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Random Warranty and Replacement Models Customizing from the Perspective of Heterogeneity

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Abstract: Driven by the wide application of industrial software integrated with digital technologies, the real information of task cycles for some products in the real world can be monitored in real time and transmitted to the management center. Monitored task cycles hide consumers' usage characteristics, which are signals of the products' usage heterogeneity because they vary from one consumer to another consumer. By classifying monitored task cycles into different categories, this paper customizes two random maintenance models to ensure the life cycle reliability of the product with monitored task cycles on the basis of usage categories. The first model is customized using usage categories, the key objective of which is, from the perspective of heterogeneity, to ensure warranty-stage reliability. In view of using minimal repair service, the first model is named a random free repair warranty with heterogeneity (RFRW-H), which is modeled from the viewpoints of cost and time measures. By calculating the limits of cost and time measures, some specific cases are presented to model other warranties. The second model is customized using the same usage categories, which aims to ensure post-warranty-stage reliability. In view of using each of 'whichever occurs first/last', the second model is named a customized random periodic replacement first (CRPRF) model or a customized random periodic replacement last (CRPRL) model, respectively, which are modeled from the viewpoint of the cost rate function. By calculating the limits of the cost rate function, the cost rate functions of other maintenance models are obtained. Finally, from the numerical viewpoint, some of the features of the customized models are mined, and the performances are compared.

Keywords: heterogeneity; task cycle; life cycle reliability; warranty; random periodic replacement

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1. Introduction

From the perspectives of the management responsibilities for ensuring the life cycle reliability of the product, management responsibilities can be divided into two types: manufacturers' management responsibilities and consumers' management responsibilities. Manufacturers use warranty models/policies to fulfill the related responsibilities, while consumers use self-maintenance models/policies to fulfill their own responsibilities, named by some studies as post-warranty maintenance models/policies. Depending on whether two types of management responsibilities are simultaneously considered, the research streams of models/policies to fulfill respective responsibilities can be divided into three categories: ① warranty models/policies from the perspective of manufacturers, which are only confined to ensuring the warranty-stage reliability; ② maintenance policies/models from the perspective of consumers, which are only confined to ensuring the post-warranty-stage reliability; ③ warranty and maintenance models from the perspectives of both, which are successively used to ensure the life cycle reliability of the product.

The warranty models of the first stream can be classified into three types of models. The first type of model is named the free repair (i.e., minimal repair) warranty (FRW) model (see Refs. [1–3]) in which minimal repair service removes the warranty-stage failures. Because minimal repair service cannot improve product reliability, introducing preventive maintenance (PM) to the warranty stage, some scholars and practitioners have designed the PM warranty (PMW) model (see Refs. [4,5]), where PM can improve the product reliability and reduce the frequency number of failures, which belongs to the second type of the present stream. From the perspective of engineering practice, by means of PM, it is very difficult to recover the used product into a completely new product. In view of this fact, some scholars and practitioners have designed a replacement warranty (RW) model where the failed product over the warranty stage is replaced as a new identical product, which is categorized as the third type of the present stream. Based on the renewal and non-renewal cases of the warranty period, RW can be categorized into renewable free replacement warranty (RFRW) with the renewable warranty period and non-renewable replacement warranty (NRRW) model with the non-renewable warranty period. The RFRW model includes an on-condition RFRW model (see Ref. [6]), wherein any of the on-condition maintenance services in Refs. [7–13] have been used to ensure the reliability of the product subject to any stochastic processes in Refs. [14–29], and the classic RFRW model (see Refs. [30,31]), wherein classic replacement services have been used to ensure the reliability of the product with self-announcing failure.

Similar to classifying the models in the first stream, here, we classify the maintenance models of the second stream into three types of models. The first type of model is self-repair action, where minimal repair is used to remove post-warranty-stage failures (see Ref. [32]). The second type of model is self-PM action, where PM is used to improve the post-warranty-stage reliability and reduce the failure frequency over the post-warranty stage (see Ref. [33]). The third type of model is self-replacement action, where block replacement is used to ensure post-warranty-stage reliability (see Ref. [34]).

The models of the third stream can be classified into three types of models to ensure life cycle reliability, which can be listed as follows. The first type is warranty and maintenance models, which successively ensure the life cycle reliability of the product subject to self-announcing failure (see Refs. [35,36]). The second type is the on-condition warranty model and on-condition maintenance model, which successively ensure the life cycle reliability of the product subject to degradation failure. For example, Ref. [6] proposed an on-condition RFRW and on-condition replacement for ensuring the life cycle reliability of the product where the process of degradation failure is modeled by an inverse Gaussian (IG) process. The third type is the random warranty model and random maintenance model, which successively ensure the life cycle reliability of the product, the working/mission/task cycles of which can be monitored in real time and transmitted to the management center. For example, integrating one of the types of cycles into the warranty and post-warranty stages, Refs. [37–46] proposed some random warranty and random maintenance models for ensuring the life cycle reliability of the product with monitored working/mission/task cycles.

From the perspective of estimating the cost of models, the product's accumulated usage, such as accumulated mileage, accumulated operating time, or both, is one of the key factors affecting the cost of each model. In view of this, integrating usages (i.e., mileages) into the warranty stage or the life cycle, some scholars and practitioners have designed warranty and/or maintenance models with usages. For example, classifying the usages during the warranty stage into three types, Ref. [47] proposed a warranty model for ensuring the warranty-stage reliability; classifying the usages during the life cycle into three types, Ref. [48] proposed warranty and maintenance models for ensuring the life cycle reliability. From the perspective of applying models, both can be applied to ensure the reliability of the vehicle because vehicle usage can be measured by accumulated mileage. For some products with monitored working data, which include but are not limited to working/mission/task cycles, their reliabilities can be ensured by means of monitored working data because these data can measure the historical and remaining reliability performance. The related contribution can be found in recent studies in Refs. [49–55]. For some non-vehicle products, the accumulated operating time produced by monitored mission/task cycles can measure their usage. For example, by means of industrial software (IS) integrated with digital technologies, the manufacturers of some intelligent air conditioners can remotely obtain the time of the consumers' unit usage, and the consumers can also see the historical usage information from the use logs, which is stored in consumers' mobile devices. The manufacturers and consumers of the intelligent electric bicycle can obtain the usage data and the operating envelope of such bicycles by means of respective ISs. Their accumulated operating time, i.e., usages, can be measured as the sum of historical cycles. During the same calendar time, the lengths of all types of usage are different from one consumer to another consumer. These signals indicate that the usages of consumers are heterogeneous. Therefore, considering usages, customizing random warranty models and random maintenance models for ensuring the life cycle reliability of the product with monitored task cycles is a very practical topic. To the best of our knowledge, on the basis of heterogeneity, customizing the corresponding models has never been developed from the perspective of ensuring the life cycle reliability of the product with monitored task cycles.

In this paper, four intervals are designed to classify consumers' usages to screen the differences in usages. According to the screening results of the differences in usages, a random warranty model is customized to help manufacturers ensure the warranty-stage reliability of the product with monitored task cycles. Similarly, a random maintenance model is customized to help consumers ensure the post-warranty-stage reliability of products with monitored task cycles. For the former model, minimal repair services with four different areas of coverage are used to remove the warranty-stage failures. Due to considering usage heterogeneity, the former model is named a random free repair warranty with heterogeneity (RFRW-H). For the latter model, four periodic replacement models are used to ensure the post-warranty-stage reliability. Because four periodic replacement models are designed on the basis of both usage classification and each of 'whichever occurs first/last', the latter model is called a customized random periodic replacement first (CRPRF) model and a customized random periodic replacement last (CRPRL) model. The RFRW-H is modeled from the perspectives of cost and time measures, and the specific cases of cost and time measures are presented for modeling other random warranties. The CRPRF and CRPRL are modeled from the viewpoint of cost rate functions by using the renewal cycle. Some specific cases related to cost rate functions are presented for modeling other maintenances to ensure the post-warranty-stage reliability of the product.

The innovative points of this paper are listed below: ① By categorizing consumers' usages, a random warranty model is customized for controlling the warranty cost based on the differences in usages, which is different from Ref. [56] where the differences in usages were used to maintain warranty fairness rather than control the warranty cost. ② In the case of categorizing consumers' usages, random replacement first and last are customized for ensuring the post-warranty-stage reliability, which differs from Ref. [57] where random replacement models to ensure the post-warranty-stage reliability were customized based on cases of warranty expiries rather than usages.

The study's structure is as follows. Section 2 customizes a random warranty with heterogeneity by screening the differences in usages and models the related warranty from the viewpoints of cost and time measures. In Section 3, two types of maintenance models are customized to ensure the post-warranty-stage reliability, the cost rate functions of both are presented from the consumers' perspectives, and some of the specific cases are obtained for modeling other maintenance models. Section 4 analyzes the sensitivities of key parameters related to the models. Section 5 concludes the paper.

2. Random Warranty Model Customizing Basing on Usage Heterogeneity

The assumptions of this study are provided as: the product implements tasks at cycles with variable lengths, called tasks cycles, and the cycles Y_i of the ith ($i = 1, 2, \ldots$) task are independent and identically distributed random variables with a memory-less distribution function given by $G(y) = \Pr\{Y_i < y\}$, where the symbol Pr represents the probability that the cycles Y_i are completed before y, similarly hereinafter; the distribution function $F(x)$ of the time X to first failure is given by $F(x) = 1 - \exp\left(-\int_0^x \lambda(u)\mathrm{d}u\right)$ with the failure rate function $\lambda(u) = \alpha(u)^\beta$, wherein α ($\alpha > 0$) and β ($\beta \geq 1$) are two parameters; and the time to repair/replacement is negligible.

2.1. The Customization of Random Warranty

Let m and n be two natural numbers, respectively, and let w_1, w_2, and w_3 be three time thresholds, respectively, which satisfy $0 < w_1 < w_2 < w_3$. Using them, the terms of a random warranty are listed below:

- *If the operating time of the product reaches the time threshold w_3 before the end of* mth *task cycle, then the warranty responsibility expires at w_3;*
- *If the operating time of the product does not reach the time threshold w_3 until the end of the* mth *task cycle, then the product reliability is ensured by any of the following extended terms:* ① *the minimal repair with coverage w_3 is triggered to ensure the product reliability if the* mth *task cycle ends in interval $[0,\ w_1)$; the minimal repair with coverage $w_3 - w_1$ is triggered to ensure the product reliability if the* mth *task cycle ends in interval $[w_1,\ w_2)$;* ③ *the minimal repair with coverage consisting of time span $w_3 - w_2$ and the end of the* nth *task cycle, whichever occurs first, is triggered to ensure the product reliability if the* mth *task cycle ends in interval $[w_2,\ w_3)$.*

Notable: ① setting four different areas of coverage, which are $[w_3,\ +\infty)$, $[0,\ w_1)$, $[w_1,\ w_2)$, and $[w_2,\ w_3)$, aims to classify the heterogeneity of the consumers' usages, which can be measured by the different lengths of the operating time S_m ($S_m = \sum\limits_{i=1}^{m} Y_i$) resulting from the end of the mth task cycle; ② based on four cases where the operating time S_m falls into four different coverages, customizing four warranty terms, which are listed as: the warranty service, which expires at w_3, the minimal repair with the coverage w_3, the minimal repair with the coverage $w_3 - w_1$, as well as the minimal repair with a coverage consisting of the time span $w_3 - w_2$ and the end of the nth task cycle. Therefore, this warranty model is named a random free repair warranty with heterogeneity (RFRW-H).

To easily model RFRW-H, hereinafter, the above four warranty terms will be called a non-extended term (i.e., the first term) and three extended terms (i.e., the last three terms). Let s_m be a realization of S_m. Then, the operating time S_m satisfies the distribution and reliability functions, which are given by

$$G^{(m)}(s_m) = \Pr\{S_m < s_m\} = \int_0^{s_m} G^{(m-1)}(s_m - u)\mathrm{d}G(u) \text{ and } \overline{G}^{(m)}(s_m) = 1 - \int_0^{s_m} G^{(m-1)}(s_m - u)\mathrm{d}G(u)$$

Similarly, the operating time S_n produced by the nth task cycle end has the related functions $G^{(n)}(s_n)$ and $\overline{G}^{(n)}(s_n)$, which will be used hereinafter.

2.2. Service Measures of the RFRW-H

In this section, two service measures of the RFRW-H, i.e., the warranty service cost and time, are derived as below.

2.2.1. The Warranty Service Cost of the RFRW-H

The event that triggers the above four terms can be modeled by four in-equations, which are $w_3 \leq S_m$, $S_m < w_1$, $w_1 \leq S_m < w_2$, and $w_2 \leq S_m < w_3$. Because S_m satisfies $\overline{G}^{(m)}(s_m)$ and $G^{(m)}(s_m)$, the probabilities that respective terms are triggered can be given by

$$P = \begin{cases} \overline{G}^{(m)}(w_3) & if\ w_3 \le S_m \\ G^{(m)}(w_1) & if\ S_m < w_1 \\ G^{(m)}(w_2) - G^{(m)}(w_1) & if\ w_1 \le S_m < w_2 \\ G^{(m)}(w_3) - G^{(m)}(w_2) & if\ w_2 \le S_m < w_3 \end{cases} \tag{1}$$

where $G^{(m)}(w_2) - G^{(m)}(w_1)$ and $G^{(m)}(w_3) - G^{(m)}(w_2)$ can be obtained by means of $\Pr\{w_1 < s_m < w_2\}$ and $\Pr\{w_2 < s_m < w_3\}$.

Summing up these four probabilities, their sum is clearly equal to 1.

In the case of using $\lambda(u)$, the failure rate functions at S_m can be presented by $\lambda(S_m + u)$. Let c_m be the unit repair cost of minimal repair. By means of $\lambda(u)$ and $\lambda(S_m + u)$, the warranty service costs $WSC(S_m)$ related to the last three terms are measured as

$$WSC(S_m) = c_m \times \begin{cases} \int_0^{S_m} \lambda(u)\mathrm{d}u + \int_0^{w_3} \lambda(S_m + u)\mathrm{d}u & if\ S_m < w_1 \\ \int_0^{S_m} \lambda(u)\mathrm{d}u + \int_0^{w_2 - w_1} \lambda(S_m + u)\mathrm{d}u & if\ w_1 \le S_m < w_2 \\ \int_0^{S_m} \lambda(u)\mathrm{d}u + \int_0^{w_3 - w_2} \overline{G}^{(n)}(s_n)\lambda(S_m + s_n)\mathrm{d}s_n & if\ w_2 \le S_m < w_3 \end{cases} \tag{2}$$

where $\int_0^{w_3 - w_2} \overline{G}^{(n)}(s_n)\lambda(S_m + s_n)\mathrm{d}s_n$ can be obtained by computing $\int_0^{w_3-w_2} \left(\int_0^{s_n} \lambda(S_m + u)\mathrm{d}u\right)\mathrm{d}G^{(n)}(s_n) + \overline{G}^{(n)}(w_3 - w_2)\int_0^{w_3-w_2} \lambda(S_m + u)\mathrm{d}u$.

The warranty service cost WSC related to the first term is presented by

$$WSC = c_m \int_0^{w_3} \lambda(u)\mathrm{d}u \tag{3}$$

The probabilities that four terms are triggered have been offered by P in Equation (1). By means of these probabilities, the total warranty service cost WSC_T of the RFRW-H is measured as

$$\begin{aligned} WSC_T &= c_m \left(\begin{array}{l} \overline{G}^{(m)}(w_3)\int_0^{w_3} \lambda(u)\mathrm{d}u + \int_0^{w_1} \left(\int_0^{s_m} \lambda(u)\mathrm{d}u + \int_0^{w_3} \lambda(s_m + u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) \\ +\int_{w_1}^{w_2} \left(\int_0^{s_m} \lambda(u)\mathrm{d}u + \int_0^{w_2 - w_1} \lambda(s_m + u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) + \\ \int_{w_2}^{w_3} \left(\int_0^{s_m} \lambda(u)\mathrm{d}u + \int_0^{w_3 - w_2} \overline{G}^{(n)}(s_n)\lambda(s_m + s_n)\mathrm{d}s_n\right)\mathrm{d}G^{(m)}(s_m) \end{array} \right) \\ &= c_m \left(\begin{array}{l} \int_0^{w_3} \overline{G}^{(m)}(s_m)\lambda(s_m)\mathrm{d}s_m + \int_0^{w_1} \left(\int_0^{w_3} \lambda(s_m + u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) + \\ \int_{w_1}^{w_2} \left(\int_0^{w_2 - w_1} \lambda(s_m + u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) + \int_{w_2}^{w_3} \left(\int_0^{w_3 - w_2} \overline{G}^{(n)}(s_n)\lambda(s_m + s_n)\mathrm{d}s_n\right)\mathrm{d}G^{(m)}(s_m) \end{array} \right) \end{aligned} \tag{4}$$

2.2.2. The Warranty Service Time of the RFRW-H

The warranty service time WST related to four terms is measured as

$$WST = \begin{cases} w_3 & if\ w_3 \le S_m \\ S_m + w_3 & if\ S_m < w_1 \\ S_m + w_2 - w_1 & if\ w_1 \le S_m < w_2 \\ S_m + \int_0^{w_3 - w_2} \overline{G}^{(n)}(s_n)\mathrm{d}s_n & if\ w_2 \le S_m < w_3 \end{cases} \tag{5}$$

where $\int_0^{w_3 - w_2} \overline{G}^{(n)}(s_n)\mathrm{d}s_n$ can be obtained by calculating $\int_0^{w_3 - w_2} s_n \mathrm{d}G^{(n)}(s_n) + \overline{G}^{(n)}(w_3 - w_2)(w_3 - w_2)$.

By means of P in Equation (1), the total warranty service time WS_T of the RFRW-H is similarly given by

$$\begin{aligned} WS_T &= \overline{G}^{(m)}(w_3)w_3 + \int_0^{w_1} (s_m + w_3)\mathrm{d}G^{(m)}(s_m) + \int_{w_1}^{w_2} (s_m + w_2 - w_1)\mathrm{d}G^{(m)}(s_m) + \int_{w_2}^{w_3} \left(s_m + \int_0^{w_3 - w_2} \overline{G}^{(n)}(s_n)\mathrm{d}s_n\right)\mathrm{d}G^{(m)}(s_m) \\ &= \int_0^{w_3} \overline{G}^{(m)}(s_m)\mathrm{d}s_m + G^{(m)}(w_1)w_3 + \left(G^{(m)}(w_2) - G^{(m)}(w_1)\right)(w_2 - w_1) + \left(G^{(m)}(w_3) - G^{(m)}(w_2)\right)\int_0^{w_3 - w_2} \overline{G}^{(n)}(s_n)\mathrm{d}s_n \end{aligned} \tag{6}$$

2.3. Specific Models of the RFRW-H

Obviously, RFRW-H is a generalized warranty model. This signals that the RFRW-H includes some underlying variant models. By means of some simple algebraic operations, the variant models of the RFRW-H can be mined below.

Specific model 1: when $m \to \infty$, the model WSC_T of Equation (4) can be reduced to:

$$\lim_{m\to\infty} WSC_T = c_m \int_0^{w_3} \lambda(s_m) \mathrm{d}s_m \tag{7}$$

where the heterogeneity of usages is removed.

$m \to \infty$ signals that the end of the mth task cycle no longer plays a role in identifying the heterogeneities of consumer usages, and all extended terms are removed. Therefore, $m \to \infty$ reduces the RFRW-H to a classic FRW model with a total warranty service cost given by Equation (7). Similarly, the total warranty service time $\lim_{m\to\infty} WS_T$ of the FRW is obtained by calculating the limit; i.e.,

$$\lim_{m\to\infty} WS_T = w_3 \tag{8}$$

Specific model 2: when $n \to \infty$, the model WSC_T of Equation (4) can be reduced to:

$$\lim_{n\to\infty} WSC_T = c_m \left(\begin{array}{c} \int_0^{w_3} \overline{G}^{(m)}(s_m)\lambda(s_m)\mathrm{d}s_m + \int_0^{w_1} \left(\int_0^{w_3} \lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) + \\ \int_{w_1}^{w_2} \left(\int_0^{w_2-w_1} \lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) + \int_{w_2}^{w_3} \left(\int_0^{w_3-w_2} \lambda(s_m+s_n)\mathrm{d}s_n\right)\mathrm{d}G^{(m)}(s_m) \end{array} \right) \tag{9}$$

$n \to \infty$ signals that the end of the nth task cycle no longer plays a role in limiting the warranty coverage and does not affect any of the other warranty terms. Therefore, $n \to \infty$ reduces the RFRW-H to a free repair warranty with heterogeneity (FRW-H), the total warranty service cost of which is given by Equation (9). Furthermore, the total warranty service time $\lim_{n\to\infty} WS_T$ of the FRW-H is measured as THE EQUATION

$$\lim_{n\to\infty} WS_T = G^{(m)}(w_1)w_3 + \left(G^{(m)}(w_2) - G^{(m)}(w_1)\right)(w_2 - w_1) + \left(G^{(m)}(w_3) - G^{(m)}(w_2)\right)(w_3 - w_2) + \int_0^{w_3} \overline{G}^{(m)}(s_m)\mathrm{d}s_m \tag{10}$$

Specific model 3: when $w_1 \to 0$, $w_2 \to 0$, and $n \to 0$, the model WSC_T of Equation (4) can be reduced to:

$$\lim_{w_1, w_2,\, n\to 0} WSC_T = c_m \int_0^{w_3} \overline{G}^{(m)}(s_m)\lambda(s_m)\mathrm{d}s_m \tag{11}$$

which is the same as the result in Ref. [42] and in which the heterogeneity of usages is removed.

$w_1 \to 0$, $w_2 \to 0$, and $n \to 0$ signal that the coverage of the last three terms is completely removed and never appears, which means that the RFRW-H model is reduced to an FRW model with two warranty limits, i.e., w_3 and the end of the mth task cycle, called a 2DFRWF in Ref. [42]. Therefore, the model given by Equation (11) is the total warranty service cost of the 2DFRWF. Similarly, the total warranty service time $\lim_{\substack{w_1 \text{ and } w_2 \to 0 \\ n \to 0}} WS_T$ of the 2DFRWF is given by

$$\lim_{\substack{w_1,\, w_2 \to 0 \\ n \to 0}} WS_T = \int_0^{w_3} \overline{G}^{(m)}(s_m)\mathrm{d}s_m \tag{12}$$

3. Random Maintenance Model Customizing Based on Usage Heterogeneity

The above RFRW-H has been customized by identifying the heterogeneity of usages and dividing usage heterogeneity into different categories. Enlightened by such a thought, two consumers' random maintenance models will be customized to ensure the post-warranty-stage reliability by assuming that the FRW-H in Equation (9) is used to ensure the warranty-stage reliability.

3.1. The Customization of Random Maintenance Model 1

Denote decision variables T and N by a post-warranty service time and a natural number; denote M_1, M_2, and M_3 by three natural numbers where $0 < M_1 < M_2 < M_3$. On the basis of the above usage categories, the terms of random maintenance model 1 are listed as follows.

- *The product the FRW-H of which expires at the time threshold w_3 before the end of the mth task cycle will undergo the minimal repair schedule with coverage consisting of the post-warranty service time T and the end of the Nth task cycle, whichever occurs first;*
- *The product that undergoes the first type of extended term will undergo the minimal repair schedule with coverage consisting of the post-warranty service time T and the end of the $(N + M_1)$th task cycle, whichever occurs first;*
- *The product that undergoes the second type of extended terms will undergo the minimal repair schedule with coverage consisting of the post-warranty service time T and the end of the $(N + M_2)$th task cycle, whichever occurs first;*
- *The product that undergoes the third type of extended terms will undergo the minimal repair schedule with coverage consisting of the post-warranty service time T and the end of the $(N + M_3)$th task cycle, whichever occurs first.*

Notably, four replacement coverages are considered in such a model, where the relationship among them is in ascending order (see Ref. [43]). In addition, periodic replacement is used as a key term to ensure post-warranty-stage reliability, and 'whichever occurs first' is used as a constraint. Therefore, such a random maintenance model is referred to as a customized random periodic replacement first (CRPRF) model.

In the reliability field, there exist two types of objective functions, which are the expected/average cost rate (see Refs. [44–48,58–60]) and availability (see Refs. [61–69]), which are based on the renewal process in Ref. [70], and the renewal process related to Markov processes can be found in Refs. [71,72]. To seek optimal solutions of decision variables T and N, i.e., seeking T^* and N^*, the average cost rate model will be built by defining the time span from a new identical product sold with an FRW-H to replacing it with another new identical product sold with an FRW-H is a renewal cycle, which can be obtained by calculating two types of measures; i.e., total service cost during the renewal cycle and total service time during the renewal cycle.

3.1.1. Total Service Cost during the Renewal Cycle

In the case of using FRW-H, the failure rate functions $r(u)$ at all expiries of the FRW-H can be given by

$$r(u) = \begin{cases} \lambda(w_3 + u) & if\ w_3 \le S_m \\ \lambda(S_m + w_3 + u) & if\ S_m < w_1 \\ \lambda(S_m + w_2 - w_1 + u) & if\ w_1 \le S_m < w_2 \\ \lambda(S_m + w_3 - w_2 + u) & if\ w_2 \le S_m < w_3 \end{cases} \tag{13}$$

Let c_M be the unit failure cost including the unit minimal repair, where $c_M > c_m$. Then, under the case of using CRPRF, the post-warranty service cost $PWSC_f(T, N)$ is measured by

$$PWSC_f(T, N) = c_M \times \begin{cases} \int_0^T \overline{G}^{(N)}(u)\lambda(w_3 + u)\mathrm{d}u & if\ w_3 \le S_m \\ \int_0^T \overline{G}^{(N+M_1)}(u)\lambda(S_m + w_3 + u)\mathrm{d}u & if\ S_m < w_1 \\ \int_0^T \overline{G}^{(N+M_2)}(u)\lambda(S_m + w_2 - w_1 + u)\mathrm{d}u & if\ w_1 \le S_m < w_2 \\ \int_0^T \overline{G}^{(N+M_3)}(u)\lambda(S_m + w_3 - w_2 + u)\mathrm{d}u & if\ w_2 \le S_m < w_3 \end{cases} \tag{14}$$

where each term can be obtained by means of a similar method to calculate $\int_0^{w_3-w_2} \overline{G}^{(n)}(s_n)\lambda(S_m+s_n)\mathrm{d}s_n$ in Equation (2).

By means of P in Equation (1), the total post-warranty service cost $PWSC_f^T(T,N)$ during the renewal cycle is modeled as THE EQUATION

$$PWSC_f^T(T,N) = c_M\begin{pmatrix} \overline{G}^{(m)}(w_3)\int_0^T \overline{G}^{(N)}(u)\lambda(w_3+u)\mathrm{d}u + \int_0^{w_1}\left(\int_0^T \overline{G}^{(N+M_1)}(u)\lambda(s_m+w_3+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) + \\ \int_{w_1}^{w_2}\left(\int_0^T \overline{G}^{(N+M_2)}(u)\lambda(s_m+w_2-w_1+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) + \\ \int_{w_2}^{w_3}\left(\int_0^T \overline{G}^{(N+M_3)}(u)\lambda(s_m+w_3-w_2+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) \end{pmatrix} + c_r \tag{15}$$

where c_r is the unit replacement cost.

By algebraic operation, the total service cost $SC_f^T(T,N)$ during the renewal cycle is modeled as

$$\begin{aligned} SC_f^T(T,N) = \; & c_f\begin{pmatrix} \int_0^{w_3} \overline{G}^{(m)}(s_m)\lambda(s_m)\mathrm{d}s_m + \int_0^{w_1}\left(\int_0^{w_3}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) + \\ \int_{w_1}^{w_2}\left(\int_0^{w_2-w_1}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) + \int_{w_2}^{w_3}\left(\int_0^{w_3-w_2}\overline{G}^{(n)}(s_n)\lambda(s_m+s_n)\mathrm{d}s_n\right)\mathrm{d}G^{(m)}(s_m) \end{pmatrix} + \\ & c_M\begin{pmatrix} \overline{G}^{(m)}(w_3)\int_0^T \overline{G}^{(N)}(u)\lambda(w_3+u)\mathrm{d}u + \int_0^{w_1}\left(\int_0^T \overline{G}^{(N+M_1)}(u)\lambda(s_m+w_3+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) + \\ \int_{w_1}^{w_2}\left(\int_0^T \overline{G}^{(N+M_2)}(u)\lambda(s_m+w_2-w_1+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) + \\ \int_{w_2}^{w_3}\left(\int_0^T \overline{G}^{(N+M_3)}(u)\lambda(s_m+w_3-w_2+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) \end{pmatrix} + c_r \end{aligned} \tag{16}$$

where the first term represents the total failure cost, which can be obtained by replacing c_m in $\lim\limits_{n\to\infty} WSC_T$ of Equation (9) as the unit failure cost c_f.

3.1.2. Total Service Time during the Renewal Cycle

When the product sold with an FRW-H is replaced in the form of the CRPRF, the related post-warranty service time $PWST_f(T,N)$ can be expressed as

$$PWST_f(T,N) = \begin{cases} \int_0^T \overline{G}^{(N)}(u)\mathrm{d}u & if\ w_3 \le S_m \\ \int_0^T \overline{G}^{(N+M_1)}(u)\mathrm{d}u & if\ S_m < w_1 \\ \int_0^T \overline{G}^{(N+M_2)}(u)\mathrm{d}u & if\ w_1 \le S_m < w_2 \\ \int_0^T \overline{G}^{(N+M_3)}(u)\mathrm{d}u & if\ w_2 \le S_m < w_3 \end{cases} \tag{17}$$

By means of P in Equation (1), the total post-warranty service time $PWST_f^T(T,N)$ during the renewal cycle is modeled as

$$PWST_f^T(T,N) = \begin{pmatrix} \overline{G}^{(m)}(w_3)\int_0^T \overline{G}^{(N)}(u)\mathrm{d}u + G^{(m)}(w_1)\int_0^T \overline{G}^{(N+M_1)}(u)\mathrm{d}u + \\ \left(G^{(m)}(w_2) - G^{(m)}(w_1)\right)\int_0^T \overline{G}^{(N+M_2)}(u)\mathrm{d}u + \left(G^{(m)}(w_3) - G^{(m)}(w_2)\right)\int_0^T \overline{G}^{(N+M_3)}(u)\mathrm{d}u \end{pmatrix} \tag{18}$$

By summing the total warranty service time $\lim\limits_{n\to\infty} WS_T$ given by Equation (10) and the total post-warranty service time $PWST_f^T(T,N)$ given by Equation (18), the total service time $ST_f^T(T,N)$ during the renewal cycle can be modeled as

$$\begin{aligned} ST_f^T(T,N) &= \begin{pmatrix} \int_0^{w_3} \overline{G}^{(m)}(s_m)\mathrm{d}s_m + G^{(m)}(w_1)w_3 + \left(G^{(m)}(w_2) - G^{(m)}(w_1)\right)(w_2 - w_1) + \\ \left(G^{(m)}(w_3) - G^{(m)}(w_2)\right)\int_0^{w_3-w_2} \overline{G}^{(n)}(s_n)\mathrm{d}s_n \end{pmatrix} + \\ & \begin{pmatrix} \overline{G}^{(m)}(w_3)\int_0^T \overline{G}^{(N)}(u)\mathrm{d}u + G^{(m)}(w_1)\int_0^T \overline{G}^{(N+M_1)}(u)\mathrm{d}u + \\ \left(G^{(m)}(w_2) - G^{(m)}(w_1)\right)\int_0^T \overline{G}^{(N+M_2)}(u)\mathrm{d}u + \left(G^{(m)}(w_3) - G^{(m)}(w_2)\right)\int_0^T \overline{G}^{(N+M_3)}(u)\mathrm{d}u \end{pmatrix} \\ &= \begin{pmatrix} \int_0^{w_3} \overline{G}^{(m)}(s_m)\mathrm{d}s_m + \overline{G}^{(m)}(w_3)\int_0^T \overline{G}^{(N)}(u)\mathrm{d}u + \\ G^{(m)}(w_1)\left(\int_0^T \overline{G}^{(N+M_1)}(u)\mathrm{d}u + w_3\right) + \left(G^{(m)}(w_2) - G^{(m)}(w_1)\right)\left(w_2 - w_1 + \int_0^T \overline{G}^{(N+M_2)}(u)\mathrm{d}u\right) + \\ \left(G^{(m)}(w_3) - G^{(m)}(w_2)\right)\left(\int_0^{w_3-w_2} \overline{G}^{(n)}(s_n)\mathrm{d}s_n + \int_0^T \overline{G}^{(N+M_3)}(u)\mathrm{d}u\right) \end{pmatrix} \end{aligned} \tag{19}$$

3.1.3. Average Cost Rate

By algebraic operation, the average cost rate $CR_f(T,N)$ can be given by

$$CR_f(T,N)=\frac{c_f\left(\begin{array}{l}\int_0^{w_3}\overline{G}^{(m)}(s_m)\lambda(s_m)\mathrm{d}s_m+\int_0^{w_1}(\int_0^{w_3}\lambda(s_m+u)\mathrm{d}u)\mathrm{d}G^{(m)}(s_m)+\\ \int_{w_1}^{w_2}\left(\int_0^{w_2-w_1}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\int_{w_2}^{w_3}\left(\int_0^{w_3-w_2}\overline{G}^{(n)}(s_n)\lambda(s_m+s_n)\mathrm{d}s_n\right)\mathrm{d}G^{(m)}(s_m)\end{array}\right)+c_r+c_M\left(\begin{array}{l}\overline{G}^{(m)}(w_3)\int_0^T\overline{G}^{(N)}(u)\lambda(w_3+u)\mathrm{d}u+\int_0^{w_1}\left(\int_0^T\overline{G}^{(N+M_1)}(u)\lambda(s_m+w_3+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\\ \int_{w_1}^{w_2}\left(\int_0^T\overline{G}^{(N+M_2)}(u)\lambda(s_m+w_2-w_1+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\\ \int_{w_2}^{w_3}\left(\int_0^T\overline{G}^{(N+M_3)}(u)\lambda(s_m+w_3-w_2+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)\end{array}\right)}{\begin{array}{l}\int_0^{w_3}\overline{G}^{(m)}(s_m)\mathrm{d}s_m+\overline{G}^{(m)}(w_3)\int_0^T\overline{G}^{(N)}(u)\mathrm{d}u+\\ G^{(m)}(w_1)\left(\int_0^T\overline{G}^{(N+M_1)}(u)\mathrm{d}u+w_3\right)+\left(G^{(m)}(w_2)-G^{(m)}(w_1)\right)\left(w_2-w_1+\int_0^T\overline{G}^{(N+M_2)}(u)\mathrm{d}u\right)+\\ \left(G^{(m)}(w_3)-G^{(m)}(w_2)\right)\left(\int_0^{w_3-w_2}\overline{G}^{(n)}(s_n)\mathrm{d}s_n+\int_0^T\overline{G}^{(N+M_3)}(u)\mathrm{d}u\right)\end{array}} \quad (20)$$

3.2. The Customization of Random Maintenance Model 2

By revising 'whichever occurs first' as 'whichever occurs last', the above CRPRF can be rewritten as a customized random periodic replacement last (CRPRL) model. Similar to obtaining the average cost rate $ACR_f(T,N)$ of the CRPRF, we next derive the average cost rate of the CRPRL.

3.2.1. Total Service Cost during the Renewal Cycle

When the CRPRL is used to manage the post-warranty-stage reliability of the product sold with the FRW-H model, the related post-warranty service cost $PWSC_l(T,N)$ is given by

$$PWSC_l(T,N)=c_M\times\begin{cases}\int_0^T\lambda(w_3+u)\mathrm{d}u+\int_0^T\overline{G}^{(N)}(u)\lambda(w_3+u)\mathrm{d}u & if\ w_3\le S_m\\ \int_0^T\lambda(S_m+w_3+u)\mathrm{d}u+\int_0^T\overline{G}^{(N+M_1)}(u)\lambda(S_m+w_3+u)\mathrm{d}u & if\ S_m<w_1\\ \int_0^T\lambda(S_m+w_2-w_1+u)\mathrm{d}u+\int_0^T\overline{G}^{(N+M_2)}(u)\lambda(S_m+w_2-w_1+u)\mathrm{d}u & if\ w_1\le S_m<w_2\\ \int_0^T\lambda(S_m+w_3-w_2+u)\mathrm{d}u+\int_0^T\overline{G}^{(N+M_3)}(u)\lambda(S_m+w_3-w_2+u)\mathrm{d}u & if\ w_2\le S_m<w_3\end{cases} \quad (21)$$

where the first term $\int_0^T\lambda(w_3+u)\mathrm{d}u+\int_0^T\overline{G}^{(N)}(u)\lambda(w_3+u)\mathrm{d}u=G^{(N)}(T)\int_0^T\lambda(w_3+u)\mathrm{d}u+\int_T^\infty\left(\int_0^s\lambda(w_3+u)\mathrm{d}u\right)\mathrm{d}G^{(N)}(s)$ and other terms can be obtained by means of a similar method.

By means of P in Equation (1), the total post-warranty service cost $PWSC_l^T(T,N)$ of the CRPRL is modeled as

$$PWSC_l^T(T,N)=c_M\left(\begin{array}{l}\overline{G}^{(m)}(w_3)\left(\int_0^T\lambda(w_3+u)\mathrm{d}u+\int_T^\infty\overline{G}^{(N)}(u)\lambda(w_3+u)\mathrm{d}u\right)+\\ \int_0^{w_1}\left(\int_0^T\lambda(s_m+w_3+u)\mathrm{d}u+\int_T^\infty G^{(N+M_1)}(u)\lambda(s_m+w_3+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\\ \int_{w_1}^{w_2}\left(\int_0^T\lambda(s_m+w_2-w_1+u)\mathrm{d}u+\int_T^\infty\overline{G}^{(N+M_2)}(u)\lambda(s_m+w_2-w_1+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\\ \int_{w_2}^{w_3}\left(\int_0^T\lambda(s_m+w_3-w_2+u)\mathrm{d}u+\int_T^\infty\overline{G}^{(N+M_3)}(u)\lambda(s_m+w_3-w_2+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)\end{array}\right)+c_r \quad (22)$$

Similarly, the total service cost $SC_l^T(T,N)$ during the renewal cycle is modeled as

$$SC_l^T(T,N)=c_f\left(\begin{array}{l}\int_0^{w_3}\overline{G}^{(m)}(s_m)\lambda(s_m)\mathrm{d}s_m+\int_0^{w_1}(\int_0^{w_3}\lambda(s_m+u)\mathrm{d}u)\mathrm{d}G^{(m)}(s_m)+\\ \int_{w_1}^{w_2}\left(\int_0^{w_2-w_1}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\int_{w_2}^{w_3}\left(\int_0^{w_3-w_2}\lambda(s_m+s_n)\mathrm{d}s_n\right)\mathrm{d}G^{(m)}(s_m)\end{array}\right)+ c_M\left(\begin{array}{l}\overline{G}^{(m)}(w_3)\left(\int_0^T\lambda(w_3+u)\mathrm{d}u+\int_T^\infty\overline{G}^{(N)}(u)\lambda(w_3+u)\mathrm{d}u\right)+\\ \int_0^{w_1}\left(\int_0^T\lambda(s_m+w_3+u)\mathrm{d}u+\int_T^\infty G^{(N+M_1)}(u)\lambda(s_m+w_3+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\\ \int_{w_1}^{w_2}\left(\int_0^T\lambda(s_m+w_2-w_1+u)\mathrm{d}u+\int_T^\infty\overline{G}^{(N+M_2)}(u)\lambda(s_m+w_2-w_1+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\\ \int_{w_2}^{w_3}\left(\int_0^T\lambda(s_m+w_3-w_2+u)\mathrm{d}u+\int_T^\infty\overline{G}^{(N+M_3)}(u)\lambda(s_m+w_3-w_2+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)\end{array}\right)+c_r \quad (23)$$

3.2.2. Total Service Time during the Renewal Cycle

When the CRPRL is used as a random replacement model of the product through FRW-H, the related post-warranty service time $PWST_l(T,N)$ can be expressed as

$$TM_l(N,T)=\begin{cases} T+\int_T^{\infty}\overline{G}^{(N)}(u)\mathrm{d}u & if\ w_3\le S_m \\ T+\int_T^{\infty}\overline{G}^{(N+M_1)}(u)\mathrm{d}u & if\ S_m<w_1 \\ T+\int_T^{\infty}\overline{G}^{(N+M_2)}(u)\mathrm{d}u & if\ w_1\le S_m<w_2 \\ T+\int_T^{\infty}\overline{G}^{(N+M_3)}(u)\mathrm{d}u & if\ w_2\le S_m<w_3 \end{cases} \tag{24}$$

where the first term $T+\int_T^{\infty}G^{(N)}(u)\mathrm{d}u$ can be obtained by calculating $G^{(N)}(T)T+\int_T^{\infty}u\mathrm{d}G^{(N)}(u)$, and other terms can be obtained by means of a similar method.

By means of P in Equation (1), the total post-warranty service time $PWST_l^T(T,N)$ during the renewal cycle is modeled as

$$\begin{aligned} PWST_l^T(T,N) &= \overline{G}^{(m)}(w_3)\Big(T+\int_T^{\infty}\overline{G}^{(N)}(u)\mathrm{d}u\Big)+G^{(m)}(w_1)\Big(T+\int_T^{\infty}\overline{G}^{(N+M_1)}(u)\mathrm{d}u\Big)+ \\ &\Big(G^{(m)}(w_2)-G^{(m)}(w_1)\Big)\Big(T+\int_T^{\infty}\overline{G}^{(N+M_2)}(u)\mathrm{d}u\Big)+\Big(G^{(m)}(w_3)-G^{(m)}(w_2)\Big)\Big(T+\int_T^{\infty}\overline{G}^{(N+M_3)}(u)\mathrm{d}u\Big) \\ &= T+\overline{G}^{(m)}(w_3)\int_T^{\infty}\overline{G}^{(N)}(u)\mathrm{d}u+G^{(m)}(w_1)\int_T^{\infty}\overline{G}^{(N+M_1)}(u)\mathrm{d}u+ \\ &\Big(G^{(m)}(w_2)-G^{(m)}(w_1)\Big)\int_T^{\infty}\overline{G}^{(N+M_2)}(u)\mathrm{d}u+\Big(G^{(m)}(w_3)-G^{(m)}(w_2)\Big)\int_T^{\infty}\overline{G}^{(N+M_3)}(u)\mathrm{d}u \end{aligned} \tag{25}$$

By summing the total warranty service time WS_T and the total post-warranty service time $PWST_l^T(T,N)$, the total service time $ST_l^T(T,N)$ during the renewal cycle can be modeled as

$$ST_l^T(T,N)=\left(\begin{array}{l} \int_0^{w_3}\overline{G}^{(m)}(s_m)\mathrm{d}s_m+G^{(m)}(w_1)w_3+\Big(G^{(m)}(w_2)-G^{(m)}(w_1)\Big)(w_2-w_1)+ \\ \Big(G^{(m)}(w_3)-G^{(m)}(w_2)\Big)\int_0^{w_3-w_2}\overline{G}^{(n)}(s_n)\mathrm{d}s_n \end{array}\right)+ \\ \left(\begin{array}{l} T+\overline{G}^{(m)}(w_3)\int_T^{\infty}\overline{G}^{(N)}(u)\mathrm{d}u+G^{(m)}(w_1)\int_T^{\infty}\overline{G}^{(N+M_1)}(u)\mathrm{d}u+ \\ \Big(G^{(m)}(w_2)-G^{(m)}(w_1)\Big)\int_T^{\infty}\overline{G}^{(N+M_2)}(u)\mathrm{d}u+\Big(G^{(m)}(w_3)-G^{(m)}(w_2)\Big)\int_T^{\infty}\overline{G}^{(N+M_3)}(u)\mathrm{d}u \end{array}\right) \tag{26}$$

where the first term of the right-hand side of the first equation is the total warranty service time $\lim_{n\to\infty}WS_T$ given by Equation (10).

3.2.3. Average Cost Rate

By algebraic operation, the average cost rate $CR_l(T,N)$ of the CRPRL can be represented as

$$CR_l(T,N)=\frac{\begin{array}{l} c_f\left(\begin{array}{l} \int_0^{w_1}\big(\int_0^{w_3}\lambda(s_m+u)\mathrm{d}u\big)\mathrm{d}G^{(m)}(s_m)+\int_{w_1}^{w_2}\Big(\int_0^{w_2-w_1}\lambda(s_m+u)\mathrm{d}u\Big)\mathrm{d}G^{(m)}(s_m)+ \\ \int_{w_2}^{w_3}\Big(\int_0^{w_3-w_2}\lambda(s_m+s_n)\mathrm{d}s_n\Big)\mathrm{d}G^{(m)}(s_m)+\int_0^{w_3}\overline{G}^{(m)}(s_m)\lambda(s_m)\mathrm{d}s_m \end{array}\right)+c_r+ \\ c_M\left(\begin{array}{l} \overline{G}^{(m)}(w_3)\Big(\int_0^{T}\lambda(w_3+u)\mathrm{d}u+\int_T^{\infty}\overline{G}^{(N)}(u)\lambda(w_3+u)\mathrm{d}u\Big)+ \\ \int_0^{w_1}\Big(\int_0^{T}\lambda(s_m+w_3+u)\mathrm{d}u+\int_T^{\infty}\overline{G}^{(N+M_1)}(u)\lambda(s_m+w_3+u)\mathrm{d}u\Big)\mathrm{d}G^{(m)}(s_m)+ \\ \int_{w_1}^{w_2}\Big(\int_0^{T}\lambda(s_m+w_2-w_1+u)\mathrm{d}u+\int_T^{\infty}\overline{G}^{(N+M_2)}(u)\lambda(s_m+w_2-w_1+u)\mathrm{d}u\Big)\mathrm{d}G^{(m)}(s_m)+ \\ \int_{w_2}^{w_3}\Big(\int_0^{T}\lambda(s_m+w_3-w_2+u)\mathrm{d}u+\int_T^{\infty}\overline{G}^{(N+M_3)}(u)\lambda(s_m+w_3-w_2+u)\mathrm{d}u\Big)\mathrm{d}G^{(m)}(s_m) \end{array}\right) \end{array}}{\begin{array}{l} \left(\begin{array}{l} \int_0^{w_3}\overline{G}^{(m)}(s_m)\mathrm{d}s_m+G^{(m)}(w_1)w_3+\Big(G^{(m)}(w_2)-G^{(m)}(w_1)\Big)(w_2-w_1)+ \\ \Big(G^{(m)}(w_3)-G^{(m)}(w_2)\Big)\int_0^{w_3-w_2}\overline{G}^{(n)}(s_n)\mathrm{d}s_n \end{array}\right)+ \\ \left(\begin{array}{l} T+\overline{G}^{(m)}(w_3)\int_T^{\infty}\overline{G}^{(N)}(u)\mathrm{d}u+G^{(m)}(w_1)\int_T^{\infty}\overline{G}^{(N+M_1)}(u)\mathrm{d}u+ \\ \Big(G^{(m)}(w_2)-G^{(m)}(w_1)\Big)\int_T^{\infty}\overline{G}^{(N+M_2)}(u)\mathrm{d}u+\Big(G^{(m)}(w_3)-G^{(m)}(w_2)\Big)\int_T^{\infty}\overline{G}^{(N+M_3)}(u)\mathrm{d}u \end{array}\right) \end{array}} \tag{27}$$

3.3. The Specific Models of the Average Cost Rates

The above two average cost rates $CR_f(T,N)$ and $CR_l(T,N)$ have been derived by using the FRW-H as a warranty model and each of the CRPRF and CRPRL to ensure the life cycle reliability of the product with monitored task cycles. Similar to obtaining specific models of the RFRW-H, some specific models of the average cost rates are presented here.

Specific model A: when $m \to \infty$, the average cost rate $CR_f(T,N)$ is reduced to:

$$\lim_{m\to\infty} CR_f(T,N) = \frac{c_f \int_0^{w_3} \lambda(s_m)\mathrm{d}s_m + c_M \int_0^T \overline{G}^{(N)}(u)\lambda(w_3+u)\mathrm{d}u + c_r}{w_3 + \int_0^T \overline{G}^{(N)}(u)\mathrm{d}u} \tag{28}$$

$m \to \infty$ signals that the end of the cycle number (i.e., m) never occurs. This means that all extended terms are removed, and the FRW-H model is reduced to a classic FRW model, which has been mentioned above. In addition, $m \to \infty$ removes the last three replacement terms in the CRPRF, which signals that the CRPRF is reduced to a random periodic replacement first (RPRF) model in Ref. [42]. Therefore, the average cost rate $\lim_{m\to\infty} CR_f(T,N)$ of Equation (28) is a model where FRW and RPRF models are used to ensure the life cycle reliability. In this model, heterogeneity during the life cycle has been completely ignored; namely, usages are not classified (similarly hereinafter).

Specific model B: when $N \to 1$, the average cost rate $CR_f(T,N)$ is reduced to:

$$CR_f(T,1) = \frac{c_f\left(\begin{array}{l}\int_0^{w_3} \overline{G}^{(m)}(s_m)\lambda(s_m)\mathrm{d}s_m + \int_0^{w_1}\left(\int_0^{w_3}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) + \\ \int_{w_1}^{w_2}\left(\int_0^{w_2-w_1}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) + \int_{w_2}^{w_3}\left(\int_0^{w_3-w_2}\overline{G}^{(n)}(s_n)\lambda(s_m+s_n)\mathrm{d}s_n\right)\mathrm{d}G^{(m)}(s_m)\end{array}\right) + c_r + c_M\left(\begin{array}{l}\overline{G}^{(m)}(w_3)\int_0^T \overline{G}(u)\lambda(w_3+u)\mathrm{d}u + \int_0^{w_1}\left(\int_0^T \overline{G}^{(1+M_1)}(u)\lambda(s_m+w_3+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) + \\ \int_{w_1}^{w_2}\left(\int_0^T \overline{G}^{(1+M_2)}(u)\lambda(s_m+w_2-w_1+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) + \\ \int_{w_2}^{w_3}\left(\int_0^T \overline{G}^{(1+M_3)}(u)\lambda(s_m+w_3-w_2+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)\end{array}\right)}{\begin{array}{l}\int_0^{w_3}\overline{G}^{(m)}(s_m)\mathrm{d}s_m + \overline{G}^{(m)}(w_3)\int_0^T \overline{G}(u)\mathrm{d}u + \\ G^{(m)}(w_1)\left(\int_0^T \overline{G}^{(1+M_1)}(u)\mathrm{d}u + w_3\right) + \left(G^{(m)}(w_2) - G^{(m)}(w_1)\right)\left(w_2 - w_1 + \int_0^T \overline{G}^{(1+M_2)}(u)\mathrm{d}u\right) + \\ \left(G^{(m)}(w_3) - G^{(m)}(w_2)\right)\left(\int_0^{w_3-w_2}\overline{G}^{(n)}(s_n)\mathrm{d}s_n + \int_0^T \overline{G}^{(1+M_3)}(u)\mathrm{d}u\right)\end{array}} \tag{29}$$

$N \to 1$ signals that CRPRF is reduced to a special customized random periodic replacement first (CRPRF) model where the post-warranty service time T becomes a unique decision variable and other terms are still kept. Therefore, the average cost rate $CR_f(T,1)$ of Equation (29) is a model where FRW and special CRPRF models are used to ensure the life cycle reliability. In this model, heterogeneity during the life cycle is still considered.

Specific model C: when $M_1 \to 0$, $M_2 \to 0$, and $M_3 \to 0$, the average cost rate $CR_f(T,N)$ is simplified as

$$\lim_{M_1,M_2,M_3\to 0} CR_f(T,N) = \frac{c_f\left(\begin{array}{l}\int_0^{w_1}\left(\int_0^{w_3}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) + \int_{w_1}^{w_2}\left(\int_0^{w_2-w_1}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) + \\ \int_{w_2}^{w_3}\left(\int_0^{w_3-w_2}\lambda(s_m+s_n)\mathrm{d}s_n\right)\mathrm{d}G^{(m)}(s_m) + \int_0^{w_3}\overline{G}^{(m)}(s_m)\lambda(s_m)\mathrm{d}s_m\end{array}\right) + c_r + c_M\left(\begin{array}{l}\overline{G}^{(m)}(w_3)\int_0^T \overline{G}^{(N)}(u)\lambda(w_3+u)\mathrm{d}u + \int_0^{w_1}\left(\int_0^T \overline{G}^{(N)}(u)\lambda(s_m+w_3+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) + \\ \int_{w_1}^{w_2}\left(\int_0^T \overline{G}^{(N)}(u)\lambda(s_m+w_2-w_1+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s) + \\ \int_{w_2}^{w_3}\left(\int_0^T \overline{G}^{(N)}(u)\lambda(s_m+w_3-w_2+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)\end{array}\right)}{\begin{array}{l}G^{(m)}(w_1)\left(\int_0^T \overline{G}^{(N)}(u)\mathrm{d}u + w_3\right) + \left(G^{(m)}(w_2) - G^{(m)}(w_1)\right)\left(w_2 - w_1 + \int_0^T \overline{G}^{(N)}(u)\mathrm{d}u\right) \\ + \left(G^{(m)}(w_3) - G^{(m)}(w_2)\right)\left(w_3 - w_2 + \int_0^T \overline{G}^{(N)}(u)\mathrm{d}u\right) + \int_0^{w_3}\overline{G}^{(m)}(s_m)\mathrm{d}s_m + \overline{G}^{(m)}(w_3)\int_0^T \overline{G}^{(N)}(u)\mathrm{d}u\end{array}} \tag{30}$$

$M_1 \to 0$, $M_2 \to 0$, and $M_3 \to 0$ reduce the CRPRF to an RPRF in Ref. [42]. Therefore, the average cost rate $\lim_{M_1,M_2,M_3\to 0} CR_f(T,N)$ of Equation (30) is a model where FRW and RPRF models are used to ensure the life cycle reliability. In this model, all heterogeneities during the warranty stage are still kept, heterogeneity during the post-warranty stage has been completely ignored, and RPRF is a uniform model to ensure the post-warranty-stage reliability.

Specific model D: when $M_1 \to 0$, $M_2 \to 0$, $M_3 \to 0$, and $N \to 1$, the average cost rate $CR_f(T,N)$ is simplified as

$$\lim_{M_1,M_2,M_3\to 0} CR_f(T,1) = \frac{c_f\begin{pmatrix} \int_0^{w_1}\left(\int_0^{w_3}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\int_{w_1}^{w_2}\left(\int_0^{w_2-w_1}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+ \\ \int_{w_2}^{w_3}\left(\int_0^{w_3-w_2}\lambda(s_m+s_n)\mathrm{d}s_n\right)\mathrm{d}G^{(m)}(s_m)+\int_0^{w_3}\overline{G}^{(m)}(s_m)\lambda(s_m)\mathrm{d}s_m \end{pmatrix}+c_r+ c_M\begin{pmatrix} \overline{G}^{(m)}(w_3)\int_0^T\overline{G}^{(N)}(u)\lambda(w_3+u)\mathrm{d}u+\int_0^{w_1}\left(\int_0^T\overline{G}^{(N)}(u)\lambda(s_m+w_3+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+ \\ \int_{w_1}^{w_2}\left(\int_0^T\overline{G}^{(N)}(u)\lambda(s_m+w_2-w_1+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s)+ \\ \int_{w_2}^{w_3}\left(\int_0^T\overline{G}^{(N)}(u)\lambda(s_m+w_3-w_2+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) \end{pmatrix}}{\int_0^{w_3}\overline{G}^{(m)}(s_m)\mathrm{d}s_m+G^{(m)}(w_1)w_3+\left(G^{(m)}(w_2)-G^{(m)}(w_1)\right)(w_2-w_1)+ \left(G^{(m)}(w_3)-G^{(m)}(w_2)\right)(w_3-w_2)+\int_0^T\overline{G}^{(N)}(u)\mathrm{d}u} \tag{31}$$

In this model, all usage heterogeneities during the warranty stage are still kept, heterogeneity during the post-warranty stage has been ignored because $M_1 = M_2 = M_3 = 0$, and RPRF is a uniform model to ensure the post-warranty-stage reliability.

Specific model E: when $M_1 \to 0$, $M_2 \to 0$, $M_3 \to 0$, and $N \to \infty$, the average cost rate $CR_f(T,N)$ is simplified as

$$\lim_{\substack{M_1,M_2,M_3\to 0\\ N\to\infty}} CR_f(T,N) = \frac{c_f\begin{pmatrix} \int_0^{w_3}\overline{G}^{(m)}(s_m)\lambda(s_m)\mathrm{d}s_m+\int_0^{w_1}\left(\int_0^{w_3}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+ \\ \int_{w_1}^{w_2}\left(\int_0^{w_2-w_1}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\int_{w_2}^{w_3}\left(\int_0^{w_3-w_2}\lambda(s_m+s_n)\mathrm{d}s_n\right)\mathrm{d}G^{(m)}(s_m) \end{pmatrix}+c_r+ c_M\begin{pmatrix} \overline{G}^{(m)}(w_3)\int_0^T\lambda(w_3+u)\mathrm{d}u+\int_0^{w_1}\left(\int_0^T\lambda(s_m+w_3+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+ \\ \int_{w_1}^{w_2}\left(\int_0^T\lambda(s_m+w_2-w_1+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\int_{w_2}^{w_3}\left(\int_0^T\lambda(s_m+w_3-w_2+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) \end{pmatrix}}{\begin{pmatrix} G^{(m)}(w_1)w_3+\left(G^{(m)}(w_2)-G^{(m)}(w_1)\right)(w_2-w_1)+ \\ \left(G^{(m)}(w_3)-G^{(m)}(w_2)\right)(w_3-w_2)+\int_0^{w_3}\overline{G}^{(m)}(s_m)\mathrm{d}s_m \end{pmatrix}+T} \tag{32}$$

$M_1 \to 0$, $M_2 \to 0$, $M_3 \to 0$, and $N \to \infty$ reduce CRPRF to a classic periodic replacement (PR) model in Ref. [73]. Therefore, the average cost rate $\lim_{\substack{M_1,M_2,M_3\to 0\\ N\to\infty}} CR_f(T,N)$ of Equation (32) is a model where the FRW-H and PR models are used to ensure the life cycle reliability. In this model, heterogeneity during the warranty stage is still kept, while heterogeneity during the post-warranty stage has been completely removed.

Specific model a: when $m \to \infty$, the average cost rate $CR_l(T,N)$ is reduced to:

$$\lim_{m\to\infty} CR_l(T,N) = \frac{c_f\int_0^{w_3}\lambda(s_m)\mathrm{d}s_m + c_M\left(\int_0^T\lambda(w_3+u)\mathrm{d}u+\int_T^\infty\overline{G}^{(N)}(u)\lambda(w_3+u)\mathrm{d}u\right)+c_r}{w_3+T+\int_T^\infty\overline{G}^{(N)}(u)\mathrm{d}u} \tag{33}$$

$m \to \infty$ reduces the FRW-H to a classic FRW model, which has been mentioned in Equation (33). Similarly, $m \to \infty$ reduces the CRPRL to a random periodic replacement first (RPRL) model in Ref. [42]. Therefore, the average cost rate $\lim_{m\to\infty} CR_l(T,N)$ of Equation (33) is a model where FRW and RPRF models are used to ensure the life cycle reliability. In this model, heterogeneity during the life cycle has also been completely removed.

Specific model b: when $N \to 1$, the average cost rate $CR_l(T,N)$ is reduced to:

$$CR_l(T,1)=\frac{c_f\begin{pmatrix}\int_0^{w_1}\left(\int_0^{w_3}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\int_{w_1}^{w_2}\left(\int_0^{w_2-w_1}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\\ \int_{w_2}^{w_3}\left(\int_0^{w_3-w_2}\lambda(s_m+s_n)\mathrm{d}s_n\right)\mathrm{d}G^{(m)}(s_m)+\int_0^{w_3}\overline{G}^{(m)}(s_m)\lambda(s_m)\mathrm{d}s_m\end{pmatrix}+c_r+c_M\begin{pmatrix}\overline{G}^{(m)}(w_3)\left(\int_0^T\lambda(w_3+u)\mathrm{d}u+\int_T^\infty\overline{G}^{(N)}(u)\lambda(w_3+u)\mathrm{d}u\right)+\\ \int_0^{w_1}\left(\int_0^T\lambda(s_m+w_3+u)\mathrm{d}u+\int_T^\infty\overline{G}^{(1+M_1)}(u)\lambda(s_m+w_3+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\\ \int_{w_1}^{w_2}\left(\int_0^T\lambda(s_m+w_2-w_1+u)\mathrm{d}u+\int_T^\infty\overline{G}^{(1+M_2)}(u)\lambda(s_m+w_2-w_1+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\\ \int_{w_2}^{w_3}\left(\int_0^T\lambda(s_m+w_3-w_2+u)\mathrm{d}u+\int_T^\infty\overline{G}^{(1+M_3)}(u)\lambda(s_m+w_3-w_2+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)\end{pmatrix}}{\begin{pmatrix}\int_0^{w_3}\overline{G}^{(m)}(s_m)\mathrm{d}s_m+G^{(m)}(w_1)w_3+\left(G^{(m)}(w_2)-G^{(m)}(w_1)\right)(w_2-w_1)+\\ \left(G^{(m)}(w_3)-G^{(m)}(w_2)\right)\int_0^{w_3-w_2}\overline{G}^{(n)}(s_n)\mathrm{d}s_n\end{pmatrix}+\begin{pmatrix}T+\overline{G}^{(m)}(w_3)\int_T^\infty\overline{G}^{(N)}(u)\mathrm{d}u+G^{(m)}(w_1)\int_T^\infty\overline{G}^{(1+M_1)}(u)\mathrm{d}u+\\ \left(G^{(m)}(w_2)-G^{(m)}(w_1)\right)\int_T^\infty\overline{G}^{(1+M_2)}(u)\mathrm{d}u+\left(G^{(m)}(w_3)-G^{(m)}(w_2)\right)\int_T^\infty\overline{G}^{(1+M_3)}(u)\mathrm{d}u\end{pmatrix}} \quad (34)$$

$N\to 1$ signals that CRPRL is reduced to a special customized random periodic replacement last (CRPRL) model where the post-warranty service time T becomes a unique decision variable and other terms are still kept. Therefore, the average cost rate $CR_l(T,1)$ of Equation (34) is a model where FRW and special CRPRL models are used to ensure the life cycle reliability. In this model, heterogeneity during the life cycle is still considered.

Specific model c: when $M_1\to 0$, $M_2\to 0$, and $M_3\to 0$, the average cost rate $CR_l(T,N)$ is simplified as

$$\lim_{M_1,M_2,M_3\to 0}CR_l(T,N)=\frac{c_f\begin{pmatrix}\int_0^{w_3}\overline{G}^{(m)}(s_m)\lambda(s_m)\mathrm{d}s_m+\int_0^{w_1}\left(\int_0^{w_3}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\\ \int_{w_1}^{w_2}\left(\int_0^{w_2-w_1}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\int_{w_2}^{w_3}\left(\int_0^{w_3-w_2}\lambda(s_m+s_n)\mathrm{d}s_n\right)\mathrm{d}G^{(m)}(s_m)\end{pmatrix}+c_r+c_M\begin{pmatrix}\overline{G}^{(m)}(w_3)\left(\int_0^T\lambda(w_3+u)\mathrm{d}u+\int_T^\infty\overline{G}^{(N)}(u)\lambda(w_3+u)\mathrm{d}u\right)+\\ \int_0^{w_1}\left(\int_0^T\lambda(s_m+w_3+u)\mathrm{d}u+\int_T^\infty\overline{G}^{(N)}(u)\lambda(s_m+w_3+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\\ \int_{w_1}^{w_2}\left(\int_0^T\lambda(s_m+w_2-w_1+u)\mathrm{d}u+\int_T^\infty\overline{G}^{(N)}(u)\lambda(s_m+w_2-w_1+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\\ \int_{w_2}^{w_3}\left(\int_0^T\lambda(s_m+w_3-w_2+u)\mathrm{d}u+\int_T^\infty\overline{G}^{(N)}(u)\lambda(s_m+w_3-w_2+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)\end{pmatrix}}{\begin{pmatrix}G^{(m)}(w_1)w_3+\left(G^{(m)}(w_2)-G^{(m)}(w_1)\right)(w_2-w_1)+\\ \left(G^{(m)}(w_3)-G^{(m)}(w_2)\right)(w_3-w_2)+\int_0^{w_3}\overline{G}^{(m)}(s_m)\mathrm{d}s_m\end{pmatrix}+T+\int_T^\infty\overline{G}^{(N)}(u)\mathrm{d}u} \quad (35)$$

$M_1\to 0$, $M_2\to 0$, and $M_3\to 0$ reduce the CRPRL to an RPRL in Ref. [42]. Therefore, the average cost rate $\lim\limits_{M_1,M_2,M_3\to 0}CR_l(T,N)$ of Equation (35) is a model where FRW and RPRL models are used to ensure the life cycle reliability. In this model, heterogeneity during the warranty stage is still maintained, while heterogeneity during the post-warranty stage is completely ignored.

Specific model d: when $M_1\to 0$, $M_2\to 0$, $M_3\to 0$, and $N\to 1$, the average cost rate $CR_l(T,N)$ is reduced to:

$$\lim_{M_1,M_2,M_3\to 0}CR_l(T,1)=\frac{c_f\begin{pmatrix}\int_0^{w_3}\overline{G}^{(m)}(s_m)\lambda(s_m)\mathrm{d}s_m+\int_0^{w_1}\left(\int_0^{w_3}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\\ \int_{w_1}^{w_2}\left(\int_0^{w_2-w_1}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\int_{w_2}^{w_3}\left(\int_0^{w_3-w_2}\lambda(s_m+s_n)\mathrm{d}s_n\right)\mathrm{d}G^{(m)}(s_m)\end{pmatrix}+c_r+c_M\begin{pmatrix}\overline{G}^{(m)}(w_3)\left(\int_0^T\lambda(w_3+u)\mathrm{d}u+\int_T^\infty\overline{G}(u)\lambda(w_3+u)\mathrm{d}u\right)+\\ \int_0^{w_1}\left(\int_0^T\lambda(s_m+w_3+u)\mathrm{d}u+\int_T^\infty\overline{G}(u)\lambda(s_m+w_3+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\\ \int_{w_1}^{w_2}\left(\int_0^T\lambda(s_m+w_2-w_1+u)\mathrm{d}u+\int_T^\infty\overline{G}(u)\lambda(s_m+w_2-w_1+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\\ \int_{w_2}^{w_3}\left(\int_0^T\lambda(s_m+w_3-w_2+u)\mathrm{d}u+\int_T^\infty\overline{G}(u)\lambda(s_m+w_3-w_2+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)\end{pmatrix}}{\begin{pmatrix}G^{(m)}(w_1)w_3+\left(G^{(m)}(w_2)-G^{(m)}(w_1)\right)(w_2-w_1)+\\ \left(G^{(m)}(w_3)-G^{(m)}(w_2)\right)(w_3-w_2)+\int_0^{w_3}\overline{G}^{(m)}(s_m)\mathrm{d}s_m\end{pmatrix}+T+\int_T^\infty\overline{G}(u)\mathrm{d}u} \quad (36)$$

The average cost rate $\lim_{M_1,M_2,M_3\to 0} CR_l(T,1)$ of Equation (36) is a model where FRW and special CRPRL models are used to ensure the life cycle reliability. In this model, heterogeneity during the life cycle is still considered.

Specific model e: when $M_1 \to 0,\ M_2 \to 0,\ M_3 \to 0$, and $N \to 0$, the average cost rate $CR_l(T,N)$ is simplified as

$$\lim_{\substack{M_1,M_2,M_3\to 0\\ N\to 0}} CR_l(T,N) = \frac{c_f\begin{pmatrix} \int_0^{w_1}\left(\int_0^{w_3}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\int_{w_1}^{w_2}\left(\int_0^{w_2-w_1}\lambda(s_m+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\\ \int_{w_2}^{w_3}\left(\int_0^{w_3-w_2}\lambda(s_m+s_n)\mathrm{d}s_n\right)\mathrm{d}G^{(m)}(s_m)+\int_0^{w_3}\overline{G}^{(m)}(s_m)\lambda(s_m)\mathrm{d}s_m \end{pmatrix}+c_r+ \\ c_M\begin{pmatrix} \overline{G}^{(m)}(w_3)\left(\int_0^{T}\lambda(w_3+u)\mathrm{d}u+\int_T^{\infty}\overline{G}^{(N)}(u)\lambda(w_3+u)\mathrm{d}u\right)+\int_0^{w_1}\left(\int_0^{T}\lambda(s_m+w_3+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)\\ +\int_{w_1}^{w_2}\left(\int_0^{T}\lambda(s_m+w_2-w_1+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m)+\int_{w_2}^{w_3}\left(\int_0^{T}\lambda(s_m+w_3-w_2+u)\mathrm{d}u\right)\mathrm{d}G^{(m)}(s_m) \end{pmatrix}}{\begin{pmatrix} G^{(m)}(w_1)w_3+\left(G^{(m)}(w_2)-G^{(m)}(w_1)\right)(w_2-w_1)+\\ \left(G^{(m)}(w_3)-G^{(m)}(w_2)\right)(w_3-w_2)+\int_0^{w_3}\overline{G}^{(m)}(s_m)\mathrm{d}s_m \end{pmatrix}+T} \quad (37)$$

$M_1 \to 0,\ M_2 \to 0,\ M_3 \to 0$, and $N \to 0$ reduce CRPRL to a classic periodic replacement (PR) model in Ref. [73]. Therefore, the average cost rate $\lim_{\substack{M_1,M_2,M_3\to 0\\ N\to 0}} CR_l(T,N)$ of Equation (37) is a model where FRW and RPRF models are used to ensure the life cycle reliability. Similar to the case of Equation (36), heterogeneity during the warranty stage is still maintained, while heterogeneity during the post-warranty stage has been completely ignored.

4. Numerical Experiments

In this paper, we have presented three warranty models and four maintenance models. In this section, the RFRW-H provided in Section 2.1, the CRPRF mentioned in (29), and the CRPRL mentioned in (34) will be subjected to sensitivity analysis, and other models can be similarly analyzed from the perspective of sensitivity, which will no longer be provided hereinafter.

The latest boiler of company X has been integrated into IS, which aims to send task cycles to manufacturers and consumers. Therefore, such a type of boiler is the potential application carrier of the proposed models. In view of this, the latest boiler of X company is considered as a case study for executing numerical analysis. Assume that the cycles Y_i are independent and identically distributed with an exponential distribution function given by $G(y) = 1 - \exp(-\mu y)$, which is a nonnegative constant; i.e., $\mu > 0$. Similarly, in this paper, the latest boiler of X company is still used to execute numerical analysis. Some common parameters are set as $\mu = 2$, $\beta = 2$, and $c_m = 0.3$.

4.1. Sensitivity Analysis of the RFRW-H

To explore how m, w_1, and w_2 impact the total warranty service cost of the RFRW-H, Figure 1 has been plotted using $\alpha = 0.5$, $w_3 = 1$, and $n = 3$. Figure 1A, where $w_2 = 0.6$ shows that the increase in m can make the total warranty service cost of the RFRW-H decrease to a constant, which is the total warranty service cost of the FRW and is provided by Equation (7); under the case where m is smaller, the total warranty service cost of the RFRW-H is increasing with respect to w_1. Figure 1B, where $w_1 = 0.3$, shows that the increase in m can make the total warranty service cost of the RFRW-H first increase and then decrease to a constant, which is likewise the total warranty service cost of the FRW; under the case where m is smaller, the total warranty service cost of the RFRW-H is increasing with respect to w_2. The above increasing phenomena under the case of m being smaller have a common cause in which the increase in any of the warranty terms m, w_1, and w_2 can enlarge the warranty coverage, which enhances the number of warranty-stage failures.

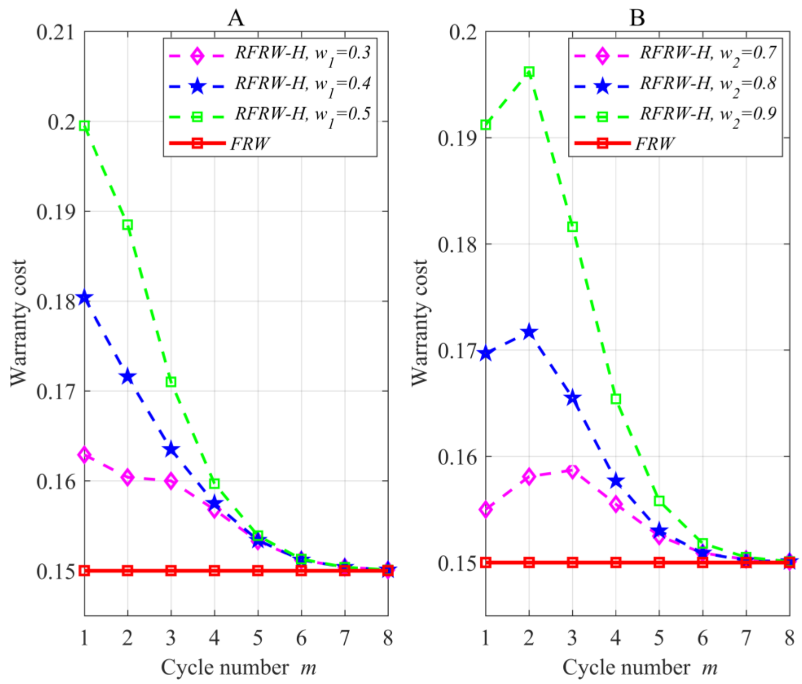

Figure 1. m, w_1, and w_2 versus the cost measure of the RFRW-H. (**A**) The effects of m and w_1; (**B**) The effects of m and w_2.

To mine how n, w_1, and w_2 impact the total warranty service cost of the RFRW-H, Figure 2 has been offered by means of $\alpha = 0.5$, $w_3 = 1$, and $m = 2$. Figure 2 shows that the increase in n can make the total warranty service cost of the RFRW-H increase to be a constant, which can be measured by $\lim_{n\to\infty} WSC_T$ in Equation (9); the total warranty service cost of the RFRW-H is increasing with respect to any of the warranty terms w_1 and w_2, and the related cause is similar to the cause in Figure 1.

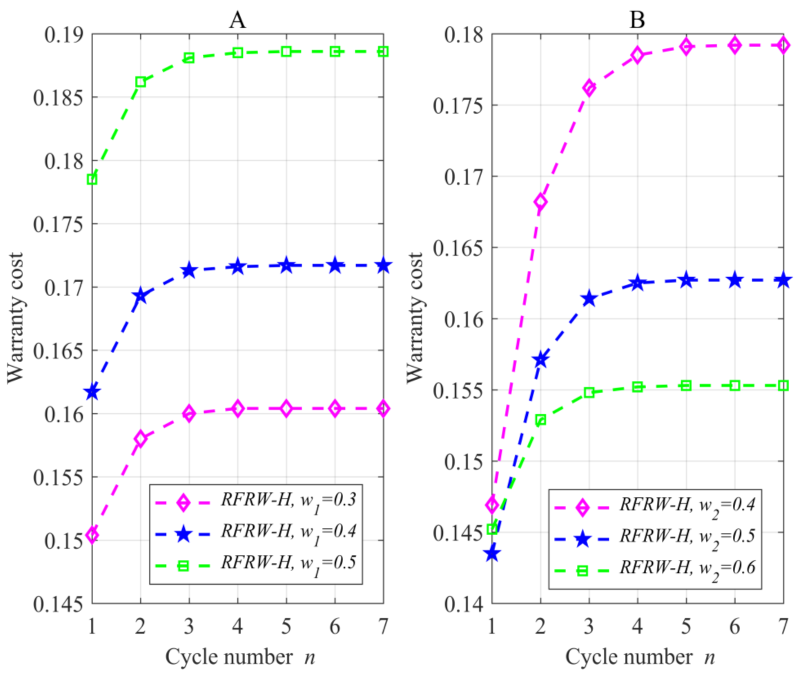

Figure 2. n, w_1, and w_2 versus the cost measure of the RFRW-H. (**A**) The effects of n and w_1; (**B**) The effects of n and w_2.

To obtain the impacts of m on the service measures of the RFRW-H, Figure 3 is shown setting $\alpha = 1.5$, $w_1 = 0.3$, $w_2 = 0.6$, $w_3 = 1$, and $n = 2$.

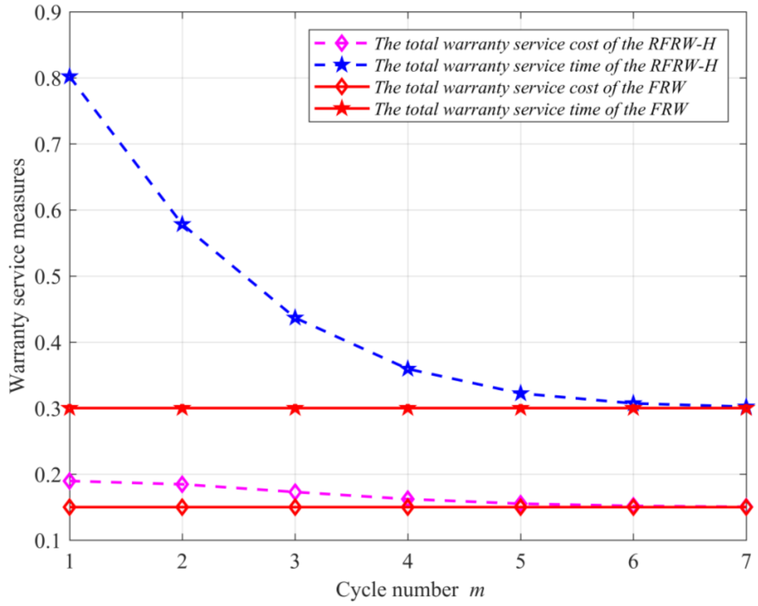

Figure 3. m versus the service measures of the RFRW-H.

Figure 3 shows that m can make the total warranty service cost of the RFRW-H increase to the total warranty service cost of the FRW; under the case of m increasing, the total warranty service time of the RFRW-H decreases to the total warranty service time of the FRW. The simultaneous increases in two service measures of the RFRW-H cannot mine the performance of the RFRW-H. In addition, the dimensions of cost and time are not the same; thus, the performance of the RFRW-H cannot be mined by the above two service measures.

By translating the dimensions of both into the same dimension, Ref. [46] compared the performances of warranty models. By using this method and setting $\alpha = 1.5$, $w_1 = 0.3$, $w_2 = 0.6$, $w_3 = 1$, and $n = 2$, Table 1 has been plotted to compare the performances of the RFRW-H and the FRW. Table 1 shows that the total warranty service time of the FRW is shorter than the total warranty service time of the RFRW-H, i.e., $T_{FRW} < T_{RFRW-H}$, which implies that the RFRW-H is superior to the FRW.

Table 1. Comparing the RFRW-H and the FRW.

m	RFRW-H		FRW		Time Measures		Relationships
	WCS_T	WS_T	$\lim_{m\to\infty} WCS_T$	$\lim_{m\to\infty} WS_T$	T_{RFRW-H}	T_{FRW}	
1	0.1896	0.8021	0.1500	0.3000	0.1203	0.0569	$T_{RFRW-H} > T_{FRW}$
2	0.1846	0.5783	0.1500	0.3000	0.0867	0.0554	$T_{RFRW-H} > T_{FRW}$
3	0.1729	0.4367	0.1500	0.3000	0.0655	0.0519	$T_{RFRW-H} > T_{FRW}$
4	0.1622	0.3593	0.1500	0.3000	0.0539	0.0487	$T_{RFRW-H} > T_{FRW}$

Note that: ① time measures T_{RFRW-H} and T_{FRW} can be obtained by calculating $T_{RFRW-H} = WS_T \times \lim_{m\to\infty} WCS_T$ and $T_{FRW} = \lim_{m\to\infty} WS_T \times WCS_T$, respectively; ② under the case where manufacturers absorb all cost during the warranty stage, the service time measure of a warranty model is a key factor affecting consumers' tendencies to products, and thus, using the service time measure to compare warranty models is more practical than using the cost measure to compare warranty models for manufacturers.

4.2. Sensitivity Analysis of Random Maintenance Models

In this section, the sensitivities of the CRPRF mentioned in (29) and the CRPRL mentioned in (34) will be analyzed from consumers' perspectives.

4.2.1. Sensitivity Analysis of the CRPRF

To numerically prove that the optimal CRPRF uniquely exists, Figure 4 has been offered setting $\alpha = 1.5$, $m = 2$, $w_1 = 0.5$, $w_2 = 1$, $w_3 = 1.5$, $c_m = 0.3$, $c_M = 0.4$, $c_f = 0.1$, $M_1 = 1$, $M_2 = 2$, and $M_3 = 3$.

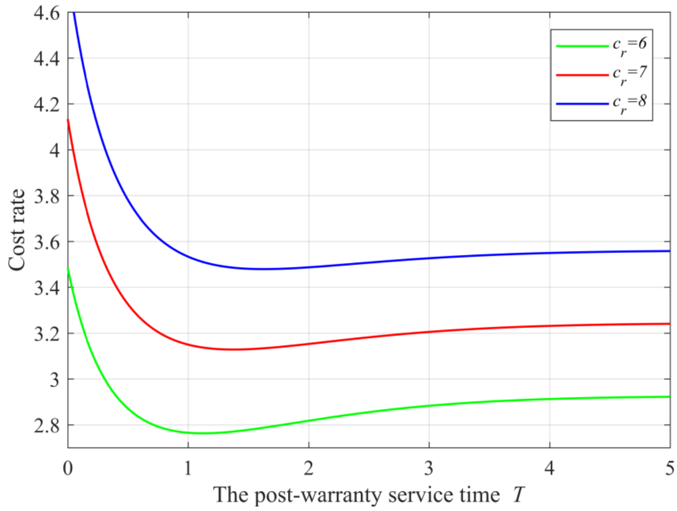

Figure 4. c_r versus the optimal CRPRF.

Figure 4 shows that the optimal CRPRF uniquely exists because the optimal post-warranty service time T^* and the optimal value $CR_f(T^*, 1)$ of the cost rate (i.e., vertical coordinates) are unique; the increase in the replacement cost c_r can lengthen the optimal post-warranty service time but cannot reduce the optimal value of the cost rate.

To display the impacts of replacement terms M_i $(i = 1, 2, 3)$ on the optimal CRPRF, Figure 5 has been offered setting $\alpha = 1.5$, $w_1 = 0.5$, $w_2 = 1$, $w_3 = 1.5$, $c_m = 0.3$, $c_M = 0.4$, $c_f = 0.1$, $m = 2$, and $c_r = 12$.

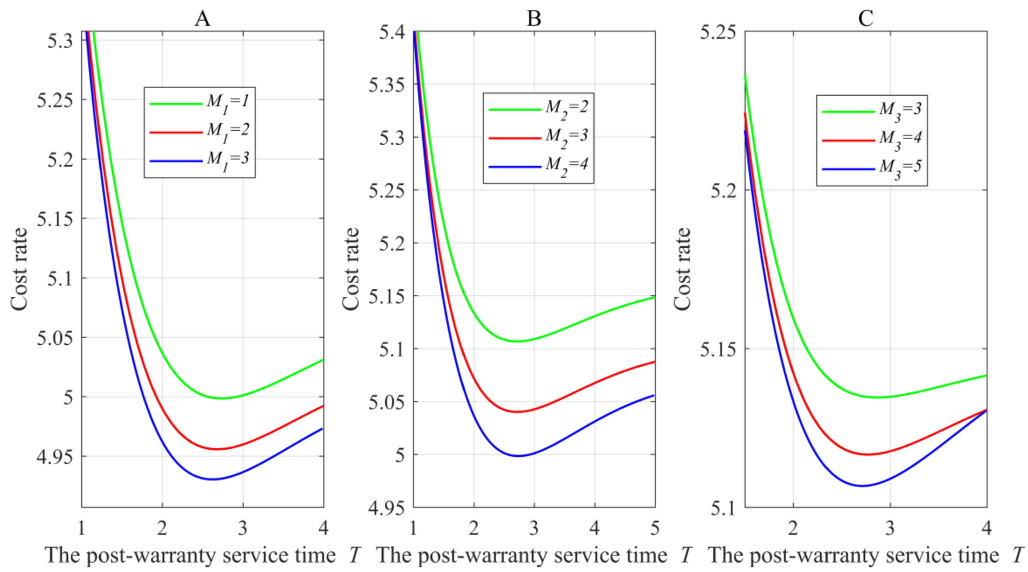

Figure 5. M_i where $i = 1, 2, 3$ versus the optimal CRPRF. (**A**) M_1 versus the optimal CRPRF; (**B**) M_2 versus the optimal CRPRF; (**C**) M_3 versus the optimal CRPRF.

Figure 5A, where $M_2 = 4$ and $M_3 = 5$, shows that the increase in M_1 can reduce the optimal value of the cost rate but cannot lengthen the optimal post-warranty service time; Figure 5B, where $M_1 = 1$ and $M_3 = 5$, shows that the increase in M_2 can reduce the optimal value of the cost rate and lengthen the optimal post-warranty service time, which is completely different from that of Figure 5A; Figure 5C, where $M_1 = 1$ and $M_2 = 2$, shows that the increase in M_3 can reduce the optimal value of the cost rate and shorten the optimal post-warranty service time, which is completely the same as that of Figure 5A. Because these changes are not completely in the same direction, they cannot illustrate the performance of the CRPRF.

To illustrate the performance of the CRPRF, Table 2 has been obtained by setting $\alpha = 1.5$, $w_1 = 0.3$, $w_2 = 0.7$, $w_3 = 1$, $c_m = 0.3$, $c_M = 0.4$, $c_f = 0.1$, $M_1 = 1$, $M_2 = 1$, $M_3 = 1$, $m = 2$, and $c_r = 6$. Table 2 shows that the CRPRF is superior to the RPRF because $T_{CRPRF} > T_{RPRF}$, where $T_{CRPRF} = ST_f^T(T^*, 1) \times \lim_{\overline{G}(\cdot)\to 1} SC_f^T(T^*, 1)$ and $T_{RPRF} = \lim_{\overline{G}(\cdot)\to 1} ST_f^T(T^*, 1) \times SC_f^T(T^*, 1)$.

Table 2. The replacement terms M_i versus the optimal CRPRF.

m	Measures of the Optimal CRPRF		Measures of the Optimal RPRF		Time Measures		Relationships
	$SC_f^T(T^*,1)$	$ST_f^T(T^*,1)$	$\lim_{\overline{G}(\cdot)\to 1} SC_f^T(T^*,1)$	$\lim_{\overline{G}(\cdot)\to 1} ST_f^T(T^*,1)$	T_{CRPRF}	T_{RPRF}	
1	8.5157	2.2834	7.0847	1.6857	16.1772	14.3549	$T_{CRPRF} > T_{RPRF}$
2	8.3693	2.2057	7.0696	1.6827	15.5934	14.0830	$T_{CRPRF} > T_{RPRF}$
3	7.9315	1.9572	7.0645	1.6250	13.8266	12.8887	$T_{CRPRF} > T_{RPRF}$
4	7.5005	1.7270	7.0568	1.5643	12.1871	11.7330	$T_{CRPRF} > T_{RPRF}$

To obtain the results of how w_1 and m affect the optimal CRPRF, Table 3 has been offered setting $\alpha = 1.5$, $w_2 = 0.5$, $w_3 = 1$, $c_m = 0.3$, $c_M = 0.4$, $c_f = 0.1$, $M_1 = 1$, $M_2 = 2$, $M_3 = 3$, and $c_r = 6$. Table 3 shows that the increase in w_1 can reduce the optimal value of the cost rate and shorten the optimal post-warranty service time for a given m; the increase in m can lengthen the optimal post-warranty service time and enhance the optimal value of the cost rate for a given w_1.

Table 3. m and w_1 versus the optimal CRPRF.

m	w_1=0.2		w_1=0.3		w_1=0.4	
	T^*	$CR_f(T^*,1)$	T^*	$CR_f(T^*,1)$	T^*	$CR_f(T^*,1)$
1	2.2076	3.7591	2.1324	3.7242	2.0387	3.6811
2	2.1380	3.7812	2.1074	3.7678	2.0452	3.7378
3	2.2512	4.0180	2.2449	4.0165	2.2163	4.0047
4	2.4464	4.3152	2.4467	4.3158	2.4363	4.3127

Similar to the objective of Table 3, Table 4 has been offered setting $\alpha = 1.5$, $w_1 = 0.3$, $w_3 = 1$, $c_m = 0.3$, $c_M = 0.4$, $c_f = 0.1$, $M_1 = 1$, $M_2 = 2$, $M_3 = 3$, and $c_r = 6$.

Table 4 shows that the increase in w_2 can lengthen the post-warranty service time and reduce the cost rate for a given m; the changes produced by the increase in m are similar to the changes produced by the increase in m for Table 3.

Table 4. m and w_2 versus the optimal CRPRF.

m	w_2=0.5		w_2=0.6		w_2=0.7	
	T^*	$CR_f(T^*,1)$	T^*	$CR_f(T^*,1)$	T^*	$CR_f(T^*,1)$
1	2.1324	3.7242	2.1820	3.7369	2.1795	3.7295
2	2.1074	3.7678	2.1850	3.7924	2.2111	3.7944
3	2.2449	4.0165	2.3317	4.0420	2.3809	4.0525
4	2.4467	4.3158	2.5280	4.3327	2.5861	4.3432

4.2.2. Sensitivity Analysis of the CRPRL

To prove whether the optimal CRPRL exists, Figure 6 has been offered setting $\alpha = 1.5$, $w_1 = 0.3$, $w_2 = 0.7$, $w_3 = 1$, $c_m = 0.3$, $c_M = 0.4$, $c_f = 0.1$, $m = 2$, $M_1 = 1$, $M_2 = 2$, and $M_3 = 3$.

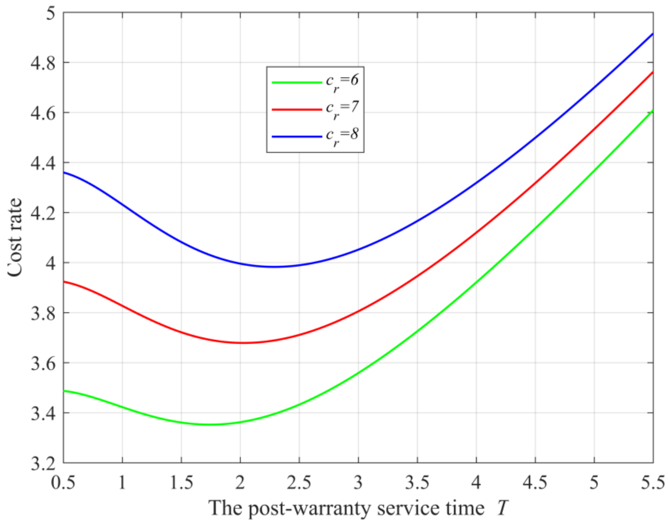

Figure 6. c_r versus the optimal CRPRL.

Figure 6 shows that the optimal CRPRL is unique; the increase in the replacement cost can lengthen the optimal post-warranty service time but cannot reduce the optimal value of the cost rate, which is similar to that of Figure 4.

Note that to control the length of the paper, we do not present other sensitivity analyses of the CRPRL because they can be obtained and analyzed by means of the same method.

5. Conclusions

In the case of being able to monitor task cycles of the product that implements tasks, it has been discovered that the lengths of the task cycles are heavily dependent on consumers' usage situations. This fact signals that task cycles are heterogeneous time data, which have heterogeneous effects on the life cycle reliability. In view of these, classifying usages into the different categories, this study modeled two types of random models, which can be applied to ensure the life cycle reliability of the product with monitored task cycles. The first type of model is suitable for ensuring warranty-stage reliability, which has been named a random free repair warranty with heterogeneity (RFRW-H) because minimal repair can remove all failures over the warranty stage. Each model in the second type of model is suitable for ensuring post-warranty-stage reliability, which has been named a customized random periodic replacement first (CRPRF) model and a customized random periodic replacement last (CRPRL) model because periodic replacements with different coverage and each of 'whichever occurs first/last' are combined to customize these two models. The models mentioned above are modeled from the perspectives of cost and time measures or the cost rate function, and some specific cases are presented by calculating the limits of mathematical models. Some models have been numerically illustrated for valuable mining management. Taking the CRPRF model as a typical model, the performance of the CRPRF is numerically compared as well.

Usage heterogeneity and the differences in reliability have a significant effect on controlling the life cycle cost of the product with monitored task cycles. By classifying usage and screening the differences in reliability, modeling random models for ensuring the life cycle reliability of the product with monitored task cycles is a more practical topic, which is being investigated by the present authors.

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References

1. Chen, C.-K.; Lo, C.-C.; Weng, T.-C. Optimal production run length and warranty period for an imperfect production system under selling price dependent on warranty period. *Eur. J. Oper. Res.* **2017**, *259*, 401–412. [CrossRef]
2. Ye, Z.-S.; Murthy, D.P.; Xie, M.; Tang, L.-C. Optimal burn-in for repairable products sold with a two-dimensional warranty. *IIE Trans.* **2013**, *45*, 164–176. [CrossRef]
3. Yeh, R.H.; Ho, W.-T.; Tseng, S.-T. Optimal production run length for products sold with warranty. *Eur. J. Oper. Res.* **2000**, *120*, 575–582. [CrossRef]
4. Wang, X.; Zhao, X.; Liu, B. Design and pricing of extended warranty menus based on the multinomial logit choice model. *Eur. J. Oper. Res.* **2020**, *287*, 237–250. [CrossRef]
5. Li, X.; Liu, Z.; Wang, Y.; Li, M. Optimal burn-in strategy for two-dimensional warranted products considering preventive maintenance. *Int. J. Prod. Res.* **2019**, *57*, 5414–5431. [CrossRef]
6. Shang, L.; Si, S.; Sun, S.; Jin, T. Optimal warranty design and post-warranty maintenance for products subject to stochastic degradation. *IISE Trans.* **2018**, *50*, 913–927. [CrossRef]
7. Wang, J.; Qiu, Q.; Wang, H. Joint optimization of condition-based and age-based replacement policy and inventory policy for a two-unit series system. *Reliab. Eng. Syst. Saf.* **2020**, *205*, 107251. [CrossRef]
8. Wang, J.; Qiu, Q.; Wang, H.; Lin, C. Optimal condition-based preventive maintenance policy for balanced systems. *Reliab. Eng. Syst. Saf.* **2021**, *211*, 107606. [CrossRef]
9. Zhu, W.; Fouladirad, M.; Bérenguer, C. Condition-based maintenance policies for a combined wear and shock deterioration model with covariates. *Comput. Ind. Eng.* **2015**, *85*, 268–283. [CrossRef]
10. Qiu, Q.; Maillart, L.M.; Prokopyev, O.A.; Cui, L. Optimal Condition-Based Mission Abort Decisions. *IEEE Trans. Reliab.* **2022**, *72*, 408–425. [CrossRef]
11. Zhang, N.; Tian, S.; Cai, K.; Zhang, J. Condition-based maintenance assessment for a deteriorating system considering stochastic failure dependence. *IISE Trans.* **2022**, *55*, 687–697. [CrossRef]
12. Chen, Y.; Qiu, Q.; Zhao, X. Condition-based opportunistic maintenance policies with two-phase inspections for continuous-state systems. *Reliab. Eng. Syst. Saf.* **2022**, *228*, 108767. [CrossRef]
13. Yang, A.; Qiu, Q.; Zhu, M.; Cui, L.; Chen, W.; Chen, J. Condition-based maintenance strategy for redundant systems with arbitrary structures using improved reinforcement learning. *Reliab. Eng. Syst. Saf.* **2022**, *225*, 108643. [CrossRef]
14. Qiu, Q.; Cui, L.; Dong, Q. Preventive maintenance policy of single-unit systems based on shot-noise process. *Qual. Relia. Eng. Int.* **2019**, *35*, 550–560. [CrossRef]
15. Zhao, X.; Sun, J.; Qiu, Q.; Chen, K. Optimal inspection and mission abort policies for systems subject to degradation. *Eur. J. Oper. Res.* **2020**, *292*, 610–621. [CrossRef]
16. Zhao, X.; Fan, Y.; Qiu, Q.; Chen, K. Multi-criteria mission abort policy for systems subject to two-stage degradation process. *Eur. J. Oper. Res.* **2021**, *295*, 233–245. [CrossRef]
17. Zhao, X.; Li, R.; Cao, S.; Qiu, Q. Joint modeling of loading and mission abort policies for systems operating in dynamic environments. *Reliab. Eng. Syst. Saf.* **2023**, *230*, 108948. [CrossRef]
18. Zhao, X.; Chai, X.; Sun, J.; Qiu, Q. Joint optimization of mission abort and protective device selection policies for multistate systems. *Risk Anal.* **2022**, *42*, 2823–2834. [CrossRef]
19. Ye, S.Z.; Xie, M. Stochastic modelling and analysis of degradation for highly reliable products. *Appl. Stoch. Model. Bus. Ind.* **2015**, *31*, 16–32. [CrossRef]

20. Zhao, X.; Li, R.; Fan, Y.; Qiu, Q. Reliability and optimal triggering policy for multi-state systems subject to shocks and supported by a protective device. *Comput. Ind. Eng.* **2021**, *156*, 107232. [CrossRef]
21. Zhao, X.; Li, R.; Fan, Y.; Qiu, Q. Reliability modeling for multi-state systems with a protective device considering multiple triggering mechanism. *Proc. Inst. Mech. Eng. Part O J. Risk Reliab.* **2022**, *236*, 173–193. [CrossRef]
22. Yang, L.; Chen, Y.; Qiu, Q.; Wang, J. Risk Control of Mission-Critical Systems: Abort Decision-Makings Integrating Health and Age Conditions. *IEEE Trans. Ind. Inform.* **2022**, *18*, 6887–6894. [CrossRef]
23. Qiu, Q.; Cui, L. Gamma process based optimal mission abort policy. *Reliab. Eng. Syst. Saf.* **2019**, *190*, 106496. [CrossRef]
24. Qiu, Q.; Cui, L. Reliability evaluation based on a dependent two-stage failure process with competing failures. *Appl. Math. Model.* **2018**, *64*, 699–712. [CrossRef]
25. Qiu, Q.; Cui, L.; Wu, B. Dynamic mission abort policy for systems operating in a controllable environment with self-healing mechanism. *Reliab. Eng. Syst. Saf.* **2020**, *203*, 107069. [CrossRef]
26. Qiu, Q.; Kou, M.; Chen, K.; Deng, Q.; Kang, F.; Lin, C. Optimal stopping problems for mission oriented systems considering time redundancy. *Reliab. Eng. Syst. Saf.* **2020**, *205*, 107226. [CrossRef]
27. Qiu, Q.; Cui, L. Optimal mission abort policy for systems subject to random shocks based on virtual age process. *Reliab. Eng. Syst. Saf.* **2019**, *189*, 11–20. [CrossRef]
28. Zhao, X.; Chai, X.; Sun, J.; Qiu, Q. Joint optimization of mission abort and component switching policies for multistate warm standby systems. *Reliab. Eng. Syst. Saf.* **2021**, *212*, 107641. [CrossRef]
29. Wang, J.; Zhou, S.; Peng, R.; Qiu, Q.; Yang, L. An inspection-based replacement planning in consideration of state-driven imperfect inspections. *Reliab. Eng. Syst. Saf.* **2023**, *232*, 109064. [CrossRef]
30. Wang, X.; He, K.; He, Z.; Li, L.; Xie, M. Cost analysis of a piece-wise renewing free replacement warranty policy. *Comput. Ind. Eng.* **2019**, *135*, 1047–1062. [CrossRef]
31. Liu, B.; Wu, J.; Xie, M. Cost analysis for multi-component system with failure interaction under renewing free-replacement warranty. *Eur. J. Oper. Res.* **2015**, *243*, 874–882. [CrossRef]
32. Jung, K.M.; Park, M.; Park, D.H. Cost optimization model following extended renewing two-phase warranty. *Comput. Ind. Eng.* **2015**, *79*, 188–194. [CrossRef]
33. Park, M.; Jung, K.M.; Park, D.H. Optimal maintenance strategy under renewable warranty with repair time threshold. *Appl. Math. Model.* **2017**, *43*, 498–508. [CrossRef]
34. Park, M.; Pham, H. Cost models for age replacement policies and block replacement policies under warranty. *Appl. Math. Model.* **2016**, *40*, 5689–5702. [CrossRef]
35. Shang, L.; Si, S.; Cai, Z. Optimal maintenance–replacement policy of products with competing failures after expiry of the warranty. *Comput. Ind. Eng.* **2016**, *98*, 68–77. [CrossRef]
36. Shang, L.; Cai, Z.; Chen, H.; Zhang, S. Post-warranty maintenance optimization for products with deterioration depending on aging and shock. *Qual. Technol. Quant. Manag.* **2018**, *16*, 651–671. [CrossRef]
37. Shang, L.; Shang, G.; Du, Y.; Qiu, Q.; Yang, L.; Dong, Q. Post-Warranty Replacement Models for the Product under a Hybrid Warranty. *Mathematics* **2022**, *10*, 1644. [CrossRef]
38. Shang, L.; Du, Y.; Wu, C.; Ma, C. A Bivariate Optimal Random Replacement Model for the Warranted Product with Job Cycles. *Mathematics* **2022**, *10*, 2225. [CrossRef]
39. Shang, L.; Yu, X.; Du, Y.; Zou, A.; Qiu, Q. An Optimal Random Hybrid Maintenance Policy of Systems under a Warranty with Rebate and Charge. *Mathematics* **2022**, *10*, 3229. [CrossRef]
40. Shang, L.; Yu, X.; Wang, L.; Du, Y. Design of Random Warranty and Maintenance Policy: From a Perspective of the Life Cycle. *Mathematics* **2022**, *10*, 3723. [CrossRef]
41. Shang, L.; Qiu, Q.; Wu, C.; Du, Y. Random replacement policies to sustain the post-warranty reliability. *J. Qual. Maint. Eng.* **2022**, *29*, 481–508. [CrossRef]
42. Shang, L.; Qiu, Q.; Wang, X. Random periodic replacement models after the expiry of 2D-warranty. *Comput. Ind. Eng.* **2022**, *164*, 107885. [CrossRef]
43. Shang, L.; Liu, B.; Cai, Z.; Wu, C. Random maintenance policies for sustaining the reliability of the product through 2D-warranty. *Appl. Math. Model.* **2022**, *111*, 363–383. [CrossRef]
44. Shang, L.; Zou, A.; Qiu, Q.; Du, Y. A random maintenance last model with preventive maintenance for the product under a random warranty. *Eksploat. Niezawodn.* **2022**, *24*, 544–553. [CrossRef]
45. Shang, L.; Wang, H.; Wu, C.; Cai, Z. The post-warranty random maintenance policies for the product with random working cycles. *Eksploat. Niezawodn.—Maint. Reliab.* **2021**, *23*, 726–735. [CrossRef]
46. Shang, L.; Liu, B.; Qiu, Q.; Yang, L.; Du, Y. Designing warranty and maintenance policies for products subject to random working cycles. *Reliab. Eng. Syst. Saf.* **2023**, *234*, 109187. [CrossRef]
47. Huang, Y.-S.; Huang, C.-D.; Ho, J.-W. A customized two-dimensional extended warranty with preventive maintenance. *Eur. J. Oper. Res.* **2017**, *257*, 971–978. [CrossRef]
48. Shang, L.; Shang, G.; Qiu, Q. A Bivariate Post-Warranty Maintenance Model for the Product under a 2D Warranty. *Mathematics* **2022**, *10*, 1957. [CrossRef]
49. Yang, L.; Wei, F.; Qiu, Q. Mission Risk Control via Joint Optimization of Sampling and Abort Decisions. *Risk Anal.* **2023**. [CrossRef]

50. Yang, L.; Chen, Y.; Zhou, S.; Ma, X.; Qiu, Q.; Peng, R. A Prognosis-centered Intelligent Maintenance Optimization Framework under Uncertain Failure Threshold. *IEEE Trans. Reliab.* **2023**. [CrossRef]
51. Zhuang, L.; Xu, A.; Wang, X.-L. A prognostic driven predictive maintenance framework based on Bayesian deep learning. *Reliab. Eng. Syst. Saf.* **2023**, *234*, 109181. [CrossRef]
52. de Pater, I.; Reijns, A.; Mitici, M. Alarm-based predictive maintenance scheduling for aircraft engines with imperfect Remaining Useful Life prognostics. *Reliab. Eng. Syst. Saf.* **2022**, *221*, 108341. [CrossRef]
53. Lu, B.; Chen, Z.; Zhao, X. Data-driven dynamic adaptive replacement policy for units subject to heterogeneous degradation. *Comput. Ind. Eng.* **2022**, *171*, 108478. [CrossRef]
54. Yang, L.; Zhao, Y.; Peng, R.; Ma, X. Opportunistic maintenance of production systems subject to random wait time and multiple control limits. *J. Manuf. Syst.* **2018**, *47*, 12–34. [CrossRef]
55. Yang, L.; Chen, Y.; Ma, X. A State-age-dependent Opportunistic Intelligent Maintenance Framework for Wind Turbines Under Dynamic Wind Conditions. *IEEE Trans. Ind. Inform.* **2023**, 1–9. [CrossRef]
56. Shang, L.; Zhang, N.; Yang, L.; Shang, L. Study of a Random Warranty Model Maintaining Fairness and a Random Replacement Next Model Sustaining Post-Warranty Reliability. *Axioms* **2023**, *12*, 258. [CrossRef]
57. Shang, L.; Liu, B.; Yang, L.; Gao, K. Customizing Random Replacement Model and Flexible Warranty Model from the Perspective of Screening Reliability. *Axioms* **2023**, *12*, 358. [CrossRef]
58. Shang, L.; Liu, B.; Qiu, Q.; Yang, L. Three-dimensional warranty and post-warranty maintenance of products with monitored mission cycles. *Reliab. Eng. Syst. Saf.* **2023**, *109506*. [CrossRef]
59. Wang, L.; Song, Y.; Zhang, W.; Ling, X. Condition-based inspection, component reallocation and replacement optimization of two-component interchangeable series system. *Reliab. Eng. Syst. Saf.* **2023**, *230*, 108907. [CrossRef]
60. Wu, T.; Wei, F.; Yang, L.; Ma, X.; Hu, L. Maintenance Optimization of k-Out-of-n Load-Sharing Systems Under Continuous Operation. *IEEE Trans. Syst. Man, Cybern. Syst.* **2023**, 1–13. [CrossRef]
61. Qiu, Q.; Cui, L.; Gao, H. Availability and maintenance modelling for systems subject to multiple failure modes. *Comput. Ind. Eng.* **2017**, *108*, 192–198. [CrossRef]
62. Qiu, Q.; Liu, B.; Lin, C.; Wang, J. Availability analysis and maintenance optimization for multiple failure mode systems considering imperfect repair. *Proc. Inst. Mech. Eng. Part O J. Risk Reliab.* **2021**, *235*, 982–997. [CrossRef]
63. Qiu, Q.; Cui, L.; Shen, J. Availability and maintenance modeling for systems subject to dependent hard and soft failures. *Appl. Stoch. Model. Bus. Ind.* **2018**, *34*, 513–527. [CrossRef]
64. Qiu, Q.; Cui, L.; Shen, J.; Yang, L. Optimal maintenance policy considering maintenance errors for systems operating under performance-based contracts. *Comput. Ind. Eng.* **2017**, *112*, 147–155. [CrossRef]
65. Qiu, Q.; Cui, L.; Kong, D. Availability and maintenance modeling for a two-component system with dependent failures over a finite time horizon. *Proc. Inst. Mech. Eng. Part O J. Risk Reliab.* **2018**, *233*, 200–210. [CrossRef]
66. Qiu, Q.; Cui, L. Availability analysis for general repairable systems with repair time threshold. *Commun. Stat.—Theory Methods* **2019**, *48*, 628–647. [CrossRef]
67. Qiu, Q.; Cui, L.; Shen, J. Availability analysis and maintenance modelling for inspected Markov systems with down time threshold. *Qual. Technol. Quant. Manag.* **2019**, *16*, 478–495. [CrossRef]
68. Qiu, Q.; Cui, L. Availability analysis for periodically inspected systems subject to multiple failure modes. *Int. J. Syst. Sci. Oper. Logist.* **2019**, *6*, 258–271. [CrossRef]
69. Qiu, Q.; Cui, L.; Kong, D. Availability analysis and optimal inspection policy for systems with neglected down time. *Commun. Stat.—Theory Methods* **2019**, *48*, 2787–2809. [CrossRef]
70. Barlow, R.E.; Proschan, F. *Mathematical Theory of Reliability*; John Wiley & Sons: Hoboken, NJ, USA, 1965.
71. Limnios, N.; Oprişan, G. *Semi-Markov Processes and Reliability*; Springer: Berlin/Heidelberg, Germany, 2001. [CrossRef]
72. Limnios, N. Reliability Measures of Semi-Markov Systems with General State Space. *Methodol. Comput. Appl. Probab.* **2012**, *14*, 895–917. [CrossRef]
73. Nakagawa, T.; Yasui, K. Periodic-replacement models with threshold levels. *IEEE Trans. Reliab.* **1991**, *40*, 395–397. [CrossRef]