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Sharp Existence of Ground States Solutions for a Class of Elliptic Equations with Mixed Local and Nonlocal Operators and General Nonlinearity

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Abstract: In this paper, we study the existence/non-existence of ground states for the following type of elliptic equations with mixed local and nonlocal operators and general nonlinearity: $(-\triangle)^s u - \triangle u + \lambda u = f(u),\ x \in \mathbb{R}^N$, which is driven by the superposition of Brownian and Lévy processes. By considering a constrained variational problem, under suitable assumptions on f, we manage to establish a sharp existence of the ground state solutions to the equation considered. These results improve the ones in the existing reference.

Keywords: constrained variational method; mixed local and nonlocal operators; sharp existence; general nonlinearity

MSC: 35J50; 35Q41; 35Q55; 37K45

1. Introduction

In this paper, we are concerned with the following nonlinear elliptic equation with mixed local and nonlocal operators and general nonlinearity:

$$i\partial_t \psi(t,x) + (-\triangle)^s \psi(t,x) - \triangle \psi(t,x) = f(\psi(t,x)), \tag{1}$$

where $N \geq 2, 0 < s < 1$ and $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N$, $(-\triangle)^s$ is the fractional Laplacian which will be given below. Such a class of models with mixed local and nonlocal operators arises as the superposition of a classical random walk and a Lévy flight. For instance, in [1], it has been used to describe a biological species whose individuals diffuse either by a random walk or by a jump process, with given probabilities. Moreover, other different types of mixed operators motivated by biological questions has appeared in [2–4].

Motivated by various applications, we are interested in the standing wave solutions for (1), which are solutions of the form $\psi(t,x) := e^{-i\lambda t}u(x)$, where $\lambda \in \mathbb{R}$ denotes the frequency. The function $u(x)$ solves the following stationary equation:

$$(-\triangle)^s u - \triangle u + \lambda u = f(u),\ x \in \mathbb{R}^N, \tag{2}$$

where $N \geq 2, 0 < s < 1, \lambda \in \mathbb{R}$. The fractional Laplacian is given by

$$(-\triangle)^s u := C_{N,s} \int_{R^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy,$$

with $C_{N,s} := 2^{2s} \pi^{-\frac{N}{2}} s \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(1-s)}$, where $\Gamma(\cdot)$ is the Gamma function; see [5].

We remark that the problem as (2) with mixed operators has recently received great attention from different points of view, such as existence and non-existence results [6–9], eigenvalue problems [10,11], optimization and calculus of variations [12,13], symmetry and

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Citation: Luo, T.; Xie, Q. Sharp Existence of Ground States Solutions for a Class of Elliptic Equations with Mixed Local and Nonlocal Operators and General Nonlinearity. *Mathematics* **2023**, *11*, 3464. https://doi.org/10.3390/math11163464

Academic Editor: Mieczysław Cichoń

Received: 12 July 2023
Revised: 7 August 2023
Accepted: 7 August 2023
Published: 10 August 2023

rigidity results [14], and regularity theory [15–17]. We refer the readers to these references and the references therein.

Clearly, the Equation (2) can be treated by a variational approach. To begin with, we introduce the fractional Sobolev Spaces

$$H^s(\mathbb{R}^N) := \Big\{u : \mathbb{R}^N \to \mathbb{R} \mid u \in L^2(\mathbb{R}^N), \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y))|^2}{|x-y|^{N+2s}}dxdy < \infty\Big\}, s > 0,$$

and for $u \in H^s(\mathbb{R}^N)$, denote

$$\|\nabla_s u\|_2^2 := \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy.$$

According to [18] (Remark 1.4.1), we know that for $0 < s < 1$, $H^1(\mathbb{R}^N) \hookrightarrow H^s(\mathbb{R}^N)$. Hence,

$$H^1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N) = H^1(\mathbb{R}^N).$$

For more information about fractional Sobolev spaces and the fractional Laplacian operator; see, e.g., [18,19]. In this paper, we assume that the function f satisfies the following conditions:

(F1) $f \in C(\mathbb{R}, \mathbb{R})$, and $f(0) = 0$.
(F2) $\lim\limits_{|s|\to\infty} \frac{f(s)}{|s|} = 0$.
(F3) $\lim\limits_{|s|\to\infty} \frac{f(s)}{|s|^{l-1}} = 0$, where $l = 2 + \frac{4}{N}$.
(F4) There exists $s_0 > 0$ such that $F(s_0) > 0$, where $F(s) = \int_0^s f(\tau)d\tau$, for $s \in \mathbb{R}$.

To find a ground state solution of the Equation (2), we consider the following constrained variational problem:

$$m(c) := \inf_{u\in S(c)} E(u),\ c > 0, \tag{3}$$

where the energy functional $E(u)$ is given as

$$E(u) := \frac{1}{2}\|\nabla_s u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \int_{\mathbb{R}^N} F(u)dx, \tag{4}$$

and

$$S(c) := \{u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |u|^2dx = c\}.$$

It is standard to verify that under the assumptions (F1)–(F4), the functional $E(u)$ is well-defined and of class C^1 on $S(c)$; see, e.g., [20].

In Theorem 3, we shall prove that for the given $c > 0$, a minimizer of $m(c)$ is a ground state solution to (2). Since the L^2-norm of solutions is prescribed, the parameter $\lambda \in \mathbb{R}$ will be part of the unknown, which appears as a Lagrange Multiplier. Such types of solutions are referred to as *Normalized solutions*, which are interesting by themselves but also have particular meaning from the physical point of view, namely the conservation of mass. In addition, the variational characterization of such solutions is often a strong help to analyze their orbital stability/instability. See, e.g., [21,22].

Let us recall a special case $f(s) := |s|^{p-2}s$, where $p \in (2, 2+\frac{4}{N})$, which is referred to as the power nonlinearity with mass-subcritical growth condition. Clearly the power nonlinearity satisfies the assumptions (F1)–(F4). In such a case, the authors [6] established a sharp existence result for $m(c)$. Precisely, there is $c_0 > 0$, such that,

- When $2 < p < 2 + \frac{4s}{N}$, for all $c > 0$, $m(c)$ admits a minimizer;
- When $p = 2 + \frac{4s}{N}$, $m(c)$ has a minimizer if and only if $c > c_0$;
- When $2 + \frac{4s}{N} < p < 2 + \frac{4}{N}$, $m(c)$ has a minimizer if and only if $c \geq c_0$.

There are more details in [6] (Theorem 1.1, Remark 1.6). The aim of this paper is to extend this sharp existence result to the case where f is a general nonlinearity satisfying $(F1)$–$(F4)$. Let us note that such extension work is not highly trivial. The proofs in [6] strongly rely on some scaling arguments, which cannot be applied to the case where f is not homogenous. New ideas are needed in this situation.

To prove that $m(c)$ is attained by some function $u_c \in S(c)$, one significant tool is the concentration compactness principle of Lions [23]. Namely, given a minimizing sequence of $m(c)$, one needs to rule out the possible vanishing and dichotomy, which then derives its compactness. In this process, in most cases, the condition "$m(c) < 0$" plays a key role. Hence, as in [6], due to the non-increasing of $m(c)$ (see Lemma 2), it is natural to denote c_0 by

$$c_0 := \inf\{c > 0 \mid m(c) < 0\}. \tag{5}$$

By Lemma 2 (4), we know that $0 \leq c_0 < +\infty$, and also observe easily from the definition of c_0 that

$$\begin{cases} m(c) = 0, & 0 \leq c \leq c_0, \\ m(c) < 0, & c > c_0. \end{cases} \tag{6}$$

Here, we make the convention that $m(0) = 0$.

Our first result is stated as the following.

Theorem 1. *Suppose that (F1)–(F4) hold, and c_0 is given in (5). Then, we have,*

(1) When $c > c_0$ and $\{u_n\} \subset S(c)$ is a minimizing sequence of $m(c)$, and up to a subsequence if necessary, there exists $u \in S(c)$ and a family $\{y_n\} \subset \mathbb{R}^N$ such that $u_n(\cdot - y_n) \to u$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. In particular, $m(c)$ admits at least one minimizer.

(2) If $c_0 > 0$, then for any $0 < c < c_0$, $m(c)$ has no minimizers.

(3) If $c_0 > 0$, then $m(c_0)$ admits at least one minimizer.

Remark 1. *According to [22], the point that a minimizing sequence of $m(c)$ is relatively compact modulo translations, is crucial to derive the orbital stability of standing waves to (1), if the corresponding Cauchy problem of (1) is locally well-posed. This theorem provides a sharp description of the existence/non-existence of minimizers of $m(c)$. To prove this theorem, our main approach is the concentration compactness principle of Lions. However, since scaling arguments in [6] cannot be applied, we will propose some new arguments to overcome the difficulty caused by the general nonlinearity.*

To determine whether the case $c_0 > 0$ holds, we need to know more about the behavior of f near 0. We have the following theorem:

Theorem 2. *Suppose (F1)–(F4).*

(1) If $\underline{\lim}_{s\to 0} \frac{F(s)}{|s|^l} = \infty$ holds, then $c_0 = 0$ holds.

(2) If $\overline{\lim}_{s\to 0} \frac{F(s)}{|s|^l} < \infty$ holds, then $c_0 > 0$ holds.

Finally, we prove that a minimizer of $m(c)$ is indeed a ground state solution of (2). Here, standardly, a ground state solution of (2) means a solution whose action functional $I_\lambda(u)$ has the least energy value among all non-trivial solutions of (2), where

$$\begin{aligned} I_\lambda(u) :&= \frac{1}{2}\|\nabla_s u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{\lambda}{2}\int_{\mathbb{R}^N} |u|^2 dx - \int_{\mathbb{R}^N} F(u) dx \\ &= E(u) + \frac{\lambda}{2}\int_{\mathbb{R}^N} |u|^2 dx. \end{aligned} \tag{7}$$

Namely, $u \in H^1(\mathbb{R}^N)$ is a ground state solution of (2) if and only if

$$I'_\lambda(u) = 0, \text{ and } I_\lambda(u) = \inf\{I_\lambda(v) \mid v \in H^1(\mathbb{R}^N) \setminus \{0\},\ I'_\lambda(v) = 0\}.$$

Then, we can prove that

Theorem 3. *For given $c > 0$, let $u_c \in S(c)$ be a minimizer of $m(c)$, then there exists $\lambda_c > 0$, such that u_c is a ground state solution of (2) with $\lambda = \lambda_c$.*

We remark that when one removes the fractional Laplacian in (2), similar existences as the above theorems have been observed in [24,25]. However, since our equation involves fractional Laplacian, together with a general nonlinearity, more careful analyses are needed in our proofs. Most importantly, we provide a different proof from the one in [24,25]. Ours looks more simple and direct, which we believe could be applied in other equations with mixed operators, which may be the main novelty of this paper.

This paper is organized as follows: in Section 2, we provide some preliminary results which are necessary for the proofs of the main results. In Section 3, we show the the existence of minimizers for $m(c)$, to prove Theorem 1. Section 4 is devoted to prove the remaining theorems: Theorems 2 and 3. Section 5 is the conclusion of the whole paper.

2. Preliminary Results

In this section, we provide some preliminary results that will be used in the sequel. Firstly, let us introduce some notations. In this paper, $L^p(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ are the usual *Lebesgue* space and *Sobolev* space. For convenience, we denote the norm of $L^p(\mathbb{R}^N)$ by $\|\cdot\|_p$, and the norm of $H^1(\mathbb{R}^N)$ by $\|\cdot\|_{H^1}$. Denote $B(0,R)$ as a ball centered at $x = 0$, with the radius being R.

Lemma 1. *Assume that (F1)–(F4) hold, and $l = 2 + \frac{4}{N}$. Then*

(1) Let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^N)$. Then, for any $q \in (2, 2^*)$, if

$$\lim_{n\to\infty} \|u_n\|_q = 0,$$

then $\lim\limits_{n\to\infty} \int_{\mathbb{R}^N} F(u_n)dx = 0$.

(2) For any $c > 0$, there holds that $m(c) > -\infty$.

Proof. (1) By the assumptions (F1)–(F4), one observes that for any $\varepsilon > 0$, there exists a constant $C(f,\varepsilon) > 0$, which depends on f and ε, such that

$$|F(\cdot)| \le \varepsilon|\cdot|^2 + C(f,\varepsilon)|\cdot|^l, \quad |F(\cdot)| \le \varepsilon|\cdot|^l + C(f,\varepsilon)|\cdot|^2. \tag{8}$$

Thus, using the first inequality in (8), we have

$$\left|\int_{\mathbb{R}^N} F(u)dx\right| \le \varepsilon\|u\|_2^2 + C(f,\varepsilon)\|u\|_l^l, \quad \forall\, u \in H^1(\mathbb{R}^N). \tag{9}$$

Let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^N)$, with $\lim\limits_{n\to\infty} \|u_n\|_q = 0$. Then, by the interpolation inequality, we deduce that $\lim\limits_{n\to\infty} \|u_n\|_l = 0$. Thus, it follows from (9) that

$$\varlimsup_{n\to\infty} \Big|\int_{\mathbb{R}^N} F(u_n)dx\Big| \le \varepsilon \varlimsup_{n\to\infty} \|u_n\|_2^2.$$

Since $\varepsilon > 0$ is arbitrary, then we conclude that $\lim\limits_{n\to\infty} \int_{\mathbb{R}^N} F(u_n)dx = 0$.

(2) Applying the Gagliardo–Nirenberg inequality [18] (Theorem 1.3.7), we have

$$\|u\|_l^l \leq C(N)\|\nabla u\|_2^2 \cdot \|u\|_2^{\frac{4}{N}}, \ \forall\, u \in H^1(\mathbb{R}^N), \tag{10}$$

where $C(N)$ is a positive constant which depends on N. Thus, as (9), using the second inequality in (8), we have from (10) that

$$\left|\int_{\mathbb{R}^N} F(u)dx\right| \leq C(f,\varepsilon)\|u\|_2^2 + \varepsilon C(N)\|\nabla u\|_2^2\|u\|_2^{\frac{4}{N}}, \quad \forall\, u \in H^1(\mathbb{R}^N). \tag{11}$$

Choosing $\varepsilon > 0$ as $\varepsilon C(N)c^{\frac{2}{N}} = \frac{1}{4}$, then, for any $u \in S(c)$, we obtain

$$\begin{aligned} E(u) &= \frac{1}{2}\|\nabla_s u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \int_{\mathbb{R}^N} F(u)dx \\ &\geq \frac{1}{2}\|\nabla_s u\|_2^2 + \frac{1}{4}\|\nabla u\|_2^2 - C(f,\varepsilon)\cdot c \\ &\geq -C(f,\varepsilon)\cdot c. \end{aligned} \tag{12}$$

From the above, we obtain immediately that $m(c) > -\infty$. □

Next, we prove the properties of $m(c)$, which are interesting by themselves but also important for the proof of the existence results.

Lemma 2. *Suppose that (F1)–(F4) hold, then*

(1) $m(c) \leq 0$, $\forall\, c > 0$;
(2) $m(c+\mu) \leq m(c) + m(\mu)$, $\forall c, \mu > 0$;
(3) $c \mapsto m(c)$ *is non-increasing on* $(0, \infty)$;
(4) $m(c) < 0$ *holds provided* $c > 0$ *is sufficiently large;*
(5) $c \mapsto m(c)$ *is continuous on* $(0, \infty)$.

The following two lemmas, which are well-known, are also important for our proofs.

Lemma 3 ([26])**.** *Suppose that* $f_n \to f$ *is almost everywhere and* $\{f_n\}$ *is a bounded sequence in* $L^p(\mathbb{R}^N)$, *then*

$$\lim_{n\to\infty}\left(\|f_n\|_p^p - \|f_n - f\|_p^p - \|f\|_p^p\right) = 0,$$

where $0 < p < \infty$.

Lemma 4 ([6] (Lemma 5.1))**.** *Suppose that* $\{u_n\} \subset H^s(\mathbb{R}^N)$ *and* $u \in H^s(\mathbb{R}^N)$, *being such that*

$$u_n \rightharpoonup u, \text{ in } H^s(\mathbb{R}^N),$$

then

$$\|\nabla_s u_n\|_2^2 - \|\nabla_s(u_n - u)\|_2^2 - \|\nabla_s u\|_2^2 = o_n(1), \text{ as } n \to +\infty. \tag{13}$$

Proof of Lemma 2. (1) For any $c > 0$, given $u \in S(c)$, we consider the scaling $u_t(x) = t^{\frac{N}{2}}u(tx), t > 0$. Then, $u_t \in S(c)$, $\|u_t\|_l^l = t^2\|u\|_l^l$, and $\lim_{t\to 0}\|u_t\|_l^l = 0$. Thus, by (9), we have

$$\lim_{t\to 0}\int_{\mathbb{R}^N} F(u_t)dx = 0. \tag{14}$$

In addition, $\|\nabla u_t\|_2^2 = t^2\|\nabla u\|_2^2$, $\|\nabla_s u_t\|_2^2 = t^{2s}\|\nabla_s u\|_2^2$, which implies that

$$\lim_{t\to 0}\|\nabla u_t\|_2^2 = 0, \ \lim_{t\to 0}\|\nabla_s u_t\|_2^2 = 0. \tag{15}$$

Hence, $\lim_{t\to 0} E(u_t) = 0$. Then, by the definition of $m(c)$, it follows that $m(c) \leq 0$.

(2) From the definition of $m(c)$ and $m(\mu)$, for given $\varepsilon > 0$, there exists $u_\varepsilon \in S(c) \cap C_0^\infty(\mathbb{R}^N)$ and $v_\varepsilon \in S(\mu) \cap C_0^\infty(\mathbb{R}^N)$, such that

$$E(u_\varepsilon) \leq m(c) + \varepsilon,\ E(v_\varepsilon) \leq m(\mu) + \varepsilon.$$

Choose $R > 0$ large enough such that $supp\ u_\varepsilon \subset B_R(0)$ and $supp\ v_\varepsilon \subset B_R(0)$. Take $x_0 \in \mathbb{R}^N$ with $|x_0| > 2R$, and define

$$w := u_\varepsilon + v_\varepsilon(\cdot + x_0).$$

Then, $w \in m(c+\mu)$ and

$$m(c+\mu) \leq E(w) = E(u_\varepsilon) + E(v_\varepsilon) \leq m(c) + m(\mu) + 2\varepsilon. \tag{16}$$

Let $\varepsilon \to 0^+$; then, we obtain $m(c+\mu) \leq m(c) + m(\mu)$.

(3) From (1) and (2), it follows immediately that $m(c)$ is non-increasing.

(4) To prove $m(c) < 0$, it is enough to find a function $u \in H^1(\mathbb{R}^N)$ satisfying $E(u) < 0$. For this purpose, let s_0 be provided in $(F4)$, and for $R > 0$, we define

$$u_R(x) = \begin{cases} s_0, & |x| \leq R, \\ s_0(R+1-|x|), & R < |x| \leq R+1, \\ 0, & |x| > R+1. \end{cases} \tag{17}$$

Then, a direct calculation yields that

$$\begin{aligned} E(u_R) &= \frac{1}{2}\|\nabla_s u_R\|_2^2 + \frac{1}{2}\|\nabla u_R\|_2^2 - \int_{\mathbb{R}^N} F(u_R)dx \\ &= \frac{1}{2}\int_{B(0,R+1)\times B(0,R+1)} \frac{|u_R(x)-u_R(y)|^2}{|x-y|^{N+2s}}dxdy \\ &+ \frac{1}{2}\int_{B(0,R+1)} |\nabla u_R|^2 dx - \int_{B(0,R+1)} F(u_R)dx. \end{aligned}$$

Define

$$\begin{aligned} I &:= \frac{1}{2}\int_{B(0,R+1)\times B(0,R+1)} \frac{|u_R(x)-u_R(y)|^2}{|x-y|^{N+2s}}dxdy, \\ II &:= \frac{1}{2}\int_{B(0,R+1)} |\nabla u_R|^2 dx - \int_{B(0,R+1)} F(u_R)dx. \end{aligned}$$

Then, we note that

$$\begin{aligned} I &= \frac{1}{2}\int_{B(0,R)\times B(0,R)} \frac{|u_R(x)-u_R(y)|^2}{|x-y|^{N+2s}}dxdy + \frac{1}{2}\int_{B(0,R)\times [B(0,R+1)\setminus B(0,R)]} \frac{|u_R(x)-u_R(y)|^2}{|x-y|^{N+2s}}dxdy \\ &+ \frac{1}{2}\int_{[B(0,R+1)\setminus B(0,R)]\times B(0,R)} \frac{|u_R(x)-u_R(y)|^2}{|x-y|^{N+2s}}dxdy \\ &+ \frac{1}{2}\int_{[B(0,R+1)\setminus B(0,R)]\times [B(0,R+1)\setminus B(0,R)]} \frac{|u_R(x)-u_R(y)|^2}{|x-y|^{N+2s}}dxdy \\ &= \frac{1}{2}\int_{[B(0,R)\times [B(0,R+1)\setminus B(0,R)]} \frac{s_0^2||y|-R|^2}{|x-y|^{N+2s}}dxdy + \frac{1}{2}\int_{[B(0,R+1)\setminus B(0,R)]\times B(0,R)} \frac{s_0^2|R-|x||^2}{|x-y|^{N+2s}}dxdy \\ &+ \frac{1}{2}\int_{[B(0,R+1)\setminus B(0,R)]\times [B(0,R+1)\setminus B(0,R)]} \frac{s_0^2||x|-|y||^2}{|x-y|^{N+2s}}dxdy \\ &\leq \frac{3}{2}\int_{B(0,R+1)\times B(0,R+1)} \frac{s_0^2||x|-|y||^2}{|x-y|^{N+2s}}dxdy \leq \frac{3}{2}\int_{B(0,R+1)\times B(0,R+1)} \frac{s_0^2}{|x-y|^{N+2s-2}}dxdy. \end{aligned}$$

Denote ω_N of the surface area of the unit sphere in the N-dimension. Then

$$
\begin{aligned}
II =& \int_{B(0,R)} \Big(\frac{1}{2}|\nabla u_R|^2 - F(u_R)\Big)dx + \int_{B(0,R+1)\setminus B(0,R)} \Big(\frac{1}{2}|\nabla u_R|^2 - F(u_R)\Big)dx \\
\le& -\int_{B(0,R)} F(s_0)dx + \int_{B(0,R+1)\setminus B(0,R)} \Big(\frac{s_0^2}{2} + \sup_{0\le s\le s_0}|F(s)|\Big)dx \\
=& -F(s_0)\omega_N \int_0^R r^{N-1}dr + \Big(\frac{s_0^2}{2} + \sup_{0\le s\le s_0}|F(s)|\Big)\omega_N \int_R^{R+1} r^{N-1}dr \\
=& -F(s_0)\omega_N \cdot \frac{R^N}{N} + \Big(\frac{s_0^2}{2} + \sup_{0\le s\le s_0}|F(s)|\Big)\omega_N \cdot \frac{(R+1)^N - R^N}{N} \\
=& -F(s_0)\omega_N \cdot \frac{R^N}{N} + \Big(\frac{s_0^2}{2} + \sup_{0\le s\le s_0}|F(s)|\Big)\omega_N \cdot \frac{1}{N}(C_N^1 R^{N-1} + C_N^2 R^{N-2} + \cdots + 1).
\end{aligned}
$$

Combine the above, then we obtain

$$
\begin{aligned}
E(u_R) \le& \frac{3}{2}\int_{B(0,R+1)\times B(0,R+1)} \frac{s_0^2}{|x-y|^{N+2s-2}}dxdy \\
&- F(s_0)\omega_N \cdot \frac{R^N}{N} + \Big(\frac{s_0^2}{2} + \sup_{0\le s\le s_0}|F(s)|\Big)\omega_N \cdot \frac{1}{N}(C_N^1 R^{N-1} + C_N^2 R^{N-2} + \cdots + 1).
\end{aligned}
$$

Since $0 < N+2s-2 < N$ when $N \ge 2, 0 < s < 1$, then clearly

$$
0 < \int_{B(0,R+1)\times B(0,R+1)} \frac{s_0^2}{|x-y|^{N+2s-2}}dxdy < \infty. \tag{18}
$$

Therefore, by letting $R > 0$ be sufficiently large, we have $E(u_R) < 0$. Moreover, we observe that

$$
\|u_R\|_2^2 \ge \int_{B(0,R)} |u_R|^2dx = \int_{B(0,R)} s_0^2dx = \frac{s_0^2\omega_N R^N}{N}. \tag{19}
$$

Hence, there exists $c_1 > 0$ large enough, such that $m(c_1) < 0$. Thus, by the non-increasing of $m(c)$, we know that $m(c) < 0$ for all $c \ge c_1$.

(5) To show the continuity, we first let $c_n \to c^-$, then by the definition of $m(c)$, for any $\varepsilon > 0$, there exists $u \in S(c)$, such that $E(u) < m(c) + \varepsilon$. Denote $b_n := \sqrt{\frac{c_n}{c}}$, then $b_n \to 1^-$, and

$$
\begin{aligned}
0 \le m(c_n) - m(c) &\le E(\sqrt{b_n}u) - E(u) + \varepsilon \\
&= \frac{b_n - 1}{2}\Big(\|\nabla_s u\|_2^2 + \|\nabla u\|_2^2\Big) - \int_{\mathbb{R}^N} \Big(F(\sqrt{b_n}u) - F(u)\Big)dx + \varepsilon.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \Big(F(\sqrt{b_n}u) - F(u)\Big)dx \\
=& \int_{\mathbb{R}^N} \left(\int_0^1 f(|u| + (\sqrt{b_n}-1)\theta|u|)(\sqrt{b_n}-1)|u|d\theta\right)dx \\
=& (\sqrt{b_n}-1)\int_{\mathbb{R}^N} \left(\int_0^1 f(|u| + (\sqrt{b_n}-1)\theta|u|)|u|d\theta\right)dx,
\end{aligned}
$$

and that $0 \leq |u| + (\sqrt{b_n} - 1)\theta|u| \leq |u|$, together with $|f(s)| \leq |s| + C(f)|s|^{l-1}$—see (8); then, we have

$$\left| \int_{\mathbb{R}^N} \left(\int_0^1 f(|u| + (\sqrt{b_n} - 1)\theta|u|)|u| d\theta \right) dx \right|$$
$$\leq \int_{\mathbb{R}^N} \left(\int_0^1 |u|^2 + C(f)|u|^l d\theta \right) dx = \int_{\mathbb{R}^N} |u|^2 + C(f)|u|^l dx.$$

Hence, we have

$$0 \leq m(c_n) - m(c) \leq o_n(1) + \varepsilon, \ \text{ as } \ n \to \infty. \tag{20}$$

By letting $\varepsilon \to 0$, then $\lim_{c_n \to c^-} m(c_n) = m(c)$. Similarly, we can prove that $\lim_{c_n \to c^+} m(c_n) = m(c)$. Hence, the continuity is verified. □

3. Proof of Theorem 1

In this section, we will use the concentration-compactness argument to prove Theorem 1. To begin with, we show a fundamental Lemma which is crucial for our proof.

Lemma 5. *Assume that for some $a > b > 0$, $m(a)$ and $m(b)$ are both attained, and namely, there exists $u_a \in S(a)$ and $u_b \in S(b)$, such that $E(u_a) = m(a), E(u_b) = m(b)$. Then, we have*

(i) For any $d > a$, $m(d) < m(a)$;

(ii) There holds that,

$$m(a + b) < m(a) + m(b). \tag{21}$$

Proof. Let u_a be given as in this lemma, and for any $t > 1$, we define $u_a^t := u_a(t^{-\frac{1}{N}} x)$, then $\|u_a^t\|_2^2 = ta$ and

$$E(u_a^t) = t\Big(\frac{t^{-\frac{2s}{N}}}{2}\|\nabla_s u_a\|_2^2 + \frac{t^{-\frac{2}{N}}}{2}\|\nabla u_a\|_2^2 - \int_{\mathbb{R}^N} F(u_a)dx\Big) < tE(u_a) = tm(a), \tag{22}$$

which implies that $m(ta) < tm(a)$, $\forall\, t > 1$. Since by Lemma 2 (1), $m(a) \leq 0$, then

$$m(d) = m(\frac{d}{a} \cdot a) < \frac{d}{a} m(a) \leq m(a), \tag{23}$$

which proves point (i). Similar to the above, we can also obtain that $m(tb) < tm(b)$, $\forall\, t > 1$. Thus,

$$m(a+b) = m(\frac{a+b}{a} \cdot a) < \frac{a+b}{a} m(a) = m(a) + \frac{b}{a} m(a) = m(a) + \frac{b}{a} m(\frac{a}{b} \cdot b) < m(a) + m(b).$$

This ends the proof. □

Now we prove the following proposition.

Proposition 1. *Suppose (F1)–(F4), and $c > 0$ is such that*

$$-\infty < m(c) < 0, \tag{24}$$

then any minimizing sequence of $m(c)$ is relatively compact modulo translations. In particular, in this case $m(c)$ admits a minimizer.

Proof. Let $\{u_n\} \subset S(c)$ be an arbitrary minimizing sequence of $m(c)$, namely,

$$\|u_n\|_2^2 = c \ \text{ and } \ E(u_n) \to m(c) < 0, \ \text{ as } n \to \infty.$$

Then, by (12), we observe that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Presently, we claim that

$$\int_{\mathbb{R}^N} |u_n|^l dx \nrightarrow 0, \ \text{ as } n \to \infty. \tag{25}$$

Indeed, if $\int_{\mathbb{R}^N} |u_n|^l dx \to 0$, then by Lemma 1,

$$m(c) = E(u_n) + o_n(1) = \frac{1}{2}\|\nabla_s u_n\|_2^2 + \frac{1}{2}\|\nabla u_n\|_2^2 + o_n(1),$$

which contradicts with the fact that $m(c) < 0$. Thus, by (25) and the vanishing Lemma of Lions [23] (Lemma I.1), there exists a constant $\delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B(y_n,1)} |u_n|^2 dx \geq \delta > 0,$$

or equivalently

$$\int_{B(0,1)} |u_n(\cdot + y_n)|^2 dx \geq \delta > 0. \tag{26}$$

Presently, let $v_n(\cdot) := u_n(\cdot + y_n)$, then obviously $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, and thus, up to the subsequence (still denoted by $\{v_n\}$), there exists $u \in H^1(\mathbb{R}^N)$, such that

$$v_n \rightharpoonup u \ \text{ weakly in } H^1(\mathbb{R}^N), \ \text{ and } \ v_n \to u \ \text{ in } L^2_{loc}(\mathbb{R}^N).$$

Thus, by (26), we conclude that $u \neq 0$, since

$$0 < \delta \leq \lim_{n\to\infty} \int_{B(0,1)} |v_n|^2 dx = \int_{B(0,1)} |u|^2 dx. \tag{27}$$

Next, we prove that $v_n \to u$ in $H^1(\mathbb{R}^N)$. Since by Lemmas 3 and 4, we have

$$c = \|v_n\|_2^2 = \|v_n - u\|_2^2 + \|u\|_2^2 + o_n(1). \tag{28}$$

$$m(c) = E(v_n) = E(v_n - u) + E(u) + o_n(1). \tag{29}$$

If $\|u\|_2^2 < c$, then by (28), $\|v_n - u\|_2^2 \to d > 0$. Denote

$$w_n := \frac{\sqrt{d}}{\|v_n - u\|_2}(v_n - u),$$

then $\|w_n\|_2^2 = d$, $\|u\|_2^2 = c - d > 0$, and using the proof as in Lemma 2 (5), one can verify that $E(w_n) = E(v_n - u) + o_n(1)$. Therefore, using (29) and Lemma 2 (3), we have

$$\begin{aligned} m(c) &= E(w_n) + E(u) + o_n(1) \\ &\geq m(d) + m(c-d) \\ &\geq m(c), \end{aligned} \tag{30}$$

from which one observes the following facts:

(1) $m(c-d)$ is reached by $u \in S(c-d)$;
(2) $\{w_n\} \subset S(d)$ is a minimizing sequence of $m(d)$;
(3) $m(c) = m(d) + m(c-d)$.

In particular, from Lemma 5 (i), $m(c) < m(c-d)$, therefore, $m(d) < 0$. Thus, applying the above arguments to the minimizing sequence $\{w_n\}$ of $m(d) < 0$, we deduce that up to a subsequence and translation, there exists $w \in H^1(\mathbb{R}^N)$, such that $w_n \rightharpoonup w \neq 0$ in $H^1(\mathbb{R}^N)$ and either $\|w\|_2^2 = d$, or $\|w\|_2^2 < d$, and there exist $d_1 > 0$ and $d_2 > 0$, such that $d = d_1 + d_2$,

$$m(c) = m(d_1) + m(d_2) + m(c-d),$$

where $m(d_2)$ and $m(c-d)$ are reached. If $\|w\|_2^2 = d$, then $m(d)$ is reached by $w \in S(d)$, thus, using Lemma 5 (ii), we have

$$m(c) = m(d) + m(c-d) > m(d+c-d) = m(c),$$

which is obviously a contradiction. In addition, if $\|w\|_2^2 < d$, then $m(c) = m(d_1) + m(d_2) + m(c-d)$, since $m(d_2)$ and $m(c-d)$ are reached, then, using Lemma 5 (ii) and Lemma 2 (2), we have

$$m(c) = m(d_1) + m(d_2) + m(c-d) > m(d_1) + m(d_2+c-d) \geq m(d_1+d_2+c-d) = m(c),$$

which is also a contradiction.

Hence, we have proved from the above that $\|u\|_2^2 = c$, then $v_n \to u$ in $L^2(\mathbb{R}^N)$. Thus, by the interpolation inequality and Lemma 1, we have

$$\int_{\mathbb{R}^N} F(v_n)dx \to \int_{\mathbb{R}^N} F(u)dx.$$

Furthermore, since $u \in S(c)$, then

$$m(c) \leq E(u) \leq \lim_{n\to\infty} E(v_n) = m(c),$$

and $\|v_n\|_{H^1} \to \|u\|_{H^1}$, $\|\nabla_s v_n\|_2 \to \|\nabla_s u\|_2$. This implies that $v_n \to u$ in $H^1(\mathbb{R}^N)$; then, $u \in S(c)$ is a minimizer of $m(c)$. □

Combining Lemma 5 and Proposition 1, one immediately obtains the following conclusion:

Remark 2. *Assume that (F1)–(F4) hold, and let $a > 0$ be such that $m(a) < 0$, then for any $b > a$, we have*

$$m(b) < m(a), \text{ and } m(a+b) < m(a) + m(b). \tag{31}$$

Proof of Theorem 1. (1) When $c > c_0$, since $m(c) < 0$, then using Proposition 1, we immediately obtain the conclusion of Theorem 1 (1).

(2) If $c_0 > 0$, then for any $0 < c < c_0$, we have $m(c) = 0$, in which case, if we supposed that $m(c)$ admits a minimizer, then from Lemma 5 (i), we have

$$m(c_0) < m(c) = 0,$$

which is a contradiction since $m(c_0) = 0$. Hence, for any $0 < c < c_0$, $m(c)$ has no minimizers.

(3) If $c_0 > 0$, then to show the existence of minimizers of $m(c_0)$, we use an approximated method. Let $c_n := c_0 + \frac{1}{n}$, then $c_n \to c_0^+$ as $n \to \infty$. Since $c_0 > 0$, then by (6) and Lemma 2, $m(c_n) < 0, \forall\, n \in \mathbb{N}^+$ and $\lim_{n\to\infty} m(c_n) = m(c_0) = 0$. Furthermore, Proposition 1 asserts that there exists a minimizer $u_n \in S(c_n)$, such that $E(u_n) = m(c_n)$. Now, we claim that $\|u_n\|_l \nrightarrow 0$ as $n \to \infty$. Indeed, when using Lemma 1 (1), we have $\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n)dx = 0$. This implies that

$$m(c_n) = E(u_n) = \frac{1}{2}\|\nabla_s u_n\|_2^2 + \frac{1}{2}\|\nabla u_n\|_2^2 + o_n(1) \geq 0,$$

which clearly contradicts with the fact that $m(c_n) < 0$ for all $n \in \mathbb{N}$.

Thus, applying the Vanishing lemma of Lions [23], there exists a constant $\delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$, such that

$$\int_{B(y_n,1)} |u_n|^2 dx \geq \delta > 0. \tag{32}$$

or equivalently

$$\int_{B(0,1)} |u_n(\cdot + y_n)|^2 dx \geq \delta. \tag{33}$$

Presently, let $v_n(\cdot) = u_n(\cdot + y_n)$. As the proof of Proposition 1, one can verify that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Thus, up to a subsequence of $\{v_n\}$, there exists $v_0 \in H^1(\mathbb{R}^N)$ such that

$$v_n \rightharpoonup v_0 \ \text{ in } H^1(\mathbb{R}^N) \quad \text{ and } \quad v_n \to v_0 \ \text{ in } L^2_{loc}(\mathbb{R}^N).$$

We note that $v_0 \neq 0$, since by (33)

$$\int_{B(0,1)} |v_0|^2 dx = \lim_{n\to\infty} \int_{B(0,1)} |v_n|^2 dx \geq \delta > 0.$$

Presently, we prove that v_0 is a minimizer of $m(c_0)$. Recall from Lemmas 3 and 4 that,

$$c_0 = \|v_n\|_2^2 = \|v_n - v_0\|_2^2 + \|v_0\|_2^2 + o_n(1). \tag{34}$$

$$0 = m(c_0) = E(v_n) + o_n(1) = E(v_n - v_0) + E(v_0) + o_n(1). \tag{35}$$

This implies that

$$0 = m(c_0) \geq m(\|v_n - v_0\|_2^2) + m(\|v_0\|_2^2) + o_n(1).$$

Due to the fact that $0 < \|v_0\|_2^2 \leq c_0$, then $m(\|v_0\|_2^2) = 0$. Now, if $\|v_0\|_2^2 < c_0$, then

$$\|v_n - v_0\|_2^2 + o_n(1) = \|v_n\|_2^2 - \|v_0\|_2^2 = c_0 - \|v_0\|_2^2 > 0. \tag{36}$$

Thus, using Lemma 2 (5), we have

$$\lim_{n\to\infty} E(v_n - v_0) \geq \lim_{n\to\infty} m(||v_n - v_0||_2^2) = m(c_0 - ||v_0||_2^2) = 0. \tag{37}$$

Therefore, from (35) and (37) and the fact that $m(||v_0||_2^2) = 0$, we then obtain that $E(v_0) = 0$. This shows that v_0 is a minimizer of $m(||v_0||_2^2)$. Since we assume that $||v_0||_2^2 < c_0$, then using Lemma 5 (i), we have

$$m(c_0) < m(||v_0||_2^2) = 0,$$

which is impossible. Thus, necessarily, $\|v_0\|_2^2 = c_0$. Hence, $v_n \to v_0$ in $L^2(\mathbb{R}^N)$, and following the arguments in the proof of Proposition 1, we deduce that $v_0 \in S(c)$ is a minimizer of $m(c_0)$. This ends the proof. □

4. Proof of Theorems 2 and 3

In this section, we shall prove the rest of the Theorems. In particular, we shall need to use some ideas from [25] to prove Theorem 2.

Proof of Theorem 2. (1) For any given $c > 0$, we fix a function $u_0 \in S(c) \cap C_0^\infty(\mathbb{R}^N)$, and denote $u_t(x) := t^{\frac{N}{2}} u_0(tx),\ t > 0$, then $u_t \in S(c)$ for all $t > 0$. Due to the assumption of $\underline{\lim}_{s \to 0} \frac{F(s)}{|s|^l} = \infty$, there exists a constant $\delta > 0$, such that

$$F(s) \geq C|s|^l, \ \text{ if } |s| < \delta, \tag{38}$$

where C is given as

$$C := \frac{\int_{\mathbb{R}^N} |\nabla_s u_0|^2 dx + \int_{\mathbb{R}^N} |\nabla u_0|^2 dx}{\int_{\mathbb{R}^N} |u_0|^l dx}. \tag{39}$$

Choosing $t > 0$ small enough such that $|u_t| < \delta$, then $F(u_t) \geq C|u_t|^l$ holds. Thus,

$$\begin{aligned} E(u_t) &= \frac{1}{2}\|\nabla_s u_t\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 - \int_{\mathbb{R}^N} F(u_t)dx \\ &\leq \frac{1}{2}\|\nabla_s u_t\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 - C\int_{\mathbb{R}^N} |u_t|^l dx \\ &= -\frac{1}{2}\|\nabla_s u_t\|_2^2 - \frac{1}{2}\|\nabla u_t\|_2^2 \\ &= -\frac{t^{2s}}{2}\|\nabla_s u_0\|_2^2 - \frac{t^2}{2}\|\nabla u_0\|_2^2 < 0. \end{aligned}$$

This implies that $m(c) \leq E(u_t) < 0$. Namely, $m(c) < 0$ for all $c > 0$. Then, from the definition of $c_0 = \inf\{c > 0, m(c) < 0\}$, we conclude that $c_0 = 0$.

(2) From $\overline{\lim}_{s \to 0} \frac{F(s)}{s^l} < \infty$, we deduce that there exists a constant $C(f)$ depending on f, such that

$$F(s) \leq C(f)|s|^l, \ \text{ if } s \geq 0. \tag{40}$$

Then, for $u \in S(c)$, by the Gagliardo–Nirenberg inequality (10), we compute that

$$\begin{aligned} \int_{\mathbb{R}^N} F(u)dx &\leq C(f)\|u\|_l^l \\ &\leq C(f)C(N)\|\nabla u\|_2^2 \cdot \|u\|_2^{\frac{4}{N}} \\ &= C(f)C(N)\|\nabla u\|_2^2 \cdot c^{\frac{2}{N}}. \end{aligned}$$

Taking $c > 0$ small enough, such that $C(f)C(N)c^{\frac{2}{N}} = \frac{1}{2}$, then for any $u \in S(c)$, we have

$$\begin{aligned} E(u) &= \frac{1}{2}\|\nabla_s u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \int_{\mathbb{R}^N} F(u)dx \\ &\geq \frac{1}{2}\|\nabla_s u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\|\nabla u\|_2^2 \\ &= \frac{1}{2}\|\nabla_s u\|_2^2 > 0. \end{aligned}$$

This implies $m(c) \geq 0$ as $c > 0$ small, since Lemma 2 shows that $m(c) \leq 0$ for any $c > 0$. Hence, $m(c) = 0$ as $c > 0$ small. Therefore, from the definition of c_0, we conclude that $c_0 > 0$. Then, the proof is completed. □

Proof of Theorem 3. Let $u_c \in S(c)$ be a minimizer of $m(c)$, then standardly, there exists a Lagrange multiplier $\lambda_c \in \mathbb{R}$, such that u_c solves weakly

$$(-\triangle)^s u - \triangle u + \lambda_c u = f(u),\ x \in \mathbb{R}^N, \tag{41}$$

namely $I'_{\lambda_c}(u_c) = 0$ in $H^{-1}(\mathbb{R}^N)$, where $I_\lambda(u)$ is given by (7). Now, we prove that $\lambda_c > 0$. Indeed, we recall that u_c is a critical point of $E(u)$ on $S(c)$, then as [6] (Lemma 2.2) and [26] (Proposition 1), there holds necessarily that $Q(u_c) = 0$, where

$$\begin{aligned} Q(u) := & \frac{d}{dt}E(u_t)|_{t=1} \\ & = s\|\nabla_s u\|_2^2 + \|\nabla u\|_2^2 - \frac{N}{2}\int_{\mathbb{R}^N}\Big(f(u)u - 2F(u)\Big)dx, \end{aligned} \tag{42}$$

where $u_t = t^{\frac{N}{2}}u(tx), t > 0$. Thus, $Q(u_c) = 0$ leads to

$$\int_{\mathbb{R}^N}\Big(f(u_c)u_c - 2F(u_c)\Big)dx = \frac{2}{N}\Big[s\|\nabla_s u_c\|_2^2 + \|\nabla u_c\|_2^2\Big]. \tag{43}$$

Hence, after multiplying (41) by u_c and integrating by part, we have from (43) that

$$\begin{aligned} \lambda_c\|u_c\|_2^2 &= \int_{\mathbb{R}^N} f(u_c)u_c dx - \|\nabla_s u_c\|_2^2 - \|\nabla u_c\|_2^2 \\ &= \int_{\mathbb{R}^N}\Big(f(u_c)u_c - 2F(u_c)\Big)dx - 2E(u_c) \\ &= \frac{2}{N}\Big[s\|\nabla_s u_c\|_2^2 + \|\nabla u_c\|_2^2\Big] - 2E(u_c) \\ &\geq \frac{2}{N}\Big[s\|\nabla_s u_c\|_2^2 + \|\nabla u_c\|_2^2\Big] > 0, \end{aligned}$$

where the inequality $E(u_c) = m(c) \leq 0$ is used. Thus, we obtain that $\lambda_c > 0$.

Now, to prove that u_c is a ground state, we denote

$$d := \inf\{I_{\lambda_c}(v) \mid v \in H^1(\mathbb{R}^N) \setminus \{0\},\ I'_{\lambda_c}(v) = 0\}, \tag{44}$$

then it is enough to show that

$$I_{\lambda_c}(u_c) = d. \tag{45}$$

First, clearly, we have $I_{\lambda_c}(u_c) \geq d$. On the other hand, we claim that for any $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ with $I'_{\lambda_c}(u) = 0$, then

$$I_{\lambda_c}(u) = \max_{t>0}\{I_{\lambda_c}(u^t)\},$$

where $u^t := u(t^{-\frac{1}{N}}x), t > 0$. Indeed, note that

$$I_{\lambda_c}(u^t) = \frac{t^{1-\frac{2s}{N}}}{2}\|\nabla_s u\|_2^2 + \frac{t^{1-\frac{2}{N}}}{2}\|\nabla u\|_2^2 + \frac{\lambda_c t}{2}\|u\|_2^2 - t\int_{\mathbb{R}^N} F(u)dx.$$

Since $I'_{\lambda_c}(u) = 0$, then by [6] (Lemma 2.2) and [26] (Proposition 1), we have the following Pohozave identity:

$$\frac{N-2s}{2N}\|\nabla_s u\|_2^2 + \frac{N-2}{2N}\|\nabla u\|_2^2 + \frac{\lambda_c}{2}\|u\|_2^2 - \int_{\mathbb{R}^N} F(u)dx = 0, \tag{46}$$

which also can be obtained by multiplying the Equation (41) with $x \cdot \nabla u$ and integrating by part; see [6] (Lemma 2.2) or [5] for more details. Using (46), then we have

$$I_{\lambda_c}(u^t) = \frac{t^{1-\frac{2s}{N}}}{2}\|\nabla_s u\|_2^2 + \frac{t^{1-\frac{2}{N}}}{2}\|\nabla u\|_2^2 - t\Big[\frac{N-2s}{2N}\|\nabla_s u\|_2^2 + \frac{N-2}{2N}\|\nabla u\|_2^2\Big],$$

and

$$\frac{d}{dt}I_{\lambda_c}(u^t) = (1-\frac{2s}{N})\frac{t^{-\frac{2s}{N}}}{2}\|\nabla_s u\|_2^2 + (1-\frac{2}{N})\frac{t^{-\frac{2}{N}}}{2}\|\nabla u\|_2^2 - \Big[\frac{N-2s}{2N}\|\nabla_s u\|_2^2 + \frac{N-2}{2N}\|\nabla u\|_2^2\Big].$$

Since $0 < s < 1, N \geq 2$, then the claim follows easily from the above.

Now for any $v \in H^1(\mathbb{R}^N) \setminus \{0\}$ with $I'_{\lambda_c}(v) = 0$, choose $t_c > 0$ such that $v^{t_c} \in S(c)$, then we have

$$\begin{aligned} I_{\lambda_c}(v) &= \max_{t>0}\{I_{\lambda_c}(v^t)\} \\ &\geq I_{\lambda_c}(v^{t_c}) \\ &= E(v^{t_c}) + \frac{\lambda_c}{2}\int |v^{t_c}|^2 dx \\ &\geq E(u_c) + \frac{\lambda_c}{2}\int |u_c|^2 dx = I_{\lambda_c}(u_c). \end{aligned}$$

By taking a infimum, this proves that $d \geq I_{\lambda_c}(u_c)$. Hence, it follows that $I_{\lambda_c}(u_c) = d$, and then the proof is completed. □

5. Conclusions

In sum, in this paper, by considering a global variational problem on a L^2-norm-constrained manifold, we manage to establish a sharp existence of ground state solutions to the stationary Equation (2), which then provides the existence of the standing wave solutions to the time-dependent Equation (1). Due to the appearance of the general nonlinearity in the equations, scaling arguments in some classical references cannot be applied, and we propose some new ideas to overcome the difficulties caused by both the mixed operators and the general nonlinearity. In this process, we observe the great difference in treating elliptic equations with power nonlinearity and general nonlinearity. Compared with the existing references, we believe the approaches and arguments proposed in this paper can also be applied to other equations with mixed operators and general nonlinearity, for example the equations with mixed fractional Laplacians [6] and the ones with both the bi-harmonic and Laplaican operators [27,28].

Author Contributions: Data curation, Q.X.; Writing—original draft, T.L. All authors have read and agreed to the published version of the manuscript.

Funding: This work is partially supported by the National Natural Science Foundation of China (12271116).

Acknowledgments: The authors wish to sincerely thank the anonymous referees for their valuable comments and suggestions which have helped a lot for the revised version of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

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