], \\ v(t, 0) = v(t, l) = 0, t \in [-d, \infty), \end{cases} \tag{12}$$

where $\varphi(t, x), a(x)$ are given sufficiently smooth functions (SSFs) and a delta greater than zero is a significant enough number. Suppose that $a(x) \geq a > 0$.

We can reduce the IBVP (12) to IVP (1) in $E = C[0, l]$ with the strong positive operator A^x in $C[0, l]$ according to the following formula:

$$A^x v = -a(x)\frac{d^2v}{dx^2} + \delta v \tag{13}$$

with domain $D(A^x) = \left\{v \in C^{(2)}[0, l] : v(0) = v(l) = 0\right\}$ [23]. Moreover, we have the following estimates:

$$\| \exp\{-tA^x\}\|_{C[0,l] \to C[0,l]} \leq P, \ t \in [0, \infty), \|tA^x \exp\{-tA^x\}\|_{C[0,l] \to C[0,l]} \leq P, t \in (0, \infty).$$

Therefore, from that and abstracting Theorem 1, we have the following:

Theorem 2. *Suppose the hypotheses below:*

1. $\varphi : [-d, 0] \times [0, l] \times C^{(1)}[0, l] \to C[0, l]$ *is a continuous function and*

$$\|\varphi_x(t, .)\|_{C[0,l]} \leq H. \tag{14}$$

2. $f : [0,\infty) \times (0,l) \times C^{(1)}[0,l] \times C^{(1)}[0,l] \to C[0,l]$ *is a bounded and continuous function, i.e.,*

$$\|f(t,.,v_x,u_x))\|_{C[0,l]} \le \overline{H} \tag{15}$$

and with respect to z, the Lipschitz condition holds:

$$\|f(t,.,v_x,z_x) - f(t,.,u_x,z_x)\|_{C[0,l]} \le L\|v_x - u_x\|_{C[0,l]}, \tag{16}$$

where $L, H, \overline{H}$, *are positive constants and* $L < \frac{1}{2Pd^{\frac{1}{2}}}$. *Then, problem (12) has a unique BS in* $[0,\infty) \times C[0,l]$.

In addition, we consider the IBVP for semi-linear one-dimensional DPPDEs with the Neumann condition:

$$\begin{cases} v_t(t,x) - a(x)v_{xx}(t,x) + \delta v(t,x) = f(t,x,v_x(t,x),v_x(t-d,x)), \\ t \in (0,\infty), x \in (0,l) \\ v(t,x) = \varphi(t,x), \varphi_x(t,0) = \varphi_x(t,l) = 0, t \in [-d,0], x \in [0,l], \\ v_x(t,0) = v_x(t,l) = 0, t \in [-d,\infty), \end{cases} \tag{17}$$

where $\varphi(t,x), a(x)$ are given SSFs and delta greater than zero is a significant enough number. We suppose that $a(x) \ge a > 0$.

We can reduce the IBVP (17) to IVP (1) in $E = C[0,l]$ with the strong positive operator A^x in $C[0,l]$ according to the formula (13) with domain [23]:

$$D(A^x) = \left\{v \in C^{(2)}[0,l] : v'(0) = v'(l) = 0\right\}.$$

Moreover, we have the following estimates:

$$\|\exp\{-tA^x\}\|_{C[0,l]\to C[0,l]} \le P,\ t \in [0,\infty), \|tA^x \exp\{-tA^x\}\|_{C[0,l]\to C[0,l]} \le P, t \in (0,\infty).$$

Therefore, from that and abstracting Theorem 1, we have the following:

Theorem 3. *Suppose that assumptions (14)–(16) hold. Then, problem (17) has a unique BS in* $[0,\infty) \times C[0,l]$.

Furthermore, we consider the IBVP for semi-linear one-dimensional DPPDEs with nonlocal conditions:

$$\begin{cases} v_t(t,x) - a(x)v_{xx}(t,x) + \delta v(t,x) = f(x,v_x(t,x),v_x(t-d,x)), \\ t \in (0,\infty), x \in (0,l), \\ v(t,x) = \varphi(t,x), \varphi(t,0) = \varphi(t,l), \varphi_x(t,0) = \varphi_x(t,l), \\ t \in [-d,0], x \in [0,l], \\ v(t,0) = v(t,l), v_x(t,0) = v_x(t,l), t \in [-d,\infty), \end{cases} \tag{18}$$

where $\varphi(t,x), a(x)$ are given SSFs and a delta greater than zero is a significant enough number. We suppose that $a(x) \ge a > 0$.

We can reduce the IBVP (18) to IVP (1) in $E = C[0,l]$ with the strong positive operator A^x in $C[0,l]$ according to the Formula (13) with domain [23]:

$$D(A^x) = \left\{ v \in C^{(2)}[0,l] : v(0) = v(l), v'(0) = v'(l) \right\}.$$

Moreover, we have the following estimates:

$$\| \exp\{-tA^x\}\|_{C[0,l]\to C[0,l]} \le P, t \in [0,\infty), \|tA^x \exp\{-tA^x\}\|_{C[0,l]\to C[0,l]} \le P, t \in (0,\infty).$$

Therefore, from that and abstracting Theorem 1, we have the following:

Theorem 4. *Suppose that assumptions (14)–(16) hold. Then, problem (18) has a unique BS in* $[0,\infty) \times C[0,l]$.

Finally, we consider the IBVP for semi-linear one-dimensional DPPDEs with Robin condition:

$$\begin{cases} v_t(t,x) - (a(x)v_x(t,x))_x + \delta v(t,x) = f(x, v_x(t,x), v_x(t-d,x)), \\ t \in (0,\infty), x \in (0,l) \\ v(t,x) = \varphi(t,x), \varphi(t,0) = b\varphi_x(t,0), -\varphi(t,l) = c\varphi_x(t,l), t \in [-d,0], x \in [0,l], \\ v(t,0) = bv_x(t,0), -v(t,l) = cv_x(t,l), t \in [-d,0], \end{cases} \tag{19}$$

where $\varphi(t,x), a(x)$ are given SSFs. Here, $a(x) \ge a > 0$ and b, c, δ are positive constants.

We can reduce the IBVP (19) to IVP (1) in $E = L_2[0,l]$ with the self-adjoint positive-definite operator A^x in $L_2[0,l]$ according to the following formula:

$$Az = -\frac{d}{dx}\left(a(x)\frac{dv(x)}{dx}\right) + \delta v(x) \tag{20}$$

with domain $D(A^x) = \{v : v, v_2'[0,l], v(0) = bv^{'}(0), -v(l) = cv^{'}(l)\}$ [24]. Moreover, we have the following estimates:

$$\| \exp\{-tA^x\}\|_{L_2[0,l]\to L_2[0,l]} \le 1,\ t \in [0,\infty), \|tA^x \exp\{-tA^x\}\|_{L_2[0,l]\to L_2[0,l]} \le 1, t \in (0,\infty).$$

Therefore, from that and abstracting Theorem 1, we have the following:

Theorem 5. *Suppose the hypotheses below:*

1. $\varphi : [-d,0] \times [0,l] \times L_2[0,l] \to C[0,l]$ *is a continuous function and*
$$\|\varphi_x(t,.)\|_{W_2^1[0,l]} \le H. \tag{21}$$
2. $f : [0,\infty) \times (0,l) \times W_2^1[0,l] \times W_2^1[0,l] \to L_2[0,l]$ *is a bounded and continuous function, i.e.,*
$$\|f(t,.,v_x,u_x))\|_{L_2[0,l]} \le \overline{H} \tag{22}$$
and with respect to z, the Lipschitz condition holds:
$$\|f(t,.,v_x,z_x) - f(t,.,u_x,z_x)\|_{L_2[0,l]} \le L\|v_x - u_x\|_{L_2[0,l]}, \tag{23}$$
where $L, H, \overline{H}$, *are positive constants and* $L < \frac{1}{2Pd^{\frac{1}{2}}}$. *Then, problem (19) has a unique BS in* $[0,\infty) \times L_2[0,l]$.

4. Numerical Results

Generally speaking, a semi-linear equation cannot be solved precisely. Henceforth, the FSADSs for the solution of semi-linear one-dimensional DPPDE are described. The numerical results are given. Consider the IBVP

$$\begin{cases} v_t(t,x) - v_{xx}(t,x) = v_x(t,x)\{v([t-1],x)\cos x - v_x([t-1],x)\sin x\}, \\ t \in (0,\infty),\ x \in (0,\pi), \\ v(0,x) = \sin x, x \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0,\ t \in [0,\infty) \end{cases} \quad (24)$$

for the semi-linear DPPDE. Here, $[\cdot]$ is notation of an integer function. The ES of this problem is $v(t,x) = e^{-t}\sin x$.

We obtain the following iterative FADS for the approximate solution of the IBVP (24)

$$\begin{cases} \frac{{}_rv_n^k - {}_rv_n^{k-1}}{\tau} - \frac{{}_rv_{n+1}^k - 2{}_rv_n^k + {}_rv_{n-1}^k}{h^2} \\ = \frac{{}_{r-1}v_{n+1}^k - {}_{r-1}v_{n-1}^k}{2h}\left\{{}_rv_n^{[k-N]}\cos x_n - \frac{{}_rv_{n+1}^{[k-N]} - {}_rv_{n-1}^{[k-N]}}{2h}\sin x_n\right\}, \\ t_k = k\tau, x_n = nh, k \in \overline{1,\infty}, n \in \overline{1,M-1}, \\ {}_rv_n^0 = \sin x_n, x_n = nh, n \in \overline{0,M}, \\ {}_rv_0^k = {}_rv_M^k = 0, k \in \overline{0,\infty} \end{cases} \quad (25)$$

for the numerical solution of the semi-linear delay parabolic equation.

Here, r stands for the iteration number, ${}_0v_n^k$, $k \in \overline{0,N}$, and $n \in \overline{0,M}$ is the initial starting value. Numerically, we use the steps listed below to solve the difference scheme (25). For $k \in \overline{0,N}$, $n \in \overline{0,M}$

- $r = 1$
- ${}_{r-1}v_n^k$ is known;
- ${}_rv_n^k$ is determined;
- $r = r+1$ is taken, and we proceed to step 2 if the maximum absolute error between ${}_{r-1}v_n^k$ and ${}_rv_n^k$ is more than the specified tolerance value. If not, stop the iteration process and use ${}_rv_n^k$ as the solution to the given problem.

We write (25) in matrix form:

$$\begin{gathered} A_r V^k + B_r V^{k-1} = R\varphi({}_{r-1}v^k, {}_rv^{k-N}), k \in \overline{1,N}, \\ {}_rV^0 = \{\sin x_n\}_{n=0}^M, n \in \overline{0,M}, \end{gathered} \quad (26)$$

Additionally, using the SADS for the AS of problem (24), we have the following SEs:

$$\begin{cases} \frac{{}_r v_n^k - {}_r v_n^{k-1}}{\tau} - \frac{{}_r v_{n+1}^k - 2{}_r v_n^k + {}_r v_{n-1}^k}{h^2} + \tau \frac{{}_r v_{n+2}^k - 4{}_r v_{n+1}^k + 6{}_r v_n^k - 4{}_r v_{n-1}^k + {}_r v_{n-2}^k}{2h^4} \\ = \frac{1}{2}\left\{ \frac{{}_{r-1}v_{n+1}^k - {}_{r-1}v_{n-1}^k}{2h} \right\}\left\{ {}_r v_n^{k-N} \cos x_n - \frac{{}_r v_{n+1}^{k-N} - {}_r v_{n-1}^{k-N}}{2h} \sin x_n \right\} \\ + \frac{1}{2}\left\{ \frac{{}_{r-1}v_{n+1}^{k-1} - {}_{r-1}v_{n-1}^{k-1}}{2h} \right\}\left\{ {}_r v_n^{k-1-N} \cos x_n - \frac{{}_r v_{n+1}^{k-1-N} - {}_r v_{n-1}^{k-1-N}}{2h} \sin x_n \right\} \\ - \frac{\tau}{4}\left\{ \frac{{}_{r-1}v_{n+2}^k - {}_{r-1}v_n^k}{2h} \right\} \frac{{}_r v_{n+1}^{k-N} \cos x_{n+1} - \frac{{}_r v_{n+2}^{k-N} - {}_r v_n^{k-N}}{2h} \sin x_{n+1}}{h^2} \\ - \frac{\tau}{4}\left\{ \frac{{}_{r-1}v_{n+1}^k - {}_{r-1}v_{n-1}^k}{2h} \right\} \frac{-2{}_r v_n^{k-N} \cos x_n + 2\frac{{}_r v_{n+1}^{k-N} - {}_r v_{n-1}^{k-N}}{2h} \sin x_n}{h^2} \\ - \frac{\tau}{4}\left\{ \frac{{}_{r-1}v_n^k - {}_{r-1}v_{n-2}^k}{2h} \right\} \frac{{}_r v_{n-1}^{k-N} \cos x_{n-1} - \frac{{}_r v_n^{k-N} - {}_r v_{n-2}^{k-N}}{2h} \sin x_{n-1}}{h^2} \\ - \frac{\tau}{4}\left\{ \frac{{}_r v_{n+2}^{k-1} - {}_r v_n^{k-1}}{2h} \right\} \frac{{}_{r-1}v_{n+1}^{k-1-N} \cos x_{n+1} - \frac{{}_{r-1}v_{n+2}^{k-1-N} - {}_{r-1}v_n^{k-1-N}}{2h} \sin x_{n+1}}{h^2} \\ - \frac{\tau}{4}\left\{ \frac{{}_r v_{n+1}^{k-1} - {}_r v_{n-1}^{k-1}}{2h} \right\} \frac{-2{}_{r-1}v_n^{k-1-N} \cos x_n + 2\frac{{}_{r-1}v_{n+1}^{k-1-N} - {}_{r-1}v_{n-1}^{k-1-N}}{2h} \sin x_n}{h^2} \\ - \frac{\tau}{4}\left\{ \frac{{}_r v_n^{k-1} - {}_r v_{n-2}^{k-1}}{2h} \right\} \frac{{}_{r-1}v_{n-1}^{k-1-N} \cos x_{n-1} - \frac{{}_{r-1}v_n^{k-1-N} - {}_{r-1}v_{n-2}^{k-1-N}}{2h} \sin x_{n-1}}{h^2}, \\ t_k = k\tau, x_n = nh, k \in \overline{1,N}, n \in \overline{2, M-2}, \\ {}_r v_n^0 = \varphi(x_n) = \sin x_n, n \in \overline{0,M}, {}_r v_0^k = {}_r v_M^k = 0,\ k \in \overline{0,N}, \\ {}_r v_3^k = 4{}_r v_2^k - 5{}_r v_1^k, {}_r v_{M-3}^k = 4{}_r v_{M-2}^k - 5{}_r v_{M-1}^k,\ k \in \overline{0,N}. \end{cases} \tag{27}$$

We obtain again $(M+1) \times (M+1)$ SLEs, and we reformat them into matrix form (26).

Consequently, we obtain a second-order difference equation with respect to k matrix coefficients. Using (26), we can obtain this difference scheme's solution. The initial guess in computations for both FSADSs is set as ${}_0 v_n^k = e^{-t_k} \sin x_n$, and the iterative procedure is stopped when the maximum errors between two successive outcomes of the difference schemes (25) and (27) become less than 10^{-8}.

For various values of M and N, we provide numerical results and ${}_r v_n^k$ represents the numerical solutions of these difference schemes at (t_k, x_n). Tables are constructed for $M = N = 30, 60, 120$ in that order for $t \in [r, r+1]$, $r = 0, 1, 2$ and the errors are calculated using the following formula:

$${}_r\left(E_M^N\right)_p = \max_{\substack{pN+1 \le k \le (p+1)N, p=0,1,\ldots \\ 1 \le n \le M-1}} \left| v(t_k, x_n) - {}_r v_n^k \right|. \tag{28}$$

To finish iteration process, we used the following condition in each sub-interval:

$$\max_{\substack{pN+1 \le k \le (p+1)N, p=0,1,\ldots \\ 1 \le n \le M-1}} \left| {}_r v_n^k - {}_{r-1}v_n^k, \right| < 10^{-8} \tag{29}$$

These numerical experiments back up the theoretical claims, as shown in Tables 1–4. With more grid points, the maximum errors and the number of iterations are reduced.

Table 1. Errors and number r of iterations to difference schemes (25) in $t \in [r, r+1]$, $r = 0, 1, 2$ for different steps of discreatization.

$N = M$	$(E_M^N)_0^r$	r for $[0,1]$	$(E_M^N)_1^r$	r for $[1,2]$	$(E_M^N)_2^r$	r for $[2,3]$
30	6.3783×10^{-3}	2	2.3464×10^{-3}	3	8.6321×10^{-4}	3
60	3.1279×10^{-2}	2	1.1507×10^{-3}	3	4.2332×10^{-4}	2
120	1.5485×10^{-3}	2	5.6964×10^{-4}	2	2.0956×10^{-4}	2

Table 2. Errors and number r of iterations to difference schemes (27) in $t \in [r, r+1]$, $r = 0, 1, 2$ for different steps of discreatization.

$N = M$	$(E_M^N)_0^r$	r for $[0,1]$	$(E_M^N)_1^r$	r for $[1,2]$	$(E_M^N)_2^r$	r for $[2,3]$
30	4.5864×10^{-4}	3	1.6358×10^{-4}	3	5.3201×10^{-5}	2
60	1.1212×10^{-4}	3	4.2149×10^{-5}	2	1.3581×10^{-5}	2
120	4.3398×10^{-5}	2	1.0698×10^{-5}	2	3.4122×10^{-6}	2

We also consider the IBVP

$$\begin{cases} v_t(t,x) - v_{xx}(t,x) + \sin(v(t,x)) \\ = v_x(t,x)\{2v([t-1],x)\cos 2x - v_x([t-1],x)\sin 2x\} + f(t,x), \\ \ t \in (0,\infty),\ x \in (0,\pi), \\ \\ v(0,x) = \sin 2x,\ x \in [0,\pi], \\ \\ v(t,0) = v(t,\pi), v_x(t,0) = v_x(t,\pi),\ t \in [0,\infty) \end{cases} \tag{30}$$

for the semi-linear DPPDE. The ES of this problem is $v(t,x) = e^{-4t}\sin 2x$ and $f(t,x) = \sin\left(e^{-4t}\sin 2x\right)$.

We obtain the following FADS for the approximate solution of the IBVP (30)

$$\begin{cases} \frac{{}_rv_n^k - {}_rv_n^{k-1}}{\tau} - \frac{{}_rv_{n+1}^k - 2{}_rv_n^k + {}_rv_{n-1}^k}{h^2} = 2\left\{\frac{{}_{r-1}v_{n+1}^k - {}_{r-1}v_{n-1}^k}{2h}\right\}{}_rv_n^{[k-N]}\cos 2x_n \\ -\left\{\frac{{}_{r-1}v_{n+1}^k - {}_{r-1}v_{n-1}^k}{2h}\right\}\frac{{}_rv_{n+1}^{[k-N]} - {}_rv_{n-1}^{[k-N]}}{2h}\sin 2x_n - \sin\left({}_{r-1}v_n^k\right) + f(t_k,x_n), \\ \\ t_k = k\tau, x_n = nh, k \in \overline{1,N}, n \in \overline{1,M-1}, \\ \\ {}_rv_n^0 = \sin 2x_n, x_n = nh, n \in \overline{0,M}, \\ \\ {}_rv_0^k = {}_rv_M^k, {}_rv_1^k - {}_rr_0^k = {}_rv_M^k - {}_rv_{M-1}^k, \\ pN+1 \le k \le (p+1)N, p = 0,1,\ldots \end{cases} \tag{31}$$

for the numerical solution of the delay parabolic equation with nonlocal conditions.

We write (31) in matrix form:

$$\begin{gathered} A_r V^k + B_r V^{k-1} = R_{r-1}\theta^k, k \in \overline{p(N+1),(p+1)N}, p = 0,1,\ldots, \\ {}_rV^0 = \{\sin 2x_n\}_{n=0}^M, \end{gathered} \tag{32}$$

where

$${}_rV^k = \left\{{}_rv_n^k\right\}_{n=0}^M,$$

$$
\begin{aligned}
{}_{r-1}\theta_n^k = & -\sin\left({}_{r-1}v_n^k\right) + f(t_k, x_n) + 2\left\{\frac{{}_{r-1}v_{n+1}^k - {}_{r-1}v_{n-1}^k}{2h}\right\}{}_r v_n^{[k-N]} \cos 2x_n \\
& - \left\{\frac{{}_{r-1}v_{n+1}^k - {}_{r-1}v_{n-1}^k}{2h}\right\}\frac{{}_r v_{n+1}^{[k-N]} - {}_r v_{n-1}^{[k-N]}}{2h} \sin 2x_n, \\
& n = 0, \dots, M, k \in \overline{p(N+1),(p+1)N}, p = 0,1,\dots,
\end{aligned}
$$

Furthermore, using the SADS for the AS of problem (30), we obtain the following SEs:

$$
\begin{cases}
\frac{{}_r v_n^k - {}_r v_n^{k-1}}{\tau} - \frac{{}_r v_{n+1}^k - 2{}_r v_n^k + {}_r v_{n-1}^k}{h^2} + \tau \frac{{}_r v_{n+2}^k - 4{}_r v_{n+1}^k + 6{}_r v_n^k - 4{}_r v_{n-1}^k + {}_r v_{n-2}^k}{2h^4} \\
-\frac{1}{2}\left\{\frac{{}_{r-1}v_{n+1}^k - {}_{r-1}v_{n-1}^k}{2h}\right\}\left[{}_r v_n^{[k-N]} \cos 2x_n - \frac{{}_r v_{n+1}^{[k-N]} - {}_r v_{n-1}^{[k-N]}}{2h} \sin 2x_n\right] \\
-\frac{1}{2}\left\{\frac{{}_{r-1}v_{n+1}^{k-1} - {}_{r-1}v_{n-1}^{k-1}}{2h}\right\}\left[{}_r v_n^{[k-1-N]} \cos 2x_n - \frac{{}_r v_{n+1}^{[k-1-N]} - {}_r v_{n-1}^{[k-1-N]}}{2h} \sin 2x_n\right] \\
+\frac{\tau}{4}\left\{\frac{{}_{r-1}v_{n+2}^k - {}_{r-1}v_n^k}{2h}\right\}\frac{{}_r v_{n+1}^{[k-N]} \cos 2x_{n+1} - \frac{{}_r v_{n+2}^{[k-N]} - {}_r v_n^{[k-N]}}{2h} \sin 2x_{n+1}}{h^2} \\
+\frac{\tau}{4}\left\{\frac{{}_{r-1}v_{n+1}^k - {}_{r-1}v_{n-1}^k}{2h}\right\}\frac{-2{}_r v_n^{[k-N]} \cos 2x_n + 2\frac{{}_r v_{n+1}^{[k-N]} - {}_r v_{n-1}^{[k-N]}}{2h} \sin 2x_n}{h^2} \\
+\frac{\tau}{4}\left\{\frac{{}_{r-1}v_n^k - {}_{r-1}v_{n-2}^k}{2h}\right\}\frac{{}_r v_{n-1}^{[k-N]} \cos 2x_{n-1} - \frac{{}_r v_n^{[k-N]} - {}_r v_{n-2}^{[k-N]}}{2h} \sin 2x_{n-1}}{h^2} \\
+\frac{\tau}{4}\left\{\frac{{}_{r-1}v_{n+2}^{k-1} - {}_{r-1}v_n^{k-1}}{2h}\right\}\frac{{}_r v_{n+1}^{[k-1-N]} \cos 2x_{n+1} - \frac{{}_r v_{n+2}^{[k-1-N]} - {}_r v_n^{[k-1-N]}}{2h} \sin 2x_{n+1}}{h^2} \\
+\frac{\tau}{4}\left\{\frac{{}_{r-1}v_{n+1}^{k-1} - {}_{r-1}v_{n-1}^{k-1}}{2h}\right\}\frac{-2{}_r v_n^{[k-1-N]} \cos 2x_n + 2\frac{{}_r v_{n+1}^{[k-1-N]} - {}_r v_{n-1}^{[k-1-N]}}{2h} \sin 2x_n}{h^2} \\
+\frac{\tau}{4}\left\{\frac{{}_{r-1}v_n^{k-1} - {}_{r-1}v_{n-2}^{k-1}}{2h}\right\}\frac{{}_r v_{n-1}^{[k-1-N]} \cos 2x_{n-1} - \frac{{}_r v_n^{[k-1-N]} - {}_r v_{n-2}^{[k-1-N]}}{2h} \sin 2x_{n-1}}{h^2} \\
+\sin\left({}_{r-1}v_n^k\right) = f(t_k, x_n), \\
\\
t_k = k\tau, x_n = nh, k \in \overline{1,N}, n \in \overline{2,M-2}, \\
{}_r v_n^0 = \sin 2x_n, 0 \le n \le M, \\
{}_r v_0^k = {}_r r_M^k, -{}_r v_2^k + 4{}_r v_1^k - 3{}_r v_0^k = 3{}_r v_M^k - 4{}_r v_{M-1}^k + {}_r v_{M-2}^k, \\
-{}_r v_3^k + 4{}_r v_2^k - 5{}_r v_1^k + 2{}_r v_0^k \\
= 2{}_r v_M^k - 5{}_r v_{M-1}^k + 4{}_r v_{M-2}^k - {}_r v_{M-3}^k, \\
-3{}_r v_4^k + 14{}_r v_3^k - 24{}_r v_2^k + 18{}_r v_1^k - 5{}_r v_0^k \\
= 5{}_r v_M^k - 18{}_r v_{M-1}^k + 24{}_r v_{M-2}^k - 14{}_r v_{M-3}^k + 3{}_r v_{M-4}^k, \\
k \in \overline{p(N+1),(p+1)N}, p = 0,1,\dots
\end{cases}
\quad (33)
$$

We obtain another $(M+1)\times(M+1)$ SLE; they are then rewritten in matrix form (32).

For a range of M and N values, we provide numerical results, and ${}_r v_n^k$ represents the numerical solutions of these difference schemes at (t_k, x_n). Tables 3 and 4 are constructed for $M = N = 30$, 60, 120 in that order for $t \in [r, r+1]$, $r = 0,1,2$, and the errors are calculated using Formulas (28) and (29).

Table 3. Errors and number r of iterations to difference schemes (31) in $t \in [r, r+1]$, $r = 0,1,2$ for different steps of discreatization.

$N = M$	$(E_M^N)_0^r$	r for $[0,1]$	$(E_M^N)_1^r$	r for $[1,2]$	$(E_M^N)_2^r$	r for $[2,3]$
30	2.4431×10^{-2}	2	5.3731×10^{-3}	9	1.0838×10^{-4}	7
60	1.2259×10^{-2}	2	2.5664×10^{-3}	8	4.9176×10^{-5}	6
120	6.1304×10^{-3}	2	1.2517×10^{-3}	8	2.3435×10^{-5}	6

Table 4. Errors and number r of iterations to difference schemes (33) in $t \in [r, r+1]$, $r = 0, 1, 2$ for different steps of discreatization.

$N = M$	$(E_M^N)_0^r$	r for $[0, 1]$	$(E_M^N)_1^r$	r for $[1, 2]$	$(E_M^N)_2^r$	r for $[2, 3]$
30	2.0589×10^{-3}	8	3.0514×10^{-4}	8	1.5588×10^{-5}	7
60	5.4628×10^{-4}	8	7.5756×10^{-5}	7	2.0085×10^{-6}	5
120	1.3865×10^{-4}	7	1.9241×10^{-5}	6	4.8130×10^{-7}	3

As we doubled the values of N and M each time, beginning with $N = M = 30$, in the first-order accuracy difference schemes (25) and (31) in Tables 1 and 3, the errors decrease roughly by a proportion of $1/2$, while in the second-order accuracy difference schemes (27) and (33) in Tables 2 and 4, the errors decrease roughly by a proportion of $1/4$. The errors shown in the tables demonstrate the consistency of the different schemes and the reliability of the findings. Accordingly, the SADS increases more quickly than the FADS.

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Abbreviations

In this manuscript, the acronyms used are as follows:

IBVP	Initial Boundary Value Problem
IVP	Initial Value Problem
BS	Bounded Solution
DE	Differential Equation
DPPDE	Delay Parabolic Partial Differential Equation
SSFs	Sufficiently Smooth Functions
FSADSs	First- and Second-order Accuracy Difference Schemes
ES	Exact Solution
FADS	First-order Accuracy Difference Scheme
SADS	Second-order Accuracy Difference Scheme
AS	Approximate Solution
SE	System of Equation
SLEs	System of Linear Equations

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