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A Novel Conditional Connectivity and Hamiltonian Connectivity of BCube with Various Faulty Elements

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Abstract: BCube is one of the main data center networks because it has many attractive features. In practical applications, the failure of components or physical connections is inevitable. In data center networks in particular, switch failures are unavoidable. Fault-tolerance capability is one main aspect to measure the performance of data center networks. Connectivity, fault tolerance Hamiltonian connectivity, and fault tolerance Hamiltonicity are important parameters that assess the fault tolerance of networks. In general, the distribution of fault elements is scattered, and it is necessary to consider the distribution of fault elements in different dimensions. We research the fault tolerance of BCube when considering faulty switches and faulty links/edges that distribute in different dimensions. We also investigate the connectivity, fault tolerance Hamiltonian connectivity, and Hamiltonicity. This study better evaluates the fault-tolerant performance of data center networks.

Keywords: connectivity; Hamiltonicity; fault tolerance; BCube; network

MSC: 05C85; 68W15

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1. Introduction

With the rapid growth of network resources and data, cloud computing has risen rapidly in the field of computer applications [1,2]. The purpose of cloud computing is to reduce the computing task burden of end users and complete the majority of computing in the cloud by large data center networks. Data center networks are infrastructures of cloud computing and innovation platforms of next-generation networks. Data center network research has become a popular aspect in academia and industry. Scholars have proposed many data center networks, for example, Fat-Tree [3], DCell [4–6], BCube [7,8], VL2 [9], CamCube [10], Ficonn [11], FSquare [12], BCDC [13,14], and so on. Because of its desirable features, such as symmetry, small diameter, high fault tolerance, and so on, BCube has become a main data center network. It supports one-to-one and one-to-several traffic patterns [15]. It has good communication performances because it can build serveral vertex disjoint paths of shorter lengths [16]. The embedding of the path or the cycle is one of the main research topics in networks because many effective algorithms for solving various graph problems have been developed on the basis of paths and cycles [17–20] and some parallel applications [21,22]. , Hamiltonian path and Hamiltonian cycle embeddings are important properties because the occurrence of congestion and deadlock can be effectively reduced or even avoided by multi-cast algorithms based on Hamiltonian paths and Hamiltonian cycles [23]. Consequently, there are a great number of research findings on

Hamiltonian properties on particular network topologies, such as hypercube [24], cross cube [25,26], twist cube [27–29], extended cube [30], k-ary n-cube [31–33], and DCell [34,35].

In practical applications, the failure of components or physical connections in data center networks is inevitable. Fault tolerance is a vital aspect to measure the performance of networks [36,37]. Edge connectivity is the main feature used to assess the fault tolerance of networks, which is often exactly equivalent to a network's minimum degree. and the set of faulty edges that make it disconnected is often connected to a node whose degree is exactly equivalent to this network's minimum degree. In general, that all the faulty edges are concentrated on the adjacent edges of a certain node is almost impossible. Harary [38] proposed the concept of edge connectivity under some conditions in 1983. There are several related studies on this topic, such as conditional edge connectivity [39,40], extra edge connectivity [41,42], and component edge connectivity [43]. In practical applications, the distribution of fault elements may be scattered. The fault elements may be distributed in different dimensions in networks. It is necessary to study the fault situation in terms of dimensions. It is inevitable that switches in data center networks will fail. If a switch is faulty, the servers connecting to it are disconnected from each other. Faulty switches will have a more destructive effect on the stability of the networks. So, we research the fault tolerance of BCube when considering faulty switches and faulty links/edges that distribute in different dimensions.

A data center network can be denoted with a graph, where nodes are servers, edges are links connecting servers, and switches can be considered transparent devices. The topological properties of data center networks are critical to data center performance. $BCube(n,k)$ is a k-dimensional BCube that is constructed with n-port switches, where $n \geq 2$ and $k \geq 0$ [7]. The graph $BC_{n,k}$ can be viewed as the topological structure of $BCube(n,k)$, where switches are considered to be transparent [44]. In this paper, we give the corresponding relation between $BC_{n,k}$ and $BCube(n,k)$ and research the connectivity, fault-tolerant Hamiltonian connectivity, and Hamiltonicity of $BC_{n,k}$ when the faulty elements distribute in different dimensions.

We give the definitions of $BCube(n,k)$ and $BC_{n,k}$. We research the connectivity, fault-tolerant Hamiltonian connectivity, and Hamiltonicity of $BC_{n,k}$ in Section 3. We analyze the performance of the results through computer simulation experiments in Section 4. Finally, in Section 5, we draw some conclusions.

2. Preliminaries

In this section, we begin by introducing some notations. Next, we give the definitions and properties of $BCube(n,k)$ and $BC_{n,k}$. We also show the corresponding relations between $BC_{n,k}$ and $BCube(n,k)$.

The graph-theoretical terminologies and notations mainly follow [45]. Given an undirected simple graph $G = (V(G), E(G))$, $V(G)$ denotes the *node set*, and $E(G) = \{(u,v)|u,v \in V(G)\}$ represents the *edge set*. For any node u in G, let $N_G(u)$ be the set of its neighbors. The degree of u, marked as $deg_G(u)$, is the number of neighbors of u. A *path* $P(v_1, v_n) = \langle v_1, v_2, \cdots, v_n \rangle$ is a sequence of neighboring nodes in which all nodes are different except possibly $v_1 = v_n$. If a path travels through each node of G precisely once, it is called a Hamiltonian path. A path that starts and finishes at the same node is said to be a cycle. A cycle containing all of G's nodes is known as a Hamiltonian cycle. If G has a Hamiltonian cycle, G is Hamiltonian. If there is a Hamiltonian path linking any two different nodes in G, then G is said to be Hamiltonian-connected. For $n \geq 2$, let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $\cdots$, $G_n = (V_n, E_n)$ be n disjoint graphs. The union of $G_1, G_2, \cdots, G_n$, represented by $\bigcup_{i=1}^{n}(G_i)$, is the graph with the node set $V_1 \cup V_2 \cup \cdots \cup V_n$ and the edge set $E_1 \cup E_2 \cup \cdots \cup E_n \cup \{(u,v)|u \in V_i, v \in V_j\}$ for any two positive integers i and j with $1 \leq i \neq j \leq n$. The graph G_1 is isomorphic to the graph G_2 if there exists a bijection θ: $V(G_1) \to V(G_2)$ such that $(x,y) \in E(G_1)$ if and only if $(\theta(x), \theta(y)) \in E(G_2)$, represented by $G_1 \cong G_2$. $[i,j]$ is used to represent the integer set $\{d|i \leq d \leq j\}$ for any two positive integers i and j with $i \leq j$.

The BCube can be recursively defined, which contains three types of elements, namely switches, servers, and links. Multi-port servers are connected to switches with a fixed number of ports by links. For any integers $k \geq 0$ and $n \geq 2$, a server y of $BCube(n,k)$ can be denoted by $y_k y_{k-1} \cdots y_0$. Each switch in $BCube(n,k)$ can be represented by $< l, s_{k-1}s_{k-2} \cdots s_0 >$, where l is the level(or dimension) of the switch and $0 \leq l \leq k$. A link is represented by $\{(y_k y_{k-1} \cdots y_0), < l, s_{k-1}s_{k-2} \cdots s_0 >\}$. Each server with $k+1$ ports is linked to one switch at every level. These levels are recorded from Level 0 to Level k. Obviously, there exist $k+1$ switch levels and n^{k+1} servers in $BCube(n,k)$. Each level has n^k switches. Following [7], we define $BCube(n,k)$ recursively.

Definition 1 ([7])**.** *The definition of $BCube(n,k)$ is as below.*

(1) For $k=0$, $BCube(n,0)$ contains one switch with an n-port and n servers, which are connected to the switch.

(2) For $k \geq 1$, $BCube(n,k)$ contains n^k switches with an n-port and n disjoint copies of $BCube(n,k-1)$, where:

α For every $0 \leq j \leq n-1$, we obtain the subgraph $BCube^j_{n,k-1}$ by prefixing the label of every server with j in one copy of $BCube(n,k-1)$;

β For any $0 \leq j \leq n-1$, a server $jx_{k-1} \cdots x_0$ is connected to the switch $< k, s_{k-1} \cdots s_0 >$ if and only if $x_{k-1} \cdots x_0 = s_{k-1} \cdots s_0$, where j denotes the j-th port of the switch to which the server is connected to.

$BCube(4,1)$ has four servers and one four-port switch (see Figure 1). $BCube(4,1)$ contains four disjoint copies of $BCube(4,0)$, which are connected by four four-port Level 1 switches (see Figure 1). Each server has two ports in $BCube(4,1)$.

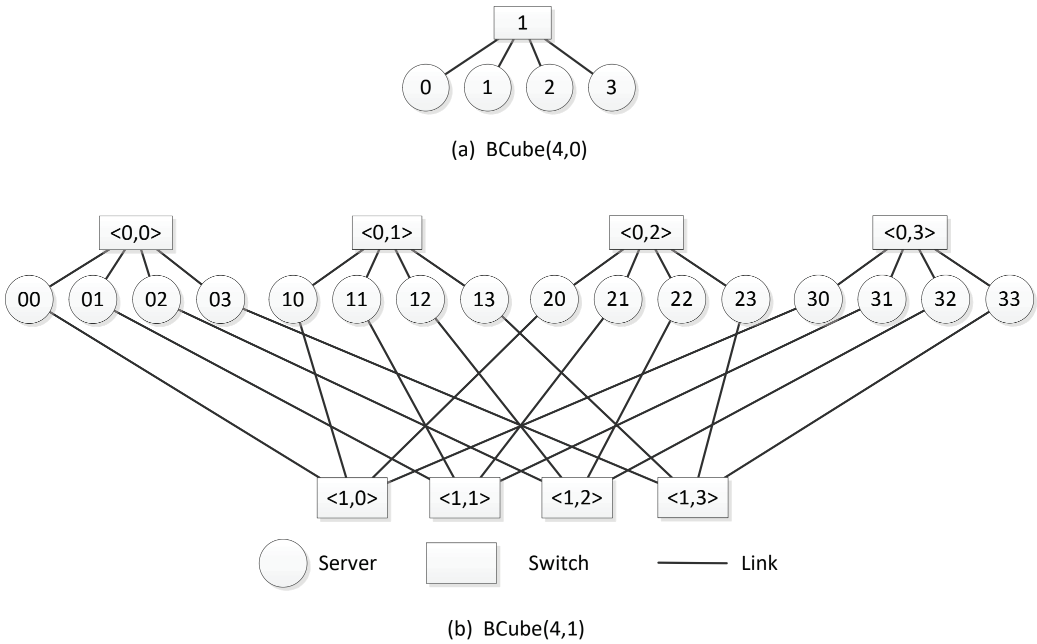


Figure 1. $BCube(4,0)$ and $BCube(4,1)$.

By making all switches transparent, we can obtain the topological structure of BCube. The topological structure of $BCube(n,k)$ is represented by $BC_{n,k}$, which is defined as follows.

Definition 2. *Given two integers k, n, $k \geq 0$ and $n \geq 2$, $BC_{n,k}$ is represented by a simple undirect graph $BC_{n,k} = (V(BC_{n,k}), E(BC_{n,k}))$, where $V(BC_{n,k}) = \{u_k u_{k-1} \cdots u_0 | u_i \in [0, n-1]$ and $i \in [0,k]\}$. For any two nodes $x = x_k x_{k-1} \cdots x_0$ and $y = y_k y_{k-1} \cdots y_0$, $(x,y) \in E(BC_{n,k})$ if and only if there exists an integer $i \in [0,k]$ such that $x_i \neq y_i$ and $x_l = y_l$ for all $l \in [0,k] - \{i\}$. We* *say (x, y) is an i-dimensional edge, denoted by i-edge. We set $E^i(BC_{n,k})$ being the set of all i edges of $BC_{n,k}$.*

A graph $BC_{n,k}$ can be decomposed into n disjoint subgraphs: $BC^0_{n,k-1}$, $BC^1_{n,k-1}$, $\ldots$, $BC^{n-1}_{n,k-1}$ along dimension k, where $BC^j_{n,k-1}$, for every $0 \le j \le n-1$, is a subgraph of $BC_{n,k}$ induced by $\{ju_{k-1}u_{k-2}|ju_{k-1}u_{k-2}\cdots u_0 \in V(BC_{n,k})\}$. Any two subgraphs are connected by the k-edges, which correspond to the Level k switches in $BCube(n,k)$. Clearly, each $BC^j_{n,k-1}$ is isomorphic to $BC_{n,k-1}$ for $0 \le j \le n-1$. The subgraph $BC^j_{n,k-1}$ of $BC_{n,k}$, $0 \le j \le n-1$, is also the topological structure of the subgraph $BCube^j_{n,k-1}$ of $BCube(n,k)$ making switches transparent. Along k different dimensions, $BC_{n,k}$ can be decomposed into n copies of $BC_{n,k-1}$.

$BC_{4,0}$ and $BC_{4,1}$ are shown in Figure 2. $BC_{n,k}$ is a kind of generalized hypercube. The graph $BC_{2,k}$ is isomorphic to the hypercube Q_{k+1}. For any node y in $BC_{n,k}$, the degree of y is $(n-1)(k+1)$. $BC_{n,k}$ is a highly symmetric network with vertex symmetry and edge symmetry.

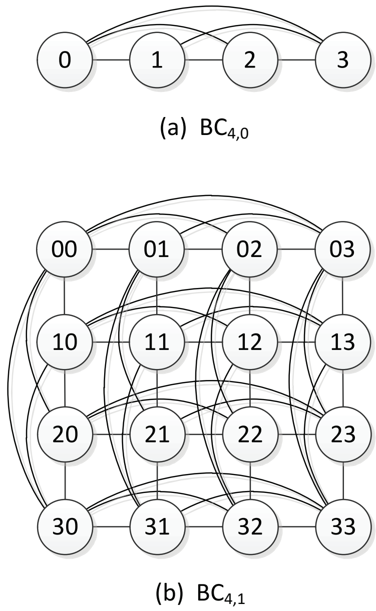
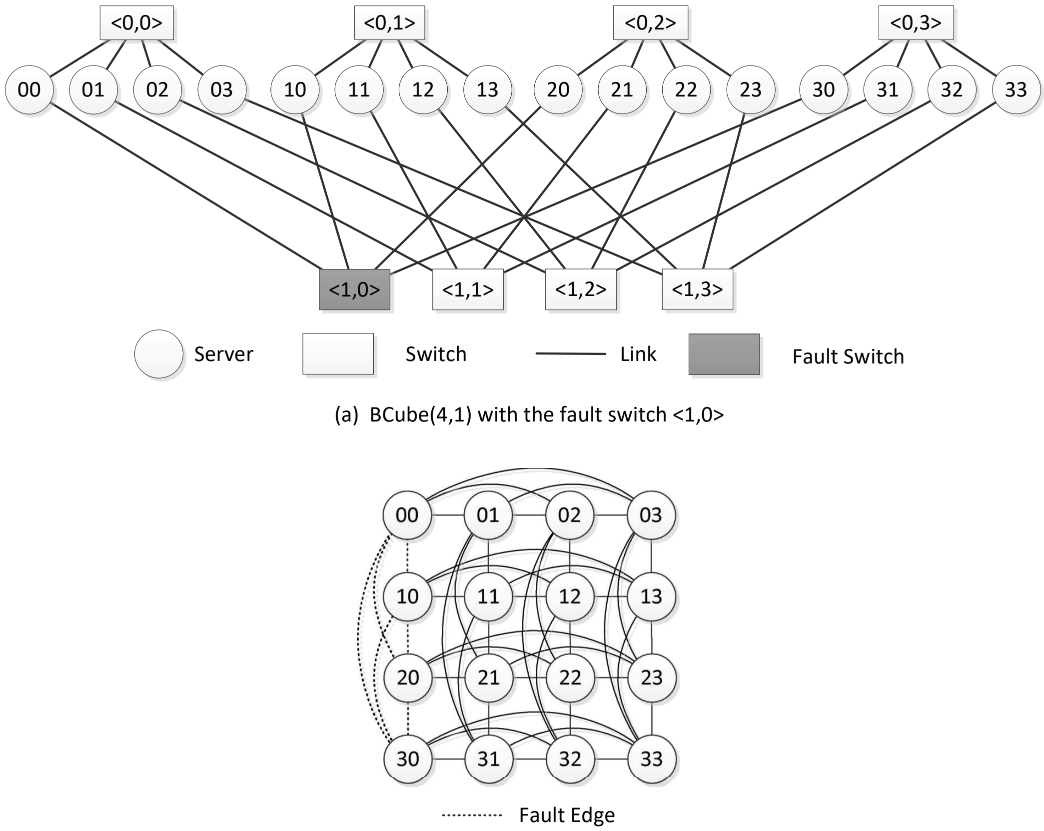


Figure 2. $BC_{4,0}$ and $BC_{4,1}$.

The corresponding relation between the elements in $BC_{n,k}$ and the elements in $BCube(n,k)$ will be discussed below. Let z be one of the elements in $BCube(n,k)$. If the element z is a server, it corresponds to a node in $BC_{n,k}$. If the element z is a switch, it corresponds to an edge subset $\{(x,y)|x$ and y are two distinct nodes, which correspond to two distinct servers of $BCube(n,k)$ connected by the switch $z\}$. If the element z is a link, it corresponds to an edge subset $\{(u,v)|u$ and v are two distinct nodes, which correspond to two servers of $BCube(n,k)$ connected through the link $z\}$. Given an integer l, $0 \le l \le k$, if the element z is a switch of $BCube(n,k)$ in Level l, it corresponds to the edge subset of $E^l(BC_{n,k})$, which has $\frac{n(n-1)}{2}$ edges. As shown in Figure 3, the corresponding element of the fault switch $< 1, 0 >$ in $BCube(4,1)$ is the edge subset $\{(00,10),(00,20),(00,30),(10,20),(10,30),(20,30)\}$ in $BC_{4,1}$.

We let F_s be a set, each element of which is a faulty edge subset in $BC_{n,k}$, which is affected by a broken switch in $BCube(n,k)$. And let F^i_s be the set, each element of which is a faulty edge subset in $BC_{n,k}$, which is caused by a broken switch of $BCube(n,k)$ in Level i, with $0 \le i \le k$. Clearly, $F_s = F^0_s \cup F^1_s \cup \cdots \cup F^k_s$. Let F_e be the set of the faulty edges in $BC_{n,k}$ which is not caused by faulty switches. Let u and v be two servers in $BC_{n,k}$. If a faulty edge (u,v) exists, the server u is unable to communicate with the server v. Furthermore, let

F_e^i be the set of the i-edges in the faulty edge set F_e, $0 \leq i \leq k$. We let $F = F_s \cup F_e$, $f = |F|$, $f_s = |F_s|$, $f_e = |F_e|$, $f_s^i = |F_s^i|$, $f_e^i = |F_e^i|$, $F^i = F_s^i \cup F_e^i$, $f^i = |F^i|$, $0 \leq i \leq k$. Because $BC_{n,k}$ is able to reflect the characteristics of $BCube(n,k)$, we will carry out the below study on $BC_{n,k}$.



(b) $BC_{4,1}$ with the fault edges set
{(00.10),(00,20),(00,30),(10,20),(10,30),(20,30)}

Figure 3. $BCube(4,1)$ with the fault switch and $BC_{4,1}$ with the fault edges set.

3. Fault-Tolerant Properties of $BC_{n,k}$

Considering that $BC_{n,k}$ is edge symmetric and node symmetric, we assume $f^0 \leq f^1 \leq \cdots \leq f^{k-1} \leq f^k$. For $1 \leq t \leq k$, we set $S_{n,k}^t = \{u_k u_{k-1} \cdots u_0 | u_i = 0$ for each $i \geq t$ and $u_j \in [0, n-1]$ for each $j \in [0, t-1]\}$. Note that the subgraph of $BC_{n,k}$ induced by $S_{n,k}^t$ is isomorphic to $BC_{n,t-1}$. The graph $BC_{2,k}$ is isomorphic to the $(k+1)$-dimensional hypercube Q_{k+1}. The relevant conclusions have been drawn and will be presented in another paper. So, we discuss the properties of $BC_{n,k}$ for $n \geq 3$ in this paper.

Theorem 1. *For any faulty set F of $BC_{n,k}$, $F = F_s \cup F_e$, $BC_{n,k} - F$ is connected if $f \leq \frac{n^{k+1}-n}{n-1} - k$ and $f^i \leq n^i - 1$ for each $0 \leq i \leq k$.*

Proof. The proof of this theorem is by induction on k. Obviously, $BC_{n,0}$ is connected if $f = 0$. Suppose that this theorem holds on $BC_{n,k-1}$, where $n \geq 3$ and $k \geq 1$. Since $BC_{n,k}$ is edge symmetric, we assume that $|F^k| = max\{|F^i|| i \in [0,k]\}$. Then, $|F| - |F^k| = \sum_{i=0}^{k-1} |F^i| = \sum_{i=0}^{k-1} f^i \leq \sum_{i=0}^{k-1} (n^i - 1) = \frac{n^k - n}{n-1} - (k-1)$. For $0 \leq j \leq n-1$, let $F_s^i(j)$ be the set, each element of which is a faulty edge subset in $BC_{n,k-1}^j$, which is caused by a faulty switch in $BCube_{n,k-1}^j$ in Level i with $0 \leq i \leq k-1$. $F_s(j) = F_s^0(j) \cup F_s^1(j) \cup \cdots \cup F_s^{k-1}(j) = \bigcup_{i=0}^{k-1} F_s^i(j)$. Let $F_e^i(j)$ be the edge subset of the faulty i-edges in $F_e \bigcap E(BC_{n,k-1}^j)$. $F_e(j) = \cup_{i=0}^{k-1} F_e^i(j)$. Let $F^i(j) = F_s^i(j) \cup F_e^i(j)$, $f^i(j) = |F^i(j)|$, $F(j) = F_s(j) \cup F_e(j)$. Since $F_s^i(j) \subseteq F_s^i$, $F_e^i(j) \subseteq F_e^i$ for each j. Hence, $|F_s^i(j)| \leq |F_s^i|$, $|F_e^i(j)| \leq |F_e^i|$ for each $0 \leq j \leq n-1$; that is, $|F_s^i(j) + F_e^i(j)| \leq |F_s^i + F_e^i| = |F^i| = f^i \leq n^i - 1$. $|F^i(j)| \leq n^i - 1$, $\sum_{i=0}^{k-1} f^i(j) \leq \sum_{i=0}^{k-1}(n^i - 1) = \frac{n^k - n}{n-1} - (k-1)$. By induction hypothesis, $BC_{n,k-1}^j - F(j)$ is connected for each $j \in [0, n-1]$. Since $f^k \leq n^k - 1$, there is an edge e between $BC_{n,k-1}^{\alpha}$ and $BC_{n,k-1}^{\beta}$ such that $e \in E(BC_{n,k} - F)$ for each $0 \leq \alpha, \beta \leq n-1$. Hence, $BC_{n,k} - F$ is connected. □

We use the following example to show that the bound is tight.

Example 1. *Let us consider that $f^t \geq n^t$ for some $t \in [0, n-1]$ with fixed t. We discuss two cases.*

Case 1. $t = 0$. We set $u = u_{k-1}u_{k-2}\cdots u_0 \in V(BC_{n,k})$ and assume that all the switches are faulty, which are connected with the node u and $|F_e| = 0$. Obviously, $|F_s^i| = 1$, $|F_e^i| = 0$ for $0 \leq i \leq k$. Then, $BC_{n,k} - F$ is disconnected since $deg_{BC_{n,k}-F}(u) = 0$ and one component of $BC_{n,k} - F$ is the node u.

Case 2. $t \geq 1$. Let B be the connected subgraph of $BC_{n,k}$ which is induced by $S_{n,k}^t$. Obviously, B is isomorphic to $BC_{n,t-1}$. Then, we have $|F^i| = |F_s^i| = n^t$ for each $t \leq i \leq k$, and $|F^i| = |F_s^i| = 0$ for each $0 \leq i < t$. We set $F = \cup_{i=0}^{k} F^i$. We have (1) $|F| = (k-t+1)n^t \leq \frac{n^{k+1}-n}{n-1} - k$, (2) $|F^t| = n^t > n^t - 1$, and (3) $|F^i| \leq n^i - 1$ for each $i \neq t$. Then, $BC_{n,k} - F$ is not connected and one component of it is the subgraph B.

Theorem 2 ([46]). *For $2 \leq m \leq n$, let $A = \{BC_{n,k-1}^{j_1}, BC_{n,k-1}^{j_2}, \ldots, BC_{n,k-1}^{j_m}\}$ with $j_i \in [0, n-1]$ and $i \in [1, m]$. Let $F(BC_{n,k-1}^{j_i})$ be the set of faulty elements in $BC_{n,k-1}^{j_i}$. For any two nodes $x \in V(BC_{n,k-1}^{j_1} - F(BC_{n,k-1}^{j_1}))$ and $y \in V(BC_{n,k-1}^{j_m} - F(BC_{n,k-1}^{j_m}))$, there is a fault-free Hamiltonian path $HP(x,y)$ in $\bigcup_{i=1}^{m}(BC_{n,k-1}^{j_i} - F(BC_{n,k-1}^{j_i}))$ where (1) For any integer $t \in \{j_1, j_2, \ldots, j_m\}$, $BC_{n,k-1}^t - F(BC_{n,k-1}^t)$ is Hamiltonian-connected. (2) There exist at least three fault-free k-edges between any two distinct graphs in the subgraph set A.*

Theorem 3. *For $n \geq 3$, let F be any faulty set of $BC_{n,1}$, $F = F_s \cup F_e$, $BC_{n,1} - F$ is Hamiltonian-connected if $f^0 = 0$ and $f^1 \leq n-3$.*

Proof. $BC_{3,1}$ is Hamiltonian-connected if $f^0 = 0$ and $f^1 = n - 3 = 0$. So, we discuss the case $n \geq 4$.

$BC_{n,1}$ can be divided into n subgraphs $BC_{n,0}^j$ for $0 \leq j \leq n-1$. For each $j \in [0, n-1]$, $BC_{n,0}^j$ is Hamiltonian-connected because it is a complete graph with n nodes. Since $f^0 = 0$, there is no fault element in $BC_{n,0}^j$, $0 \leq j \leq n-1$. Since $f^1 \leq n-3$, there are at least three fault-free switches in Level 1 in $BCube(n,1)$. We consider any three fault-free switches. We assume these switches individually connect with the nodes $\{a_0, a_1, \ldots, a_{n-1}\}$, $\{b_0, b_1, \ldots, b_{n-1}\}$ and $\{c_0, c_1, \ldots, c_{n-1}\}$, $a_j, b_j, c_j \in V(BC_{n,0}^j)$ for $0 \leq j \leq n-1$. For any two nodes $u \in V(BC_{n,0}^{\alpha})$, $v \in V(BC_{n,0}^{\beta})$, $0 \leq \alpha, \beta \leq n-1$, we divide into two cases to discuss the existence of a Hamiltonian path connecting the nodes u and v in $BC_{n,1} - F$.

Case 1. $\alpha \neq \beta$.

By Theorem 2, there exists a fault-free Hamiltonian path connecting u and v in $\bigcup_{j=0}^{n-1}(BC_{n,0}^j - F^1)$.

Case 2. $\alpha = \beta$.

W.L.O.G., we suppose $\alpha = \beta = 0$. We have two subcases.

Case 2.1. $|\{u, v\} \bigcap \{a_0, b_0, c_0\}| \leq 1$.

We assume that $\{u, v\} \bigcap \{a_0, b_0\} = \emptyset$. Since $BC_{n,0}^0$ is a complete graph, there is an edge (u, a_0) and a path $P(b_0, v)$, which contains all the nodes in $V(BC_{n,0}^0) - \{u, a_0\}$. By Theorem 2, a fault-free Hamiltonian path $P(a_1, b_{n-1})$ exists, which connects a_1 and b_{n-1} in $\bigcup_{j=1}^{n-1}(BC_{n,0}^j - F^1)$. So, $< u, a_0,\ a_1,\ P(a_1, b_{n-1}),\ b_{n-1},\ b_0,\ P(b_0, v),\ v >$ is a fault-free Hamiltonian path connecting u and v in $BC_{n,1} - F$ (see Figure 4).

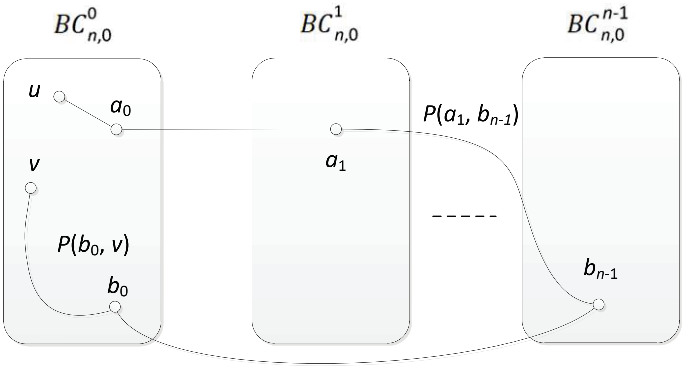


Figure 4. The illustration for Case 2.1 of Theorem 3.

Case 2.2. $|\{u, v\} \bigcap \{a_0, b_0, c_0\}| = 2$.

We assume that $u = a_0$, $v = b_0$. Since $BC^0_{n,0}$ is a complete graph, $BC^0_{n,0} - \{a_0\}$ is also a complete graph. In $BC^0_{n,0} - \{a_0\}$, there is a Hamiltonian path $P(c_0, b_0)$ connecting c_0 and b_0. By Theorem 2, a Hamiltonian path $P(a_1, c_{n-1})$ exists, which connects a_1 and c_{n-1} in $\bigcup_{j=1}^{n-1}(BC^j_{n,0} - F^1)$. So, $< a_0, a_1, P(a_1, c_{n-1}), c_{n-1}, c_0, P(c_0, b_0), b_0 >$ is a Hamiltonian path connecting u and v in $BC_{n,1} - F$. So, $BC_{n,1} - F$ is Hamiltonian-connected if $f^0 = 0$ and $f^1 \leq n - 3$ (see Figure 5). □

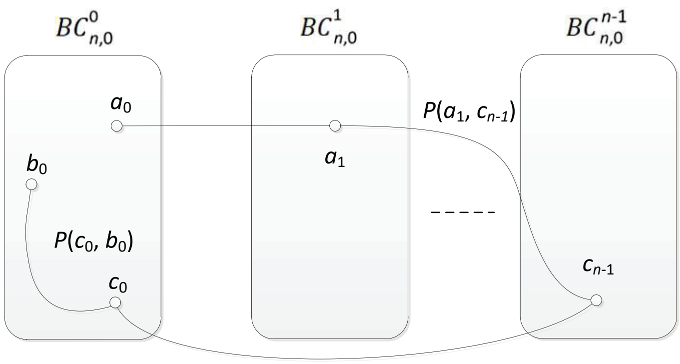


Figure 5. The illustration for Case 2.2 of Theorem 3.

Theorem 4. *For $n \geq 3$ and $k \geq 2$, let F be any faulty set of $BC_{n,k}$, $F = F_s \cup F_e$, $BC_{n,k} - F$ is Hamiltonian-connected if $f^i \leq \lfloor n^i/2 \rfloor - 1$ for each $2 \leq i \leq k$ and $f^0 = 0$, $f^1 \leq n - 3$.*

Proof. The proof of this theorem is by induction on k. By Theorem 3, $BC_{n,1}$ is Hamiltonian-connected if $f^0 = 0$ and $f^1 \leq n - 3$. Assume that this theorem holds on $BC_{n,k-1}$ with $n \geq 3$, $k \geq 2$.

For $0 \leq j \leq n - 1$, let $F^i_s(j)$ be the set, each element of which is a faulty edge set in $BC^j_{n,k-1}$, which is caused by a faulty switch in $BCube^j_{n,k-1}$ in Level i with $0 \leq i \leq k - 1$. $F_s(j) = F^0_s(j) \cup F^1_s(j) \cup \cdots \cup F^{k-1}_s(j) = \bigcup_{i=0}^{k-1} F^i_s(j)$. Let $F^i_e(j)$ be the edge subset of the faulty i-edges in $F_e \bigcap E(BC^j_{n,k-1})$. $F_e(j) = \bigcup_{i=0}^{k-1} F^i_e(j)$. $F(j) = F_s(j) \cup F_e(j)$. Since $F^i_s(j) \subseteq F^i_s$, $F^i_e(j) \subseteq F^i_e$ for each $j \in [0, n-1]$. Hence, $|F^i_s(j) + F^i_e(j)| \leq |F^i_s + F^i_e| = |F^i| = f^i \leq \lfloor n^i/2 \rfloor - 1$ for $2 \leq i \leq k - 1$, and $|F^0(j)| = 0$, $|F^1(j)| \leq n - 3$. By induction hypothesis, $BC^j_{n,k-1} - F(j)$ is Hamiltonian-connected for each $j \in [0, n-1]$. Since $f^k \leq \lfloor n^k/2 \rfloor - 1$, there are more than three fault-free edges between $BC^\alpha_{n,k-1}$ and $BC^\beta_{n,k-1}$ for $0 \leq \alpha, \beta \leq n - 1$ in $BC_{n,k} - F$. By Theorem 2, for any two nodes $u \in V(BC^\alpha_{n,k-1})$, $v \in V(BC^\beta_{n,k-1})$, $0 \leq \alpha, \beta \leq n - 1$ and $\alpha \neq \beta$, a Hamiltonian path exists, which connects u and v in $\bigcup_{j=0}^{n-1}(BC^j_{n,k-1} - F)$.

Here, we consider the situation $\alpha = \beta$. W.L.O.G., we suppose $\alpha = \beta = 0$. So, $u, v \in V(BC^0_{n,k-1})$. In $BC^0_{n,k-1} - F^0$, there is a Hamiltonian path $HP_0(u, v)$ of length n^k. Since $f^k \leq \lfloor n^k/2 \rfloor - 1$, there exists an edge (w_0, z_0) on the Hamiltonian path such that the two Level k switches are fault-free in $BCube(n, k)$, which connect with the nodes u and v individually. Let $HP_0(u, v) =< u, HP_1(u, w_0), w_0, z_0, HP_2(z_0, v), v >$. Let w_1 be the node that connects to the same Level k switch with the node w_0. Let z_{n-1} be the node that connects to the same Level k switch with the node z_0. By Theorem 2, a Hamiltonian path $HP(w_1, z_{n-1})$ exists, which connects w_1 and z_{n-1} in $\bigcup_{j=1}^{n-1}(BC^j_{n,k-1} - F)$. Then, $< u, HP_1(u, w_0), w_0, w_1, HP(w_1, z_{n-1}), z_{n-1}, z_0, HP_2(z_0, v), v >$ is a fault-free Hamiltonian path connecting u and v in $BC_{n,k} - F$. So, $BC_{n,k} - F$ is Hamiltonian-connected if $f^i \leq \lfloor n^i/2 \rfloor - 1$ for each $2 \leq i \leq k$ and $f^0 = 0$, $f^1 \leq n - 3$. □

Theorem 5. *For $n \geq 4$ and n mod $2 = 0$, $k \geq 2$, let F be any faulty set of $BC_{n,k}$, $F = F_s \cup F_e$, $BC_{n,k} - F$ is Hamiltonian if $f^i \leq \lfloor n^i/2 \rfloor - 1$ for each $2 \leq i \leq k-1$ and $f^0 = 0$, $f^1 \leq n - 3$, $f^k \leq n^k - 2$.*

Proof. By Theorem 4, $BC^j_{n,k-1} - F(j)$ is Hamiltonian-connected for each $j \in [0, n-1]$. Since $f^k \leq n^k - 2$, there are at least two fault-free switches in Level k in $BCube(n, k)$. So, we assume that one switch connects with the nodes $\{a_0, a_1, \ldots, a_{n-1}\}$, and the other connects with the nodes $\{b_0, b_1, \ldots, b_{n-1}\}$, $a_j, b_j \in V(BC^j_{n,k-1})$, $0 \leq j \leq n-1$. By Theorem 4, $BC^j_{n,k-1} - F(j)$ is Hamiltonian-connected for each $j \in [0, n-1]$. So, there is a Hamiltonian path $P_j(a_j, b_j)$ or $P_j(b_j, a_j)$ between a_j and b_j in $BC^j_{n,k-1} - F(j)$. Since n is even, the cycle $< a_0, P_0(a_0, b_0), b_0, b_1, P_1(b_1, a_1), a_1, b_2, \ldots, b_{n-1}, P_{n-1}(b_{n-1}, a_{n-1}), a_{n-1}, a_0 >$ is a Hamiltonian cycle in $BC_{n,k} - F$. So, $BC_{n,k} - F$ is Hamiltonian for $n \geq 4$ and n mod $2 = 0$, as shown in Figure 6. □

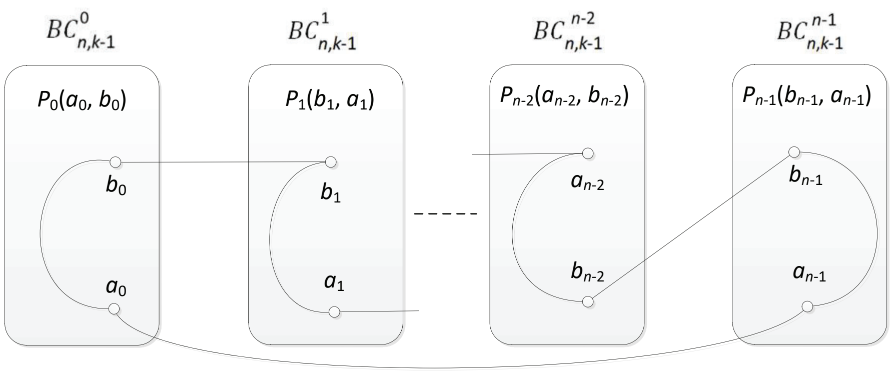


Figure 6. The illustration of Theorem 5.

Note that there is no Hamiltonian cycles in $BC_{n,k} - F$ for $f^k = n^k - 2$ if n is odd. We have the theorem below for odd number n.

Theorem 6. *For $n \geq 3$ and n mod $2 \neq 0$, $k \geq 2$, let F be any faulty set F of $BC_{n,k}$, $F = F_s \cup F_e$, $BC_{n,k} - F$ is Hamiltonian if $f^i \leq \lfloor n^i/2 \rfloor - 1$ for each $2 \leq i \leq k-1$ and $f^0 = 0$, $f^1 \leq n - 3$, $f^k \leq n^k - 3$.*

Proof. By Theorem 4, $BC^j_{n,k-1} - F(j)$ is Hamiltonian-connected for each $j \in [0, n-1]$. Since $f^k \leq n^k - 3$, there are at least three fault-free switches in Level k in $BCube(n, k)$. So, we assume that one switch connects with the nodes $\{a_0, a_1, \ldots, a_{n-1}\}$, one switch connects with the nodes $\{b_0, b_1, \ldots, b_{n-1}\}$, and the other connects with the nodes $\{c_0, c_1, \ldots, c_{n-1}\}$, $a_j, b_j, c_j \in V(BC^j_{n,k-1})$, $0 \leq j \leq n-1$. By Theorem 4, $BC^j_{n,k-1} - F(j)$ is Hamiltonian-connected for each $j \in [0, n-1]$.

So, there is a Hamiltonian path $P_j(a_j, b_j)$, $P_j(b_j, c_j)$ or $P_j(c_j, a_j)$ between any two nodes of $\{a_j, b_j, c_j\}$ in $BC^j_{n,k-1} - F(j)$. Since n is odd, the cycle $< a_0$, $P_0(a_0, b_0)$, b_0, b_1, $P_1(b_1, c_1)$, c_1, c_2, $P_2(c_2, a_2)$, a_2, a_3, $P_3(a_3, b_3)$, $\ldots$, c_{n-1}, $P_{n-1}(c_{n-1}, a_{n-1})$, $a_{n-1}, a_0 >$ is a Hamiltonian cycle in $BC_{n,k} - F$. So, $BC_{n,k} - F$ is Hamiltonian for $n \geq 4$ and $n \bmod 2 \neq 0$ if $f^i \leq \lfloor n^i/2 \rfloor - 1$ for each $2 \leq i \leq k-1$ and $f^0 = 0$, $f^1 \leq n-3$, $f^k \leq n^k - 3$, as shown in Figure 7. □

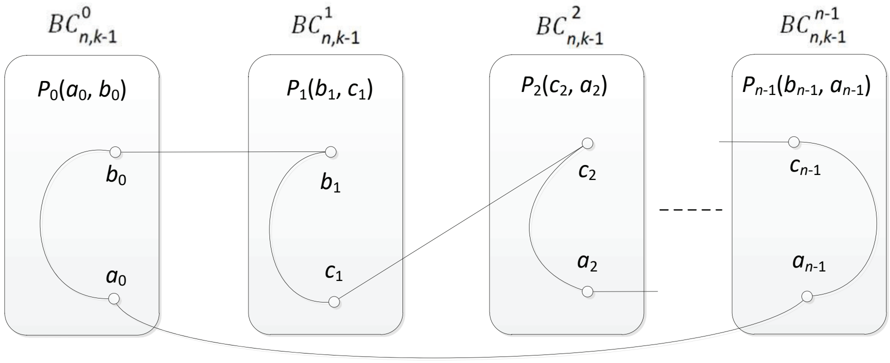


Figure 7. The illustration of Theorem 6.

4. Performance Analysis

Up to now, we have shown $BCube(n, k)$ is connected when faulty switches and faulty links distributing in different dimensions are considered. In this section, we are going to demonstrate the superiority of our results from two aspects. Compared with link faults, switch faults are more destructive, so we assume that all the fault elements are switches when analyzing performance. We discuss the maximum number of faulty switches that the network $BCube(n, k)$ can tolerate while maintaining connectivity. We also investigate the maximum distance between any two nodes in $BC_{n,k}$ when the number of faulty elements reaches the maximum.

4.1. Number of Faulty Switches

According to the proof, the maximum number of faulty switches that $BCube(n, k)$ can tolerate is $\frac{n^{k+1}-n}{n-1} - k$ when $BCube(n, k)$ is still connected. We list the maximum number of faulty switches for $n \in \{3, 4, 5\}$ and $k \in \{1, 2, 3, 4, 5, 6\}$ that $BCube(n, k)$ can tolerate in Table 1. These results indicate that $BCube(n, k)$ still has good properties while there are more faulty elements compared with the traditional method.

Table 1. Maximum Number of Faulty Switches that $BCube(n, k)$ Can Tolerate When $BCube(n, k)$ is still Connected.

	$n = 3$	$n = 4$	$n = 5$
$BCube(n, 1)$	2	3	4
$BCube(n, 2)$	10	18	28
$BCube(n, 3)$	36	81	152
$BCube(n, 4)$	116	336	776
$BCube(n, 5)$	358	1359	3900
$BCube(n, 6)$	1086	5454	19,524

4.2. The Average Value of The Maximum Distance between Any Two Nodes

In this subsection, we investigate the maximum distance between any two nodes in $BC_{n,k}$ when $\frac{n^{k+1}-n}{n-1} - k$ switches become faulty in $BC_{n,k}$. The fault switches are distributed in different levels of $BC_{n,k}$ and each level i has $f^i = n^i - 1$ faulty switches where $0 \leq i \leq k$. We design an algorithm $AverageMaxDistance(n, k)$ to calculate the maximum distance between any two nodes in $BC_{n,k}$. The faulty switches are distributed randomly in $BC_{n,k}$.

We repeat the algorithm 100 times to obtain the average value of the maximum distance between any two nodes in $BC_{n,k}$.

$BC_{n,k}$ has $k+1$ switch levels where there exist n^k n-port switches in each level. To remove a switch s of level i, we need to disconnect all the servers adjacent to s. If two servers μ and ν are connected to the same switch of level i, they are connected by an i-dimensional edge, and ν is an i-dimensional neighbor of μ. We use $N_i(\mu)$ to denote the i-dimensional neighbor set of μ in $BC_{n,k}$. In $BC_{n,k}$, the switches are transparent. To randomly remove a switch of level i, we can randomly select a node μ, then remove all the i-dimensional edges between any two nodes in $N_i(\mu) \cup \{\mu\}$. Please see Algorithm 1 for an illustration. The results obtained from Algorithm 2 are shown in Table 2. These results indicate that the distance between any two nodes is still small in $BC_{n,k}$ while there are more faulty elements.

Algorithm 1 removeSwitches(g,n,k)

Input: g: a k-dimensional $BC_{n,k}$; n: the port number of a switch in the BCube; k: the dimension of the BCube;

List $nodesList = null$;
for $i = 1; i <= k; i++$ **do**
    nodesList = new ArrayList();
    for $j = 1; j <= Math.pow(n, i) - 1; j++$ **do**
        select a random vertex x from $BC_{n,k}$;
        if $(!nodesList.contains(\mathrm{x}))$ **then**
            $nodesList.add(\mathrm{x})$;
            remove the i-dimensional edge of $N_i(x) \cup \{x\}$;
            add all i-dimensional nodes of x into nodesList;
        else
            $j--$;
        end if
    end for
end for

Algorithm 2 AverageMaxDistance(n, k)

Input: n: the port number of a switch in the BCube; k: the dimension of the BCube;
Output: the average value of the maximum distance between any two nodes in $BC_{n,k}$;

sum = 0.0;
for $i = 1; i <= 100; i++$ **do**
    g $\leftarrow$ createBCube(n, k);
    removeSwitches(g, n, k);
    obtain the maximum distance d between any two nodes in graph g.
    sum $\leftarrow$ sum + d;
end for
return sum/100;

Table 2. The average value of the maximum distance between any two nodes.

	Nodes	Faulty Switches	Average Values of the Maximum Distance
$BC_{3,1}$	9	2	3
$BC_{3,2}$	27	10	5.72
$BC_{3,3}$	81	36	7.34
$BC_{4,1}$	16	3	3
$BC_{4,2}$	64	18	5.79
$BC_{4,3}$	256	81	7.48
$BC_{5,1}$	25	4	3
$BC_{5,2}$	125	28	5.75
$BC_{5,3}$	625	152	7.61

5. Conclusions

In this work, we investigate the fault tolerance of BCube while faulty links and faulty switches distribute in different dimensions. We reveal the properties of BCube in its topological graph $BC_{n,k}$ for $k \geq 1$ and $n \geq 3$. This paper shows that (1) $BC_{n,k} - F$ is connected if $f \leq \frac{n^{k+1}-n}{n-1} - k$ and $f^i \leq n^i - 1$ for each $0 \leq i \leq k$; (2) $BC_{n,k} - F$ is Hamiltonian if $f^i \leq \lfloor n^i/2 \rfloor - 1$ for each $2 \leq i \leq k-1$ and $f^0 = 0$, $f^1 \leq n-3$, $f^k \leq n^k - 2$; (3) If $n \bmod 2 = 0$, $BC_{n,k} - F$ is Hamiltonian if $f^i \leq \lfloor n^i/2 \rfloor - 1$ for each $2 \leq i \leq k-1$ and $f^0 = 0$, $f^1 \leq n-3$, $f^k \leq n^k - 2$; (4) If $n \bmod 2 \neq 0$, $BC_{n,k} - F$ is Hamiltonian if $f^i \leq \lfloor n^i/2 \rfloor - 1$ for each $2 \leq i \leq k-1$ and $f^0 = 0$, $f^1 \leq n-3$, $f^k \leq n^k - 3$. These results indicate that compared with the traditional method, BCube still has good properties while there are more faulty elements. Based on the results obtained here, we will consider several properties such as fault tolerant routing, diameter of BCube as future research directions. In addition, our results can be extended to other data center networks.

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