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Review

Survey of Hermite Interpolating Polynomials for the Solution of Differential Equations

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Abstract: With progress on both the theoretical and the computational fronts, the use of Hermite interpolation for mathematical modeling has become an established tool in applied science. This article aims to provide an overview of the most widely used Hermite interpolating polynomials and their implementation in various algorithms to solve different types of differential equations, which have important applications in different areas of science and engineering. The Hermite interpolating polynomials, their generalization, properties, and applications are provided in this article.

Keywords: Hermite interpolation; orthogonal polynomial; Hermite basis function

MSC: 65Lxx; 65Mxx; 65Nxx

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1. Introduction

Special functions are solutions to a large class of equations having physical and mathematical applications. These functions, with a long history, and having a wide range of literature, are immensely popular both within and outside of mathematics. More precisely, such functions appear in probability theory, electro-optics, quantum mechanics, electromagnetic theory, communication systems, and nonlinear wave propagation, etc. Special functions are a crucial part of the formalisation of mathematical physics and serve as one-of-a-kind tools for developing models of real-life problems that are both straightforward and accurate. The theory of these functions, developed by classical authors such as Euler, Chebyshev, Gauss, Hardy, Hermite, Legendre, Ramanujan, and others, has been an extensive area of study in mathematics. Particularly, orthogonal polynomials [1] have significant applications in applied sciences that fall under the analytical and computational areas, with applications in quantum and electro-dynamics [2,3]. The polynomial structure reveals analytical features [4], but it lacks the computational robustness required for higher orders. By employing the Gram–Schmidt orthogonalization technique and starting with $1, x, x^2, \ldots,$ orthogonal polynomials can be generated. Among the Hermite, Laguerre, Legendre, Chebyshev, etc., polynomials, a vast literature is available on the Hermite interpolation of polynomials. A summary of the properties, error bounds, and solutions of differential equations using Hermite interpolating polynomials is reviewed in this article.

Piecewise Hermite polynomials have been used for solving differential equations. Wide ranging numerical application of these polynomials is found in the Lagrangian or action integral of the analysis of discrete elements that require continuity of derivatives of functions. Different types of Hermite polynomials may be produced to ensure $C^{(n)}$ continuity over elements. The several forms that guarantee $C^{(n)}$ continuity over an element of a certain dimension vary in terms of the quantity of nodes present in each element, the kind of data evaluated on every node, and the order of the derivative of the polynomials. The properties of these approximation sets have an impact on the computing effectiveness. An element with nodes only at its vertices makes a mesh's adjacency matrix simpler, thus reducing the bandwidth of matrices used in finite element calculations and potentially impacting the computational cost of a problem.

A French mathematician by the name of Charles Hermite [5] performed research on algebra, orthogonal polynomials, quadratic forms, elliptic functions, invariant theory, number theory, and quadratic forms. Several mathematical structures bear his name, including Hermite normal form, Hermite polynomials, Hermitian operators, and Hermite cubic splines. These polynomials have applications in areas such as quantum simple harmonic oscillators [6], brain signal analysis for the detection of epileptic seizures [7], image contraction [8], image processing (identification and expression of image specifications) [9,10], sound contraction [11,12], computer algebra [13], effective colour–texture image segmentation [14], scanning electron microscope images [15] etc.

The interpolation of functions with various variables plays an important role in solving both theoretical and applied problems. It is often used, in particular, for approximating the representation and computation of functions, numerical integration, numerical differentiation, and the development of approximation techniques for solving different kinds of linear and nonlinear problems. The problem of function interpolation is significantly extended by operator interpolation, which serves as the basis for the establishment of approximation techniques and algorithms for tackling real-world problems. It is one of the parts of applied functional analysis and the general theory of approximate methods. Hermite interpolation is a technique that permits the consideration of both the data themselves and specified derivatives at data points. It is closely related to the Newton divided difference method. The approximation will produce a polynomial with a degree that is less than or equal to the number and derivatives of these data points.

This survey article gives a comprehensive analysis of the Hermite interpolating polynomials and its use in different algorithms like collocation method, orthogonal collocation on finite element, Galerkin method, finite element method, etc. to solve various types of differential and integral equations. The development of the cubic, quintic and septic Hermite interpolation polynomials are discussed in this article. The type of error bounds, generalization, properties and applications of the Hermite interpolating polynomials are also reviewed. The Hermite basis functions are of class C^d, $d = 1, 2, 3$. Because of the continuity condition, the double calculation at mesh locations is avoided. As a result, the computational time is drastically reduced. The paper is organized as: in Section 2, the formation of Hermite interpolation polynomial is given. Literature review on Hermite polynomial is presented in Section 3. Application of Hermite as a basis function is discussed in Section 4. Author's contribution, conclusion and future application/advancements are discussed in Sections 5, 6 and 7.

2. Hermite Interpolation Polynomials

In this section, a detailed introduction about the cubic, quintic, and septic Hermite interpolation polynomials is given [16–18]. Through a discrete set of points and their derivative sets, piecewise Hermite interpolation is a widely used technique for finding a continuously differentiable curve. The Hermite polynomial is a generalisation of both the Taylor and Lagrange polynomials, and, therefore, it is also referred to as an "osculating polynomial". Let $a = x_0, x_1, \ldots, x_n = b$ be the discretization of a given domain $[a, b]$. Let $\{\mathbf{U}_{2d+1}^{I_e}, \ d = 1, 2, 3\}$ be the element-specific $I_e (= [x_{e-1}, x_e])$ collection of all algebraic expressions having a degree not exceeding n and B_{2d+1} denotes the collection of the basis functions expressed as:

$$B_{2d+1} = \{v \in C^d(a, b) : v \,|_{I_e} \in \mathbf{U}_{d+1}^{I_e}, e = 1, 2, \ldots, n\}, \tag{1}$$

which contains $(d+1)(n+1)$ functions, where B_{2d+1} represents the polynomial vector space of degree $\leq 2d+1$. On I_e, find the basis function for B_{2d+1}. Assume that the value of $f(x)$ and its first d derivatives are known at n nodes, such that:

$$f_k = f(x_k), \ \ f'_k = f'(x_k), \ldots, \ f_k^{(d)} = f^{(d)}(x_k), \ \ d = 1, 2, 3, \ \ k = 0, 1, \ldots, n.$$

It is necessary to seek an interpolating polynomial $H_{2d+1}(x)$ which interpolates $f(x)$ such that

$$H_{2d+1}(x_k) = f(x_k),\ H'_{2d+1}(x_k) = f'(x_k), \ldots, H^{(d)}_{2d+1}(x_k) = f^{(d)}(x_k),\ k = 0, 1, \ldots, n. \quad (2)$$

The above-mentioned $(d+1)(n+1)$ conditions are known as the interpolatory conditions.

2.1. Cubic Hermite Polynomial

In terms of $2(n+1)$ coefficients, the cubic Hermite polynomial $H_3(x)$ may be defined as follows for $d = 1$:

$$H_3(x) = \sum_{k=0}^{n} [p_k(x) f(x_k) + q_k(x) f'(x_k)]. \quad (3)$$

The functions p_k and q_k are evaluated by applying interpolatory conditions (2) and Lagrange interpolation polynomials. Four of these functions are used to interpolate the function and its first derivative at each end point of the elements $[x_{e-1}, x_e]$. For the e^{th} element, basis functions in terms of cubic Hermite polynomials are defined as follows:

$$p_e(x) = \begin{cases} \left(\dfrac{x - x_{e-1}}{h_e}\right)^2 \left(3 - \dfrac{2(x - x_{e-1})}{h_e}\right); & x \in [x_{e-1}, x_e] \\ \left(1 - \dfrac{x - x_e}{h_e}\right)^2 \left(1 - \dfrac{2(x - x_e)}{h_e}\right); & x \in [x_e, x_{e+1}] \\ 0; & \text{Otherwise} \end{cases}$$

$$q_e(x) = \begin{cases} -h_e \left(\dfrac{x - x_{e-1}}{h_e}\right)^2 \left(1 - \dfrac{(x - x_{e-1})}{h_e}\right); & x \in [x_{e-1}, x_e] \\ h_e \left(1 - \dfrac{x - x_e}{h_e}\right)^2 \left(\dfrac{x - x_e}{h_e}\right); & x \in [x_e, x_{e+1}] \\ 0; & \text{Otherwise} \end{cases}$$

having $p_k(x_k) = 1$, $q'_k(x_k) = 1$ for all $k = 0, 1, \ldots, n$ and $h_e = x_e - x_{e-1}$. The values of $p_e(x)$ and $q_e(x)$ and their first derivatives at the remaining points are zero. These functions are C^1 continuous and

$$B_3 = span\{p_{e-1}, q_{e-1}, p_e, q_e, e = 1, \ldots, n\}, \quad (4)$$

forms a basis. For more details and graphical representations the reader can refer to [16,19].

2.2. Quintic Hermite Polynomial

For the value of $d = 2$, the unique quintic Hermite interpolation polynomial in the form of $3(n+1)$ parameters is given as follows:

$$H_5(x) = \sum_{k=0}^{n} [p_k(x) f(x_k) + q_k(x) f'(x_k) + r_k(x) f''(x_k)], \quad (5)$$

and undetermined functions p_k, q_k, and r_k are computed by Equation (2) and Lagrange polynomials. The quintic Hermite basis function at the e^{th} element is written as follows:

$$p_e(x) = \begin{cases} 6\frac{(x-x_{e-1})^5}{h_e^5} - 15\frac{(x-x_{e-1})^4}{h_e^4} + 10\frac{(x-x_{e-1})^3}{h_e^3}; & x_{e-1} \le x \le x_e \\ 6\frac{(x_{e+1}-x)^5}{h_e^5} - 15\frac{(x_{e+1}-x)^4}{h_e^4} + 10\frac{(x_{e+1}-x)^3}{h_e^3}; & x_e \le x \le x_{e+1} \\ 0; & \text{Otherwise} \end{cases}$$

$$q_e(x) = \begin{cases} -3\frac{(x-x_{e-1})^5}{h_e^4} + 7\frac{(x-x_{e-1})^4}{h_e^3} - 4\frac{(x-x_{e-1})^3}{h_e^2}; & x_{e-1} \le x \le x_e \\ 3\frac{(x_{e+1}-x)^5}{h_e^4} - 7\frac{(x_{e+1}-x)^4}{h_e^3} + 4\frac{(x_{e+1}-x)^3}{h_e^2}; & x_e \le x \le x_{e+1} \\ 0; & \text{Otherwise} \end{cases}$$

$$r_e(x) = \begin{cases} \frac{1}{2}\frac{(x-x_{e-1})^5}{h_e^3} - \frac{(x-x_{e-1})^4}{h_e^2} + \frac{1}{2}\frac{(x-x_{e-1})^3}{h_e}; & x_{e-1} \le x \le x_e \\ \frac{1}{2}\frac{(x_{e+1}-x)^5}{h_e^3} - \frac{(x_{e+1}-x)^4}{h_e^2} + \frac{1}{2}\frac{(x_{e+1}-x)^3}{h_e}; & x_e \le x \le x_{e+1} \\ 0; & \text{Otherwise} \end{cases}$$

with

$$\begin{aligned} p_e(x_k) = \delta_{ek},\ p_e'(x_k) = 0,\ p_e''(x_k) = 0, \\ q_e(x_k) = 0,\ q_e'(x_k) = \delta_{ek},\ q_e''(x_k) = 0, \\ r_e(x_k) = 0,\ r_e'(x_k) = 0,\ r_e''(x_k) = \delta_{ek}. \end{aligned}$$

The above-mentioned functions are known as quintic Hermite basis functions, and the reader can refer to [17] for further details.

2.3. Septic Hermite Polynomial

The unique septic Hermite interpolating polynomial, i.e., for $d = 3$, is defined as below:

$$H_7(x) = \sum_{k=0}^{n} [p_k(x)f(x_k) + q_k(x)f'(x_k) + r_k(x)f''(x_k) + s_k(x)f'''(x_k)]. \tag{6}$$

Similarly, the Lagrange polynomials and Equation (2) are used to calculate the unknown functions $p_e(x_k)$, $q_e(x_k)$, $r_e(x_k)$, and $s_e(x_k)$ in Equation (6). For the e^{th} element, the basis function is defined as follows:

$$p_e(x) = \begin{cases} 35\frac{(x-x_{e-1})^4}{h_e^4} - 84\frac{(x-x_{e-1})^5}{h_e^5} + 70\frac{(x-x_{e-1})^6}{h_e^6} - 20\frac{(x-x_{e-1})^7}{h_e^7}; & x_{e-1} \le x \le x_e \\ 35\frac{(x_{e+1}-x)^4}{h_e^4} - 84\frac{(x_{e+1}-x)^5}{h_e^5} + 70\frac{(x_{e+1}-x)^6}{h_e^6} - 20\frac{(x_{e+1}-x)7}{h_e^7}; & x_e \le x \le x_{e+1} \\ 0; & \text{Otherwise} \end{cases}$$

$$q_e(x) = \begin{cases} -15\frac{(x-x_{e-1})^4}{h_e^3} + 39\frac{(x-x_{e-1})^5}{h_e^4} - 34\frac{(x-x_{e-1})^6}{h_e^5} + 10\frac{(x-x_{e-1})^7}{h_e^6}; & x_{e-1} \le x \le x_e \\ -15\frac{(x_{e+1}-x)^4}{h_e^3} + 39\frac{(x_{e+1}-x)^5}{h_e^4} - 34\frac{(x_{e+1}-x)^6}{h_e^5} + 10\frac{(x_{e+1}-x)^7}{h_e^6}; & x_e \le x \le x_{e+1} \\ 0; & \text{Otherwise} \end{cases}$$

$$r_e(x) = \begin{cases} \frac{5}{2}\frac{(x-x_{e-1})^4}{h_e^2} - 7\frac{(x-x_{e-1})^5}{h_e^3} + \frac{13}{2}\frac{(x-x_{e-1})^6}{h_e^4} - 2\frac{(x-x_{e-1})^7}{h_e^5}; & x_{e-1} \le x \le x_e \\ \frac{5}{2}\frac{(x_{e+1}-x)^4}{h_e^2} - 7\frac{(x_{e+1}-x)^5}{h_e^3} + \frac{13}{2}\frac{(x_{e+1}-x)^6}{h_e^4} - 2\frac{(x_{e+1}-x)^7}{h_e^5}; & x_e \le x \le x_{e+1} \\ 0; & \text{Otherwise} \end{cases}$$

$$s_e(x) = \begin{cases} -\frac{1}{6}\frac{(x-x_{e-1})^4}{h_e} + \frac{1}{2}\frac{(x-x_{e-1})^5}{h_e^2} - \frac{1}{2}\frac{(x-x_{e-1})^6}{h_e^3} + \frac{1}{6}\frac{(x-x_{e-1})^7}{h_e^4}; & x_{e-1} \le x \le x_e \\ -\frac{1}{6}\frac{(x_{e+1}-x)^4}{h_e} + \frac{1}{2}\frac{(x_{e+1}-x)^5}{h_e^2} - \frac{1}{2}\frac{(x_{e+1}-x)^6}{h_e^3} + \frac{1}{6}\frac{(x_{e+1}-x)^7}{h_e^4}; & x_e \le x \le x_{e+1} \\ 0; & \text{Otherwise} \end{cases}$$

provided $p_k(x_k) = 1$, $q_k'(x_k) = 1$, $r_k''(x_k) = 1$, $s_k'''(x_k) = 1$, $\forall\, k = 0, 1, \ldots, n$. The values of p_e, q_e, r_e, and s_e and their first three derivatives are zero at the rest of the points. These basis functions are C^3 continuous and

$$B_7 = span\{p_{e-1}, q_{e-1}, r_{e-1}, s_{e-1}, p_e, q_e, r_e, s_e, e = 1, \ldots, n\}, \tag{7}$$

forms a basis. The septic Hermite and its graphical representation is discussed in detail in [18].

3. Literature Review on Hermite Polynomials

In the literature, Hermite interpolation polynomials have been studied from various points of view. This section contains an extensive review of their properties, error bounds, formula generalisation, etc.

Varma and Prasad [20] studied the mean convergence of H_{2n+1}, which is based on the roots of the Chebyshev polynomials. It is shown that the rate at which H_{2n+1} converges to $f(x) \in C^1$ in weighted L^P norms is E_{2n+1}. At the interpolating nodes, the Hermite–Fejer interpolating polynomial, which has a degree of no more than $(2n+1)$, coincides with $f(x)$ and its first-order derivative vanishes. It resembles the Hermite interpolating polynomial in certain aspects. The weighted mean convergence of the Hermite–Fejer interpolating polynomial was investigated by Nevai and Vertesi [21,22]. For the Hermite interpolation, Nevai and Yuan [23] interpolated the specified function and its first-order derivative at the n roots of the modified Chebyshev polynomials by using the algebraic polynomials with a degree not exceeding $2n-1$. For weighted L^P norms, a convergence study of the polynomials and their first-order derivatives was performed.

Al-Khaled and Khalil [24,25] worked with the interpolation of Hermite types and presented norm estimates for different interpolating operators on the space of continuous functions whose derivatives were also continuous on I. Agarwal [26] provided outcomes of extended Hermite approximation derivatives on the roots of the Chebyshev polynomial in the case of weighted L^P convergence. Agarwal and Wong [27] obtained the piecewise Hermite interpolates' explicit error bounds in L_2 norm. These bounds were improvements of the results given by Schultz [28]. The results obtained by the authors were a supplement of the explicit bounds for $\|D^p(f-H_n)\|_\infty$ given in [26,29,30]. In terms of $\|D^{n+j}f\|_q$, sharp upper bounds for $\|D^p(f-H_n)\|_\infty$ were also estimated.

Al-Khaled and Alquran [24,31] investigated the simultaneous interpolation of function $f(x)$ and its derivative $f'(x)$ by using the Chebyshev-polynomial-based Hermite interpolation operator H_{2n+1}. In the case of Hermite interpolation, a theorem on extreme nodes was established by the authors that agreed with Pottinger's results [32] and was an improvement of the results obtained in [25]. Pottinger [33] used the Chebyshev nodes to prove that the convergence condition depends on the norms of H_{2n+1}. The author also proved that the growth of the operator norms is of n^{th} order:

$$\|H_n(x) - f(x)\| = O(n)E_{2n}(f'),$$

where $E_n(f)$ is the best approximation of the function $f(x)$. Szabados and Varma [34] presented a norm for higher-order derivatives of Hermite interpolation polynomials as:

$$\|H_n^{(p)}\| = sup\{\|H_n^{(p)}f(x)\| : |f^{(m)}(x_k)| \leq n^m(1-x_k^2)^{p-m/2}, k = 1,2,\ldots,n,\ m = 0,1\}.$$

The authors [34] showed that for any system of nodes:

$$\|H_n^{(p)}\| \geq C_p n^p \ln(n),\ C_p > 0.$$

Further, they obtained the operator norm as follows:

$$\|H_n(x) - f(x)\| = O(n^p \ln(n)),$$

for the matrix nodes

$$\omega(x) = P_{n-2t+1}^{(\alpha,\alpha)}(x)\prod_{i=1}^{t}\left(x^2 - \cos^2\frac{(i-1)\pi}{3t(n-2t+1)}\right),$$

where $t = \left[\dfrac{p+3}{4}\right]$, $\alpha = 2t - \dfrac{p+1}{2}$, and $P^{(\alpha,\alpha)}_{n-2t+1}(x)$ is the ultra-spherical Jacobi polynomial of degree $(n-2t)$. Therefore, the error bounds for the nodes of a matrix are represented as:

$$\|H_n^{(p)}(x) - f^{(p)}(x)\| = O(\ln(n))\omega(f^{(p)}, 1/n).$$

The derivatives of quasi-Hermite interpolation were used by Min [35] for simultaneous approximation of the derivatives of $f(x)$ in which the roots of $(1-x^2)p_n(x)$, ($p_n(x)$ is a Legendre polynomial) are considered. The author showed that zero of $(1-x^2)p_n(x)$ are almost optimal and the corresponding simultaneous approximation is more accurate than that of the roots of the first kind of Chebyshev polynomial, i.e., based on Hermite interpolation. The approximation is defined by the below-mentioned theorems:

Theorem 1. *If $f \in C^1[-1,1]$, then*

$$\|R_n'(f,x) - f'(x)\| = O(\log n)E_{2n}(f').$$

Theorem 2. *If $f \in C^p[-1,1]$, then*

$$\|R_n'(f,x) - f'(x)\| = O(\log n)E_{2n}(f') = O(\log n/n)E_{2n-1}(f''),$$

$$\|\sqrt{1-x^2}(R_n''(f,x) - f''(x))\| = O(\log n)E_{2n-1}(f''),$$

and

$$\|R_n^{(j)}(f,x) - f^{(j)}(x)\|_{[-\sigma,\sigma]} = O(\log n)E_{2n-j+1}(f^{(j)}),\ j = 2,3,\dots,p,$$

where $0 < \sigma < 1$. The corresponding degrees of approximation are provided that show that the obtained nodal matrix is almost optimal. In Berriochoa et al. [36], a few applications of the Hermite interpolation are reported.

Refs. [29,30,37,38] developed results on norm estimation, the convergence of Hermite polynomials, and many more techniques for the interpolation of several variables. An extensive review on error estimates in Hermite interpolation is discussed below.

Ciarlet et al. [39] estimated the point to point error $e(x) = f - H_{2d+1}$ and computed the derivatives in the form of $V = \max_{a\le x\le b}|f^{(2d+2)}(x)|$ as follows:

$$\left|e^{(p)}(x)\right| \le \frac{Vh^p}{p!(2(d+1)-2p)!}\Big[(x-a)(b-x)\Big]^{(d+1)-p},\quad 0 \le p \le d+1. \tag{8}$$

From Equation (8), the upper bound on the error is given as follows:

$$\max_{a\le x\le b}\left|e^{(p)}(x)\right| \le \frac{Vh^{2(d+1)-p}}{4^{(d+1)-p}p!(2(d+1)-2p)!}. \tag{9}$$

For the cubic and quintic Hermite interpolations, as well as their derivatives, Birkhoff and Priver [40] determined the optimum error bounds. The authors' study made the assumption that Equations (8) and (9) are the best constraints for $p = 0$. However, for $p > 0$, Equations (8) and (9) are not the best ones. Error estimation on $e^{(p)}(x)$ was performed for values of $p > 0$ using Peano's Green function approach and $e(x)$ can be expressed in terms of the Green's function given by the explicit formula as follows:

$$e^{(p)}(x) = \int_a^b G(x,s)f^{(2d+2)}(s)ds, \tag{10}$$

where $G(x,s)$ is the Green's function, and for the cubic and quintic Hermite interpolation, the error bounds are listed in Table 1.

Table 1. Error bounds $\max_{a \le x \le b} e^{(p)}$ for cubic, quintic, and septic Hermite polynomials.

p	1	2	3	4	5	6	7
Cubic	$V\sqrt{3}/216$	$V/12$	$V/2$	-	-	-	-
Quintic	$V\sqrt{5}/30,000$	$V/1920$	$V/120$	$V/10$	$V/2$	-	-
Septic	$3V/3,073,280\sqrt{7}$	$V/493,920$	$\frac{(3\sqrt{5}+5\sqrt{6})\sqrt{\frac{1}{7}(15-2\sqrt{30})}V}{588,000}$	$V/1680$	$V/84$	$3V/28$	$V/2$

The authors also observed that it is easy to compute optimal error bounds for $d = 1$ and 2 from the above expression but it is unlikely that the same procedure can be followed for $d > 2$. Kumari and Kukreja [18] estimated the optimal error bounds of Hermite interpolation for $d = 3$. For septic Hermite interpolation and their derivatives, the error bounds are listed in Table 1.

At the Chebyshev nodes $\cos((2j+1)\pi/2n)_{j=1}^{N-1}, -1 \le x \le 1$, Riess [41] estimated the error bounds for Hermite interpolation of function $f(x)$ of different orders of continuity. For quintic and cubic Hermite interpolating polynomials, Hall [42] computed the explicit error bounds and proposed the following theorems:

Theorem 3. *For $f \in C^4[a,b]$,*

$$\|H_3^{(p)} - s^{(p)}\| \le \gamma_p \|f^{(4)}\| \bar{h}^{4-p},\ p = 0,1,2,3,$$

where $\bar{h} = \max\{h_e\}$, $\gamma_0 = 1/96$, $\gamma_1 = 1/24$, $\gamma_2 = \beta/4$, and $\gamma_3 = \beta^2/2$.

Theorem 4. *For $f \in C^6[a,b]$,*

$$\|H_5^{(p)} - s^{(p)}\| \le \gamma_p \|f^{(6)}\| \bar{h}^{6-p},\ p = 0,1,\ldots,5,$$

where $\bar{h} = \max\{h_e\}$, $\gamma_0 = 1/23,040$, $\gamma_1 = \sqrt{3}/12,960$, $\gamma_2 = 1/720$, $\gamma_3 = \beta/60$, $\gamma_4 = \beta^2/12$, and $\gamma_5 = \beta^3/6$. Chen and Wong [43] proposed one and two independent-variable-based discrete Hermite interpolation. Furthermore, the authors provided the explicit error estimate in L_∞ for the quintic and biquintic discrete Hermite interpolation by using the discrete Peano kernel theorem [44]. The convergence of weighted L^p space for the Hermite interpolation and their derivatives was performed by Criscuolo et al. [45] on the roots of Jacobi polynomials.

Cirillo and Hormann [46] presented an iterative method for solving problems based on the Hermite interpolation starting with the Lagrange interpolant and m corrective terms are incrementally added to interpolate the data up to the m^{th} derivative. The authors focused on the Floater–Hormann interpolants, a family of barycentric rational interpolants, which are constructed by combining local polynomials of degree d interpolants. The authors further demonstrated that the rational Hermite interpolants converge at a rate of order $O(h^{(m+1)(d+1)})$ for $m = 1,2$. Their numerical findings indicate that for $m > 2$, the rate remains the same. Cirillo et al. [47] generalised this convergence rate for any value of $m \ge 1$. Varma and Katsifarakis [48] gave the optimal error bounds for the Hermite cubic interpolating polynomial, i.e., the supplement of the bounds estimated by Birkhoff and Priver [49]. The uniform error bounds for $u(x) \in C^3[0,1]$ were given as:

$$|H_3^{(p)} - u^{(p)}| \le \alpha_p L, \tag{11}$$

where $L = \max_{0 \le x,t \le 1} \mid f'''(t) - f'''(x) \mid$, $\alpha_0 = \dfrac{1}{96}$, $\alpha_1 = \dfrac{13\sqrt{13}-46}{27}$, $\alpha_2 = \dfrac{8}{27}$, $\alpha_3 = 2$ and for $u(x) \in C^2[0,1]$:

$$|H_3^{(p)} - u^{(p)}| \le \beta_p M, \tag{12}$$

where $M = \max_{0 \le x,t \le 1} |u''(t) - u''(x)|$, $\beta_0 = \dfrac{1}{16}$, $\beta_1 = 0.251497657$, $\beta_2 = \dfrac{5}{3}$, $\beta_3 = 2$.

Wong and Agarwal [50] used the Peano kernel theorem to obtain the explicit error bounds in the norm between the quintic Hermite and $f(x) \in C^{(n)}[a,b],\ n = 2,3,\ldots,6$ as:

$$\|D^p(H_5 - f)\| \le a_{n,p} h^{n-p} \|D^p f\|,\ 0 \le p \le n-1, \tag{13}$$

where $a_{n,p}$ is given in [50]. Varma and Howell [51] estimated the error bounds for derivatives in two-point Birkhoff interpolation equations as:

$$\mid u^{(p)}(x) - H^{(p)}_{2n-1}(x) \mid \le uh^{2n-p} \max_{0\le x\le h} \mid f^{(p)}(x) \mid,\ 0 \le p \le 2n-1, \tag{14}$$

where $u = \max_{0\le x\le h} \mid u^{(2n)}(x) \mid$ and $f(x) = \dfrac{x^n(h-x)^n}{2n!}$. Birkhoff et al. [49] developed the upper bounds for the Hermite interpolation errors in one and two variables with applications to partial differential equations. For this, the authors used the Peano kernel theorem and obtained the global error bounds for Hermite interpolation polynomials $H^{(p)}$ and $f(x) \in K^{t,r}(I),\ t \ge p \ge 1$ in one variable as follows:

$$\|D^j(f - f_{p,\pi})\|_{L^q} \le c_{j,p,s,r,q} h^{n-p} \bar{\pi}^{s-j-1/r+1/q} \|D^s f\|_{L^r}, \tag{15}$$

where $s = \min(t, 2p),\ q \ge r,\ 0 \le j \le p-1$. For $j = p$ if $t > p$ or $q = r$ and

$$\|D^j(f - f_{p,\pi})\|_{L^q} \le c_{j,p,s,r,r} h^{n-p} \bar{\pi}^{s-j} (b-a)^{(r-q)/rq} \|D^s f\|_{L^r} \tag{16}$$

where $1 \le q \le r, 0 \le j \le p$, π is the partition of $[a,b]$ and $f_{p,\pi}$ is the $H^{(p)}$ interpolate of $f(x)$. The authors used the higher-dimension Peano kernel theorem [52] and obtained the upper bounds for the error in the Hermite interpolation $H^{(p)}(R_i = [a_i, b_i] \times [c_i, d_i])$ and $f(x) \in K^{t,r}(I),\ t \ge 2p$ in two variables as:

$$\|D^{h,l}(f - f_{p,\pi})\|_{L^r} \le M(v)^{2m-h-l}, \forall \pi \in C, \tag{17}$$

where C is the collection of partitions of R_i, $0 \le h, l \le p$ with $0 \le h + l \le 2p - 1$ and $1 \le i \le k$. In order to derive a sharp explicit estimation for the envelope of Hermite polynomials that represents the oscillatory area $|x|(2k - 3/2)1/2$, Foster and Krasikov [53] employed a positive quadratic-forms-based technique on polynomial inequalities. Cohn [54] demonstrated the convergence in the distribution of appropriately normalised Wick powers and developed sharp asymptotes corresponding to the L_p norm of Hermite polynomial functions. To analyse an extremal problem involving Wiener chaos, the results were utilised along with numerical integration. By using piecewise Hermite interpolation with equally spaced nodes, Xu et al. [55] established the precise constants for simultaneous L_2 approximation of Sobolev classes.

Todorov [56] developed a theory for extended Hermite polynomials and also studied various formulae for the derivatives of the function $f(x^p)$ of n^{th} order. Several interpolation schemes using PH curves, such as Hermite interpolation of spatial data [57–61], Hermite interpolation in the plane [62–65], Hermite interpolation by speed reparametrization [66], and Hermite interpolation in Minkowski space [67–69]. A degree-by-degree recursive relation of Hermite interpolants $H_{2n-1} \in C^{n-1}$ was constructed by Han [70]. For $x \in [x_k, x_{k+1}]$, the author computed the formulas for H_{2n} and H_{2n+1} as follows:

$$H_{2n}(x) = H_{2n-1}(x) + \frac{1}{n!2} \sum_{j=1}^{n} \frac{(2n-j-1)!}{(j-1)!(n-j)!} h_k^j [f_k^{(j)} - (-1)^j f_{k+1}^{(j)}] \frac{x - x_k}{h_k}, \tag{18}$$

$$H_{2n+1}(x) = H_{2n}(x) + \frac{1}{n!2} \sum_{j=0}^{n} \frac{(2n-j)!}{(j)!(n-j)!} h_k^j [f_k^{(j)} - (-1)^j f_{k+1}^{(j)}](1-v)^n v^n (1-2v), \tag{19}$$

where $k = 1, 2, \ldots, n$, $v = (x - x_k)/h_k$. A few interesting properties were also discussed in that paper and it was shown that the interpolation conditions satisfied by the polynomials H_{2n} and H_{2n-1} are the same. They are an optimal estimation of the interpolant H_{2n+1}. A parametric cubic spline interpolation approach was presented by Boor et al. [71] and is an extension of Hermite interpolation. It is based on the Sabin concept for generating C^1 bicubic parametric spline surfaces. At each knot, the curvature is specified along with the position and tangent. This guarantees that the resultant interpolating piecewise cubic curve exhibits convexity, is sixth-order accurate, and belongs to the C^2 class with respect to arc length. Borzov [72] presented a new method for obtaining generalised Hermite polynomials and Chand and Viswanathan [73] presented a fractal form of the cubic Hermite function. Holvorcem [74] developed a numerical method involving the Hermite functions $\chi_m = (2^m m! \sqrt{\pi})^{\frac{-1}{2}} e^{-x^2/2} H_m(x)$ for the evaluation of slowly convergent or even divergent series having forms such as:

$$F(x) = \sum_{m=0}^{\infty} c_m \chi_m(x/\sqrt{2}),$$

where c_m decays algebraically as $m \to \infty$, $F(x)$ is known as a Fourier–Hermite series, and

$$G(x, y) = \sum_{m=0}^{\infty} c_m \chi_m(x/\sqrt{2}) \chi_m(y/\sqrt{2}).$$

The Green's functions for problems whose eigenfunctions comprise the Hermite functions are represented by this series. Following a considerable number of terms in this series, the author established rapid convergent asymptotic expansions for the remaining terms by using a method based on Poisson summation. The series can thereafter be calculated as a partial sum plus an approximation based on asymptotic theory for the remaining part. The author also showed that the remaining terms of $G(x, y)$ reveal the nature of the singular behaviour near $x = y$. Carlitz [75] and Kashpur [76] gave a bilinear generating function for Hermite polynomials in several variables. Glasser and Shawagfeh [77] provided a new integral representation of the Hermite polynomials. They also studied the asymptotic behaviour of these functions. Szeliski and Ito [78] investigated the subject of creating a smooth two-dimensional curve from a collocation of collocation points, this procedure can be used in curve coding for transmission and curve design.

For usage in CAGD, Casciola and Romani [79] described and analysed the piecewise quintic Hermite interpolation which can accurately represent any conic arc of arbitrary length by utilising just one segment. Additionally, these polynomials provided a range of local/global shape parameters for intuitively creating free-form curves without violating the $C2$ continuity that was a feature of the original layout. Xia and Lu [80] used quintic Hermite interpolation polynomials to study new beam elements for second-order effect analysis of beam structures. Ivan [81] provided a note on the Hermite interpolation polynomial for rational functions. The author gave a proof and generalised the formula of Claude Brezinski involving the Hermite interpolation polynomial. Messaoudi et al. [82] presented a matrix recursive polynomial interpolation algorithm for computing the Hermite interpolation polynomial. By using Chebyshev polynomials, Rizk [83] derived explicit expansions of the Hermite interpolation polynomials. The author assumed that the nodes are either roots of the Chebyshev polynomial of $(n+1)^{th}$ order or extremum points of the Chebyshev polynomial of order n. A review on various properties of Hermite polynomials is reported below.

Witschel [84] discussed the integral properties of the Hermite polynomials using operator methods. Stevens [85] studied the congruence properties of Hermite polynomials. Mathur and Sharma [86] discussed some interpolatory properties, and Dette and Studden [87] gave its new asymptotic properties. Asymptotic analysis on the expansion of Hermite polynomials and their uses in Gauss quadrature was reported by Xiang [88]. The Hermite matrix polynomial expansions of a few relevant matrix functions that emerge in the solutions of differential systems were discussed by Defez and Jodar [89]. Extremal interpolants, a unique class of PH quintic interpolating curves developed by Han et al. [90], were shown to preserve planarity, i.e., planar data represents planar curves. Using modified quintic Hermite curves and applications, Millham and Meyer [91] examined point files containing curvature–tangency information. They demonstrated that the quintic curve may be produced by simply applying a few of the points and tangents in the interval. The quintic curve then passes through the rest of the points in the way indicated, is examined for goodness of fit, and, if required, is substituted with a "shorter" sector.

Cramer [92] presented Hermite interpolation polynomials and distributions of ordered data. They discussed that a certain Hermite interpolation polynomial that is determined at the origin can be used to understand cumulative distribution functions. Kassebaum et al. [93] discussed the application of group representation theory to Hermite interpolation polynomials of lower orders which assures that triangle element boundaries in two dimensions have $C(n)$ continuity but cannot be easily extended to higher dimensions. Manh et al. [94] studied the Hermite interpolation on irreducible algebraic curves in C^2 and then they showed that the Hermite interpolation polynomials are well-defined in neighbourhoods of Taylorian points and continuous with respect to the interpolation point. They also provided some applications to the study of a continuity property of certain bivariate Hermite projectors. Various other properties such as Hermite polynomials and their squares and generating functions, the rate of convergence of Hermite function series, the asymptotic coefficients of Hermite series, and summability methods for Hermite functions are discussed in [95–98].

4. Application of Hermite as a Basis Function

Due to the ease of implementation and high-order accuracy, the Hermite as a basis function has been extensively used in many methods for the numerical study of ODEs and PDEs. Dyksen et al. [99] analysed the effectiveness of the Galerkin and collocation techniques in which Hermite bicubic polynomials are used to approximate the solution. The authors showed that the collocation method requires less computer time than the Galerkin method. They also observed that in terms of computer time and error, collocation performs better than the Galerkin method because the Galerkin program uses twice as much memory as required by the collocation program. Houstis [100] used the collocation method based on cubic Hermite interpolation polynomials on rectangular domains to solve linear elliptic problems involving Neumann and Dirichlet boundary conditions:

$$aD_x^2u + 2bD_xD_yu + cD_y^2u + dD_xu + eD_yu + fu = g, \quad \text{in } \Omega = [0,1] \times [0,1],$$

having the boundary condition as:

$$\alpha\frac{\partial u}{\partial x} + \beta u = 0 \text{ on } \partial\Omega.$$

Prenter and Russell [101] used fourth-order orthogonal collocation with a bicubic Hermite polynomial for an elliptic-type PDE:

$$-D_x[p(x,y)D_xu(x,y)] - D_y[q(x,y)D_yu(x,y)] + c(x,y)u = f(x,y), \quad \text{in } \Omega = [0,1] \times [0,1],$$

and the adaptive Hermite element collocation approach was used in combination with these adaptive families by Bhuiyan et al. [102]. For the 2D transportation of a solute in an incompressible fluid field, it generates a matrix having a bandwidth greater than utilising cubic Hermite elements throughout the space domain as follows:

$$\frac{\partial}{\partial x}(D_x\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(D_y\frac{\partial u}{\partial y}) - \frac{\partial}{\partial x}(uV_x) - \frac{\partial}{\partial y}(uV_y) = -\frac{\partial u}{\partial t}, \quad \text{in } \Omega = [a,b] \times [c,d].$$

Chawla et al. [103] solved nonlinear transient 1D heat conduction models by using a method based on collocation in which Hermite cubic splines were taken as basis functions. Dyksen [104] employed the collocation method using Hermite bicubic polynomials for the study of an elliptic problem of the form:

$$D_x u + D_y u = f, \ \text{in}\ \Omega = [0,1] \times [0,1],\ u = 0 \text{ on } \partial\Omega, \tag{20}$$

where

$$\begin{aligned} D_x u &= -c_2(x)u_{xx} + c_1(x)u_x + c_0(x)u, \quad c_2 > 0, \\ D_y u &= -d_2(x)u_{yy} + d_1(x)u_y + d_0(x)u, \quad d_2 > 0, \end{aligned} \tag{21}$$

and then to solve the Hermite bicubic collocation equations' tensor product a generalised ADI iterative method was applied. The author showed that the method with Hermite bicubic collocation method was reliable, numerically stable, and converged fast. Duarte and Portugal [105] developed a moving FEM with cubic Hermite polynomials to solve front causticizing reaction models of the form:

$$\begin{aligned} \frac{\partial C_{OH^-}}{\partial t} &= \frac{1}{\varepsilon} D_e \frac{\partial^2 C_{OH^-}}{\partial \tau^2} + \frac{2D_e}{\tau\varepsilon}\frac{\partial C_{OH^-}}{\partial \tau}, \\ \frac{\partial C_{CO_3^{2-}}}{\partial t} &= \frac{1}{\varepsilon} D_e \frac{\partial^2 C_{CO_3^{2-}}}{\partial \tau^2} + \frac{2D_e}{\tau\varepsilon}\frac{\partial C_{CO_3^{2-}}}{\partial \tau}, \end{aligned}$$

where $0 < \tau < z^I(t)$ and $z^I(t) < \tau < 1$. The value of $z^I(t)$, and the initial and boundary conditions are given in [105]. The derivatives of the solutions with respect to time on the nodes and nodal velocities are determined when the square norm of the discretized residuals over the domain is minimised. For the discretization of the spatial domain, a moving FEM using Hermite polynomials is implemented which forms an ODE system, can be degenerated whenever singularities arise, and is solved by implicit integration. The result obtained by the proposed algorithm is in close agreement with results computed by the orthogonal collocation in finite elements. This work demonstrates the application of the moving finite element technique in solving front reaction models. For space discretization, Leao and Rodrigues [106] used the orthogonal collocation method in which cubic Hermite polynomials are used as a basis function and backward differentiation for the time integrator to solve transient and steady-state models for simulated moving bed processes. Bialecki [107–110] discussed Fourier analysis cyclic reduction and cyclic reduction schemes to solve the system of linear equations arising when the Hermite bicubic with orthogonal spline collocation method is implemented in the Dirichlet-type Poisson's equation:

$$-\Delta u = f(x,y) \text{ in } \Omega = (0,1) \times (0,1),\ u = 0 \text{ on } \partial\Omega, \tag{22}$$

on a rectangular domain. The preconditioned Richardson and preconditioned minimum residual iterative approaches were employed by the authors to solve the linear equations after discretization. Two distinct pseudospectral methods were developed by Schumer and Holloway [111] using Hermite polynomials and weight functions for nonlinear Vlasov–Poisson problems in 1D of the following type:

$$\begin{aligned} \frac{\partial f(x,u,t)}{\partial t} + u\frac{\partial f}{\partial x} + \frac{q_e}{m_e}E(x,t)\frac{\partial f}{\partial u} &= 0, \\ \frac{\partial E(x,t)}{\partial x} = \frac{q_e}{m_0}\int_{-\infty}^{\infty} [f(x,u,t) - f_i(u,0)]du&, \end{aligned} \tag{23}$$

where $f(x,u,t)$ is the electron distribution, $x \in [-L/2, L/2]$, $u \in (-\infty, \infty)$ is the velocity, $f_i(u,0) = \int f(x,u,0)dx$, and $m_e,\ q_e$, and m_0 are the mass, electron charge, and permittivity of free space. The authors demonstrated that the asymmetrically weighted Hermite approach is numerically unstable and fails to preserve the square integral of the distribution.

They applied the symmetrically weighted Hermite scheme for the first time for the numerical solution of the Vlasov system, showing the conservation of particles, momentum (for Nu odd), or energy (for Nu even) for $\Delta t \to 0$, where the largest Hermite mode number is denoted by Nu and conserves the square integral of the distribution with velocity scaling. It is suitable for kinetic computations of warm plasmas and is numerically stable. To predict the frequencies of warm plasma phenomena and growth/damping rates, the above two Hermite techniques on proper scaling were shown to be more precise than unscaled Hermite algorithms and better than particle-in-cell-based methods.

Sun [112] applied the Hermite bicubic collocation approximation for the numerical solution of a rectangular domain, and its discretization form in terms of the tensor product can be expressed as:

$$(A_x \otimes B_y + B_x \otimes A_y) = RHS,$$

and then the FFT method was applied to solve this system. Tse and Chasnov [113] applied the Fourier–Hermite pseudospectral technique for the numerical study of a 3D penetrative-convection-based problem in a vertical direction with an infinite domain under the Boussinesq approximation:

$$\begin{aligned} \frac{\partial \xi}{\partial t} &= -F + \nabla_h^2 v + \sqrt{\sigma/R}\nabla^2 \xi, \\ \frac{\partial \mu}{\partial t} &= -G + \sqrt{\sigma/R}\nabla^2 \mu, \\ \frac{\partial v}{\partial t} &= -H - u_2(3x_2^2 - 1) + 1/\sqrt{\sigma/R}\nabla^2 v, \end{aligned} \tag{24}$$

where F and G represent $-x_2.\nabla \times \nabla \times (u.\nabla)u$, and $x_2.\nabla \times (u.\nabla)u$, $\nabla_h^2 = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_3^2}$ $R = \dfrac{g\alpha B d^4}{\nu\kappa}$ is the Rayleigh number, and $\sigma = \dfrac{\nu}{\kappa}$ is the Prandtl number. In the vertical direction, and in the absence of motion, an S-shaped temperature profile was used and variables were expressed in the form of Fourier–Hermite basis functions. The scaling of Hermite functions was performed for the adjustment of the length of the domain in the vertical aspect. In this paper, semi-implicit algorithms such as Adam–Bashforth, and Crank–Nicolson were used for time discretization, and the Fourier pseudospectral technique with Hermite as a basis function was applied for space discretization. To demonstrate the efficacy of the algorithm, heat fluxes, variances, and their budgets were studied for various values of R.

Dijkstra [114] presented the pseudospectral collocation method for a first-order differential equation in which the point of departure was taken to be a Hermite interpolation. The method gives the $(2N + 1)$ degree of precision over a knot of $(N + 1)$ points when implemented to a first-order differential equation. Every grid point in this approach counts for two, which simultaneously collocates the differential and the differentiated differential equations. The accuracy of the solution produced by the double collocation is better than the precision achieved using the traditional technique. Additionally, compared to the pseudospectral collocation approach using Lagrange interpolation, the suggested algorithm's condition number rises at a rate of N^3 rather than N^2. Edoh et al. [115] used a higher-order Hermite collocation method with cubic Hermite as a basis function for solving nonlinear first-order PDEs with periodic boundary conditions arising on investigation of invariant tori for dynamical systems:

$$\dot{x} = F(x, \lambda), \quad x \in R^n \ \lambda \in R^1,$$

where $x := (x_1, x_2, \ldots, x_n)$. The stability and convergence of the algorithm were discussed in this paper and it was established that the technique was stable and had fourth-order convergence. A smooth shape of the torus is necessary for the method to achieve high-order convergence. The method has the potential for computing invariant tori with mixed interactivity and provides better findings as compared to the leap-frog scheme. Gheri and Marzulli [116] solved the nonlinear IVP

$$y' = f(x, y), \;\; y(x_0) = y_0, x_0 \leq x \leq X,$$

using Hermite interpolation polynomials with the collocation technique. They provided sufficient conditions for the guaranteed convergence of the proposed algorithm for the nonlinear collocation system. Luo et al. [117] used the discontinuous Galerkin method (RDG) algorithm based on a Hermite weighted essentially non-oscillatory (WENO) reconstruction to study compressible Euler equations of the form:

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial F_j(u(x,t))}{\partial x_j} = 0,$$

here, u denotes the conservative vector and F is the vector of inviscid flux, given below:

$$U = \begin{bmatrix} \rho \\ \rho u_i \\ \rho e \end{bmatrix}, F_j = \begin{bmatrix} \rho u_j \\ \rho u_i u_j + p\delta_{ij} \\ u_j(\rho e + p) \end{bmatrix}, \tag{25}$$

where e represents the density, p indicates the pressure, and ρ denotes the specific total energy of the fluid. On tetrahedral knots, the authors used HENO along with the RDG technique. The two drawbacks of the DGM were successfully addressed with the help of this strategy, which also ensured the stability of the reconstructed DG technique. They created it to prevent the spurious oscillations close to strong discontinuities as well as to reduce the high computational costs of the DGM. On tetrahedral nodes, the RDG scheme was applied to solve several flow equations to show its reliability and effectiveness. The numerical outcomes showed that the RDG approach involving Hermite WENO reconstruction was capable of attaining the desired third-order precision, which was one order more precise than the DG technique. As a result, its accuracy was significantly increased without a corresponding increase in computing costs or memory requirements.

In order to solve 1D and 2D nonlinear hyperbolic conservation law systems, Zhao et al. [118] presented a family of Hermite polynomials based on weighted essentially non-oscillatory (WENO) strategies called HWENO techniques. Finite difference, nonlinearly stable Runge–Kutta, and Hermite interpolation algorithms are the foundations upon which HWENO schemes are constructed. The compactness of HWENO systems in the re-establishing is one of their main features. For a fifth-order WENO reconstruction, for instance, five nodes in the stencil are required, but only three nodes are necessary for a fifth-order HWENO (HWENO5) redevelopment. Under the same nodes, in test situations, HWENO5 schemes' numerical errors are found to be less than those of WENO5 strategies. A drawback of HWENO in comparison to the classic WENO technique is that it uses nearly twice as much computer memory and CPU time when employing the identical number of node points.

For both the Dirichlet and Neumann problems, Dyksen [119] provided the explicit closed-form equations using cubic Hermite interpolation for the eigenvectors and eigenvalues of the Laplace operator. In addition, the author demonstrated that for the Dirichlet condition, Gauss points for the collocation points provide the best approximations and for the verification of the theoretical findings, the author solved some numerical examples. Soliman [120] used the Hermite collocation method for an isothermal tubular reactor with an axial dispersion model:

$$\frac{1}{Pe}\frac{d^2u}{dx^2} + \frac{du}{dx} = Da\ R(u),$$

having boundary conditions as:

$$\frac{1}{Pe}\frac{du}{dx}|_{x=0} = u(0) - 1, \quad \frac{du}{dx}|_{x=1} = 0.$$

Heping et al. [121] used the Petrov–Galerkin Hermite spectral method for the convection–diffusion equations on unbounded domains:

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad x \in R, \quad t > 0,$$

with the initial condition as:

$$u(x,0) = U_0(x), \quad x \in R.$$

The stability and spectral convergence of this approach were also discussed. Some experiments were performed to support the theoretical stability and convergence results. Hermite polynomials were utilised as basis functions in the development of the spectral-finite difference technique for the Fokker–Planck equation [122], which has the form:

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \nu \frac{\partial u}{\partial x} - \beta \frac{\partial(\nu u)}{\partial \nu} + F(x)\frac{\partial u}{\partial \nu} - \beta\mu \frac{\partial^2 u}{\partial \nu^2} = 0, \quad |x| < Y, \ |\nu| < \infty, \ t > 0, \\
&u(-Y,\nu,t) = b_L(\nu,t), \quad \nu \geq 0, \ t > 0, \\
&u(Y,\nu,t) = b_R(\nu,t), \quad \nu \leq 0, \ t > 0, \\
&u(\nu,x,t) = w(x,\nu), \quad |x| \leq Y, \ |\nu| < \infty.
\end{aligned}$$

It was shown that the spectral algorithms based on Hermite polynomials converge with spectral precision in weighted Sobolev space. It was shown that adding a velocity scaling parameter to the Hermite basis improves the accuracy and effectiveness of the Hermite spectral technique without adding any extra work. The Hermite spectral method was used by Luo and Stephen [123] for the 1D forward Kolmogorov equation. This paper provided helpful guidance for selecting the scaling factor of the generalised Hermite functions. Tang [124] also utilised spectral methods based on Hermite polynomials for functions of a Gaussian type. Nevenka [125] applied modified Hermite polynomials, as an orthogonal basis in spectral approximation, for the numerical solution of boundary layer problems that had the following form:

$$\begin{aligned}
&\epsilon u''(x) + f(x)u'(x) + g(x)u(x) = h(x), \ x \in [0,1], \\
&(u(0), u(1)) = (A, B).
\end{aligned}$$

In order to demonstrate the efficacy of the presented approach, the paper includes a few numerical problems as well as an upper bound for the error function. Guo et al. [126] proposed spectral and pseudospectral algorithms using Hermite functions for the Dirac equation:

$$\begin{aligned}
&\partial_t \Phi_1(x,t) + \partial_x \Phi_2(x,t) + im\Phi_1(x,t) + 2\lambda Q_1(\Phi(x,t)) = f_1(x,t), \\
&\partial_t \Phi_2(x,t) + \partial_x \Phi_1(x,t) + im\Phi_2(x,t) + 2\lambda Q_2(\Phi(x,t)) = f_2(x,t), \ x \in \omega, \ 0 \leq t \leq T, \\
&\lim_{|x| \to \infty} \Phi(x,t) = 0, \ 0 \leq t \leq T, \\
&\Phi(x,t) = \Phi_0(x), \quad x \in \omega,
\end{aligned}$$

The authors first established basic fundamental approximation findings for the projections and interpolations in the spaces defined by Hermite functions. Then, they took as an experiment of application spectral and pseudospectral algorithms using Hermite functions of the Dirac equation. The proposed algorithm preserved the essential conservation property of the Dirac equation. Guo and Xu [127] used the Hermite pseudospectral scheme for solving Burgers' equation. Additionally, the proposed method's stability and convergence analysis were established. Guo [128] estimated the error bounds of the Hermite spectral method for nonlinear PDEs. For modified Ginzburg–Landau equations for population problems, Xiang and Wang [129] provided some fundamental results on extended Hermite orthogonal approximations, which are essential in spectral methods. Iqbal et al. [130] presented cubic Hermite polynomials based on Galerkin's finite element scheme for the approximation of a third-order BVPs system associated with obstruction, unilateral, and contact problems, having the following form:

$$u'''(x) = \begin{cases} f(x); & x \in [a,c] \\ g(x)u(x) + f(x) + r; & x \in [c,d] \\ f(x); & x \in [d,b], \end{cases}$$

with the boundary conditions as:

$$u(a) = a_0, \quad u'(a) = a_1, \quad u'(b) = a_2.$$

The findings were found to have a higher degree of accuracy when compared to approaches using quartic B-splines and quartic non-polynomial, quartic, cubic, finite difference, and quintic splines. A technique for categorising and analytically computing high-order Hermite interpolating polynomials of the simplex was presented by Gusev et al. [131]. They provided a standard illustration of a triangular element that might be constructed using a high-accuracy finite element technique. Yarasca et al. [132] studied a static analysis of functionally graded single and sandwich beams by applying a seven degrees-of-freedom quasi-3D hybrid-type theory and then a finite element method was applied to solve the governing equations. In this case, the vertical deflection variables were interpolated using C^1 cubic Hermite interpolation, whereas the remaining kinematics variables were computed using C^0 linear interpolation. Convergence analysis was presented to validate the finite element algorithm. Chang et al. [133] adapted the method based on Hermite polynomials for the particular solution approximations of convection–diffusion–reaction problems depending on time, having the form:

$$\frac{\partial u}{\partial t} = D(u,x,t)\nabla^2 u + \mathbf{U}(u,x,t).\nabla U + k(u,x,t)u + s(x,t) \ \text{ in } \Omega.$$

The authors used either the Crank–Nicolson or the Adams–Moulton approach to transform the given equation into time-independent convection–diffusion–reaction problems for subsequent time steps. The traditional Hermite method to approximate the particular solutions (MAPS) and Hermite radial basis function collocation method (RBFCM) were used in solving the resulting equation. They gave the comparison between RBFCM and Hermite MAPS and demonstrated that the results obtained by Hermite MAPS were more accurate and stable for the shape parameter. Karamollahi et al. [134] and Maleknejad and Yousefi [135] approximated the solution of the following nonlinear Fredholm integral equations:

$$u(x) = f(x) + \lambda \int_a^b k(x,t,u(t))dt, \quad x \in [a,b], \tag{26}$$

by using the Hermite interpolation method. Convergence analysis and error estimation were also presented for the proposed technique. The numerical experiments confirmed that the technique is quite easy to implement and gives accurate approximations in reasonable computational times. Pandey et al. [136] presented a Hermite finite element approach for solving Maxwell's equations in complex geometries. The tables and figures demonstrate that the proposed approach is efficient in addressing scalar–vector coupled field problems, such as those involving the modeling of quantum well cavity plasmonics and lasers, while permitting multi-scale practical computations. In order to solve the steady-state convection–diffusion equations, a double boundary collocation Hermitian method was presented by [137]. The proposed method was based on the meshless radial basis Hermite interpolation polynomial. Black and Geddes [138] examined the governing equations for an actively mode-locked laser model:

$$T_M \frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} + p \frac{\partial u}{\partial x} + (g(t) - l - \mu x^2) u(t,x),$$
$$\frac{\partial g(t)}{\partial t} = \alpha - \tau g(t) - \beta g(t) \int_{-\infty}^{\infty} |u(t,x)|^2 dx,$$

by using the combination of a spectral technique and Hermite interpolation polynomials. The spatial approximation using Hermite polynomials was found to be more robust than the spatial approximation based on the finite difference scheme. Orsini et al. [139] solved the multi-zone problems using the control volume method with the Hermite radial as a basis function. They implemented the proposed algorithm on one-, two-, and three-dimensional domains. Adzic [140] obtained the recurrence relation for the Hermite series coefficients and solved polynomial-coefficients-based linear differential equations of the type:

$$\sum_{m=0}^{r} p_m(x) u^{(m)}(x) = g(x).$$

Mathelin et al. [141] utilised Hermite polynomials as the basis function in the Galerkin approximation in order to study the uncertainty quantification in CFD simulations. The numerical results show that the approach is significantly more effective than the polynomial chaos and Galerkin approach. Peirce [142] proposed the cubic Hermite collocation method for solving the coupled integral–partial differential problems directing the propagation of a hydraulic fracture in a condition of planar strain. Ganaie et al. [143–146] applied the collocation method with a cubic Hermite for solving the following PDEs:

- Kuramoto–Sivashinsky equation:

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x}, \quad x \in [a,b], \quad t > 0,$$

- One-dimensional convection–diffusion equation:

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial x}, \quad x \in [a,b] \quad t > 0, \tag{27}$$

- Washing of packed bed of porous particles model:

$$\begin{aligned} \frac{\partial^2 Q}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial Q}{\partial \eta} &= \frac{\partial Q}{\partial \tau} + \frac{1-\varepsilon}{\varepsilon} N_1 \frac{\partial N}{\partial \tau}, \\ \frac{\partial N}{\partial \tau} &= P_1 (C_1 Q (1-N) - k^* N), \\ \frac{\partial C}{\partial \tau} = \frac{\phi Bi}{Pe} \frac{\partial^2 C}{\partial \xi^2} - \phi Bi \frac{\partial C}{\partial \xi} &- \theta Bi (C - Q|_{\eta=1}). \end{aligned} \tag{28}$$

In these papers, several experiments were conducted to show the efficacy of the proposed algorithm. In order to approximate the aerodynamics, Rabbath and Corriveau [147] presented the cubic Hermite interpolating polynomial and evaluated its performance in comparison to a set of standards or metrics as well as to cubic splines and to other piecewise linear functions. Pullan and Bradley [148] calculated the potential distribution across a human torso as a function of the electrical activity of the heart by using cubic Hermite polynomials in the finite element/boundary method. The authors reported findings in two and three dimensions, demonstrating the effectiveness and accuracy of this coupled approach. Shallal et al. [149] solved Equation (27) by a cubic Hermite finite element method. By using the von Neumann algorithm, it was demonstrated that the scheme is unconditionally stable and to evaluate the effectiveness of the approach, the proposed algorithm was applied to a few test problems. The heat conduction problem in 1D:

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0, \;\; a \leq x \leq b, \;\; t > 0$$

was solved by Kutluay et al. [150] by utilising a collocation method based on the Hermite cubic B-spline finite elements. The Fourier stability technique was used to study the stability of the method. A comparison of the approximate and exact solutions was also given to demonstrate the effectiveness and correctness of the presented method. Cubic Hermite polynomials were utilised by Arora et al. [151] as a basis function in the collocation approach to solve the nonlinear advection–diffusion model, including Peclet number and intraparticle coefficient of diffusion (28). The effect of bed porosity, the intraparticle diffusion coefficient, and Peclet number, were investigated theoretically and graphically. Surface plots were used to discuss the impact of the displacement ratio, exit, and average solute concentrations. Applying model-estimated values, an industrial parameter like the displacement ratio was also determined. Rekatsinas and Saravanos [152] employed the Hermite spline layerwise temporal spectral finite element technique to approximate the solution of waves and transient problems arising in laminated composite and sandwich plates. The fitted finite difference approach and the Runge–Kutta method involving cubic Hermite approximation coupled with piecewise equispaced mesh were proposed by Subburayan and Mahendran [153] for solving singularly perturbed problems involving convection–diffusion phenomena in delay differential equations of third order having the following form:

$$\begin{aligned} \varepsilon u'''(x) + a_1(x)u''(x) + b_1(x)u'(x) + c_1(x)u(x) + d_1(x)u'(x-1) = f(x), \;\; x \in \Omega, \\ u(x) = \phi(x), \; u'(2) = l, \; \phi(x) \in C^1[-1,0], \end{aligned}$$

They used the supremum norm to derive the error bounds, and it was found that the approach was first-order convergent. Wu [154] suggested a cubic Hermite-polynomials-based Eulerian–Lagrangian single-node collocation technique for computing unsteady-state advection–diffusion transport models having the following representation:

$$\begin{aligned} \alpha(x,t)\frac{\partial u}{\partial t} + v(x,t)\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\left(s(x,t)\frac{\partial u}{\partial x}\right) = f(x,t), \;\; a \leq x \leq b, \;\; t > 0, \\ u(a,t) = g_1(t), \; u(b,t) = g_2(t), \; u(x,0) = u_0(x). \end{aligned}$$

Here, $\alpha(x,t)$ is a retardation coefficient, $v(x,t)$ is the fluid velocity, $s(x,t)$ is a diffusion coefficient, $f(x,t)$ is a given source or sink function, and $u(x,t)$ represents the concentration of the dissolved substance in the subsurface flow. $u_0(x)$, $g_1(t)$, and $g_2(t)$ are the initial and boundary data that are needed to close the system. The number of unknowns was significantly reduced by using the above technique and produced precise numerical solutions, even for very large time steps. Zhao and Wu [155] applied the cubic Hermite spline collocation technique to study the variable-order nonlinear fractional differential equation:

$$\alpha^2 u(x) - D^{\beta(x)}u(x) = f(x), \;\; a \leq x \leq b, \; u(a) = u(b) = 0.$$

The convergence is of order $O(h^{\min(4-\beta,p)})$, where the approximating polynomial belongs to $C^p(p \geq 1)$. The bicubic Hermite orthogonal collocation algorithm of two-dimensional integral differential equations in square domains and the wave-Petrovsky system with memory were investigated by Xu in [156,157], respectively. Ashpazzadeh et al. [158] solved the second kind of singular Abel's equation having the following form:

$$u(x) - \int_0^x (x-s)^{-\alpha} K(x,s,u(s))ds = f(x), \; 0 < s < 1, \; 0 < \alpha < 1,$$

by using the Galerkin method along with biorthogonal Hermite cubic spline multi-wavelets as a basis function, where $K : [0,1] \times R \to R$ is considered as a known function and $f : [0,1] \to R$ is a sufficiently smooth function with respect to u. Furthermore, the convergence analysis of the algorithm is studied in this paper. An unconditionally stable 4^{th}-order method based on Hermite cubic interpolation is proposed by Luo and Du [159] for the simulation of telegraph equations:

$$\frac{\partial^2 u}{\partial t^2} + 2v\frac{\partial u}{\partial t} + \beta^2 u = \frac{\partial^2 u}{\partial x^2} + f(x,t), \;\; a \le x \le b, \;\; t > 0,$$
$$u(a,t) = g_1(t), \; u(b,t) = g_2(t), \; u(x,0) = u_0(x), \; \frac{\partial u(x,0)}{\partial t} = u_1(x),$$

In this paper, the proposed algorithm is of order $O(\Delta t^4 + h^4)$. Vincent et al. [160] proposed the Hermite-style cubic serendipity interpolation method for a finite element simulation of the cardiac monodomain problem of the form:

$$\xi\left(C_m\frac{\partial u}{\partial t} + I_{ionic}(u)\right) = \nabla.\sigma\nabla u,$$

where σ indicates the conductivity tensor, C_m represent the specific capacitance of the cell membrane, u denotes the transmembrane potential, I_{ionic} is the current that arises as a result of the flow of ions across cell membrane channels, and ξ corresponds to the surface area to volume ratio. Jebreen and Dassios [161] proposed a biorthogonal Hermite cubic spline Galerkin algorithm for the fractional Riccati equation:

$$D_0^{\beta}u(x) = f(x) + g(x)u(x) + h(x)u^2(x), \; a \le x \le b,$$
$$u^{(\eta)}(0) = g_\eta, \; \eta = 0, 1, \dots, n-1,$$

and then Newton's iterative method was used to solve the resulting algebraic system. The linear Black–Scholes equation of the type:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0, \;\; x > 0, \;\; t \in (0, T),$$

was solved by Chihaluca [162] by using the cubic Hermite finite element method. The cubic Hermite finite element continuation approach with a predictor–corrector solver was proposed by Chien and Shih [163] for the computational analysis of von Karman equations of the following type:

$$\Delta^2 v - 1/2[u,u] = 0,$$
$$\Delta^2 u + \lambda u_{xx} - [u,v] = 0, \;\; \text{in}\; \Omega = [0,l] \times [0,1].$$

Mohammadzadeh et al. [164] solved the Lane–Emden equation using a cubic Hermite-splines-based collocation method. The Lane–Emden equation solution was converted into a system of algebraic equations using the Hermite splines' properties. Piecewise cubic Hermite polynomials were used by researchers in various algorithms; for more details, readers can refer to [165–189].

Ricciardi and Brill [190] used the optimal quintic Hermite collocation algorithm with an adaptive hybrid optimisation algorithm in order to solve a one-dimensional convection–diffusion model involving transport of contaminants dissolved in groundwater. In order to determine the appropriate refinement of the mesh for a variety of models characterised by velocity fields, a hybrid approach combining an adaptive genetic method and a hill-climbing strategy was used. As compared to mesh refinements produced using direct search techniques, optimum mesh refinements determined using this hybrid approach are either significantly better or equally as good. Arora et al. [17,191–194] developed the Hermite collocation approach by using quintic Hermite polynomials as the basis function for solving various types of PDEs of the following forms:

- Linear convection–diffusion problem:

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - p(x)\frac{\partial u}{\partial x} - q(x), \;\; a < x < b, \;\; t > 0,$$

- Benjamin–Bona–Mahony–Burgers problem:

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^2 \partial t} = \alpha \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial x} - u\frac{\partial u}{\partial x}, \;\; a < x < b, \;\; t > 0.$$

and Equations (27) and (28). In addition, they studied the presented algorithm's stability and convergence analysis. Kaur et al. [195] utilised quintic Hermite polynomials as the basis function in orthogonal collocation on the finite element technique to study the impact of interstitial velocity, Peclet number, and cake thickness on the nonlinear and linear diffusion–dispersion problems of a pulp washing model:

$$\begin{aligned} a_1 \frac{\partial^2 u}{\partial x^2} = a_2 \frac{\partial u}{\partial x} + a_3 \frac{\partial u}{\partial t}, \;\; 0 < x < 1, \;\; t > 0, \\ m_1 u + m_2 \frac{\partial u}{\partial x} = k_1 \text{ at } x = 0, \\ m_3 u + m_4 \frac{\partial u}{\partial x} = k_2, \text{ at } x = 1, \\ u(x,0) = u_0(x). \end{aligned}$$

The Peclet number was the major determining factor in the pulp washing procedure, while the cake thickness and interstitial velocity had a less significant influence. The time-marching approach for linear systems provided by Kolsti and Kunz [196] depends on Hermite quintic polynomial interpolation, a fully implicit one-step collocation scheme that imposes acceleration and jerk restrictions at a point in time selected by the user. The method converges at a rate of 4 and it was demonstrated that the suggested algorithm is unconditionally stable, even for events involving a harmonic external force and viscous damping. Marasi and Derakhshan [197] developed the approximate technique involving finite difference and Hermite quintic collocation algorithms for a variable-order time fractional mobile–immobile advection–dispersion model:

$$\gamma_1 \frac{\partial u}{\partial t} + \gamma_2 D_t^{\alpha(x,t)} u = -\rho \frac{\partial u}{\partial x} + \rho_1 \frac{\partial^2 u}{\partial x^2} + f(x,t), \; 0 < x < L, \; t > 0.$$

A method with Hermite polynomials is implemented on the spatial derivative and the temporal derivative is discretized using a weighted finite difference method. In contrast to various other schemes that are available in the literature, the results obtained by the collocation method on using quintic Hermite spline polynomials as the basis function show that the presented algorithm is very efficient. Zhou and Wu [198] solved the KdV equation:

$$\frac{\partial u}{\partial t} + u^p \frac{\partial u}{\partial x} + \varepsilon \frac{\partial^2 u}{\partial x^2} = 0, \; 0 < x < L, \; t > 0.$$

using the periodic boundary conditions provided by the Hermite quintic collocation technique that involves moving meshes produced on solving moving mesh PDEs. Numerical examples were used to support the study and demonstrate the effectiveness of the approach. Quintero et al. [199] constructed techniques that can efficiently command the actuators of an articulated robot. For this, the cubic and quintic Hermite finite elements were used for the time discretization method. The proposed control optimisation entailed using a conjugate-gradient-type method to directly minimise the chosen criterion. The Hermite method's superconvergence was demonstrated using a general example.

Kvitsinsky and Hu [200] used the Faddeev-components-based Hermite tri-quintic expansion for the solution of the 3D Faddeev equations for three-body Coulomb bound states. In order to handle the strain measurement across any arbitrary area of interest in digital picture correlation, Zhao et al. [201] provided the enhanced Hermite finite element smoothing technique. This approach is based on the Tikhonov regularisation and Hermite interpolation on finite elements. Sestini et al. [202] used the quintic Hermite interpolation to dicretize the space derivatives and for adjusting the two free angular parameters that define the set of probable solutions, Pythagorean-hodograph analysis of the so-called CC criterion is carried out that is presented in [203]. Singh [204] studied traveling waves behaviour, which is a part of the KdV equation, by using orthogonal collocation on finite elements (OCFEs) with a quintic Hermite spline. A robust high-order superconvergent approach is produced by collocation using Gauss points and quintic polynomials. OCFE utilising the quintic Hermite basis is more precise than the B-splines basis and computationally more effective than collocation techniques employing piecewise polynomials. Brill [205] found the analytical solution of the self-adjoint ordinary differential Equation (29) by the Hermite collocation method.

Mkhize et al. [206] presented a collocation method on finite elements by utilising the heptic (septic) Hermite polynomials as basis functions. The illustration of the superconvergence phenomenon is achieved at the nodes. The results produced by Carl R. de Boor in 1973 are significantly supported by the global and nodal rates of convergence.

The authors have tried to include as many papers as possible on the Hermite interpolation polynomials and their properties and applications. It is possible that some papers might be unintentionally left out of this review article. Now, the overall conclusions are summarised below.

5. Author's Contribution

There is no doubting that singularly perturbed linear and nonlinear differential problems are harder to solve since the convective coefficient's sign changes, and this is especially true when the solutions involve boundary layers. Due to the abrupt changes in the solutions, particularly in the layer region where the perturbation parameter tends to zero, these problems are not easy to solve using a conventional method. Although sophisticated schemes are available for handling numerical problems outside layer boundaries, methods for tackling such problems inside layer boundaries are still limited. The authors [18,207–212] proposed the orthogonal collocation method based on the finite element method in which septic Hermite polynomials are utilised as basis functions to carry out the study of different types of singularly perturbed linear and nonlinear ordinary and partial differential equations:

- Singularly perturbed differential problem:

$$\varepsilon u'' + c(x)u' + d(x)u = m(x), \quad x \in (a,b), \tag{29}$$

- Modified Burgers' equation:

$$\frac{\partial u}{\partial t} = \varepsilon\frac{\partial^2 u}{\partial x^2} - u^p\frac{\partial u}{\partial x}, \;\; a < x < b, \;\; t > 0,$$

- Modified regularised long wave equation:

$$\frac{\partial u}{\partial t} - \mu\frac{\partial^3 u}{\partial^2 x\partial t} + \alpha u\frac{\partial u}{\partial x} - \beta\frac{\partial u}{\partial x} + \gamma u^2\frac{\partial u}{\partial x}, \;\; a < x < b, \;\; t > 0.$$

- Hodgkin–Huxley equation:

$$\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} = \beta(1-u^p)(u^p-\gamma), \;\; a < x < b, \;\; t > 0.$$

For the temporal and spatial domains, respectively, the septic Hermite collocation technique and Crank–Nicolson scheme are utilised in this paper. The stability and convergence analysis are investigated by the authors. The scheme has sixth-order convergence and is proven to be unconditionally stable. The method is implemented on different test equations to show the efficacy and robustness of the proposed algorithm. The septic Hermite interpolating polynomials have the property that not only is the solution continuous but also all of its first three derivatives are continuous over the entire domain. An explicit expression for $e(x)$ in terms of the Green's function provided by Peano's theorem is used to calculate the error bounds for septic Hermite interpolation that are given in Table 1. For more detail, the reader can refer to [18].

6. Conclusions

This review provides a comprehensive analysis of state-of-the art Hermite interpolating polynomials that are used as a basis function in a variety of techniques such as the collocation method, orthogonal collocation on finite elements, the Galerkin method, the finite element method, etc., to solve real-life phenomena occurring in the fields of science and engineering. An introduction of cubic, quintic and septic Hermite interpolation polynomials and their formation is presented. The various type of error bounds, generalization, properties and applications of the Hermite interpolating polynomials are also reviewed in this survey paper. These basis functions are of class $C^d,\ d = 1, 2, 3$. Because the dependent variable and its first d derivatives are continuous in the Hermite interpolating polynomials over the entire solution space, there are fewer equations to be solved. The principle of continuity is also used to avoid double calculations at mesh locations. As a result, the computational time is drastically reduced. The technique for solving nonlinear equations is straightforward, conservative, and easy to implement. The combination of Hermite polynomials and splines as basis functions with different algorithms provides better accuracy and stability than those produced by other conventional methods.

When compared to quartic, cubic, quartic non-polynomial, quintic spline, finite difference, and quartic B-spline schemes, Hermite polynomials' results have demonstrated that they are more accurate in some cases. The Hermite polynomials are adopted as basis functions in different methods for solving higher-order partial differential equations and many other types of nonlinear coupled systems of partial differential equations. These basis functions can be utilised to solve various elliptic and hyperbolic equations. Additionally, these functions can also be extended to solve fractional differential equations that play a vital role in different fields such as economics, science, engineering, control theory, aerodynamics, etc. The Hermite polynomials can be used to solve integral equation of different kinds. The authors believe that this review study will be significantly helpful to researchers working in this field as they develop novel numerical methods for solving various differential equations from both theoretical and numerical perspectives, for higher order Hermite polynomials ($d > 3$) and the estimation of their error bounds.

7. Future Applications/Advancements

Further applications of the septic Hermite collocation method include higher-dimensional differential equations and numerous other kinds of nonlinear, coupled systems of ordinary and partial differential equations. The method can be used to solve various types of PDEs and higher-order equations with certain transformations or substitutions. The work can be extended to solve fractional differential equations, that have many applications in diverse areas of science such as biology, physics, chemistry, visco-elasticity, heat and mass transfer, signal and image processing, etc.

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