mathematics

Article

A Parametric Family of Fuzzy Similarity Measures for Intuitionistic Fuzzy Sets

Madiha Qayyum [1], Etienne E. Kerre [2,*] and Samina Ashraf [3]

[1] Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Lahore 54000, Pakistan; mqayyum@cuilahore.edu.pk
[2] Department of Applied Mathematics, Computer Science and Statistics, Ghent University, Krijgslaan 281, S9, B-9000 Gent, Belgium
[3] Department of Mathematics, University of Education, Bank Road Campus, Lahore 54000, Pakistan
* Correspondence: etienne.kerre@ugent.be

Abstract: Measuring the similarity between two objects and classifying them on the basis of their resemblance level has been a fundamental tool of the human mind. In an intuitionistic fuzzy environment, we find researchers that have attempted to generalize the fuzzy versions of similarity measures between fuzzy sets to their intuitionistic forms for measuring the level of similarity between the intuitionistic fuzzy sets. Though many different forms of intuitionistic fuzzy similarity measures have been introduced so far, a comparative study reveals that among all these measures, it is difficult for one to claim the existence of a single measure that alone has the capability to recognize every single pattern assigned to it. This paper presents a four-parametric family of similarity measures for intuitionistic fuzzy sets employing weighted average cardinality and intuitionistic fuzzy t-norms along with their dual t-co-norms. A combinational variation of the parameters involved in this family resulted in some of the famous similarity measures having an intuitionistic version. These new measures are analyzed for their properties, and they have shown some remarkable results. Moreover, the proposed family has a practical advantage over the other measures in the existing literature because every member not only possesses the capability of successfully recognizing any pattern assigned to it up to a fine accuracy but also a choice of different t-norms and co-norms within a single measure equips it with the capacity to portray different mindsets of a decision-maker who, besides being unbiased, can possess a deep psychology of being an optimist, pessimist, or possessing neutral behavior in general. Lastly, the members of this family are tested for their feasibility in a sensitive medical decision process of detection of COVID-19.

Keywords: intuitionistic fuzzy set (IFS); weighted average cardinality measure (WACM); fuzzy measure for IFS; pattern recognition

MSC: 28E10; 03E72; 94D05

Citation: Qayyum, M.; Kerre, E.E.; Ashraf, S. A Parametric Family of Fuzzy Similarity Measures for Intuitionistic Fuzzy Sets. *Mathematics* **2023**, *11*, 3163. https://doi.org/10.3390/math11143163

Academic Editors: Vassia Atanassova, Tania Pencheva and Peter Vassilev

Received: 25 May 2023
Revised: 6 July 2023
Accepted: 11 July 2023
Published: 19 July 2023

1. Introduction and Preliminaries

In the literature on fuzzy mathematics, we find numerous definitions of fuzzy similarity measures that naturally restrict a value in $[0, 1]$ giving two popular yet structurally different approaches. Among these, the first one comprises of an axiomatic structure-based approach [1–3], while the other one concentrates on developing new formulas for fuzzy similarity measures based on fuzzification of the existing similarity measures for ordinary sets [4–6]. This second approach to building similarity measures for fuzzy sets from ordinary sets has utilized distances between sets as well as sets of theoretic constructions and gained equal popularity in terms of practical applications in various scientific domains.

To be more specific for ordinary sets, a set theoretic-based approach to building a similarity relation between sets initially builds a feature vector based on an appropriate set of attributes/features apparently common in both the objects/sets under consideration.

This feature vector has binary entries exhibiting the existence (1) or non-existence (0) of every feature under consideration. A comparison (similarity or inclusion) between the two feature vectors are equivalent to a process of generating a comparison of the two objects themselves on the basis of each attribute one by one. This in turn, provides us with the degree of similarity between objects in context of each attribute individually and is naturally in the form of a vector with dimension the same as the feature vector.

The comparison of two objects under consideration on the basis of attributes is a local or pointwise phenomenon which, however, requires a global or overall view of similarity. This final number is called the "Similarity Degree of two objects", which is formulated by utilizing a cardinality measure of the underlying ordinary sets. Similarly, in a fuzzy environment, these measures involving cardinalities employ feature vectors that have entries from $[0,1]$ representing the fuzziness of the sets involved. The $[0,1]$-valued similarity measure called fuzzy similarity measure developed by this method originated as a generalization of some of the famous similarity measures for crisp sets [5–7].

Likewise, in an intuitionistic fuzzy environment, we find researchers working in this area that have attempted to generalize the measure of similarity of fuzzy sets to their intuitionistic forms [8–14]. Though many different forms of intuitionistic fuzzy similarity measures have been introduced so far, yet a comparative study reveals that among all these measures, it is difficult for one to claim the existence of a single measure which alone has the capability of recognizing every single pattern assigned to it. Moreover, unlike their fuzzy counterparts, most of these measures for intuitionistic fuzzy sets are distance-based and hardly any having set-theoretic constructions involved.

Therefore in this paper, we have focused on the development of a novel four-parameter class of fuzzy similarity measures for intuitionistic fuzzy sets employing a feature vector having an intuitionistic fuzzy framework, i.e., every element within the feature vector is characterized by two degrees (membership and nonmembership) exclusively from $[0,1]$ obeying the condition that demands their sum not to exceed one. Thus, we can say that a feature comparison vector that represents an intuitionistic fuzzy set has components that are naturally scaled to the lattice $\Upsilon = \{\mu = (\mu_1, \mu_2) \in [0,1]^2 \mid \mu_1 + \mu_2 \leq 1\}$. Furthermore, we have studied the properties of the members of the new class of measures and found them to obey most of the axioms of the similarity measure existing in the literature. In addition, we have discovered a special feature of this class of measures which distinct it from the other parallel similarity measures existing in the literature, i.e., its every member not only possesses the capability of successfully recognizing any pattern assigned to it up to a fine accuracy but also a choice of different t-norms and co-norms within a single member equips it with a capacity to portray the different mind sets of a decision-maker who, besides being unbiased, can possess a deep psychology of being optimist, pessimist, or possessing neutral behavior in general without disturbing the final output. Moreover, we have utilized some members of this family to build a medical diagnosis technique for COVID-19.

We have organized our work as follows: in Section 1, a brief introduction of the fuzzy and intuitionistic fuzzy concepts involved in this work are presented. Moreover, as the new similarity measures required set theoretic based operations; therefore, in an attempt to define these measures, we have generalized some of the existing crisp set-theoretic operations and identities to their respective intuitionistic fuzzy set versions. After defining these necessary operations, the new parametric family of similarity measures for intuitionistic fuzzy sets are introduced in Section 2 and studied for its properties, while in the third and last section of this work we have presented the effective utility of these new measures of similarity in initial diagnosis of COVID-19, among other close viral infections.

Let $X = \{x_1, x_2, \ldots x_n\}$ be a universe and $(\Upsilon, \leq_\Upsilon)$ be a complete bounded lattice given by set: $\Upsilon = \{\mu = (\mu_1, \mu_2) \in [0,1]^2 \mid \mu_1 + \mu_2 \leq 1\}$ with order $\leq_\Upsilon$ defined as $(\mu_1, \mu_2) \leq_\Upsilon (\nu_1, \nu_2)$ if $\mu_1 \leq \nu_1$ and $\mu_2 \geq \nu_2$.This lattice Υ has elements $0_\Upsilon = (0,1)$ and $1_\Upsilon = (1,0)$ that act as its smallest and largest elements. An intuitionistic fuzzy set (*IFS*) E on X is defined as a map $E : X \longrightarrow \Upsilon$ given by $E(x) = ((\delta_E(x), \eta_E(x)) = (\mu_1, \mu_2) \in \Upsilon$ satisfying $\mu_1 + \mu_2 \leq 1; \forall x \in X$. However, it may be defined alternatively by its founder as:

Definition 1 ([15])**.** *Let X be the universe of discourse then an intuitionistic fuzzy set (IFS) on X is given by* $E = \{(x, \delta_E(x), \eta_E(x)) \mid x \in X\}$*. Here,* $\delta_E(x)$ *and* $\eta_E(x) \in [0,1]$ *are the functions representing the membership degree and the nonmembership degree of x in the set E, such that* $(\forall x \in X)(\delta_E(x) + \eta_E(x) \leq 1)$*. The complement of E is defined as:* $E^c = \{(x, \eta_E(x), \delta_E(x)) \mid x \in X\}$*. The set of all IFS on X is denoted by* $IFS(X)$*.*

Definition 2 ([16])**.** *The weighted average cardinality of IFS is a map* $Card_\omega : IFS(X) \longrightarrow [0, \infty)$ *defined as:*

$$\begin{aligned} Card_\omega(E) &= \sum_{x \in X} \omega\delta_E(x) + (1-\omega)(1-\eta_E(x)); \\ \omega &\in [0.5, 1]. \end{aligned}$$

Throughout this study, $Card_{0.5}(E)$ will be regarded as the *scalar cardinality* of E and will be $|E|$.

Definition 3 ([17])**.** *A map* $\check{T} : (\Upsilon)^2 \longrightarrow \Upsilon$ *which is commutative, increasing and associative satisfying the condition* $\check{T}(1_\Upsilon, \mu) = \mu; \forall \mu \in \Upsilon$ *is said to be an IF-t-norm. Likewise, an IF t-conorm is a commutative, increasing, and associative* $(\Upsilon)^2 \longrightarrow \Upsilon$ *map* $\check{S}$ *satisfying* $\check{S}(0_\Upsilon, \mu) = \mu; \forall\, \mu \in \Upsilon$.

Definition 4 ([17])**.** *An IF t-norm* $\overline{T}$ *(respectively, t-conorm* $\overline{S}$*) is called t-representable if there exists a t-norm T and a t-conorm S on* $[0,1]$ *(respectively, a t-conorm* S' *and a t-norm* T' *on* $[0,1]$*) such that, for* $\mu, \nu \in \Upsilon$,

$$\begin{aligned} \overline{T}(\mu, \nu) &= (T(\mu_1, \nu_1), S(\mu_2, \nu_2)) \\ [\textit{respectively, } \overline{S}(\mu, \nu) &= (S'(\mu_1, \nu_1), T'(\mu_2, \nu_2))]. \end{aligned}$$

Remark 1. *(1) Throughout this study we have modeled the union and intersection between* $E, F \in IFS(X)$ *by IF Frank t-norms* $\overline{T}_t$ *and co-norms* $\overline{S}_t$ *given as:*

$$\begin{aligned} E \cup_{\overline{S}_t} F &= \left\{ \begin{array}{c} (x, S_t(\delta_E(x), \delta_F(x)), T_t(\eta_E(x), \eta_F(x))) \\ \mid x \in X \end{array} \right\} \\ E \cap_{\overline{T}_t} F &= \left\{ \begin{array}{c} (x, T_t(\delta_E(x), \delta_F(x)), S_t(\eta_E(x), \eta_F(x))) \\ \mid x \in X \end{array} \right\} \end{aligned}$$

such that $t = M, P, L$*. Thus, notion* T_t *will represent the three members of fuzzy Frank t-norms given as follows:*

(a) $T_M(p, q) = \min(p, q)$*; for all* $p, q \in [0,1]$;
(b) $T_P(p, q) = pq$ *for all* $p, q \in [0,1]$;
(c) $T_L(p, q) = \max(0, p + q - 1)$*; for all* $p, q \in [0,1]$;

and S_t *represents their corresponding dual t-co-norms.*

(2) Moreover, we will introduce here some of the intuitionistic fuzzy versions of the classical set-theoretic identities that we have built by employing the weighted average cardinality measure and be utilized in the construction of our newly introduced class of set theoretic-based similarity measures for intuitionistic fuzzy set:

(a) $|E^c| = n - |E|$ *will be modeled by:* $|E^c| = n - |E| = \frac{n}{2} + \frac{1}{2} \sum_{x \in X} (\delta_E(x) - \eta_E(x))$.

(b) $|E\backslash F| = |E| - |E\cap F|$ *will be modeled by:*

$$|E\backslash F| = |E| - |E\cap F| = \frac{\sum_{x\in X}\delta_E(x)-\eta_E(x)-T_t(\delta_E(x),\delta_F(x))+S_t(\eta_E(x),\eta_F(x))}{2}$$
$$= \frac{\sum_{x\in X}\delta_E(x)+\eta_F(x)-T_t(\delta_E(x),\delta_F(x))-T_t(\eta_E(x),\eta_F(x))}{2}$$
$$= \frac{\sum_{x\in X}\delta_E(x)+1-\eta_E(x)-T_t(\delta_E(x),\delta_F(x))-T_t(1-\eta_E(x),1-\eta_F(x))}{2}; t = M, P, L.$$

(c) $|E \triangle F| = |E\backslash F| + |F\backslash E| = |E| + |F| - 2|E\cap F|$ *will be modeled by:*

$$|E \triangle F| = |E| + |F| - 2|E\cap F|$$
$$= \frac{\sum_{x\in X}\delta_E(x)+\delta_F(x)-\eta_E(x)-\eta_F(x)-2T_t(\delta_E(x),\delta_F(x))+2S_t(\eta_E(x),\eta_F(x))}{2}$$
$$= \frac{\sum_{x\in X}\delta_E(x)+\delta_F(x)+\eta_E(x)+\eta_F(x)-2T_t(\delta_E(x),\delta_F(x))-2T_t(\eta_E(x),\eta_F(x))}{2}$$
$$= \frac{\sum_{x\in X}\delta_E(x)+\delta_F(x)+(1-\eta_E(x))+(1-\eta_F(x))-2T_t(\delta_E(x),\delta_F(x))-2T_t(1-\eta_E(x),1-\eta_F(x))}{2}; t = M, P, L.$$

Definition 5 ([18]). *A fuzzy equivalence relation on $FS(X)$ the class of all fuzzy sets on X is a fuzzy relation Π on $FS(X)$ satisfying the following condtions:*

(1) *Reflexivity:* $\Pi(E,E) = 1$;
(2) *Symmetry:* $\Pi(E,F) = \Pi(F,E)$;
(3) T_L-*Transitivity:* $I_L(T_L(\Pi(E,F),\Pi(F,G)),\Pi(E,G)) = \epsilon > 0$, *for all $x \in X$ and $E, F, G \in FS(X)$.*

Here, $I_L(p,q) = \min(1-p+q,1)$ for all $p, q \in [0,1]$ (Lukasiewicz fuzzy implicator).

2. Parametric Family of Fuzzy Similarity Measures for IFSs

According to Zadeh, it has always been a challenging problem to build a valid and general-purpose definition of similarity measures besides its wide applications in various domains, namely, decision-making, information retrieval, pattern recognition, chemistry, biology, and machine learning.

In this section, we will present our new class of similarity measures, the *Parametric family of Fuzzy Similarity Measures for IFSs*. This class of measures that we have introduced is close to the one introduced in [19]. The extension involves the intersections of IFS by the family of frank t-norms for IFSs and their scalar cardinality.

Definition 6. *A Parametric family of Fuzzy Similarity Measures for Intuitionistic Fuzzy Sets is a class of maps $\Psi : IFS(X) \times IFS(X) \longrightarrow [0,1]$ defined as:*

$$\Psi(E,F) = \frac{\alpha(|E\triangle F|) + \beta(|E\cap F|) + \gamma(|E\cup F)^c|)}{\lambda(|E\triangle F|) + \beta(|E\cap F|) + \gamma(|E\cup F)^c|)}$$

where α, β, γ, and λ are positive real numbers such that $0 \le \alpha \le \lambda$.

This condition on the parameters enforces $\Psi(E,F) \in [0,1]$ and hence defines a class of $[0,1]$-valued similarity measures for IFSs in the form of rational expressions. Next, we present here intuitionistic fuzzy versions of the famous similarity measures that are derived from the above-defined parametric family by setting different combinations of the parameters involved, as well as by using the expressions of set-theoretic identities, unions, and intersection for IFSs introduced in Remark 1. Moreover, in this study, we have employed the notation Ψ^t_k, $k = IJ, ISM, ID, IRT, ISK1, ISK2$ to specify these transformed members of the family Ψ^t, such that $t = M, P, L$. The results obtained here are represented as follows:

(a). Intuitionistic Jaccard$(\Psi_{IJ}^{t}) = \frac{\left|E \cap_{\overline{T}_t} F\right|}{\left|E \cup_{\overline{S}_t} F\right|}$
$= \frac{\sum_{x\in X} T_t(\delta_F(x),\delta_E(x)) + T_t(1-\eta_E(x),1-\eta_F(x))}{\sum_{x\in X} S_t(\delta_E(x),\delta_F(x)) + S_t(1-\eta_F(x),1-\eta_E(x))}$ for $(\alpha,\lambda,\beta,\gamma) = (0,1,1,0)$.

(b). Intuitionistic Simple Matching $(\Psi_{ISM}^{t}) = 1 - \frac{|E \triangle F|}{n}$
$= \frac{2n - \sum_{x\in X} \delta_F(x)+\delta_E(x)+(1-\eta_E(x))+(1-\eta_F(x))-2T_t(\delta_E(x),\delta_F(x))-2T_t(1-\eta_F(x),1-\eta_E(x))}{2n}$ for $(\alpha,\lambda,\beta,\gamma) = (0,1,1,1)$.

(c). Intuitionistic Dice $(\Psi_{ID}^{t}) = \frac{2|E\cap F|}{|E\triangle F|+2|E\cap F|}$
$= \frac{2\sum_{x\in X} T_t(\delta_F(x),\delta_E(x)) + T_t(1-\eta_E(x),1-\eta_F(x))}{\sum_{x\in X} \delta_E(x)+\delta_F(x)+(1-\eta_F(x))+(1-\eta_E(x))}$ for $(\alpha,\lambda,\beta,\gamma) = (0,1,2,0)$.

(d). Intuitionistic Rogers and Tanimoto $(\Psi_{IRT}^{t}) = \frac{n-|E\triangle F|}{n+|E\triangle F|}$
$= \frac{2n - \sum_{x\in X} \delta_E(x)+\delta_F(x)+(1-\eta_E(x))+(1-\eta_F(x))-2T_t(\delta_E(x),\delta_F(x))-2T_t(1-\eta_E(x),1-\eta_F(x))}{2n + \sum_{x\in X} \delta_E(x)+\delta_F(x)+(1-\eta_F(x))+(1-\eta_E(x))-2T_t(\delta_E(x),\delta_F(x))-2T_t(1-\eta_F(x),1-\eta_E(x))}$ for $(\alpha,\lambda,\beta,\gamma) = (0,2,1,1)$.

(e). Intuitionistic Sokal and Sneath 1 $(\Psi_{ISK1}^{t}) = \frac{|E\cap F|}{|E\cap F|+2|E\triangle F|}$
$= \frac{\sum_{x\in X} T_t(\delta_F(x),\delta_E(x)) + T_t(1-\eta_E(x),1-\eta_F(x))}{\sum_{x\in X} 2\delta_E(x)+2\delta_F(x)+2(1-\eta_E(x))+2(1-\eta_F(x))-3T_t(\delta_F(x),\delta_E(x))-3T_t(1-\eta_F(x),1-\eta_E(x))}$ for $(\alpha,\lambda,\beta,\gamma) = (0,2,1,0)$.

(f). Intuitionistic Sokal and Sneath 2 $(\Psi_{ISK2}^{t}) = 1 - \frac{|E\triangle F|}{2n-|E\triangle F|}$
$= \frac{4n - 2\sum_{x\in X} \delta_E(x)+\delta_F(x)+(1-\eta_E(x))+(1-\eta_F(x))-2T_t(\delta_F(x),\delta_E(x))-2T_t(1-\eta_E(x),1-\eta_F(x))}{4n - \sum_{x\in X} \delta_E(x)+\delta_F(x)+(1-\eta_E(x))+(1-\eta_F(x))-2T_t(\delta_E(x),\delta_F(x))-2T_t(1-\eta_F(x),1-\eta_E(x))}$ for $(\alpha,\lambda,\beta,\gamma) = (0,1,2,2)$.

Theorem 1. *For all $E, F, G \in IFS(X)$, the members Ψ_k^t, $k = IJ, ISM, ID, IRT, ISK1, ISK2$ of the family Ψ^t obey the following properties:*
(a) $\Psi_k^t(E,E) = 1$, *for* $t = M$;
(b) $\Psi_k^t(E,E^c) = 1 \Leftrightarrow \delta_E(x) = \eta_E(x), \forall\, x \in X$; $t = M$;
(c-1) $\Psi_k^t(E,E^c) = 0 \Leftrightarrow E = 1_Y$ *or* $E = 0_Y$; $t = M, P$;
(c-2) $\Psi_k^t(E,E^c) = 0 \Leftrightarrow E$ *is a fuzzy set;* $t = L$;
(d) $\Psi_k^t(E,F) = \Psi_k^t(F,E)$; $t = M, P, L$;
(e) *For* $E \subseteq F \subseteq G$, $\Psi_k^t(E,G) \leq \Psi_k^t(F,G)$ *and* $\Psi_k^t(E,G) \leq \Psi_k^t(E,F)$ *where* $k = IJ$ *and* $t = M$.

Proof. As an example, we prove *(b)*, *(c-1)*, *(e)* for $k = IJ$ and $t = M$ and (c-2) for $k = IJ$ and $t = L$
(b) $\Psi_{IJ}^M(E,E^c) = 1$
$\Leftrightarrow \min(\delta_E(x),\eta_E(x)) + \min(1-\eta_E(x),1-\delta_E(x))$
$= \max(\delta_E(x),\eta_E(x)) + \max(1-\eta_E(x),1-\delta_E(x)), \forall\, x \in X$
$\Leftrightarrow \delta_E(x) = \eta_E(x), \forall\, x \in X.$
(c-1) $\Psi_{IJ}^t(E,E^c) = 0$
$\Leftrightarrow \sum_{x\in X} T_t(\delta_E(x),\eta_E(x)) + T_t(1-\eta_E(x),1-\delta_E(x)) = 0$
$\Leftrightarrow$

$$T_t(\delta_E(x),\eta_E(x)) + T_t(1-\eta_E(x),1-\delta_E(x)) = 0\,,\ \forall\, x \in X. \tag{1}$$

Let us fix $t = M$ in (1)
then $\min(1-\eta_E(x),1-\delta_E(x)) = 0$ and $\min(\delta_E(x),\eta_E(x)) = 0, \forall\, x \in X$.
$\Leftrightarrow$ either $[\eta_E(x) = 0$ and $1-\delta_E(x) = 0]$ or $[\delta_E(x) = 0$ and $1-\eta_E(x) = 0]$ $\forall\, x \in X$
$\Leftrightarrow$ either $[\eta_E(x) = 0$ and $\delta_E(x) = 1]$ or $[\delta_E(x) = 0$ and $\eta_E(x) = 1]$ $\forall\, x \in X$
$\Leftrightarrow$ either $E = 1_Y$ or $E = 0_Y$.
(c-2) $\Psi_{IJ}^L(E,E^c) = 0$
$\Leftrightarrow \sum_{x\in X} T_L(\delta_E(x),\eta_E(x)) + T_L(1-\eta_E(x),1-\delta_E(x)) = 0$

$\Leftrightarrow$

$$T_L(\delta_E(x), \eta_E(x)) + T_L(1 - \eta_E(x), 1 - \delta_E(x)) = 0, \forall\, x \in X. \tag{2}$$

$\Leftrightarrow \max(0, \delta_E(x) + \eta_E(x) - 1) + \max(0, (1 - \eta_E(x) + (1 - \delta_E(x)) - 1) = 0, \forall\, x \in X$
$\Leftrightarrow \max(0, \delta_E(x) + \eta_E(x) - 1, 1 - \eta_E(x) - \delta_E(x)) = 0$
$\Leftrightarrow \delta_E(x) + \eta_E(x) - 1 = 0$
$\Leftrightarrow E$ is fuzzy set.
(e) Let $E \subseteq F \subseteq G$,
$\Rightarrow \delta_E(x) \leq \delta_F(x) \leq \delta_G(x)$ and $\eta_E(x) \geq \eta_F(x) \geq \eta_G(x)$
$\Rightarrow \min(\delta_E(x), \delta_F(x)) = \min(\delta_E(x), \delta_G(x))$
and $\min(1 - \eta_E(x), 1 - \eta_F(x)) = \min(1 - \eta_E(x), 1 - \eta_G(x))$
$\Rightarrow \max(\delta_E(x), \delta_F(x)) \leq \max(\delta_E(x), \delta_G(x))$
and $\max(1 - \eta_F(x), 1 - \eta_E(x)) \leq \max(1 - \eta_G(x), 1 - \eta_E(x))$
$\Rightarrow \frac{\min(1-\eta_G(x), 1-\eta_E(x)) + \min(\delta_G(x), \delta_E(x))}{\max(\delta_E(x), \delta_G(x)) + \max(1-\eta_E(x), 1-\eta_G(x))}$
$\leq \frac{\min(1-\eta_E(x), 1-\eta_F(x)) + \min(\delta_E(x), \delta_F(x))}{\max(\delta_F(x), \delta_E(x)) + \max(1-\eta_E(x), 1-\eta_F(x))}, \forall\, x \in X$
$\Rightarrow \Psi^M_{IJ}(E, G) \leq \Psi^M_{IJ}(E, F)$
similarly we can obtain $\Psi^M_{IJ}(E, G) \leq \Psi^M_{IJ}(F, G)$. □

Corollary 1. *For all $E, F, G \in IFS(X)$, the fuzzy similarity measure Ψ^t_{IJ}, $t = M$ defines a fuzzy equivalence relation on $IFS(X)$.*

3. Medical Diagnosis of Coronavirus (COVID-19) by Parametric Family of Fuzzy Similarity Measures for IFSs

In recent times, the medical industry has widely accepted fuzzy logic in areas including the construction of knowledge-based systems in medicine with objective of interpretation of medical findings, for an actual and timely monitoring of patients' records, for medicine syndrome differentiations involving diagnosis in modern medicine and the selection of the best possible medical procedure employing western/eastern medical knowledge of medicine [20,21]. However, with the advancement of such medical knowledge, the real-time situations occurred that demanded the description of a medical problem in a more generalized way rather than by mere use of a fuzzy linguistic variable involving membership function only. The Intuitionistic fuzzy set (IFS), introduced by Atanassov [15] among many generalizations of fuzzy sets originated as a real-time solution to this highly demanding situation of medical diagnosis that requires flexibility due to the possible existence of hesitation at every step of evaluation.

In this section, we present an efficient and easy method of diagnosis based on the new class of similarity measures Ψ. The method initially involves the construction of a database, i.e., a comprehensive description of symptoms collection; a set of diagnoses and the patient's medical state based on the knowledge of his/her medical test results. All this information/data is presented in the form of an IFS, which is built on the basis of different attributes of symptoms and diseases.

The complete three-stage diagnosis process is as follows:

(1) Symptom identification;
(2) Formation of medical information employing IFSs;
(3) Disease diagnosis technique by pattern recognition employing Ψ^t_k where $k = IJ, ISM, ID, IRT, ISK1, ISK2$, and $t = M, P, L$.

In this particular case study, we have a patient ρ whose initial medical examination reveals the presence of the following symptoms: Temperature, Headache, Nausea with Weakness, Sore Throat with Cough, and Stuffed Nose/Trouble Breathing, all having different levels of intensity.

In the first stage of this diagnosis procedure, the medical expert working on this case formulates a set Ω of symptoms that are an ordinary set acting as a universe of discourse.

In this particular case study, it is: $\Omega = \{\Omega_1$*(Temperature)*, Ω_2*(Headache)*, Ω_3*(Nausea with Weakness)*, Ω_4*(Sore Throat with Cough)*, Ω_5*(Stuffed Nose/Trouble Breathing)*$\}$.

In the next stage, the medical expert by utilizing all of his/her medical expertise and previous knowledge composes all possible diagnoses for this case and will formulate a set of diagnosis say $\Delta = \{\Delta_1$*(HINI)*, Δ_2*(COVID-19)*, Δ_3*(H5NI)*, Δ_4*(Hanta Virus)*, Δ_5*(SARS)*$\}$ with each of these diagnoses acting as an intuitionistic fuzzy set on Ω say given as (Table 1):

Table 1. Diseases vs. Symptoms.

	$\boldsymbol{\Omega_1}$	$\boldsymbol{\Omega_2}$	$\boldsymbol{\Omega_3}$	$\boldsymbol{\Omega_4}$	$\boldsymbol{\Omega_5}$
Δ_1	$(0.40, 0.0)$	$(0.30, 0.50)$	$(0.10, 0.70)$	$(0.40, 0.30)$	$(0.10, 0.70)$
Δ_2	$(0.70, 0.0)$	$(0.20, 0.60)$	$(0.0, 0.90)$	$(0.70, 0.0)$	$(0.10, 0.80)$
Δ_3	$(0.30, 0.30)$	$(0.60, 0.10)$	$(0.20, 0.70)$	$(0.20, 0.60)$	$(0.10, 0.90)$
Δ_4	$(0.10, 0.70)$	$(0.20, 0.40)$	$(0.80, 0.0)$	$(0.20, 0.70)$	$(0.20, 0.70)$
Δ_5	$(0.10, 0.80)$	$(0.0, 0.80)$	$(0.20, 0.80)$	$(0.20, 0.80)$	$(0.80, 0.10)$

The Table 1 relates every disease to every symptom assigning it a possible degree of belonging and a degree of non-belonging depending on medical databases collected from different sources. For instance, the first entry (0.40, 0.0) of the table states the fact the symptom Ω_1*(Temperature)* has appeared in Δ_1*(HINI)* up to 40 percent while its nonappearance is recorded as 0 percent.

In addition, according to this technique, the medical state of the patient ρ is represented by an IFS on the set of symptoms Ω. For this particular case study, let it be (Table 2):

Table 2. Patient vs. Symptoms.

	$\boldsymbol{\Omega_1}$	$\boldsymbol{\Omega_2}$	$\boldsymbol{\Omega_3}$	$\boldsymbol{\Omega_4}$	$\boldsymbol{\Omega_5}$
$\rho(Patient)$	$(0.80, 0.10)$	$(0.60, 0.10)$	$(0.20, 0.80)$	$(0.60, 0.10)$	$(0.10, 0.60)$

Clearly, in Table 2, we see that the symptom Ω_1*(Temperature)* has appeared in patient ρ up to 80 percent while its nonappearance is recorded as 10 percent.

In the last stage of this diagnosis problem, the expert applies the pattern recognition technique for the classification of the pattern ρ in one of the classes $\Delta_1, \Delta_2, \Delta_3, \Delta_4$, and Δ_5. The proper diagnosis Δ_θ among $\theta = 1, 2, 3, 4, 5$ is derived according to the formula:

$$\begin{aligned} \xi &= \arg \underset{1 \leq \theta \leq 5}{Max} \left\{ \Psi_k^t(\rho, \Delta_\theta) \right\} \qquad (3) \\ \text{where } k &= IJ, ISM, ID, IRT, ISK1, ISK2 \text{ and } t = M, P, L. \end{aligned}$$

Now, particularly for this example, we shall present here both the diagnosis table along with a graphical representation of the pattern recognition result obtained for $\Psi_k^t(\rho, \Delta_\theta)$, $k = IJ$ and $t = M, P, L$; while for the rest of the members of the family Ψ_k^t; where $k = ISM, ID, IRT, ISK1, ISK2$ and $t = M, P, L$ the calculational results are represented graphically only (see Figures 1–3) (Table 3):

Table 3. Results with $\Psi_{IJ}^t(\rho, \Delta_\theta)$, $t = M, P, L$.

$\boldsymbol{\Psi_{IJ}^t(\rho, \Delta_\theta)}$	$\boldsymbol{\theta = 1}$	$\boldsymbol{\theta = 2}$	$\boldsymbol{\theta = 3}$	$\boldsymbol{\theta = 4}$	$\boldsymbol{\theta = 5}$
$t = M$	0.67240	0.69490	0.64910	0.35210	0.31430
$t = P$	0.43280	0.50830	0.39670	0.24680	0.21370
$t = L$	0.27630	0.40850	0.27030	0.12940	0.1220

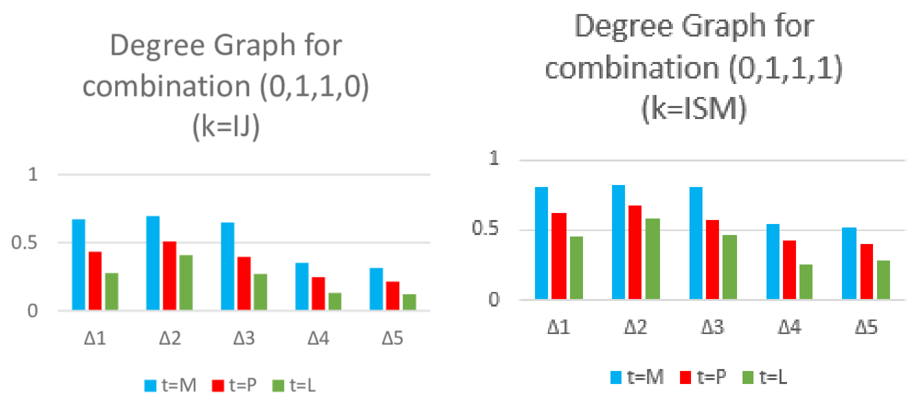

Figure 1. Degree graphs for $k = IJ, ISM$.

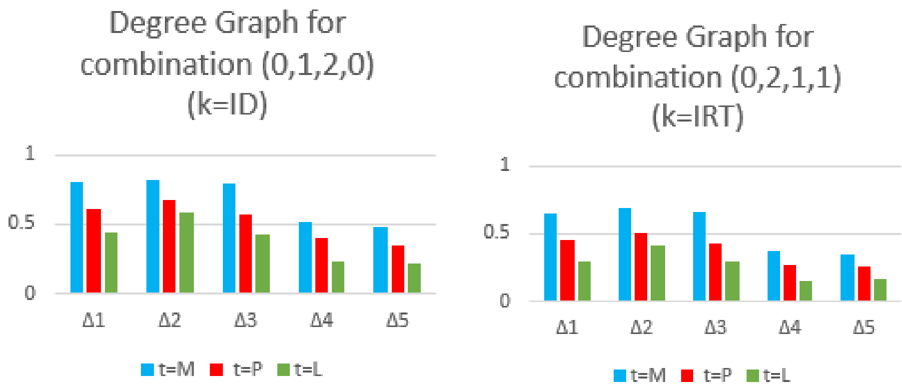

Figure 2. Degree graphs for $k = ID, IRT$.

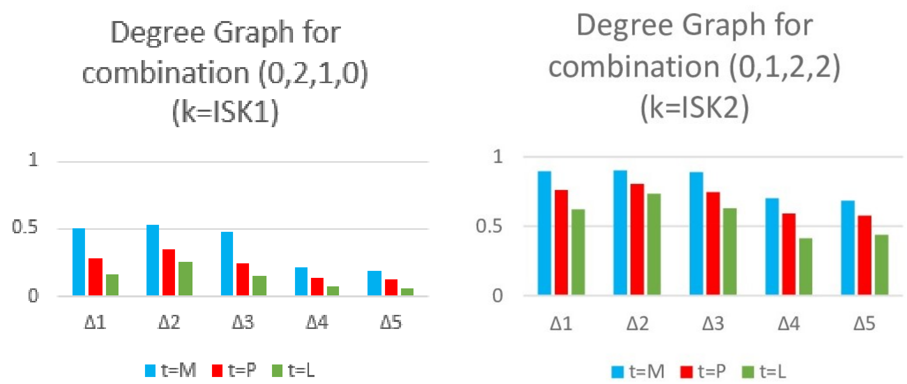

Figure 3. Degree graphs for $k = ISK1, ISK2$.

Now, when we apply the Formula (3) to every $k = IJ, ISM, ID, IRT, ISK1, ISK2$, and $t = M, P, L$ we obtain results for 18 intuitionistic fuzzy similarity measures belonging to this family and each of these remarkably assigned the same diagnosis $\Delta_2(COVID\text{-}19)$ to the patient ρ utilizing recognition principal (see Figure 4).

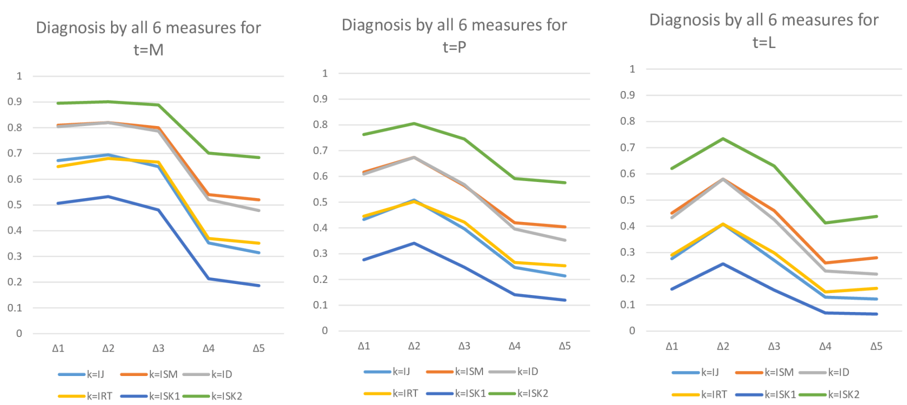

Figure 4. Diagnosis by all 6 measures for each $t = M, P, L$.

4. Conclusions

In this study, we have introduced a family of fuzzy similarity measures for intuitionistic fuzzy sets called *The Parametric family of Fuzzy similarity Measures for IFSs* involving four parameters along with a pair of IF Frank t-norms with their respective dual co-norms as the candidates of intersection and union of intuitionistic fuzzy sets. Moreover, we studied their properties that showed some interesting results, such as the IF- Jaccard coefficient obeyed most of all the axioms of a similarity measure existing in the literature. We presented a case study utilizing 18 of these measures in the field of medical diagnosis of Corona (COVID-19) and remarkably all of the measures generated the same result. Moreover, it was revealed that a variation of t-norms within the same measure resulted into different measurements of similarity between the same pair of sets. For instance, a variation of three basic t-norms within the Jaccard measure Ψ^t_{IJ} will be more practical in a decision process as this variation will generate the option of selection among all the three measures for the decision-maker who may be an optimist, pessimist, or the one always maintaining a neutral behavior according to a situation in which he is placed. Thus, we are in a position to claim that an intuitionistic Jaccard coefficient (or the other proposed measures) variant is capable of providing a more flexible, workable, yet reliable decision-making environment in real life situations.

Next, we will state some of the open problems that can lead to different future research explorations in this area:

- In this work, a scalar cardinality-based family of similarity measures involving only operations of addition/subtraction is presented. We do find other measures in the mathematical literature, such as the cosine coefficient, which are based on the multiplication of cardinalities and clearly are not members of the suggested family of similarity measures. Thus, by utilizing other mathematical operations besides addition/subtraction, we can design many new families of similarity measures that may have the ability to represent such constructions. This definitely can be regarded as an open challenge.
- Similarly, the proposed parametric family employs only scalar cardinality for IFSs. A selection of a different type of cardinality measures for intuitionistic fuzzy sets other than a scalar cardinality can produce new constructions of cardinality-based measures of similarity for the IFS.
- Likewise, a choice of only four parameters in this family seems too restrictive as it surely reduces the chance for the appearance of the intuitionistic version of many other famous crisp similarity measures existing in the literature. This particular problem can be resolved by involving more parameters, for example, six instead of four. Further research in this direction could lead to some remarkable results.

Author Contributions: M.Q. proposed the generalized parametric family of fuzzy similarity measure for intuitionistic fuzzy sets based on the weighted average cardinality measure, including the expressions of six of its members, namely Intuitionistic Jaccard, Intuitionistic Simple Matching, Intuitionistic Dice, Intuitionistic Rogers and Tanimoto, Intuitionistic Sokal and Sneath 1 and Intuitionistic Sokal and Sneath 2. E.E.K. and M.Q. explored the properties and gave main results of these six members. Both the authors further proposed a multicriteria medical diagnosis method based on the above-mentioned six members of this new family of fuzzy similarity measures for intuitionistic fuzzy sets. S.A. gave an actual example of the proposed medical diagnosis procedure to detect COVID-19 with all background information data and worked with M.Q. and E.E.K. in building the calculational results and analysis; we wrote the paper together. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The data presented in this case study is an exemplary data build with no actual patient involved.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Santini, S.; Jain, R. Similarity measures. *IEEE Trans. Pattern Anal. Mach. Intell.* **1999**, *21*, 871–883. [CrossRef]
2. Wang, X.; De Baets, B.; Kerre, E. A comparative study of similarity measures. *Fuzzy Sets Syst.* **1995**, *73*, 259–268. [CrossRef]
3. Williams, J.; Steele, N. Difference, distance and similarity as a basis for fuzzy decision support based on prototypical decision classes. *Fuzzy Sets Syst.* **2002**, *131*, 35–46. [CrossRef]
4. Dice, L.R. Measures of the amount of ecologic association between species. *Ecology* **1945**, *26*, 297–302. [CrossRef]
5. Jaccard, P. Nouvelles recherches sur la distribution florale. *Bull. Sociee Vaudoise Sci. Nat.* **1908**, *44*, 223–270.
6. Rogers, D.J.; Tanimoto, T.T. A computer program for classifying plants. *Science* **1960**, *132*, 1115–1118. [CrossRef] [PubMed]
7. Pappis, C.P.; Karacapilidis, N.I. A comparative assessment of measures of similarity of fuzzy values. *Fuzzy Sets Syst.* **1993**, *56*, 171–174. [CrossRef]
8. Hung, W.L.; Yang, M.S. Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance. *Pattern Recognit. Lett.* **2004**, *25*, 1603–1611. [CrossRef]
9. Iancu, I. Intuitionistic fuzzy similarity measures based on Frank t-norms family. *Pattern Recognit. Lett.* **2014**, *42*, 128–136. [CrossRef]
10. Meng, F.; Chen, X. Entropy and similarity measure of Atanassov's intuitionistic fuzzy sets and their application to pattern recognition based on fuzzy measures. *Pattern Anal. Appl.* **2016**, *19*, 11–20. [CrossRef]
11. Nguyen, H. A novel similarity/dissimilarity measure for intuitionistic fuzzy sets and its application in pattern recognition. *Expert Syst. Appl.* **2016**, *45*, 97–107. [CrossRef]
12. Dengfeng, L.; Chuntian, C. New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions. *Pattern Recognit. Lett.* **2002**, *23*, 221–225. [CrossRef]
13. Chen, S.M.; Randyanto, Y. A novel similarity measure between intuitionistic fuzzy sets and its applications. *Int. J. Pattern Recognit. Artif. Intell.* **2013**, *37*, 1350021. [CrossRef]
14. Beg, I.; Rashid, T. Intuitionistic fuzzy similarity measure: Theory and applications. *J. Intell. Fuzzy Syst.* **2016**, *30*, 821–829. [CrossRef]
15. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
16. Qayyum, M.; Ashraf, S.; Kerre, E.E. Measure of intuitionistic fuzzy inclusion. *C. R. Acad. Bulg. Des Sci.* **2016**, *69*, 971–980.
17. Cornelis, C.; Deschrijver, G.; Kerre, E.E. Implication in Intuitionistic fuzzy and interval-valued fuzzy set theory: Construction, classification, application. *Int. J. Approx. Reason.* **2004**, *35*, 55–95. [CrossRef]
18. Beg, I.; Ashraf, S. Fuzzy equivalence relations. *Kuwait J. Sci. Eng.* **2008**, *35*, 33.
19. Janssens, S. Bell Inequalities in Cardinality-Based Similarity Measurement. Ph.D. Thesis, Ghent University, Gent, Belgium, 2006.
20. Helgason, C.M. The application of fuzzy logic to the prescription of antithrombotic agents in the elderly. *Drugs Aging* **2004**, *21*, 731–736. [CrossRef] [PubMed]
21. Phuong, N.H.; Kreinovich, V. Fuzzy logic and its applications in medicine. *Int. J. Med. Inform.* **2001**, *62*, 165–173. [CrossRef] [PubMed]