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Neutral Differential Equations of Higher-Order in Canonical Form: Oscillation Criteria

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Abstract: This paper aims to study a class of neutral differential equations of higher-order in canonical form. By using the comparison technique, we obtain sufficient conditions to ensure that the studied differential equations are oscillatory. The criteria that we obtained are to improve and extend some of the results in previous literature. In addition, an example is given that shows the applicability of the results we obtained.

Keywords: higher-orders; neutral differential equations; oscillation criteria; canonical form

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1. Introduction

Consider the even-order DE with the neutral term

$$\left(\psi(t)\left(v^{(n-1)}(t)\right)^{\ell}\right)' + \phi(t)x^{i}(\delta(t)) = 0, \tag{1}$$

where $t \geq t_0 > 0$,

$$v(t) = x(t) + g(t)x(\sigma(t)), \tag{2}$$

and $n \geq 4$ is an even positive integer. We also suppose the following:

(M1) ℓ and i are quotients of odd positive integers;

(M2) $\psi \in C([t_0,\infty),(0,\infty))$, $\psi'(t) \geq 0$ and under the canonical form, that is

$$\int_{t_0}^{t} \frac{1}{\psi^{1/\ell}(\zeta)} d\zeta \to \infty \text{ as } t \to \infty; \tag{3}$$

(M3) $\delta, \sigma \in C([t_0,\infty),\mathbb{R})$, $\delta(t) \leq t$, $\sigma(t) \leq t$, $\sigma'(t) > 0$, and $\lim_{t\to\infty} \sigma(t) = \lim_{t\to\infty} \delta(t) = \infty$;

(M4) $\phi, g \in C([t_0,\infty),(0,\infty))$;

(M5)

$$\lim_{t\to\infty} \left(\frac{t}{\sigma(t)}\right)^{(n-1)/\varepsilon} \frac{1}{g(t)} = 0, \tag{4}$$

for some $\varepsilon \in (0,1)$.

Under the solution of (1), we mean a non-trivial function $x \in C([t_x, \infty), \mathbb{R})$, $t_x \geq t_0$, which has the properties $v(t) \in C^{n-1}([t_x, \infty), \mathbb{R})$, $\psi(t)\left(v^{(n-1)}(t)\right)^{\ell} \in C^1([t_x, \infty), \mathbb{R})$, and x satisfies (1) on $[t_x, \infty)$. Our attention is restricted to those solutions $x(t)$ of (1) satisfying $\sup\{|x(t)| : t \geq t_a\} > 0$ for all $t_a \geq t_x$, and we assume that (1) possesses such solutions.

A solution x to (1) is referred to as oscillatory or non-oscillatory depending on whether it is essentially positive or negative. If all the solutions to an equation oscillate, the equation is said to be oscillator.

Differential equations have played a critical role in different sciences for a long time, and they are expected to continue being indispensable for future investigations. However, they often provide only an initial estimate of the systems being studied. To create more realistic models, the past states of these systems must be taken into account, necessitating the use of differential equations (DEs) with time delays.

In recent times, there has been a growing interest in the theory of oscillation in functional differential equations (FDEs) due to their numerous applications in various fields of science. As a result, we recommend that readers refer to [1–11] to learn about the various contributions to the study of oscillatory and non-oscillatory behaviour of DEs with different orders.

It is known that the neutral differential equation (NDE) has many applications in various sciences, but as a general rule, we find that they have specific properties, thus studying them is difficult in both aspects of ideas and techniques. These difficulties explain the relatively small number of works devoted to the investigation of the oscillatory properties of solutions to this type of equation.

Several researchers have investigated the oscillatory behaviour of even-order DEs under various conditions. For more information, see [12–20]. We mention in some detail:

Dzurina et al. [13] investigated the oscillatory properties of the DE

$$x^{(4)}(t) + w(t)x'(t) + \phi(t)x^{i}(\delta(t)) = 0, \tag{5}$$

where $w \in C([t_0, \infty))$ and $w(t)$ is positive.

For DEs of the form

$$\left(\psi(t)\left((x(t) + g(t)x(\sigma(t)))^{(n-1)}\right)^{\ell}\right)' + \phi(t)x^{\ell}(\delta(t)) = 0$$

some oscillation criteria were established by Bazighifan et al. [14], where $0 \leq g(t) < g_0 < \infty$ and (3) hold.

The oscillatory behaviour of NDEs

$$(x(t) + g(t)x(\sigma(t)))^{(n)} + \phi(t)x(\delta(t)) = 0 \tag{6}$$

was the focus of research by Agarwal et al. [15]. They introduced some new conditions that ensure that (6) is oscillatory. For the convenience of the reader, we mention one of the theorems.

Theorem 1. *Let $n \geq 4$ be even, (M3) and (M4) hold, and*

$$\delta(t) \leq \sigma(t), \quad 1 - \frac{\Upsilon^{n-1}(t)}{g(\sigma^{-1}(\sigma^{-1}(t)))} \geq 0.$$

Assume that $\rho, \varsigma \in C^1([t_0, \infty), (0, \infty))$ such that, for some $\lambda_0 \in (0,1)$,

$$\int^{\infty}\left(g_*(t)\phi(t)\rho(t)\frac{\left(\sigma^{-1}(\delta(t))\right)^{n-1}}{t^{n-1}} - \frac{(n-2)!}{4\lambda_0}\frac{\left((\rho'(t))_+\right)^2}{t^{n-2}\rho(t)}\right)\mathrm{d}t = \infty$$

and

$$\int^{\infty}\left(\varsigma(t)\frac{\int^{\infty}(\zeta-t)^{n-3}\phi(\zeta)g^{*}(\delta(\zeta))\frac{\sigma^{-1}(\delta(\zeta))}{\zeta}\mathrm{d}\zeta}{(n-3)!}-\frac{\left((\varsigma'(t))_{+}\right)^{2}}{4\varsigma(t)}\right)\mathrm{d}t=\infty,$$

where

$$g^{*}(t)=\frac{1}{g(\sigma^{-1}(t))}\left(1-\frac{\Upsilon(t)}{g(\sigma^{-1}(\sigma^{-1}(t)))}\right),$$

$$g_{*}(t)=\frac{1}{g(\sigma^{-1}(t))}\left(1-\frac{\Upsilon^{n-1}(t)}{g(\sigma^{-1}(\sigma^{-1}(t)))}\right)$$

and

$$\Upsilon(t)=\frac{\sigma^{-1}(\sigma^{-1}(t))}{\sigma^{-1}(t)}.$$

Then (6) is oscillatory.

Muhib et al. [18] took into account the oscillatory behaviour of the NDE

$$\left(\psi(t)\left(\left(x(t)+q_{*}(t)x^{i_{*}}(\sigma_{1}(t))+g(t)x^{\gamma}(\sigma(t))\right)^{(n-1)}\right)^{\ell}\right)'+f(t,x(\delta(t)))=0, \quad (7)$$

where γ and i_* are ratios of odd positive integers with $\gamma\geq 1$, $0<i_*<1$, $q_*(t)\in C([t_0,\infty),(0,\infty))$ and $f\in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$, and there exists $\phi\in C([t_0,\infty),(0,\infty))$ such that $|f(t,x)|\geq\phi(t)|x|^{i}$. For the convenience of the reader, we mention one of the theorems.

Theorem 2. *Assume that*

$$\lim_{t\to\infty}g(t)\left(t^{n-2}\int_{t_0}^{t}\frac{1}{\psi^{1/\ell}(\zeta)}\mathrm{d}\zeta\right)^{\gamma-1}=\lim_{t\to\infty}q_{*}(t)=0 \quad (8)$$

holds. If there exists $\xi\in C^{1}([t_0,\infty),\mathbb{R}^{+})$ such that

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(\xi(\zeta)\phi(\zeta)\epsilon^{i}\Omega(\zeta)-\frac{1}{(\ell+1)^{\ell+1}}\frac{(\xi'(\zeta))^{\ell+1}}{\Theta^{\ell}(\zeta)}\right)\mathrm{d}\zeta=\infty, \quad (9)$$

for all $\lambda\in(0,1)$, $\theta>0$ and for some $\epsilon\in(0,1)$, $k_1,k_2>0$, then (7) is oscillatory, where

$$\Theta(t):=\ell\lambda\theta\delta^{n-2}(t)\psi^{-1/\ell}(t)\delta'(t)$$

and

$$\Omega(t)=\begin{cases}k_1^{i-\ell} & \text{if } i\geq\ell;\\ k_2^{i-\ell}\left(t^{n-2}\right)^{i-\ell}\left(\int_{t_1}^{t}\frac{1}{\psi^{1/\ell}(\zeta)}\mathrm{d}\zeta\right)^{i-\ell} & \text{if } i<\ell.\end{cases}$$

Based on the literature mentioned earlier, our objective is to establish criteria for oscillation in (1) by comparing it to first-order delay DEs with known oscillatory properties. In the final part of the paper, we use an example to show how our conditions improve some of the relevant findings that have been published in the literature.

2. Preliminary Lemmas

The following lemmas are needed in order to arrive at our result:

Lemma 1 ([21])**.** *Let $\varpi\in C^{n}([t_0,\infty),(0,\infty))$. Assume that the derivative $\varpi^{(n)}(t)$ is of fixed sign and not identically zero on a sub-ray of $[t_0,\infty)$, and there exists a $t_x\geq t_0$ for all $t\geq t_1$ such that $\varpi^{(n-1)}(t)\varpi^{(n)}(t)\leq 0$. If $\lim_{t\to\infty}\varpi(t)\neq 0$, then for every $\lambda\in(0,1)$ there exists $t_\lambda\geq t_1$ such that*

$$|\varpi(t)| \geq \frac{\lambda}{(n-1)!} t^{n-1} \left| \varpi^{(n-1)}(t) \right|, \tag{10}$$

for all $t \geq t_\lambda$.

Lemma 2 ([22])**.** *Let the function $\varpi(t)$ be as in Lemma 1 for $t_x \geq t_y$, and $t_* \geq t_x$ be assigned to $\varpi(t)$ by Lemma 1. Then there exists a $t_{**} \geq t_*$ such that*

$$\frac{\varpi(t)}{\varpi'(t)} \geq \varepsilon \frac{t}{\kappa} \text{ for } t \geq t_{**}, \tag{11}$$

for every $\varepsilon \in (0,1)$.

Lemma 3 ([15])**.** *Let x be a positive solution of (1), and that (3) holds. Then, $\left(\psi(t)\left(v^{(n-1)}(t)\right)^{\ell}\right)' < 0$; furthermore, we find that there are the following two possible cases eventually:*

$$\begin{aligned} \textbf{(I)}\ v(t) &> 0,\ v'(t) > 0,\ v''(t) > 0,\ v^{(n-1)}(t) > 0,\ v^{(n)}(t) \leq 0, \\ \textbf{(II)}\ v(t) &> 0,\ v^{(j)}(t) > 0,\ v^{(j+1)}(t) < 0, \text{ for all odd } j \in \{1,2,...,n-3\}, \\ v^{(n-1)}(t) &> 0,\ v^{(n)}(t) \leq 0. \end{aligned}$$

3. Main Results

The oscillation criteria for (1) will now be presented.

Theorem 3. *Let conditions (M1)–(M5) and $i \geq 1$ hold. Assume that there exists $\mu \in C^1([t_0,\infty),(0,\infty))$ such that*

$$\mu'(t) > 0,\ \mu(t) < \sigma(t),\ \mu(t) \leq \delta(t) \text{ and } \lim_{t\to\infty} \mu(t) = \infty. \tag{12}$$

If there are no positive solutions of

$$\left(\psi(t)\left(v^{(n-1)}(t)\right)^{\ell}\right)' + \epsilon_1^i c^{i-1} \phi(t) g^{-i}\left(\sigma^{-1}(\delta(t))\right) v(q(t)) \leq 0 \tag{13}$$

and

$$\left(\psi(t)\left(v^{(n-1)}(t)\right)^{\ell}\right)' + \epsilon_2^i c^{i-1} \phi(t) g^{-i}\left(\sigma^{-1}(\delta(t))\right) v(q(t)) \leq 0, \tag{14}$$

then (1) is oscillatory, where $q(t) = \sigma^{-1}(\mu(t))$, $\epsilon_2, \epsilon_1 \in (0,1)$ and $c > 0$ is constant.

Proof. Assume that (1) possesses an eventually positive solution $x(t)$, say $x(t),\ x(\delta(t)),\ x(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$. From (2), we find

$$x(t) = \frac{v\left(\sigma^{-1}(t)\right) - x\left(\sigma^{-1}(t)\right)}{g(\sigma^{-1}(t))}$$

and so

$$x(t) \geq \frac{v\left(\sigma^{-1}(t)\right)}{g(\sigma^{-1}(t))} - \frac{v(h(t))}{g(\sigma^{-1}(t))g(h(t))}, \tag{15}$$

where $h(t) = \sigma^{-1}\left(\sigma^{-1}(t)\right)$. Suppose (I) holds. Since $(n-1) \geq \kappa \geq 3$, using Lemma 2, we have

$$\frac{v(t)}{v'(t)} \geq \varepsilon \frac{t}{\kappa} \geq \varepsilon \frac{t}{(n-1)}, \tag{16}$$

Now,

$$\begin{aligned} \left(\frac{v(t)}{t^{(n-1)/\varepsilon}}\right)' &= \frac{\varepsilon t^{(n-1)/\varepsilon} v'(t) - (n-1)v(t)t^{((n-1)/\varepsilon)-1}}{\varepsilon t^{2(n-1)/\varepsilon}} \\ &= \frac{\varepsilon t v'(t) - (n-1)v(t)}{\varepsilon t^{((n-1)/\varepsilon)+1}} \leq 0 \text{ for } t \geq t_\varepsilon. \end{aligned} \tag{17}$$

Since $\sigma(t) \leq t$ and $\sigma'(t) > 0$, σ^{-1} is increasing, and therefore $t \leq \sigma^{-1}(t)$. Thus,

$$\sigma^{-1}(t) \leq \sigma^{-1}\left(\sigma^{-1}(t)\right). \tag{18}$$

By using (17) and (18), we find

$$(h(t))^{(n-1)/\varepsilon} v\left(\sigma^{-1}(t)\right) \geq \left(\sigma^{-1}(t)\right)^{(n-1)/\varepsilon} v(h(t)). \tag{19}$$

From (15) and (19), we find

$$x(t) \geq \frac{v(\sigma^{-1}(t))}{g(\sigma^{-1}(t))}\left(1 - \left(\frac{(h(t))}{(\sigma^{-1}(t))}\right)^{(n-1)/\varepsilon} \frac{1}{g(h(t))}\right), \text{ for some } t_3 \geq t_\varepsilon. \tag{20}$$

From (M5), there exists an $\epsilon_1 \in (0,1)$, such that

$$\left(\frac{(h(t))}{(\sigma^{-1}(t))}\right)^{(n-1)/\varepsilon} \frac{1}{g(h(t))} \leq 1 - \epsilon_1. \tag{21}$$

Using (21) in (20) gives

$$x(t) \geq \frac{v(\sigma^{-1}(t))}{g(\sigma^{-1}(t))}\epsilon_1 \text{ for } t \geq t_4. \tag{22}$$

Using (1) and (22), we obtain

$$\left(\psi(t)\left(v^{(n-1)}(t)\right)^{\ell}\right)' + \epsilon_1^i \phi(t) g^{-i}\left(\sigma^{-1}(\delta(t))\right) v^i\left(\sigma^{-1}(\delta(t))\right) \leq 0. \tag{23}$$

Since $\mu(t) \leq \delta(t)$ and $v'(t) > 0$, inequality (23) becomes

$$\left(\psi(t)\left(v^{(n-1)}(t)\right)^{\ell}\right)' + \epsilon_1^i \phi(t) g^{-i}\left(\sigma^{-1}(\delta(t))\right) v^i\left(\sigma^{-1}(\mu(t))\right) \leq 0,\ t \geq t_4. \tag{24}$$

Since $v(t) > 0$ and $v'(t) > 0$ on $[t_4, \infty)$, there exists a $t_5 \geq t_4$ and a constant $c > 0$ such that

$$v(t) \geq c \text{ for } t \geq t_5. \tag{25}$$

From (24), (25) and $i \geq 1$, we find

$$\left(\psi(t)\left(v^{(n-1)}(t)\right)^{\ell}\right)' + \epsilon_1^i c^{i-1} \phi(t) g^{-i}\left(\sigma^{-1}(\delta(t))\right) v(q(t)) \leq 0,\ t \geq t_5. \tag{26}$$

has a positive solution v. That is, (13) also possesses a solution that is positive, and thus we arrive at a contradiction.

Next, suppose (II) holds. Since $\kappa = 1$, using Lemma 2, we have

$$\frac{v(t)}{v'(t)} \geq \varepsilon\frac{t}{1}, t \geq t_\varepsilon, \tag{27}$$

from which we obtain

$$\left(\frac{v(t)}{t^{1/\varepsilon}}\right)' = \frac{\varepsilon t^{1/\varepsilon} v'(t) - v(t) t^{(1/\varepsilon)-1}}{\varepsilon t^{2/\varepsilon}} = \frac{\varepsilon t v'(t) - v(t)}{\varepsilon t^{1+(1/\varepsilon)}} \leq 0 \text{ for } t \geq t_\varepsilon. \tag{28}$$

By (18) and (28),

$$\left(\sigma^{-1}(t)\right)^{1/\varepsilon} v(h(t)) \leq (h(t))^{1/\varepsilon} v\left(\sigma^{-1}(t)\right), \tag{29}$$

for some $t_3 \geq t_\varepsilon$. Combining (15) and (29), we obtain

$$x(t) \geq \frac{v(\sigma^{-1}(t))}{g(\sigma^{-1}(t))}\left(1 - \left(\frac{h(t)}{\sigma^{-1}(t)}\right)^{1/\varepsilon} \frac{1}{g(h(t))}\right), \ t \geq t_3. \tag{30}$$

From (M5), for any $\epsilon_2 \in (0,1)$ there exists $t_5 \geq t_4$ such that

$$\left(\frac{h(t)}{\sigma^{-1}(t)}\right)^{1/\varepsilon} \frac{1}{g(h(t))} \leq 1 - \epsilon_2, \ t \geq t_5,$$

and using this in (30) implies

$$x(t) \geq \frac{\epsilon_2 v(\sigma^{-1}(t))}{g(\sigma^{-1}(t))}, \text{ for } t \geq t_5. \tag{31}$$

Using (31) in (1) yields

$$\left(\psi(t)\left(v^{(n-1)}(t)\right)^{\ell}\right)' + \epsilon_2^i \phi(t) g^{-i}\left(\sigma^{-1}(\delta(t))\right) v^i\left(\sigma^{-1}(\delta(t))\right) \leq 0. \tag{32}$$

Since $v'(t) > 0$ and $\mu(t) \leq \delta(t)$, (32) takes the form

$$\left(\psi(t)\left(v^{(n-1)}(t)\right)^{\ell}\right)' + \epsilon_2^i \phi(t) g^{-i}\left(\sigma^{-1}(\delta(t))\right) v^i\left(\sigma^{-1}(\mu(t))\right) \leq 0. \tag{33}$$

In view of (25) and $i \geq 1$, we find

$$\left(\psi(t)\left(v^{(n-1)}(t)\right)^{\ell}\right)' + \epsilon_2^i c^{i-1} \phi(t) g^{-i}\left(\sigma^{-1}(\delta(t))\right) v(q(t)) \leq 0, \ t \geq t_5. \tag{34}$$

has a positive solution v. That is, (14) also possesses a solution that is positive, and thus we arrive at a contradiction. Here, the proof ends. □

Theorem 4. *Let conditions (M1)–(M5) and $i < 1$ hold. Assume that there exists $\mu \in C^1([t_0,\infty),(0,\infty))$ such that (12) holds. If there are no positive solutions of*

$$\left(\psi(t)\left(v^{(n-1)}(t)\right)^{\ell}\right)' + \epsilon_1^i d_1^{i-1}\left(q^{(n-1)/\varepsilon}(t)\right)^{i-1} \phi(t) g^{-i}\left(\sigma^{-1}(\delta(t))\right) v(q(t)) \leq 0 \tag{35}$$

and

$$\left(\psi(t)\left(v^{(n-1)}(t)\right)^{\ell}\right)' + \epsilon_2^i d_2^{i-1}\left(q^{1/\varepsilon}(t)\right)^{i-1} \phi(t) g^{-i}\left(\sigma^{-1}(\delta(t))\right) v(q(t)) \leq 0, \tag{36}$$

then (1) is oscillatory, where $q(t) = \sigma^{-1}(\mu(t))$, $\epsilon_2, \epsilon_1 \in (0,1)$ and $d_1, d_2 > 0$ are constant.

Proof. Assume that (1) possesses an eventually positive solution $x(t)$, say $x(t)$, $x(\delta(t))$, $x(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$.

Suppose (I) holds. By applying the same processes used in the proof of Theorem 3, we obtain (17) and (24). By (17), there exists $t_3 \geq t_\varepsilon$ and a constant $d_1 > 0$ such that

$$\frac{v(t)}{t^{(n-1)/\varepsilon}} \leq d_1$$

and so

$$v(t) \leq d_1 t^{(n-1)/\varepsilon} \text{ for } t \geq t_3. \tag{37}$$

Using (37) in (24), we have

$$\left(\psi(t)\left(v^{(n-1)}(t)\right)^{\ell}\right)' + \epsilon_1^i d_1^{i-1}\left(q^{(n-1)/\varepsilon}(t)\right)^{i-1}\phi(t)g^{-i}\left(\sigma^{-1}(\delta(t))\right)v(q(t)) \leq 0, \text{ for some } t_4 \geq t_3. \tag{38}$$

That is, (35) possesses a solution that is positive, and thus we arrive at a contradiction.

Next, suppose (II) holds. By applying the same processes used in the proof of Theorem 3, we obtain (28) and (33). By (28), there exists $t_3 \geq t_\varepsilon$ and a constant $d_2 > 0$ such that

$$\frac{v(t)}{t^{1/\varepsilon}} \leq d_2$$

and so

$$v(t) \leq d_2 t^{1/\varepsilon} \text{ for } t \geq t_3. \tag{39}$$

Using (39) in (33), we have

$$\left(\psi(t)\left(v^{(n-1)}(t)\right)^{\ell}\right)' + \epsilon_2^i d_2^{i-1}\left(q^{1/\varepsilon}(t)\right)^{i-1}\phi(t)g^{-i}\left(\sigma^{-1}(\delta(t))\right)v(q(t)) \leq 0, \text{ for } t \geq t_5. \tag{40}$$

That is, (36) possesses a solution that is positive, and thus we arrive at a contradiction. Here, the proof ends. □

Theorem 5. *Let conditions (M1)–(M5) and $i \geq 1$ hold. Assume that there exists $\mu \in C^1([t_0,\infty),(0,\infty))$ such that (12) holds. If*

$$y'(t) + \epsilon_1^i c^{i-1}\frac{\lambda_1}{(n-1)!}\frac{q^{n-1}(t)}{\psi^{1/\ell}(q(t))}\phi(t)g^{-i}\left(\sigma^{-1}(\delta(t))\right)y^{1/\ell}(q(t)) = 0 \tag{41}$$

and

$$\omega'(t) + \epsilon_2^{i/\ell}c^{(i-1)/\ell}\varepsilon_1^{1/\ell}q^{1/\ell}(t)R_{n-3}(t)\omega^{1/\ell}(q(t)) = 0 \tag{42}$$

are oscillatory, for some constants $\lambda_1, \varepsilon_1 \in (0,1)$, then (1) is oscillatory, where

$$R_0(t) = \left(\frac{1}{\psi(t)}\int_t^\infty \phi(\zeta)g^{-i}\left(\sigma^{-1}(\delta(\zeta))\right)\mathrm{d}\zeta\right)^{1/\ell},$$

$$R_m(t) = \int_t^\infty R_{m-1}(\zeta)\mathrm{d}\zeta, \text{ for } m = 1,2,\ldots n-3,$$

$\epsilon_2, \epsilon_1 \in (0,1)$ and $c > 0$ is constant.

Proof. Assume that (1) possesses an eventually positive solution $x(t)$, say $x(t)$, $x(\delta(t))$, $x(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$.

Suppose (I) holds. By applying the same processes used in the proof of Theorem 3, we obtain (26). Now, by Lemma 1, we have

$$v(t) \geq \frac{\lambda}{(n-1)!}t^{n-1}v^{(n-1)}(t) \text{ for } t \geq t_\hbar \text{ and } t_\hbar \geq t_5, \tag{43}$$

and so

$$v(q(t)) \geq \frac{\lambda}{(n-1)!} q^{n-1}(t) v^{(n-1)}(q(t)) \text{ for } t \geq t_6, \tag{44}$$

for some $t_6 \geq t_\lambda$. From (44), (26), we obtain

$$\left(\psi(t)\left(v^{(n-1)}(t)\right)^{\ell}\right)' + \epsilon_1^i c^{i-1} \frac{\lambda}{(n-1)!} q^{n-1}(t)\phi(t) g^{-i}\left(\sigma^{-1}(\delta(t))\right) v^{(n-1)}(q(t)) \leq 0.$$

If we set $y(t) = \psi(t)\left(v^{(n-1)}(t)\right)^{\ell}$, then $y(t)$ is a positive solution of

$$y'(t) + \epsilon_1^i c^{i-1} \frac{\lambda}{(n-1)!} \frac{q^{n-1}(t)}{\psi^{1/\ell}(q(t))} \phi(t) g^{-i}\left(\sigma^{-1}(\delta(t))\right) y^{1/\ell}(q(t)) \leq 0, \ t \geq t_6. \tag{45}$$

It follows from [23] that (41) also possesses a solution that is positive, and thus we arrive at a contradiction with (41).

Next, suppose (II) holds. By applying the same processes used in the proof of Theorem 3, we obtain (27) and (34). Integrating (34) from $t \geq t_5$ to ∞ gives

$$\left(v^{(n-1)}(t)\right)^{\ell} \geq \epsilon_2^i c^{i-1} \frac{v(q(t))}{\psi(t)} \int_t^{\infty} \phi(\zeta) g^{-i}\left(\sigma^{-1}(\delta(\zeta))\right) \mathrm{d}\zeta$$

and so

$$v^{(n-1)}(t) \geq \epsilon_2^{i/\ell} c^{(i-1)/\ell} R_0(t) v^{1/\ell}(q(t)). \tag{46}$$

Integrating (46) from t to ∞ a total of $n-3$ times, we have

$$-v''(t) \geq \epsilon_2^{i/\ell} c^{(i-1)/\ell} R_{n-3}(t) v^{1/\ell}(q(t))$$

and so

$$v''(t) + \epsilon_2^{i/\ell} c^{(i-1)/\ell} R_{n-3}(t) v^{1/\ell}(q(t)) \leq 0. \tag{47}$$

Using (27) in (47) yields

$$v''(t) + \epsilon_2^{i/\ell} c^{(i-1)/\ell} \varepsilon^{1/\ell} q^{1/\ell}(t) R_{n-3}(t) \left(v'(q(t))\right)^{1/\ell} \leq 0. \tag{48}$$

If we set $\omega(t) = v'(t)$, then $\omega(t)$ is a positive solution of

$$\omega'(t) + \epsilon_2^{i/\ell} c^{(i-1)/\ell} \varepsilon^{1/\ell} q^{1/\ell}(t) R_{n-3}(t) \omega^{1/\ell}(q(t)) \leq 0, \tag{49}$$

for every $\varepsilon \in (0,1)$. We complete the proof in the same way as in case (I). Here, the proof ends. □

Corollary 1. *Let conditions (M1)–(M5), $\ell = 1$ and $i \geq 1$ hold. Assume that there exists $\mu \in C^1([t_0,\infty),(0,\infty))$ such that (12) holds. If*

$$\lim_{t\to\infty} \int_{q(t)}^{t} \frac{q^{n-1}(\zeta)}{\psi(q(\zeta))} \phi(\zeta) g^{-i}\left(\sigma^{-1}(\delta(\zeta))\right) \mathrm{d}\zeta = \infty \tag{50}$$

and

$$\lim_{t\to\infty} \int_{q(t)}^{t} q(\zeta) R_{n-3}(\zeta) \mathrm{d}\zeta = \infty, \tag{51}$$

then Equation (1) is oscillatory.

Proof. Suppose (I) holds. By applying the same processes used in the proof of Theorem 5, we obtain (45). Integrating (45) from $q(t)$ to t and using $\ell = 1$ and the fact that $y' < 0$, we obtain

$$\int_{q(t)}^{t} y'(\zeta)\mathrm{d}\zeta \leq -\epsilon_1^i c^{i-1}\frac{\lambda}{(n-1)!}y(q(t))\int_{q(t)}^{t}\frac{q^{n-1}(\zeta)}{\psi(q(\zeta))}\phi(\zeta)g^{-i}\Big(\sigma^{-1}(\delta(\zeta))\Big)\mathrm{d}\zeta$$

and so

$$-y(q(t)) \leq -\epsilon_1^i c^{i-1}\frac{\lambda}{(n-1)!}y(q(t))\int_{q(t)}^{t}\frac{q^{n-1}(\zeta)}{\psi(q(\zeta))}\phi(\zeta)g^{-i}\Big(\sigma^{-1}(\delta(\zeta))\Big)\mathrm{d}\zeta,$$

and this can be expressed as follows

$$\frac{(n-1)!}{\epsilon_1^i c^{i-1}\lambda} \geq \int_{q(t)}^{t}\frac{q^{n-1}(\zeta)}{\psi(q(\zeta))}\phi(\zeta)g^{-i}\Big(\sigma^{-1}(\delta(\zeta))\Big)\mathrm{d}\zeta,$$

which contradicts (50).

Next, suppose (II) holds. By applying the same processes used in the proof of Theorem 5, we obtain (49). Integrating (49) from $q(t)$ to t and using $\ell = 1$ and the fact that $\omega' < 0$, we obtain

$$\int_{q(t)}^{t}\omega'(\zeta)\mathrm{d}\zeta \leq -\epsilon_2^i c^{i-1}\varepsilon\omega(q(t))\int_{q(t)}^{t} q(\zeta)R_{n-3}(\zeta)\mathrm{d}\zeta$$

and so

$$-\omega(q(t)) \leq -\epsilon_2^i c^{i-1}\varepsilon\omega(q(t))\int_{q(t)}^{t} q(\zeta)R_{n-3}(\zeta)\mathrm{d}\zeta,$$

and this can be expressed as follows

$$\frac{1}{\epsilon_2^i c^{i-1}\varepsilon} \geq \int_{q(t)}^{t} q(\zeta)R_{n-3}(\zeta)\mathrm{d}\zeta,$$

which contradicts (51). Here, the proof ends. □

Theorem 6. *Let conditions (M1)–(M5) and $i < 1$ hold. Assume that there exists $\mu \in C^1([t_0,\infty),(0,\infty))$ such that (12) holds. If*

$$y'(t) + \frac{\epsilon_1^i d_1^{i-1}\lambda_1}{(n-1)!}\frac{\Big(q^{(n-1)/\varepsilon}(t)\Big)^{i-1} q^{n-1}(t)\phi(t)g^{-i}\big(\sigma^{-1}(\delta(t))\big)}{\psi^{1/\ell}(q(t))}y^{1/\ell}(q(t)) = 0 \tag{52}$$

and

$$\omega'(t) + \epsilon_2^{i/\ell}d_2^{(i-1)/\ell}\varepsilon_1^{1/\ell}q^{1/\ell}(t)F_{n-3}(t)\omega^{1/\ell}(q(t)) = 0 \tag{53}$$

are oscillatory, for some constants $\lambda_1, \varepsilon_1 \in (0,1)$, then (1) is oscillatory, where

$$F_0(t) = \left(\frac{1}{\psi(t)}\int_t^\infty \Big(q^{1/\varepsilon}(\zeta)\Big)^{i-1}\phi(\zeta)g^{-i}\Big(\sigma^{-1}(\delta(\zeta))\Big)\mathrm{d}\zeta\right)^{1/\ell},$$

$$F_m(t) = \int_t^\infty F_{m-1}(\zeta)\mathrm{d}\zeta \text{ for } m = 1,2,\ldots,n-3,$$

$\epsilon_2, \epsilon_1 \in (0,1)$ and $d_1, d_2 > 0$ are constant.

Proof. Assume that (1) possesses an eventually positive solution $x(t)$, say $x(t),\ x(\delta(t)),\ x(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$.

Suppose (I) holds. By applying the same processes used in the proof of Theorem 4, we obtain (38). Now, by Lemma 1, we see that (43) holds. Using (43) in (38) we have

$$\Big(\psi(t)\Big(v^{(n-1)}(t)\Big)^{\ell}\Big)' + \frac{\epsilon_1^i d_1^{i-1}\lambda}{(n-1)!}\Big(q^{(n-1)/\varepsilon}(t)\Big)^{i-1}q^{n-1}(t)\phi(t)g^{-i}\Big(\sigma^{-1}(\delta(t))\Big)v^{(n-1)}(q(t)) \leq 0,$$

for $t \geq t_4$. If we set $y(t) = \psi(t)\left(v^{(n-1)}(t)\right)^{\ell}$, then $y(t)$ is a positive solution of

$$y'(t) + \frac{\epsilon_1^i d_1^{i-1}\lambda}{(n-1)!} \frac{\left(q^{(n-1)/\varepsilon}(t)\right)^{i-1} q^{n-1}(t)\phi(t)g^{-i}(\sigma^{-1}(\delta(t)))}{\psi^{1/\ell}(q(t))} y^{1/\ell}(q(t)) \leq 0. \tag{54}$$

It follows from [23] that (52) also possesses a solution that is positive, thus arriving at a contradiction with (52).

Next, suppose (II) holds. Then again (27) holds. By applying the same processes used in the proof of Theorem 4, we obtain (40). Integrating (40) from $t \geq t_5$ to ∞, we obtain

$$-\psi(t)\left(v^{(n-1)}(t)\right)^{\ell} + \epsilon_2^i d_2^{i-1} v(q(t)) \int_t^{\infty} \left(q^{1/\varepsilon}(\zeta)\right)^{i-1} \phi(\zeta) g^{-i}\left(\sigma^{-1}(\delta(\zeta))\right) \mathrm{d}\zeta \leq 0$$

and so

$$v^{(n-1)}(t) \geq \epsilon_2^{i/\ell} d_2^{(i-1)/\ell} F_0(t) v^{1/\ell}(q(t)). \tag{55}$$

Integrating (55) from t to ∞ a total of $n-3$ times, we have

$$-v''(t) \geq \epsilon_2^{i/\ell} d_2^{(i-1)/\ell} F_{n-3}(t) v^{1/\ell}(q(t))$$

and so

$$v''(t) + \epsilon_2^{i/\ell} d_2^{(i-1)/\ell} F_{n-3}(t) v^{1/\ell}(q(t)) \leq 0, \ t \geq t_5. \tag{56}$$

Now, with $\omega(t) = v'(t)$ and from (27), we find that (56) becomes

$$\omega'(t) + \epsilon_2^{i/\ell} d_2^{(i-1)/\ell} \varepsilon^{1/\ell} q^{1/\ell}(t) F_{n-3}(t) \omega^{1/\ell}(q(t)) \leq 0, \tag{57}$$

with ω as a positive solution of (57). We complete the proof in the same way as in case (I). Here, the proof ends. □

Corollary 2. *Let conditions (M1)–(M5), $\ell = 1$ and $i < 1$ hold. Assume that there exists $\mu \in C^1([t_0, \infty), (0, \infty))$ such that (12) holds. If*

$$\lim_{t \to \infty} \int_{q(t)}^{t} \frac{\left(q^{(n-1)/\varepsilon}(\zeta)\right)^{i-1} q^{n-1}(\zeta)\phi(\zeta)g^{-i}(\sigma^{-1}(\delta(\zeta)))}{\psi(q(\zeta))} \mathrm{d}\zeta = \infty \tag{58}$$

and

$$\lim_{t \to \infty} \int_{q(t)}^{t} q(\zeta) F_{n-3}(\zeta) \mathrm{d}\zeta = \infty, \tag{59}$$

then Equation (1) is oscillatory.

Proof. The proof is similar to Corollary 1, and therefore the details are omitted. □

We use the example below to illustrate our results.

Example 1. *Let us consider the NDE*

$$\left(x(t) + e^t x\left(\frac{t}{A_1}\right)\right)^{(4)} + \frac{\phi_0}{t^3 e^{-A_1 t/A_2}} x\left(\frac{t}{A_2}\right) = 0, \ t \geq 1. \tag{60}$$

It is easy to verify that

$$\int_{t_0}^{\infty} \frac{1}{\psi^{1/\ell}(\zeta)} \mathrm{d}\zeta = \infty.$$

Choosing $\mu(t) = t/A_3$, where $A_3 > A_1$ and $A_3 \geq A_2$, then (12) holds. We also find that

$$\sigma^{-1}(t) = A_1 t,\ q(t) = \frac{A_1}{A_3}t,\ \text{and}\ \sigma^{-1}(\delta(t)) = \frac{A_1}{A_2}t.$$

By choosing $\varepsilon = 1/4$, we find that (4) holds.
Now, we note that Condition (50) is satisfied, where

$$\begin{aligned}\lim_{t\to\infty}\int_{q(t)}^{t}\frac{q^{n-1}(\zeta)}{\psi(q(\zeta))}\phi(\zeta)g^{-i}\left(\sigma^{-1}(\delta(\zeta))\right)\mathrm{d}\zeta &= \lim_{t\to\infty}\int_{A_1t/A_3}^{t}\left(\frac{A_1}{A_3}\zeta\right)^3\frac{\phi_0}{\zeta^3 e^{-A_1\zeta/A_2}}\frac{1}{e^{A_1\zeta/A_2}}\mathrm{d}\zeta\\ &= \lim_{t\to\infty}\int_{A_1t/A_3}^{t}\left(\frac{A_1}{A_3}\right)^3\phi_0\mathrm{d}\zeta = \infty.\end{aligned}$$

Moreover, by a simple computation, we have that

$$\begin{aligned}R_0(t) &= \left(\frac{1}{\psi(t)}\int_t^\infty \phi(\zeta)g^{-i}\left(\sigma^{-1}(\delta(\zeta))\right)\mathrm{d}\zeta\right)^{1/\ell} = \int_t^\infty \frac{\phi_0}{\zeta^3 e^{-A_1\zeta/A_2}}\frac{1}{e^{A_1\zeta/A_2}}\mathrm{d}\zeta\\ &= \frac{\phi_0}{2}\frac{1}{t^2}\end{aligned}$$

and

$$R_1(t) = \int_t^\infty R_0(\zeta)\mathrm{d}\zeta = \int_t^\infty \frac{\phi_0}{2}\frac{1}{\zeta^2}\mathrm{d}\zeta = \frac{\phi_0}{2}\frac{1}{\zeta}.$$

Thus, we note that Condition (51) is satisfied, where

$$\begin{aligned}\lim_{t\to\infty}\int_{q(t)}^{t} q(\zeta)R_{n-3}(\zeta)\mathrm{d}\zeta &= \lim_{t\to\infty}\int_{A_1t/A_3}^{t}\frac{A_1}{A_3}\zeta\frac{\phi_0}{2}\frac{1}{\zeta}\mathrm{d}\zeta\\ &= \lim_{t\to\infty}\int_{A_1t/A_3}^{t}\frac{A_1}{A_3}\frac{\phi_0}{2}\mathrm{d}\zeta = \infty.\end{aligned}$$

Thus, using Corollary 1, we find that (60) is oscillatory.

Remark 1. *In Equation (7), if $q_*(t) = 0$ and $\gamma = 1$, we find that Equation (7) is identical to Equation (1). In this case, if we apply Theorem 2 to Equation (60), we find that it fails in the oscillation test for Equation (60), while when using the results we obtained we find that Equation (60) is oscillatory.*

Remark 2. *If we set $A_1 = 5$ and $A_2 = 4$ in (60), we note that in this case Theorem 1 cannot be applied to (60), while when using the results we obtained we find that Equation (60) is oscillatory.*

Remark 3. *It is easy to see that Equation (60) in Example 1 oscillates at any value of $\phi_0 > 0$. Furthermore, through this paper, we were able to extend previous results in the literature, which can be applied more widely compared to [14,15,19].*

4. Conclusions

The focus of this paper was to investigate the oscillatory behaviour of NDEs of even-order under the Condition (3). By using comparison principles with the first-order DEs, we offer some new sufficient conditions which ensure that any solution to (1) oscillates. Further, the results in [15,18] cannot apply to the example. In future studies, we aim to establish further criteria for the oscillation of Equation (1) when

$$\int_{t_0}^\infty \frac{1}{\psi^{1/\ell}(\zeta)}\mathrm{d}\zeta < \infty.$$

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