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Posner's Theorem and ∗-Centralizing Derivations on Prime Ideals with Applications

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Abstract: A well-known result of Posner's second theorem states that if the commutator of each element in a prime ring and its image under a nonzero derivation are central, then the ring is commutative. In the present paper, we extended this bluestocking theorem to an arbitrary ring with involution involving prime ideals. Further, apart from proving several other interesting and exciting results, we established the ∗-version of Vukman's theorem. Precisely, we describe the structure of quotient ring $\mathfrak{A}/\mathfrak{L}$, where $\mathfrak{A}$ is an arbitrary ring and $\mathfrak{L}$ is a prime ideal of $\mathfrak{A}$. Further, by taking advantage of the ∗-version of Vukman's theorem, we show that if a 2-torsion free semiprime $\mathfrak{A}$ with involution admits a nonzero ∗-centralizing derivation, then $\mathfrak{A}$ contains a nonzero central ideal. This result is in the spirit of the classical result due to Bell and Martindale (Theorem 3). As the applications, we extended and unified several classical theorems. Finally, we conclude our paper with a direction for further research.

Keywords: derivation; ∗-centralizing derivation; ∗-commuting derivation; involution; prime ideal; prime ring; semiprime ring

MSC: 16N60; 16W10; 16W25

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1. Introduction

The motivation for this paper lies in an attempt to extend in some way the famous results due to Posner [1], Vukman [2] and Ali-Dar [3]. A number of authors have generalized these theorems in several ways (see, for example, [4–11], where further references can be found). Throughout this article, $\mathfrak{A}$ will represent an associative ring with center $\mathfrak{Z}(\mathfrak{A})$. The standard polynomial identity s_4 in four variables is defined as $s_4(\ell_1, \ell_2, \ell_3, \ell_4) = \sum_{\sigma \in s_4} (-1)^\sigma \ell_{\sigma(1)} \ell_{\sigma(2)} \ell_{\sigma(3)} \ell_{\sigma(4)}$, where $(-1)^\sigma$ is $+1$ or -1 according to σ being an even or odd permutation in symmetric group s_4. For any $s, t \in \mathfrak{A}$, the symbol $[s, t] = st - ts$ stands for a commutator, while the symbol $s \circ t$ will stand for the anti-commutator $st + ts$. The higher-order commutator is defined as follows: for any $s, t \in \mathfrak{A}$,

$$[s, t]_0 = s, \ [s, t]_1 = [s, t] = st - ts \text{ and } [s, t]_2 = [[s, t], t],$$

and, inductively, we write $[s, t]_k = [[s, t]_{k-1}, t]$ (where $k > 1$ is a fixed integer), which is called the commutator of order k or simply the k^{th}-commutator. It is also known as the Engel condition in the literature (viz.; [8]). Analogously, we define the higher-order anti-commutator as follows:

$$s \circ_0 t = s, \; s \circ_1 t = st + ts \text{ and } s \circ_2 t = (s \circ_1 t) \circ t,$$

and, inductively, we set $s \circ_k t = (s \circ_{k-1} t) \circ t$, (where $k > 1$ is a fixed integer), which is called the anti-commutator of order k or simply the k^{th}-anti-commutator.

Recall that a ring $\mathfrak{A}$ is called prime if, for $\ell, \vartheta \in \mathfrak{A}$, $\ell\mathfrak{A}\vartheta = (0)$ implies that $\ell = 0$ or $\vartheta = 0$. By a prime ideal of a ring $\mathfrak{A}$, we mean a proper ideal $\mathfrak{L}$ and, for $\ell, \vartheta \in \mathfrak{A}$, $\ell\mathfrak{A}\vartheta \subseteq \mathfrak{L}$ implies that $\ell \in \mathfrak{L}$ or $\vartheta \in \mathfrak{L}$. We note that, for a prime ring $\mathfrak{A}$, (0) is the prime ideal of $\mathfrak{A}$ and $\mathfrak{A}/\mathfrak{L}$ is a prime ring. An ideal $\mathfrak{L}$ of a ring $\mathfrak{A}$ is called semiprime if it is the intersection of prime ideals or, alternatively, if $\ell\mathfrak{A}\ell \subseteq \mathfrak{L}$ implies that $\ell \in \mathfrak{L}$ for any $\ell \in \mathfrak{A}$. A ring $\mathfrak{A}$ is said to be n-torsion free if $n\ell = 0$, $\ell \in \mathfrak{A}$ implies that $\ell = 0$. An additive mapping $\ell \mapsto \ell^*$ satisfying $(\ell\vartheta)^* = \vartheta^*\ell^*$ and $(\ell^*)^* = \ell$ is called an involution. A ring equipped with an involution is known as a ring with involution or $*$-ring. An element ℓ in a ring with involution $*$ is said to be Hermitian if $\ell^* = \ell$ and skew-Hermitian if $\ell^* = -\ell$. The sets of all Hermitian and skew-Hermitian elements of $\mathfrak{A}$ will be denoted by $\mathfrak{H}(\mathfrak{A})$ and $S(\mathfrak{A})$, respectively. If $\mathfrak{A}$ is 2-torsion free, then every $\ell \in \mathfrak{A}$ can be uniquely represented as $2\ell = h + k$, where $h \in \mathfrak{H}(\mathfrak{A})$ and $k \in S(\mathfrak{A})$. The involution is said to be of the first kind if $\mathfrak{H}(\mathfrak{A}) \subseteq \mathfrak{Z}(\mathfrak{A})$; otherwise, it is said to be of the second kind. We refer the reader to [12,13] for justification and amplification for the above mentioned notations and key definitions.

A map $e : \mathfrak{A} \to \mathfrak{A}$ is a derivation of a ring $\mathfrak{A}$ if e is additive and satisfies $e(\ell\vartheta) = e(\ell)\vartheta + \ell e(\vartheta)$ for all $\ell, \vartheta \in \mathfrak{A}$. A derivation e is called inner if there exists $a \in \mathfrak{A}$ such that $e(\ell) = [a, \ell]$ for all $\ell \in \mathfrak{A}$. An additive map $F : \mathfrak{A} \to \mathfrak{A}$ is called a generalized derivation if there exists a derivation e of $\mathfrak{A}$ such that $F(\ell\vartheta) = F(\ell)\vartheta + \ell e(\vartheta)$ for all $\ell, \vartheta \in \mathfrak{A}$ (see [14] for details). For a nonempty subset S of $\mathfrak{A}$, a mapping $\xi : S \to \mathfrak{A}$ is called commuting (resp. centralizing) on S if $[\xi(\ell), \ell] = 0$ (resp. $[\xi(\ell), \ell] \in \mathfrak{Z}(\mathfrak{A})$) for all $\ell \in S$. The investigation into the commuting and centralizing mappings goes back to 1955 when Divinsky [15] established a significant result. Specifically, Divinsky demonstrated that a simple Artinian ring is commutative if it has a commuting automorphism different from the identity mapping. Two years later, Posner [1] showed that a prime ring must be commutative if it admits a nonzero centralizing derivation. In 1970, Luh [16] generalized Divinsky's result for prime rings. Later, Mayne [17] established the analogous result of Posner for nonidentity centralizing automorphisms. The culminating results in this series can be found in [2,6–8,18–23]. In ([2], Theorem 1), Vukman generalized Posner's second theorem for the second-order commutator and established that, if a prime ring of characteristic different from 2 admits a nonzero derivation e such that $[e(\ell), \ell]_2 = 0$ for all $\ell \in \mathfrak{A}$, then $\mathfrak{A}$ is commutative. In this sequel, Bell and Martindale [18] generalized the result of Mayne [24] for nonzero left ideals. Precisely, they proved that if a semiprime ring $\mathfrak{A}$ admits a derivation e that is nonzero on $\mathfrak{U}$ and centralizing on $\mathfrak{U}$, where $\mathfrak{U}$ is a nonzero left ideal of $\mathfrak{A}$, then $\mathfrak{A}$ contains a nonzero central ideal. The most classical and elegant generalization of Posner's second theorem is due to Lanski [25]. Precisely, he proved that, if a prime ring $\mathfrak{A}$ admits a nonzero derivation e such that $[e(\ell), \ell]_k = 0$ for all $\ell \in L$, where L is a non-commutative Lie ideal of $\mathfrak{A}$ and $k > 0$ is a fixed integer, then $char(\mathfrak{A}) = 2$ and $\mathfrak{A} \subseteq M_2(F)$ for a field F. These results have been extended in various ways (viz.; [10,11,26–28] and references therein). The goal of this paper was to study these results in the setting of arbitrary rings with involution engaging prime ideals and to describe the structure of a quotient ring $\mathfrak{A}/\mathfrak{L}$, where $\mathfrak{A}$ is an arbitrary ring and $\mathfrak{L}$ is a prime ideal of $\mathfrak{A}$.

Let $\mathfrak{A}$ be a ring with involution $*$ and S be a nonempty subset of $\mathfrak{A}$. Following [3,29], a mapping ϕ of $\mathfrak{A}$ onto itself is called $*$-centralizing on S if $\phi(\ell)\ell^* - \ell^*\phi(\ell) \in \mathfrak{Z}(\mathfrak{A})$ for all $\ell \in S$. In the special case where $\phi(\ell)\ell^* - \ell^*\phi(\ell) = 0$ for all $\ell \in S$, the mapping ϕ is said to be $*$-commuting on S. In [3,29], the first author together with Dar initiated the study of these mappings and proved that the existence of a nonzero $*$-centralizing derivation of a prime ring with second-kind involution forces the ring to be commutative. Apart from the characterizations of these mappings of prime and semiprime rings with involution, they also proved $*$-version of Posner's second theorem and its related problems. Precisely, they established that: let $\mathfrak{A}$ be a prime ring with involution $*$ such that $char(\mathfrak{A}) \neq 2$. Let e be a nonzero derivation of $\mathfrak{A}$ such that $[e(\ell), \ell^*] \in \mathfrak{Z}(\mathfrak{A})$ for all $\ell \in \mathfrak{A}$ and $e(S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A})) \neq (0)$. Then, $\mathfrak{A}$ is commutative. Further, they showed that every $*$-commuting map $f : \mathfrak{A} \to \mathfrak{A}$ on a semiprime ring with involution of a characteristic different from two is of the form $f(\ell) = \lambda \ell^* + \mu'(\ell)$ for all $\ell \in \mathfrak{A}$, $\lambda \in C$ (the extended centroid of $\mathfrak{A}$) and that $\mu' : \mathfrak{A} \to C$ is an additive mapping. In the sequel, recently, Nejjar et al. ([4], Theorem 3.7) established that, if a 2-torsion free prime ring with involution of the second kind admits a nonzero derivation e such that $e(\ell)\ell^* - \ell^* e(\ell) \in \mathfrak{Z}(\mathfrak{A})$ for all $\ell \in \mathfrak{A}$, then $\mathfrak{A}$ is commutative. In 2020, Alahmadi et al. [30] extended the above mentioned result for generalized derivations. Over the last few years, the interest on this topic has been increased and numerous papers concerning these mappings on prime rings have been published (see [4,9,30–37] and references therein). In [38], Creedon studied the action of derivations of prime ideals and proved that if e is a derivation of a ring $\mathfrak{A}$ and $\mathfrak{L}$ is a semiprime ideal of $\mathfrak{A}$ such that $\mathfrak{A}/\mathfrak{L}$ is characteristic-free and $e^k(\mathfrak{L}) \subseteq \mathfrak{L}$, then $e(\mathfrak{L}) \subseteq \mathfrak{L}$ for some positive integer k. Very recently, Idrissi and Oukhtite [39] investigated the structure of a quotient ring $\mathfrak{A}/\mathfrak{L}$ via the action of generalized derivations on the prime ideal of $\mathfrak{L}$. For more recent works, see [40–42] and references therein. In view of the above observations and motivation, the aim of the present paper was to prove the following main theorems.

Theorem 1. *Let $\mathfrak{A}$ be a ring with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \not\subseteq \mathfrak{L}$. If e_1 and e_2 are derivations of $\mathfrak{A}$ such that $e_1(\ell)\ell^* - \ell^* e_2(\ell) \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then one of the following holds:*

1. *$char(\mathfrak{A}/\mathfrak{L}) = 2$;*
2. *$e_1(\mathfrak{A}) \subseteq \mathfrak{L}$ and $e_2(\mathfrak{A}) \subseteq \mathfrak{L}$;*
3. *$\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain.*

Theorem 2. *Let $\mathfrak{A}$ be a ring with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \not\subseteq \mathfrak{L}$. If e_1 and e_2 are derivations of $\mathfrak{A}$ such that $[e_1(\ell), \ell^*] + [\ell, e_2(\ell^*)] + [\ell, \ell^*] \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then one of the following holds:*

1. *$char(\mathfrak{A}/\mathfrak{L}) = 2$;*
2. *$\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain.*

Theorem 3. *Let $\mathfrak{A}$ be a ring with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \not\subseteq \mathfrak{L}$. If $\mathfrak{A}$ admits a derivation e such that $e(\ell\ell^*) - e(\ell^*)e(\ell) \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then one of the following holds:*

1. *$char(\mathfrak{A}/\mathfrak{L}) = 2$;*
2. *$e(\mathfrak{A}) \subseteq \mathfrak{L}$.*

Theorem 4. *Let $\mathfrak{A}$ be a ring with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \not\subseteq \mathfrak{L}$. If $\mathfrak{A}$ admits a derivation e such that $[[e(\ell), \ell^*], \ell^*] \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then one of the following holds:*

1. *$char(\mathfrak{A}/\mathfrak{L}) = 2$;*
2. *$e(\mathfrak{A}) \subseteq \mathfrak{L}$;*
3. *$\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain.*

Theorem 5. *Let $\mathfrak{A}$ be a 2-torsion free semiprime ring with involution $*$ of the second kind. If $\mathfrak{A}$ admits a nonzero $*$-centralizing derivation e, i.e., $[e(\ell), \ell^*] \in \mathfrak{Z}(\mathfrak{A})$ for all $\ell \in \mathfrak{A}$, then $\mathfrak{A}$ contains a nonzero central ideal.*

In view of $*$-centralizing mappings [3,29], Theorems 4 and 5 are recognized as the $*$-versions of well-known theorems due to Vukman [2] and Posner [1]. As the applications of Theorems A to E just mentioned above, we extended and unified several classical theorems proved in [1–4,23,29,32]. Since these results are in a new direction, there are various interesting open problems related to our work. Hence, we conclude our paper with a direction for further research in this new and exciting area of theory of rings with involution.

We performed a large amount of calculation with commutators and anti-commutators, routinely using the following basic identities: For all $s, t, w \in \mathfrak{A}$;

$$[st, w] = s[t, w] + [s, w]t \text{ and } [s, tw] = t[s, w] + [s, t]w$$

$$s \circ (tw) = (s \circ t)w - t[s, w] = t(s \circ w) + [s, t]w$$

$$(st) \circ w = s(t \circ w) - [s, w]t = (s \circ w)t + s[t, w].$$

2. Preliminary Results

Let $\mathfrak{A}$ be a $*$-ring. Following [33,43], an additive mapping $e : R \to R$ is called a $*$-derivation of $\mathfrak{A}$ if $e(\ell\vartheta) = e(\ell)\vartheta^* + \ell e(\vartheta)$ for all $\ell, \vartheta \in \mathfrak{A}$. An additive mapping $e : \mathfrak{A} \to \mathfrak{A}$ is called a Jordan $*$-derivation of $\mathfrak{A}$ if $e(\ell^2) = e(\ell)\ell^* + \ell e(\ell)$ for all $\ell \in \mathfrak{A}$. In [19], Brešar showed that if a prime ring $\mathfrak{A}$ admits nonzero derivations e_1 and e_2 of $\mathfrak{A}$ such that $e_1(\ell)\ell - \ell e_2(\ell) \in \mathfrak{Z}(\mathfrak{A})$ for all $\ell \in I$, where I is a nonzero left ideal of $\mathfrak{A}$, then $\mathfrak{A}$ is commutative. Further, this result was extended by Argac [44] as follows: let $\mathfrak{A}$ be a semiprime ring and e_1, e_2 be derivations of $\mathfrak{A}$ such that at least one is nonzero. If $e_1(\ell)\ell - \ell e_2(\ell) \in \mathfrak{Z}(\mathfrak{A})$ for all $\ell \in \mathfrak{A}$, then $\mathfrak{A}$ contains a nonzero central ideal. Motivated by the above mentioned results, the first author together with Alhazmi et al. [35] studied a more general problem in the setting of rings with involution. Precisely, they proved that if a $(m+n)!$-torsion free prime ring with involution of the second kind admit Jordan $*$-derivations e and g of $\mathfrak{A}$ such that $e(\ell^m)\ell^n \pm \ell^n g(\ell^m) = 0$ for all $\ell \in \mathfrak{A}$ (where m and n are fixed positive integers), then $e = g = 0$ or $\mathfrak{A}$ is commutative. In the sequel, very recently, Nejjar et al. ([4] Theorem 3.7) established that if a 2-torsion free prime ring with involution of the second kind admits a nonzero derivation e such that $e(\ell)(\ell)^* - (\ell)^* e(\ell) \in \mathfrak{Z}(\mathfrak{A})$ for all $\ell \in \mathfrak{A}$, then $\mathfrak{A}$ is commutative. The goal of this section is to initiate the study of a more general concept than $*$-centralizing mappings; that is, we consider the situation where the mappings ϕ and ξ of a ring $\mathfrak{A}$ satisfy $\phi(\ell)(\ell)^* - (\ell)^*\xi(\ell) \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, where $\mathfrak{A}$ is an arbitrary ring and $\mathfrak{L}$ is a prime ideal of $\mathfrak{A}$. Precisely, we prove the following theorem.

Theorem 6. *Let $\mathfrak{A}$ be a ring with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \not\subseteq \mathfrak{L}$. If e_1 and e_2 are derivations of $\mathfrak{A}$ such that $e_1(\ell)(\ell)^* - (\ell)^* e_2(\ell) \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then one of the following holds:*

1. *$char(\mathfrak{A}/\mathfrak{L}) = 2$;*
2. *$e_1(\mathfrak{A}) \subseteq \mathfrak{L}$ and $e_2(\mathfrak{A}) \subseteq \mathfrak{L}$;*
3. *$\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain.*

The following are the immediate consequences of Theorem 6. In fact, Corollary 1 is in the spirit of the result due to Posner's second theorem.

Corollary 1. *Let $\mathfrak{A}$ be a ring with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \not\subseteq \mathfrak{L}$. If $\mathfrak{A}$ admits a derivation e such that $[e(\ell), (\ell)^*] \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then one of the following holds:*

1. *$char(\mathfrak{A}/\mathfrak{L}) = 2$;*
2. *$e(\mathfrak{A}) \subseteq \mathfrak{L}$;*
3. *$\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain.*

Corollary 2. *Let $\mathfrak{A}$ be a ring with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \not\subseteq \mathfrak{L}$. If $\mathfrak{A}$ admits a derivation e such that $e(\ell) \circ (\ell)^* \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then one of the following holds:*

1. *$char(\mathfrak{A}/\mathfrak{L}) = 2$;*
2. *$e(\mathfrak{A}) \subseteq \mathfrak{L}$.*

Corollary 3. *Let $\mathfrak{A}$ be a prime ring with involution $*$ of the second kind such that $char(\mathfrak{A}) \neq 2$. If $\mathfrak{A}$ admits a $*$-commuting derivation e, then $e = 0$ or $\mathfrak{A}$ is a commutative integral domain.*

Corollary 4. *Let $\mathfrak{A}$ be a prime ring with involution $*$ of the second kind such that $char(\mathfrak{A}) \neq 2$. If $\mathfrak{A}$ admits a derivation e such that $e(\ell) \circ (\ell)^* = 0$ for all $\ell \in \mathfrak{A}$, then $e = 0$.*

For the proof of Theorem 6, we need the following lemmas, some of which are of independent interest. We begin our discussions with the following.

Lemma 1 ([42] (Lemma 2.1))**.** *Let $\mathfrak{A}$ be a ring and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$. If e is a derivation of $\mathfrak{A}$ satisfying the condition $[e(\ell), \ell] \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then $e(\mathfrak{A}) \subseteq \mathfrak{L}$ or $\mathfrak{A}/\mathfrak{L}$ is commutative.*

Lemma 2 ([45] (Lemma 1))**.** *Let $\mathfrak{A}$ be a ring, $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$, e_1 and $e_2$be derivations of $\mathfrak{A}$. Then, $e_1(\ell)\ell - \ell e_2(\ell) \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$ if and only if $e_1(\mathfrak{A}) \subseteq \mathfrak{L}$ and $e_2(\mathfrak{A}) \subseteq \mathfrak{L}$ or $\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain.*

Lemma 3. *Let $\mathfrak{A}$ be a ring with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \nsubseteq \mathfrak{L}$. If $[\ell, (\ell)^*] \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then one of the following holds:*
1. *$char(\mathfrak{A}/\mathfrak{L}) = 2$;*
2. *$\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain.*

Proof. We assume that $char(\mathfrak{A}/\mathfrak{L}) \neq 2$. By that assumption, we have

$$[\ell, (\ell)^*] \in \mathfrak{L} \tag{1}$$

for all $\ell \in \mathfrak{A}$. Direct linearization of relation (1) gives

$$[\ell, \vartheta^*] + [\vartheta, (\ell)^*] \in \mathfrak{L} \tag{2}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Replacing ℓ with ℓk in (2), where $0 \neq k \in S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A})$, we obtain

$$k[\ell, (\vartheta)^*] - k[\vartheta, (\ell)^*] \in \mathfrak{L}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Since $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \nsubseteq \mathfrak{L}$, it follows that

$$[\ell, (\vartheta)^*] - [\vartheta, (\ell)^*] \in \mathfrak{L} \tag{3}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Combining (2) and (3), we obtain

$$2[\ell, (\vartheta)^*] \in \mathfrak{L}$$

for all $\ell, \vartheta \in \mathfrak{A}$. This implies that

$$[\ell, \vartheta] \in \mathfrak{L} \tag{4}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Since elements of $\mathfrak{A}/\mathfrak{L}$ are cosets, and noticing that $\ell \in \mathfrak{L}$ implies that $\ell + \mathfrak{L} = \mathfrak{L}$, the above equation gives

$$\ell\vartheta - \vartheta\ell + \mathfrak{L} = \mathfrak{L} \tag{5}$$

for all $\ell, \vartheta \in \mathfrak{A}$; hence, we infer that

$$\ell\vartheta + \mathfrak{L} = \vartheta\ell + \mathfrak{L} \tag{6}$$

for all $\ell, \vartheta \in \mathfrak{A}$. This can be written as

$$(\ell + \mathfrak{L})(\vartheta + \mathfrak{L}) = (\vartheta + \mathfrak{L})(\ell + \mathfrak{L}) \tag{7}$$

for all $\ell, \vartheta \in \mathfrak{A}$. This implies that $\mathfrak{A}/\mathfrak{L}$ is commutative. Now, we show that $\mathfrak{A}/\mathfrak{L}$ is an integral domain. We suppose that

$$(\ell + \mathfrak{L})(\vartheta + \mathfrak{L}) = \mathfrak{L}$$

for all $\ell, \vartheta \in \mathfrak{A}$. This is equivalent to the expression

$$\ell\vartheta + \mathfrak{L} = \mathfrak{L}$$

for all $\ell, \vartheta \in \mathfrak{A}$. This implies that $\ell\vartheta \in \mathfrak{L}$ for all $\ell, \vartheta \in \mathfrak{A}$. For any $r \in \mathfrak{A}$, we have $r(\ell\vartheta) \in \mathfrak{L}$ for all $\ell, \vartheta \in \mathfrak{A}$. This gives $\ell r\vartheta \in \mathfrak{L}$. Hence, $\ell\mathfrak{A}\vartheta \subseteq \mathfrak{L}$. Thus, we obtain $\ell \in \mathfrak{L}$ or $\vartheta \in \mathfrak{L}$. This further implies that $\ell + \mathfrak{L} = \mathfrak{L}$ or $\vartheta + \mathfrak{L} = \mathfrak{L}$. This shows that $\mathfrak{A}/\mathfrak{L}$ is an integral domain. Consequently, we conclude that $\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain. This proves the lemma. □

In view of Lemmas 1 and 3, we conclude the following result.

Lemma 4. *Let $\mathfrak{A}$ be a ring and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$. If e is a derivation of $\mathfrak{A}$ satisfying the condition $[e(\ell), \ell] \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then $e(\mathfrak{A}) \subseteq \mathfrak{L}$ or $\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain.*

We are now ready to prove our first main theorem.

Proof of Theorem 6. We assume that $char(\mathfrak{A}/\mathfrak{L}) \neq 2$. By that assumption, we have

$$e_1(\ell)(\ell)^* - (\ell)^* e_2(\ell) \in \mathfrak{L} \text{ for all } \ell \in \mathfrak{A}. \tag{8}$$

Linearizing (8), we have

$$e_1(\ell)(\vartheta)^* + e_1(\vartheta)(\ell)^* - (\ell)^* e_2(\vartheta) - (\vartheta)^* e_2(\ell) \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A}. \tag{9}$$

Replacing ℓ with ℓh in (9), where $0 \neq h \in \mathfrak{H}(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A})$, we obtain

$$(e_1(\ell)(\vartheta)^* + e_1(\vartheta)(\ell)^* - (\ell)^* e_2(\vartheta) - (\vartheta)^* e_2(\ell))h + \ell(\vartheta)^* e_1(h) - (\vartheta)^* \ell e_2(h) \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A}.$$

Applying (9),

$$\ell(\vartheta)^* e_1(h) - (\vartheta)^* \ell e_2(h) \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A}.$$

This gives

$$\ell\vartheta e_1(h) - \vartheta\ell e_2(h) \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A}. \tag{10}$$

We replace h with k^2 in (10), where $0 \neq k \in S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \not\subseteq \mathfrak{L}$, to obtain

$$\ell\vartheta e_1(k) - \vartheta\ell e_2(k) \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A}. \tag{11}$$

Substituting ℓk in place of ℓ in (9), where $0 \neq k \in S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A})$, we arrive at

$$e_1(\ell)(\vartheta)^* k + \ell(\vartheta)^* e_1(k) - e_1(\vartheta)(\ell)^* k + (\ell)^* e_2(\vartheta)k - (\vartheta)^* \ell e_2(k) - (\vartheta)^* e_2(\ell)k \in \mathfrak{L} \tag{12}$$

for all $\ell, \vartheta \in \mathfrak{A}$. From (9), we have

$$e_1(\ell)(\vartheta)^* k + e_1(\vartheta)(\ell)^* k - (\ell)^* e_2(\vartheta)k - (\vartheta)^* e_2(\ell)k \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A}. \tag{13}$$

Adding (12) and (13), we obtain

$$2e_1(\ell)(\vartheta)^* k - 2(\vartheta)^* e_2(\ell)k + \ell(\vartheta)^* e_1(k) - (\vartheta)^* \ell e_2(k) \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A}$$

this implies

$$2e_1(\ell)\vartheta k - 2\vartheta e_2(\ell)k + \ell\vartheta e_1(k) - \vartheta\ell e_2(k) \in \mathfrak{L} \tag{14}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Using (11) in (14), we have

$$2e_1(\ell)\vartheta k - 2\vartheta e_2(\ell)k \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A}.$$

Since $char(\mathfrak{A}/\mathfrak{L}) \neq 2$ and $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \nsubseteq \mathfrak{L}$, we have

$$e_1(\ell)\vartheta - \vartheta e_2(\ell) \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A}. \tag{15}$$

In particular, for $\vartheta = \ell$, we obtain $e_1(\ell)\ell - \ell e_2(\ell) \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$. Therefore, from Lemma 2, we conclude that $e_1(\mathfrak{A}) \subseteq \mathfrak{L}$ and $e_2(\mathfrak{A}) \subseteq \mathfrak{L}$ or $\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain. □

Corollary 5. *Let $\mathfrak{A}$ be a prime ring with involution $*$ of the second kind such that $char(\mathfrak{A}) \neq 2$. If $\mathfrak{A}$ admits derivations e_1 and e_2 such that $e_1(\ell)(\ell)^* - (\ell)^* e_2(\ell) = 0$ for all $\ell \in \mathfrak{A}$, then $e_1 = e_2 = 0$ or $\mathfrak{A}$ is a commutative integral domain.*

We now prove another theorem in this vein.

Theorem 7. *Let $\mathfrak{A}$ be a ring with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \nsubseteq \mathfrak{L}$. If $\mathfrak{A}$ admits a derivation e such that $[e(\ell), (\ell)^*] + [\ell, (\ell)^*] \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then one of the following holds:*

1. *$char(\mathfrak{A}/\mathfrak{L}) = 2$;*
2. *$\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain.*

Proof. Suppose that $char(\mathfrak{A}/\mathfrak{L}) \neq 2$. By that assumption, we have

$$[e(\ell), (\ell)^*] + [\ell, (\ell)^*] \in \mathfrak{L} \tag{16}$$

for all $\ell \in \mathfrak{A}$. First, we assume that $e(\mathfrak{A}) \subseteq \mathfrak{L}$. Then, the result follows by Lemma 3. Henceforward, we suppose that $e(\mathfrak{A}) \nsubseteq \mathfrak{L}$. Linearizing (16), we obtain

$$[e(\ell), (\vartheta)^*] + [e(\vartheta), (\ell)^*] + [\ell, (\vartheta)^*] + [\vartheta, (\ell)^*] \in \mathfrak{L} \tag{17}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Replacing ℓ with ℓh in (17), where $0 \neq h \in \mathfrak{H}(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A})$, we obtain

$$e(h)[\ell, (\vartheta)^*] \in \mathfrak{L}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Replacing h with k^2 in the last relation, where $0 \neq k \in S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \nsubseteq \mathfrak{L}$, and using the hypothesis, we arrive at

$$e(k)[\ell, (\vartheta)^*] \in \mathfrak{L} \tag{18}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Replacing ℓ with ℓk in (17), where $0 \neq k \in S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \nsubseteq \mathfrak{L}$, we find that

$$e(k)[\ell, (\vartheta)^*] + k[e(\ell), (\vartheta)^*] - k[e(\vartheta), (\ell)^*] + k[\ell, (\vartheta)^*] - k[\vartheta, (\ell)^*] \in \mathfrak{L} \tag{19}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Using (18) and the condition $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \nsubseteq \mathfrak{L}$ in (19), we obtain

$$[e(\ell), (\vartheta)^*] - [e(\vartheta), (\ell)^*] + [\ell, (\vartheta)^*] - [\vartheta, (\ell)^*] \in \mathfrak{L} \tag{20}$$

for all $\ell, \vartheta \in \mathfrak{A}$. The addition of (17) and (20) gives

$$2([e(\ell), (\vartheta)^*] + [\ell, (\vartheta)^*]) \in \mathfrak{L}$$

for all $\ell, \vartheta \in \mathfrak{A}$. This implies that

$$[e(\ell), \vartheta] + [\ell, \vartheta] \in \mathfrak{L}$$

for all $\ell, \vartheta \in \mathfrak{A}$. In particular, for $\vartheta = \ell$, we have

$$[e(\ell), \ell] \in \mathfrak{L}$$

for all $\ell \in \mathfrak{A}$. In view of Lemma 4, we conclude that $\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain. $\square$

The following result is interesting in itself.

Theorem 8. *Let $\mathfrak{A}$ be a ring with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \not\subseteq \mathfrak{L}$. If e_1 and e_2 are derivations of $\mathfrak{A}$ such that $[e_1(\ell), (\ell)^*] + [\ell, e_2((\ell)^*)] + [\ell, (\ell)^*] \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then one of the following holds:*

1. *$char(\mathfrak{A}/\mathfrak{L}) = 2$;*
2. *$\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain.*

Proof. Assume that $char(\mathfrak{A}/\mathfrak{L}) \neq 2$. By that assumption, we have

$$[e_1(\ell), (\ell)^*] + [\ell, e_2((\ell)^*)] + [\ell, (\ell)^*] \in \mathfrak{L} \tag{21}$$

for all $\ell \in \mathfrak{A}$. We divide the proof in three cases.

Case (i): First, we assume that $e_2(\mathfrak{A}) \subseteq \mathfrak{L}$. Then, relation (21) reduces to

$$[e_1(\ell), (\ell)^*] + [\ell, (\ell)^*] \in \mathfrak{L}$$

for all $\ell \in \mathfrak{A}$. In view of Theorem 7, we obtain the required result.

Case (ii): Now, we assume that $e_1(\mathfrak{A}) \subseteq \mathfrak{L}$. Then, relation (21) reduces to

$$[\ell, e_2((\ell)^*)] + [\ell, (\ell)^*] \in \mathfrak{L}$$

for all $\ell \in \mathfrak{A}$. This can be further written as

$$[e_2((\ell)^*), \ell] + [(\ell)^*, \ell] \in \mathfrak{L} \tag{22}$$

for all $\ell \in \mathfrak{A}$. If $e_2(\mathfrak{A}) \subseteq \mathfrak{L}$, then the result follows by Lemma 3. Henceforward, we suppose that $e_2(\mathfrak{A}) \not\subseteq \mathfrak{L}$. Linearizing (22), we obtain

$$[e_2((\ell)^*), \vartheta] + [e_2((\vartheta)^*), \ell] + [(\ell)^*, \vartheta] + [(\vartheta)^*, \ell] \in \mathfrak{L} \tag{23}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Replacing ℓ with ℓh in (23), where $0 \neq h \in \mathfrak{H}(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A})$, we obtain

$$e_2(h)[(\ell)^*, \vartheta] \in \mathfrak{L}$$

for all $\ell, \vartheta \in \mathfrak{A}$. This implies that

$$e_2(h)[\ell, \vartheta] \in \mathfrak{L}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Replacing h with k^2 in the last relation, where $0 \neq k \in S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A})$, we arrive at

$$2e_2(k)[\ell, \vartheta]k \in \mathfrak{L}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Since $char(\mathfrak{A}/\mathfrak{L}) \neq 2$, the last relation gives

$$e_2(k)[\ell, \vartheta]k \in \mathfrak{L} \tag{24}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Replacing ℓ with ℓk in (23), where $0 \neq k \in S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A})$, we find that

$$-e_2(k)[(\ell)^*, \vartheta] - k[e_2((\ell)^*), \vartheta] + k[e_2((\vartheta)^*), \ell] - k[(\ell)^*, \vartheta] + k[(\vartheta)^*, \ell] \in \mathfrak{L} \tag{25}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Left multiplying in (23) by k, we obtain

$$k[e_2((\ell)^*), \vartheta] + k[e_2((\vartheta)^*), \ell] + k[(\ell)^*, \vartheta] + k[(\vartheta)^*, \ell] \in \mathfrak{L} \tag{26}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Combining (25) and (26), we obtain

$$-e_2(k)[(\ell)^*, \vartheta] + 2k[e_2((\vartheta)^*), \ell] + 2k[(\vartheta)^*, \ell] \in \mathfrak{L} \tag{27}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Replacing ϑ with ϑk in (27) and using (24), where $0 \neq k \in S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A})$, we obtain

$$2k^2([e_2(\vartheta), \ell] + [\vartheta, \ell]) \in \mathfrak{L}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Using the assumption $char(\mathfrak{A}/\mathfrak{L}) \neq 2$, we find that

$$k^2([e_2(\vartheta), \ell] + [\vartheta, \ell]) \in \mathfrak{L} \tag{28}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Application of the primeness of $\mathfrak{L}$ yields $k^2 \in \mathfrak{L}$ or $[e_2(\vartheta), \ell] + [\vartheta, \ell] \in \mathfrak{L}$. The first case $k^2 \in \mathfrak{L}$ implies that $k \in \mathfrak{L}$, which gives a contradiction. Thus, we have

$$[e_2(\vartheta), \ell] + [\vartheta, \ell] \in \mathfrak{L}$$

for all $\ell, \vartheta \in \mathfrak{A}$. In particular, for $\vartheta = \ell$, we have $[e_2(\ell), \ell] \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$. Therefore, in view of Lemma 4, we conclude that $\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain.

Case (iii): Finally, we assume that $e_1(\mathfrak{A}) \nsubseteq \mathfrak{L}$ and $e_2(\mathfrak{A}) \nsubseteq \mathfrak{L}$. Then, direct linearization of (21) gives

$$[e_1(\ell), (\vartheta)^*] + [e_1(\vartheta), (\ell)^*] + [\ell, e_2((\vartheta)^*)] + [\vartheta, e_2((\ell)^*)] + [\ell, (\vartheta)^*] + [\vartheta, (\ell)^*] \in \mathfrak{L} \tag{29}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Replacing ℓ with ℓh in (29), where $0 \neq h \in \mathfrak{H}(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A})$, and using it, we obtain

$$e_1(h)[\ell, (\vartheta)^*] + e_2(h)[\vartheta, (\ell)^*] \in \mathfrak{L} \tag{30}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Replacing ℓ with ℓk in (30), where $0 \neq k \in S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \nsubseteq\in \mathfrak{L}$, we obtain

$$e_1(h)[\ell, (\vartheta)^*] - e_2(h)[\vartheta, (\ell)^*] \in \mathfrak{L} \tag{31}$$

for all $\ell, \vartheta \in \mathfrak{A}$. The combination of (30) and (31) yields

$$2e_1(h)[\ell, (\vartheta)^*] \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A},$$

which implies that

$$e_1(h)[\ell, \vartheta] \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A}.$$

Replacing h with k^2 in the last relation and using the hypothesis of theorem, we obtain

$$e_1(k)[\ell, \vartheta] \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A}.$$

This implies either $e_1(k) \in \mathfrak{L}$ or $[\ell, \vartheta] \in \mathfrak{L}$. If $[\ell, \vartheta] \in \mathfrak{L}$, then, by Lemma 3, $\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain. On the other hand, we have $e_1(k) \in \mathfrak{L}$. Similarly, we can find $e_2(k) \in \mathfrak{L}$. Writing ℓk instead of ℓ in (29), where $0 \neq k \in S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \nsubseteq \mathfrak{L}$, and using the fact that $e_1(k), e_2(k) \in \mathfrak{L}$, we arrive at

$$[e_1(\ell), (\vartheta)^*] - [e_1(\vartheta), (\ell)^*] + [\ell, e_2((\vartheta)^*)] - [\vartheta, e_2((\ell)^*)] + [\ell, (\vartheta)^*] - [\vartheta, (\ell)^*] \in \mathfrak{L} \tag{32}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Comparing (29) and (32), we obtain

$$2([e_1(\ell), (\vartheta)^*] + [\ell, e_2((\vartheta)^*)] + [\ell, (\vartheta)^*]) \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A}.$$

This implies that

$$[e_1(\ell), \vartheta] + [\ell, e_2(\vartheta)] + [\ell, \vartheta] \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A}.$$

Now, replacing ℓ with ℓr in the above expression, we obtain

$$e_1(\ell)[r, \vartheta] + [\ell, \vartheta]e_1(\mathfrak{A}) \in \mathfrak{L} \text{ for all } \ell, \vartheta, r \in \mathfrak{A}.$$

In particular, for $\vartheta = \ell$, we have $e_1(\ell)[r, \ell] \in \mathfrak{L}$ for all $\ell, r \in \mathfrak{A}$. This gives $e_1(\ell)\mathfrak{A}[r, \ell] \subseteq \mathfrak{L}$ for all $\ell, r \in \mathfrak{A}$. The primeness of $\mathfrak{L}$ infers that $e_1(\ell) \in \mathfrak{L}$ or $[r, \ell] \in \mathfrak{L}$. Set $A = \{\ell \in \mathfrak{A} \mid e_1(\ell) \in \mathfrak{L}\}$ and $B = \{\ell \in \mathfrak{A} \mid [r, \ell] \in \mathfrak{L}\}$. Clearly, A and B are additive subgroups of $\mathfrak{A}$ such that $A \cup B = \mathfrak{A}$. But, a group cannot be written as a union of its two proper subgroups; consequently, $A = \mathfrak{A}$ or $B = \mathfrak{A}$. The first case contradicts our supposition that $e_1(\mathfrak{A}) \nsubseteq \mathfrak{L}$. Thus, we have $[r, \ell] \in \mathfrak{L}$ for all $r, \ell \in \mathfrak{A}$. Therefore, in view of Lemma 3, $\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain. This completes the proof of theorem. □

Using a similar approach with necessary variations, one can establish the following result.

Theorem 9. *Let $\mathfrak{A}$ be a ring with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \nsubseteq \mathfrak{L}$. If e_1 and e_2 are derivations of $\mathfrak{A}$ such that $[e_1(\ell), (\ell)^*] + [\ell, e_2((\ell)^*)] - [\ell, (\ell)^*] \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then one of the following holds:*

1. *$char(\mathfrak{A}/\mathfrak{L}) = 2$;*
2. *$\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain.*

In view of Theorems 8 and 9, we have the following corollaries:

Corollary 6. *Let $\mathfrak{A}$ be a prime ring with involution $*$ of the second kind such that $char(\mathfrak{A}) \neq 2$. If $\mathfrak{A}$ admits derivations e_1 and e_2 such that $[e_1(\ell), (\ell)^*] + [\ell, e_2((\ell)^*)] \pm [\ell, (\ell)^*] = 0$ for all $\ell \in \mathfrak{A}$, then $\mathfrak{A}$ is a commutative integral domain.*

Corollary 7. *Let $\mathfrak{A}$ be a ring with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A})) \nsubseteq \mathfrak{L}$. If $\mathfrak{A}$ admits a derivation e such that $e([\ell, (\ell)^*]) \pm [\ell, (\ell)^*] \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then one of the following holds:*

1. *$char(\mathfrak{A}/\mathfrak{L}) = 2$;*
2. *$\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain.*

Corollary 8. *Let $\mathfrak{A}$ be a ring with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A})) \nsubseteq \mathfrak{L}$. If $\mathfrak{A}$ admits a derivation e such that $e(\ell(\ell)^*) \pm \ell(\ell)^* \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then one of the following holds:*

1. *$char(\mathfrak{A}/\mathfrak{L}) = 2$;*
2. *$\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain.*

Corollary 9 ([29], Theorem 3.4)**.** *Let $\mathfrak{A}$ be a prime ring with involution $*$ of the second kind such that $char(\mathfrak{A}) \neq 2$. If $\mathfrak{A}$ admits a derivation e such that $e([\ell, (\ell)^*]) \pm [\ell, (\ell)^*] = 0$ for all $\ell \in \mathfrak{A}$, then $\mathfrak{A}$ is a commutative integral domain.*

We leave the question open as to whether or not the assumption $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \nsubseteq \mathfrak{L}$ (where $\mathfrak{L}$ is prime ideal of an arbitrary ring $\mathfrak{A}$) can be removed in Theorems 6 and 8. In view of Theorem 6 and Theorem 4.4 of [35], we conclude this section with the following conjecture.

Conjecture: Let m and n be fixed positive integers. Next, let $\mathfrak{A}$ be a $*$-ring with suitable torsion restrictions and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$. If $\mathfrak{A}$ admits Jordan $*$-derivations e and g of $\mathfrak{A}$ such that $e(\ell)^m(\ell)^{*n} \pm (\ell)^{*n}g(\ell^m) \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then what can we say about the structure of $\mathfrak{A}$ and the forms of e, g ?

3. Derivations Act as Homomorphisms and Anti-Homomorphisms on Prime Ideals

Ring homomorphisms are mappings between two rings that preserve both addition and multiplication. In particular, we are concerned with ring homomorphisms between two rings. If $\mathfrak{A}$ is the real number field, then the zero map and the identity are typical examples of ring homomorphisms on $\mathfrak{A}$. Let S be a nonempty subset of $\mathfrak{A}$ and e be a derivation on $\mathfrak{A}$. If $e(\ell\vartheta) = e(\ell)e(\vartheta)$ or $e(\ell\vartheta) = e(\vartheta)e(\ell)$ for all $\ell, \vartheta \in S$, then e is said to be a derivation that acts as a homomorphism or an anti-homomorphism on S, respectively. Of course, derivations that act as endomorphisms or anti-endomorphisms of a ring $\mathfrak{A}$ may behave as such on certain subsets of $\mathfrak{A}$; for example, any derivation e behaves as the zero endomorphism on the subring T consisting of all constants (i.e., elements ℓ for which $e(\ell) = 0$). In fact, in a semiprime ring $\mathfrak{A}$, e may behave as an endomorphism on a proper ideal of $\mathfrak{A}$. As an example of such $\mathfrak{A}$ and e, let S be any semiprime ring with a nonzero derivation δ, take $\mathfrak{A} = S \oplus S$ and define e by $e(r_1, r_2) = (\delta(r_1), 0)$. However in case of prime rings, Bell and Kappe [46] showed that the behavior of e is somewhat more restricted by proving that if $\mathfrak{A}$ is a prime ring and e is a derivation of $\mathfrak{A}$ that acts as a homomorphism or an anti-homomorphism on a nonzero right ideal of $\mathfrak{A}$, then $e = 0$ on $\mathfrak{A}$. Further, Ali et al. obtained [47] the above mentioned result for Lie ideals. Recently, Mamouni et al. [48] studied the above mentioned problem for prime ideals of an arbitrary ring by considering the identity $e(\ell\vartheta) - e(\ell)e(\vartheta) \in \mathfrak{L}$ for all $\ell, \vartheta \in \mathfrak{A}$ or $e(\ell\vartheta) - e(\vartheta)e(\ell) \in \mathfrak{L}$ for all $\ell, \vartheta \in \mathfrak{A}$, where $\mathfrak{L}$ is prime ideal of $\mathfrak{A}$. In the present section, our objective was to extend the above study in the setting of rings with involution involving prime ideals. In fact, we prove the following result:

Theorem 10. *Let $\mathfrak{A}$ be a ring with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \not\subseteq \mathfrak{L}$. If $\mathfrak{A}$ admits a derivation e such that $e(\ell(\ell)^*) - e((\ell)^*)e(\ell) \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then one of the following holds:*

1. *$char(\mathfrak{A}/\mathfrak{L}) = 2$;*
2. *$e(\mathfrak{A}) \subseteq \mathfrak{L}$.*

Proof. Assume that $char(\mathfrak{A}/\mathfrak{L}) \neq 2$. By that hypothesis, we have

$$e(\ell(\ell)^*) - e((\ell)^*)e(\ell) \in \mathfrak{L} \tag{33}$$

for all $\ell \in \mathfrak{A}$. Linearization of (33) gives

$$e(\ell(\vartheta)^*) + e(\vartheta(\ell)^*) - e((\ell)^*)e(\vartheta) - e((\vartheta)^*)e(\ell) \in \mathfrak{L} \tag{34}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Replacing ℓ with ℓh in (34), where $0 \neq h \in \mathfrak{H}(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A})$, we obtain

$$e(h)(\ell(\vartheta)^* + \vartheta(\ell)^* - (\ell)^*e(\vartheta) - e((\vartheta)^*)\ell) \in \mathfrak{L} \tag{35}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Taking $h = k^2$ in (35), where $0 \neq k \in S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \not\subseteq \mathfrak{L}$, and using the hypothesis of the theorem, we obtain

$$e(k)(\ell(\vartheta)^* + \vartheta(\ell)^* - (\ell)^*e(\vartheta) - e((\vartheta)^*)\ell) \in \mathfrak{L}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Invoking the primeness of $\mathfrak{A}$ yields $e(k) \in \mathfrak{L}$ or $\ell(\vartheta)^* + \vartheta(\ell)^* - (\ell)^*e(\vartheta) - e((\vartheta)^*)\ell) \in \mathfrak{L}$ for all $\ell, \vartheta \in \mathfrak{A}$. Consider the case where

$$\ell(\vartheta)^* + \vartheta(\ell)^* - (\ell)^*e(\vartheta) - e((\vartheta)^*)\ell \in \mathfrak{L} \tag{36}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Replacing ℓ with ℓk in (36) and combining with the obtained relation, we obtain

$$2(\ell(\vartheta)^* - e((\vartheta)^*)\ell) \in \mathfrak{L}$$

for all $\ell, \vartheta \in \mathfrak{A}$. This implies that

$$\ell\vartheta - e(\vartheta)\ell \in \mathfrak{L} \tag{37}$$

for all $\ell, \vartheta \in \mathfrak{A}$. In particular, for $\ell = k$, where $k \in S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A})$, we have $\vartheta - e(\vartheta) \in \mathfrak{L}$ for all $\vartheta \in \mathfrak{A}$. Substituting ϑr for ϑ in the last relation, we obtain $\vartheta e(\mathfrak{A}) \in \mathfrak{L}$. This yields $e(\mathfrak{A})\mathfrak{A}e(\mathfrak{A}) \subseteq \mathfrak{L}$ for all $r \in \mathfrak{A}$. Since $\mathfrak{L}$ is a prime ideal of $\mathfrak{A}$, we have $e(\mathfrak{A}) \subseteq \mathfrak{L}$. On the other hand, consider the case $e(k) \in \mathfrak{L}$. Replacing ℓ with ℓk in (34), where $0 \neq k \in S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \not\subseteq \mathfrak{L}$, we obtain

$$e(\ell(\vartheta)^*) - e(\vartheta(\ell)^*) + e((\ell)^*)e(\vartheta) - e((\vartheta)^*)e(\ell) \in \mathfrak{L} \tag{38}$$

The combination of (34) and (38) gives

$$2(e(\ell(\vartheta)^*) - e((\vartheta)^*)e(\ell)) \in \mathfrak{L}$$

for all $\ell, \vartheta \in \mathfrak{A}$. This implies that

$$e(\ell\vartheta) - e(\vartheta)e(\ell) \in \mathfrak{L}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Taking $\vartheta = k$ in the above relation and using $e(k) \in \mathfrak{L}$, we obtain $ke(\ell) \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$. Since $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \not\subseteq \mathfrak{L}$, one can conclude that $e(\mathfrak{A}) \subseteq \mathfrak{L}$. □

Applying an analogous argument, we have the following result.

Theorem 11. *Let $\mathfrak{A}$ be a ring with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A})) \not\subseteq \mathfrak{L}$. If $\mathfrak{A}$ admits a derivation e such that $e(\ell(\ell)^*) - e(\ell)e((\ell)^*) \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then one of the following holds:*

1. *$char(\mathfrak{A}/\mathfrak{L}) = 2$;*
2. *$e(\mathfrak{A}) \subseteq \mathfrak{L}$.*

Corollary 10. *Let $\mathfrak{A}$ be a prime ring with involution $*$ of the second kind such that $char(\mathfrak{A}) \neq 2$. If $\mathfrak{A}$ admits a derivation e such that $e(\ell(\ell)^*) = e((\ell)^*)e(\ell)$ or $e(\ell(\ell)^*) = e(\ell)e((\ell)^*)$ for all $\ell \in \mathfrak{A}$, then $e = 0$.*

Theorem 12. *Let $\mathfrak{A}$ be a 2-torsion free semiprime ring with involution $*$ of the second kind. If $\mathfrak{A}$ admits a derivation e such that $e(\ell(\ell)^*) - e((\ell)^*)e(\ell) = 0$ for all $\ell \in \mathfrak{A}$, then $e = 0$.*

Proof. Assume that $char(\mathfrak{A}/\mathfrak{L}) \neq 2$. By that assumption, we have

$$e(\ell(\ell)^*) - e((\ell)^*)e(\ell) = 0 \text{ for all } \ell \in \mathfrak{A}.$$

By the semiprimeness of $\mathfrak{A}$, there exists a family $\mathcal{L} = \{\mathfrak{L}_\alpha : \alpha \in \wedge\}$ of prime ideals such that $\bigcap_\alpha \mathfrak{L}_\alpha = (0)$ (see [49] for details). For each $\mathfrak{L}_\alpha$ in $\mathcal{L}$, we have

$$e(\ell(\ell)^*) - e((\ell)^*)e(\ell) \in \mathfrak{L}_\alpha \text{ for all } \ell \in \mathfrak{A}.$$

Invoking Theorem 10, we conclude that $e(\mathfrak{A}) \subseteq \mathfrak{L}_\alpha$. Consequently, we obtain $e(\mathfrak{A}) \subseteq \bigcap_\alpha \mathfrak{L}_\alpha = (0)$ and hence the result follows. Thereby, the proof is completed. □

Analogously, we can prove the following result.

Theorem 13. *Let $\mathfrak{A}$ be a 2-torsion free semiprime ring with involution $*$ of the second kind. If $\mathfrak{A}$ admits a derivation e such that $e(\ell(\ell)^*) - e(\ell)e((\ell)^*) = 0$ for all $\ell \in \mathfrak{A}$, then $e = 0$.*

4. Applications

In this section, we present some applications of the results proved in Section 2. Vukman ([2] Theorem 1) generalized the classical result due to Posner (Posner's second theorem) [1] and proved that, if e is a derivation of a prime ring $\mathfrak{A}$ of a characteristic different from 2, such that $[[e(\ell),\ell],\ell] = [e(\ell),\ell]_2 = 0$ for all $\ell \in \mathfrak{A}$, then $e = 0$ or $\mathfrak{A}$ is commutative. In fact, in view of Posner's second theorem, he merely showed that e is commutin; that is, $[e(\ell),\ell] = 0$ for all $\ell \in \mathfrak{A}$. In [50], Deng and Bell extended the above mentioned result for a semiprime ring and established that if a 6-torsion free semiprime ring admits a derivation e such that $[[e(\ell),\ell],\ell] = 0$ for all $\ell \in I$ with $e(I) \neq (0)$, where I is a nonzero left ideal of $\mathfrak{A}$, then $\mathfrak{A}$ contains a nonzero central ideal. These results were further refined and extended by a number of algebraists (see, for example, [10,26–28,51]). It is our aim in this section to study and extend Vukman's and Posner's results for arbitrary rings with involution involving prime ideals. In fact, we prove the $*$-versions of these theorems. Moreover, our approach is somewhat different from those employed by other authors. Precisely, we prove the following result.

Theorem 14. *Let $\mathfrak{A}$ be a ring with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \not\subseteq \mathfrak{L}$. If $\mathfrak{A}$ admits a derivation e such that $[[e(\ell),(\ell)^*],(\ell)^*] \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then one of the following holds:*

1. *$char(\mathfrak{A}/\mathfrak{L}) = 2$;*
2. *$e(\mathfrak{A}) \subseteq \mathfrak{L}$;*
3. *$\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain.*

A derivation $e : \mathfrak{A} \to \mathfrak{A}$ is said to be $*$-centralizing if $[e(\ell),(\ell)^*] \in \mathfrak{Z}(\mathfrak{A})$ for all $\ell \in \mathfrak{A}$. The last expression can be written as $[[e(\ell),(\ell)^*],(\ell)^*] = [e(\ell),(\ell)^*]_2 = 0$ for all $\ell \in \mathfrak{A}$. Consequently, Theorem 14 is regarded as the $*$-version of Vukman's theorem [2]. Applying Theorem 14, we also prove that if a 2-torsion free semiprime ring $\mathfrak{A}$ with involution $*$ of the second kind admits a nonzero $*$-centralizing derivation, then $\mathfrak{A}$ must contain a nonzero central ideal. In fact, we prove the following result.

Theorem 15. *Let $\mathfrak{A}$ be a 2-torsion free semiprime ring with involution $*$ of the second kind. If $\mathfrak{A}$ admits a nonzero $*$-centralizing derivation e, i.e., $[e(\ell),(\ell)^*] \in \mathfrak{Z}(\mathfrak{A})$ for all $\ell \in \mathfrak{A}$, then $\mathfrak{A}$ contains a nonzero central ideal.*

As an immediate consequence of Theorem 15, we obtain the following result.

Corollary 11. *Let $\mathfrak{A}$ be a 2-torsion free semiprime ring with involution $*$ of the second kind. If $\mathfrak{A}$ admits a nonzero $*$-commuting derivation e, i.e., $[e(\ell),(\ell)^*] = 0$ for all $\ell \in \mathfrak{A}$, then $\mathfrak{A}$ contains a nonzero central ideal.*

In order to prove Theorem 15, we need the proof of Theorem 14.

Proof of Theorem 14. Assume that $char(\mathfrak{A}/\mathfrak{L}) \neq 2$. By that hypothesis, we have

$$[[e(\ell),(\ell)^*],(\ell)^*] \in \mathfrak{L} \text{ for all } \ell \in \mathfrak{A}. \tag{39}$$

A linearization of (39) yields

$$\begin{aligned} &[[e(\ell),(\vartheta)^*],(\vartheta)^*] + [[e(\ell),(\vartheta)^*],(\ell)^*] + [[e(\vartheta),(\vartheta)^*],(\ell)^*] + [[e(\vartheta),(\ell)^*],(\ell)^*] \\ &\quad + [[e(\ell),(\ell)^*],(\vartheta)^*] + [[e(\vartheta),(\ell)^*],(\vartheta)^*] \in \mathfrak{L} \end{aligned} \tag{40}$$

for all $\ell,\vartheta \in \mathfrak{A}$. Putting $\ell = -\ell$ in (40), we obtain

$$\begin{aligned} &-[[e(\ell),(\vartheta)^*],(\vartheta)^*] + [[e(\ell),(\vartheta)^*],(\ell)^*] - [[e(\vartheta),(\vartheta)^*],(\ell)^*] + [[e(\vartheta),(\ell)^*],(\ell)^*] \\ &\quad + [[e(\ell),(\ell)^*],(\vartheta)^*] - [[e(\vartheta),(\ell)^*],(\vartheta)^*] \in \mathfrak{L} \end{aligned} \tag{41}$$

for all $\ell,\vartheta \in \mathfrak{A}$. Combining (40) and (41), we obtain

$$[[e(\ell),(\vartheta)^*],(\ell)^*] + [[e(\vartheta),(\ell)^*],(\ell)^*] + [[e(\ell),(\ell)^*],(\vartheta)^*] \in \mathfrak{L} \tag{42}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Replacing ϑ with ϑh in (42), where $h \in \mathfrak{H}(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A})$, we deduce that

$$e(h)[[\vartheta, (\ell)^*], (\ell)^*] \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A}.$$

Taking $h = k^2$, where $k \in S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \nsubseteq \mathfrak{L}$, and using the hypothesis, we have

$$e(k)[[\vartheta, (\ell)^*], (\ell)^*] \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A}. \tag{43}$$

Now, substituting ϑk in place of ϑ in (42), where $k \in S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A})$, we obtain

$$-k[[e(\ell), (\vartheta)^*], (\ell)^*] + k[[e(\vartheta), (\ell)^*], (\ell)^*] + e(k[[\vartheta, (\ell)^*], (\ell)^*]) - k[[e(\ell), (\ell)^*], (\vartheta)^*] \in \mathfrak{L}$$

for all $\ell, \vartheta \in \mathfrak{A}$. The application of (43) and the condition $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \nsubseteq \mathfrak{L}$ yields

$$-[[e(\ell), (\vartheta)^*], (\ell)^*] + [[e(\vartheta), (\ell)^*], (\ell)^*] - [[e(\ell), (\ell)^*], (\vartheta)^*] \in \mathfrak{L} \tag{44}$$

for all $\ell, \vartheta \in \mathfrak{A}$. From (42) and (44), we can obtain

$$2([[e(\vartheta), (\ell)^*], (\ell)^*]) \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A}.$$

This implies that

$$[[e(\vartheta), \ell], \ell] \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A}.$$

Writing $\ell + z$ instead of ℓ, we obtain

$$[[e(\vartheta), \ell], z] + [[e(\vartheta), z], \ell] \in \mathfrak{L} \text{ for all } \ell, \vartheta, z \in \mathfrak{A}. \tag{45}$$

Replacing z with zr in (45), we find that

$$[z, \ell][e(\vartheta), r] + [e(\vartheta), z][r, \ell] \in \mathfrak{L} \text{ for all } r, \ell, \vartheta, z \in \mathfrak{A}.$$

In particular, for $r = \ell$, we have

$$[z, \ell][e(\vartheta), \ell] \in \mathfrak{L} \text{ for all } \ell, \vartheta, z \in \mathfrak{A}.$$

This gives

$$[z, \ell]\mathfrak{A}[e(\vartheta), \ell] \subseteq \mathfrak{L} \text{ for all } \ell, \vartheta, z \in \mathfrak{A}. \tag{46}$$

Since $\mathfrak{L}$ is a prime ideal of $\mathfrak{A}$, we have $[z, \ell] \in \mathfrak{L}$ for all $z \in \mathfrak{A}$ or $[e(\vartheta), \ell] \in \mathfrak{L}$ for all $\vartheta \in \mathfrak{A}$. Let us set $A = \{\ell \in \mathfrak{A} \mid [\ell, z] \in \mathfrak{L}\}$ and $B = \{\ell \in \mathfrak{A} \mid [e(\vartheta), \ell] \in \mathfrak{L}\}$. Clearly, A and B are additive subgroups of $\mathfrak{A}$ whose union is $\mathfrak{A}$. Because a group cannot be written as a union of its two proper subgroups, it follows that either $A = \mathfrak{A}$ or $B = \mathfrak{A}$. In the first case, $\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain from Lemma 3. On the other hand, if $[e(\vartheta), \ell] \in \mathfrak{L}$ for all $\ell, \vartheta \in \mathfrak{A}$, then we obtain $[e(\ell), (\ell)^*] \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$. Hence, in view of Corollary 1, we conclude that $e(\mathfrak{A}) \subseteq \mathfrak{L}$ or $\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain. This completes the proof of theorem. □

Proof of Theorem 15. We are given that $e : \mathfrak{A} \to \mathfrak{A}$ is a $*$-centralizing derivation; that is, $[e(\ell), (\ell)^*] \in \mathfrak{Z}(\mathfrak{A})$ for all $\ell \in \mathfrak{A}$. This implies that $[[e(\ell), (\ell)^*], \vartheta] = 0$ for all $\ell, \vartheta \in \mathfrak{A}$. This gives

$$[[e(\ell), (\ell)^*], (\ell)^*] = 0 \text{ for all } \ell \in \mathfrak{A}.$$

In view of the semiprimeness of $\mathfrak{A}$, there exists a family $\mathcal{P} = \{\mathfrak{L}_\alpha : \alpha \in \wedge\}$ of prime ideals such that $\bigcap_\alpha \mathfrak{L}_\alpha = (0)$ (see [49] for more details). Let $\mathfrak{L}$ denote a fixed one of the $\mathfrak{L}_\alpha$. Thus, we have

$$[[e(\ell), (\ell)^*], (\ell)^*] \in \mathfrak{L} \text{ for all } \ell \in \mathfrak{A} \text{ and for all } \mathfrak{L} \in \mathcal{P}.$$

From the proof of Theorem 14, we observe that, for each ℓ, either

$$[z,\ell] \in \mathfrak{L} \text{ for all } z \in \mathfrak{A} \qquad (I)$$

or

$$[e(\vartheta),\ell] \in \mathfrak{L} \text{ for all } \vartheta \in \mathfrak{A} \qquad (II)$$

Define A_I to be the set of $z \in \mathfrak{A}$ for which (I) holds and A_{II} to be the set of $\vartheta \in \mathfrak{A}$ for which (II) holds. Note that both are additive subgroups of $\mathfrak{A}$ and that their union is equal to $\mathfrak{A}$. Thus, either $A_I = \mathfrak{A}$ or $A_{II} = \mathfrak{A}$, and hence $\mathfrak{L}$ satisfies one of the following:

$$[z,\ell] \in \mathfrak{L} \text{ for all } \ell, z \in \mathfrak{A} \qquad (I')$$

or

$$[e(\vartheta),\ell] \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A} \qquad (II')$$

Call a prime ideal in $\mathcal{P}$ a type-one prime if it satisfies (I'), and call all other members of $\mathcal{P}$ type-two primes. Define $\mathfrak{L}_1$ and $\mathfrak{L}_2$ as the intersection of all type-one primes and the intersection of all type-two primes, respectively, and note that

$$\mathfrak{L}_1\mathfrak{L}_2 \;=\; \mathfrak{L}_2\mathfrak{L}_1 \;=\; \mathfrak{L}_1 \cap \mathfrak{L}_2 = \{0\}.$$

Clearly, from both cases, we can conclude that $[e(\ell),\ell] \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$ for all $\mathfrak{L} \in \mathcal{P}$. This implies that $[e(\ell),\ell] \in \bigcap\limits_{\mathfrak{L}\in\mathcal{T}} \mathfrak{L} = \{0\}$ for all $\ell \in \mathfrak{A}$; that is, $[e(\ell),\ell] = 0$ for all $\ell \in \mathfrak{A}$. Hence, in view of ([18] Theorem 3), $\mathfrak{A}$ contains a nonzero central ideal. $\square$

The Jordan product version of Theorem 14 is the following.

Theorem 16. *Let $\mathfrak{A}$ be a ring with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \not\subseteq \mathfrak{L}$. If $\mathfrak{A}$ admits a derivation e such that $(e(\ell) \circ (\ell)^*) \circ (\ell)^* \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then one of the following holds:*

1. *$char(\mathfrak{A}/\mathfrak{L}) = 2$;*
2. *$e(\mathfrak{A}) \subseteq \mathfrak{L}$.*

Proof. Assume that $char(\mathfrak{A}/\mathfrak{L}) \neq 2$. By that hypothesis, we have

$$(e(\ell) \circ (\ell)^*) \circ (\ell)^* \in \mathfrak{L} \text{ for all } \ell \in \mathfrak{A}. \tag{47}$$

A linearization of (47) yields

$$\begin{aligned}(e(\ell) \circ (\vartheta)^*) \circ (\vartheta)^* + (e(\ell) \circ (\vartheta)^*) \circ (\ell)^* + (e(\vartheta) \circ (\vartheta)^*) \circ (\ell)^* + (e(\vartheta) \circ (\ell)^*) \circ (\ell)^* \\ +(e(\ell) \circ (\ell)^*) \circ (\vartheta)^* + (e(\vartheta) \circ (\ell)^*) \circ (\vartheta)^* \in \mathfrak{L}\end{aligned} \tag{48}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Putting $\ell = -\ell$ into (48), we obtain

$$\begin{aligned}-((e(\ell) \circ (\vartheta)^*) \circ (\vartheta)^*) + ((e(\ell) \circ (\vartheta)^*) \circ (\ell)^*) - ((e(\vartheta) \circ (\vartheta)^*) \circ (\ell)^*) \\ +((e(\vartheta) \circ (\ell)^*) \circ (\ell)^*) + ((e(\ell) \circ (\ell)^*) \circ (\vartheta)^*) - ((e(\vartheta) \circ (\ell)^*) \circ (\vartheta)^*) \in \mathfrak{L}\end{aligned} \tag{49}$$

for all $\ell, \vartheta \in \mathfrak{A}$. Combining (48) and (49), we obtain

$$(e(\ell) \circ (\vartheta)^*) \circ (\ell)^* + (e(\vartheta) \circ (\ell)^*) \circ (\ell)^* + (e(\ell) \circ (\ell)^*) \circ (\vartheta)^* \in \mathfrak{L} \tag{50}$$

for all $\ell, \vartheta \in \mathfrak{A}$. The substitution of ϑh with ϑ in (50), where $h \in \mathfrak{H}(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A})$, produces

$$e(h)((\vartheta \circ (\ell)^*) \circ (\ell)^*) \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A}.$$

Taking $h = k^2$, where $k \in S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \nsubseteq \mathfrak{L}$, and using the hypothesis, we have

$$e(k)((\vartheta \circ (\ell)^*) \circ (\ell)^*) \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A}. \tag{51}$$

Next, substituting ϑk in place of ϑ in (50), where $k \in S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A})$, we obtain

$$-k((e(\ell) \circ (\vartheta)^*) \circ (\ell)^*) + k((e(\vartheta) \circ (\ell)^*) \circ (\ell)^*) + e(k)((\vartheta \circ (\ell)^*) \circ (\ell)^*)) - k((e(\ell) \circ (\ell)^*) \circ (\vartheta)^*) \in \mathfrak{L}$$

for all $\ell, \vartheta \in \mathfrak{A}$. The application of (51) and the condition $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \nsubseteq \mathfrak{L}$ yields

$$-((e(\ell) \circ (\vartheta)^*) \circ (\ell)^*) + ((e(\vartheta) \circ (\ell)^*) \circ (\ell)^*) - ((e(\ell) \circ (\ell)^*) \circ (\vartheta)^*) \in \mathfrak{L} \tag{52}$$

for all $\ell, \vartheta \in \mathfrak{A}$. From (50) and (52), we can obtain

$$2((e(\vartheta) \circ (\ell)^*) \circ (\ell)^*) \in \mathfrak{L} \text{ for all } \ell, \vartheta \in \mathfrak{A};$$

that is,

$$(e(\vartheta) \circ \ell) \circ \ell \in \mathfrak{L} \tag{53}$$

for all $\ell, \vartheta \in \mathfrak{A}$. A linearization for ℓ in (53) yields that

$$(e(\vartheta) \circ \ell) \circ t + (e(\vartheta) \circ t) \circ \ell \in \mathfrak{L} \tag{54}$$

for all $\ell, \vartheta, t \in \mathfrak{A}$. Replacing ℓ with ℓt in (54), we have

$$((e(\vartheta) \circ \ell)t - \ell[e(\vartheta), t]) \circ t + ((e(\vartheta) \circ t) \circ \ell)t - \ell[e(\vartheta) \circ t, t] \in \mathfrak{L} \text{ for all } \ell, \vartheta, t \in \mathfrak{A},$$

which can be written as

$$((e(\vartheta) \circ \ell) \circ t)t - (\ell[e(\vartheta), t]) \circ t + ((e(\vartheta) \circ t) \circ \ell)t - \ell[e(\vartheta) \circ t, t] \in \mathfrak{L} \text{ for all } \ell, \vartheta, t \in \mathfrak{A}.$$

Using (54), we have

$$-(\ell[e(\vartheta), t]) \circ t - \ell[e(\vartheta) \circ t, t] \in \mathfrak{L} \text{ for all } \ell, \vartheta, t \in \mathfrak{A}.$$

This implies that

$$\ell([e(\vartheta), t] \circ t + [e(\vartheta) \circ t, t]) - [\ell, t][e(\vartheta), t] \in \mathfrak{L} \text{ for all } \ell, \vartheta, t \in \mathfrak{A}. \tag{55}$$

Replacing ℓ with ℓr in the last relation, we have

$$\ell r([e(\vartheta), t] \circ t + [e(\vartheta) \circ t, t]) - \ell[r, t][e(\vartheta), t] - [\ell, t]r[e(\vartheta), t] \in \mathfrak{L} \text{ for all } \ell, \vartheta, t, r \in \mathfrak{A}.$$

The application of (55) gives

$$[\ell, t]r[e(\vartheta), t] \in \mathfrak{L} \text{ for all } \ell, \vartheta, t, r \in \mathfrak{A};$$

that is,

$$[\ell, t]\mathfrak{A}[e(\vartheta), t] \subseteq \mathfrak{L} \text{ for all } \ell, \vartheta, t \in \mathfrak{A}. \tag{56}$$

The above relation is the same as (46). Therefore, using the same arguments as we used after (46), we obtain that $e(\mathfrak{A}) \subseteq \mathfrak{L}$ or $\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain. If $e(\mathfrak{A}) \subseteq \mathfrak{L}$, then the proof is achieved. On the other hand, if $\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain, then (54) reduces as

$$8\ell t e(\vartheta) \in \mathfrak{L} \text{ for all } \ell, \vartheta, t \in \mathfrak{A}.$$

Since $char(\mathfrak{A}/\mathfrak{L}) \neq 2$, the above relation becomes

$$e(\vartheta)\mathfrak{A}e(\vartheta) \subseteq \mathfrak{L} \text{ for all } \vartheta \in \mathfrak{A}.$$

The primeness of $\mathfrak{L}$ forces that $e(\mathfrak{A}) \subseteq \mathfrak{L}$. Thus, the proof is complete now. □

The following results are immediate corollaries of Theorems 14 and 15.

Corollary 12. *Let $\mathfrak{A}$ be a ring with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \nsubseteq \mathfrak{L}$. If $\mathfrak{A}$ admits a derivation e such that $[[e(\ell), (\ell)^*], \vartheta] \in \mathfrak{L}$ for all $\ell, \vartheta \in \mathfrak{A}$, then one of the following holds:*

1. *$char(\mathfrak{A}/\mathfrak{L}) = 2$;*
2. *$e(\mathfrak{A}) \subseteq \mathfrak{L}$;*
3. *$\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain.*

Corollary 13 ([4], Theorem 3.7)**.** *Let $\mathfrak{A}$ be a prime ring with involution $*$ of the second kind such that $char(\mathfrak{A}) \neq 2$. If $\mathfrak{A}$ admits a derivation e such that $[e(\ell), (\ell)^*] \in \mathfrak{Z}(\mathfrak{A})$ for all $\ell \in \mathfrak{A}$, then $e = 0$ or $\mathfrak{A}$ is a commutative integral domain.*

Corollary 14. *Let $\mathfrak{A}$ be a prime ring with involution $*$ of the second kind such that $char(\mathfrak{A}) \neq 2$. If $\mathfrak{A}$ admits a derivation e such that $(e(\ell) \circ (\ell)^*) \circ (\ell)^* = 0$ for all $\ell \in \mathfrak{A}$, then $e = 0$.*

Theorem 17. *Let $\mathfrak{A}$ be a 2-torsion free semiprime ring with involution $*$ of the second kind. If $\mathfrak{A}$ admits a derivation e such that $(e(\ell) \circ (\ell)^*) \circ (\ell)^* = 0$ for all $\ell \in \mathfrak{A}$, then $e = 0$.*

Proof. Given that

$$(e(\ell) \circ (\ell)^*) \circ (\ell)^* = 0 \text{ for all } \ell \in \mathfrak{A},$$

by the semiprimeness of $\mathfrak{A}$, there exists a family $\mathcal{L} = \{\mathfrak{L}_\alpha : \alpha \in \wedge\}$ of prime ideals such that $\bigcap_\alpha \mathfrak{L}_\alpha = (0)$. For each $\mathfrak{L}_\alpha$ in $\mathcal{L}$, we have

$$(e(\ell) \circ (\ell)^*) \circ (\ell)^* \in \mathfrak{L}_\alpha \text{ for all } \ell \in \mathfrak{A}.$$

The application of Theorem 16 gives that $e(\mathfrak{A}) \subseteq \mathfrak{L}_\alpha$. Thus, $e(\mathfrak{A}) \subseteq \bigcap_\alpha \mathfrak{L}_\alpha = (0)$ and hence $e = 0$. Thereby, the proof is complete. □

We feel that Theorem 14 (resp. Theorem 16) can be proved without the assumption $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \nsubseteq \mathfrak{L}$ for any prime ideal $\mathfrak{L}$ of an arbitrary ring $\mathfrak{A}$, but, unfortunately, we are unable to carry this out. Hence, Theorem 14 leads to the following conjecture.

Conjecture: Let $\mathfrak{A}$ be a ring with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$. If $\mathfrak{A}$ admits a derivation e such that $[[e(\ell), (\ell)^*], (\ell)^*] \in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$, then one of the following holds:

1. $char(\mathfrak{A}/\mathfrak{L}) = 2$;
2. $e(\mathfrak{A}) \subseteq \mathfrak{L}$;
3. $\mathfrak{A}/\mathfrak{L}$ is a commutative integral domain.

5. A Direction for Further Research

Throughout this section, we assume that k_1, k_2, m and n are fixed positive integers. Several papers in the literature show evidence of how the behavior of some additive mappings is closely related to the structure of associative rings and algebras (cf.; [3,6,7,9,19,28,30,34]. A well-known result proved by Posner [1] states that a prime ring must be commutative if $[e(\ell), \ell] = 0$ for all $\ell \in \mathfrak{A}$, where e is a nonzero derivation of $\mathfrak{A}$. In [2,23], Vukman extended Posner's theorem for commutators of order two and three and described the structure of prime rings whose characteristic is not two and satisfes $[[e(\ell), \ell], \ell] = 0$

for every $\ell \in \mathfrak{A}$. The most famous and classical generalization of Posner's and Vukman's results is the following theorem due to Lanski [8] for k^{th}-commutators.

Theorem 18 ([8] (Theorem 1))**.** *Let m, n and k be fixed positive integers and $\mathfrak{A}$ be a prime ring. If a derivation e of $\mathfrak{A}$ satisfies $[e(\ell^m), \ell^m]_k = 0$ for all $\ell \in I$, where I is a nonzero left ideal of $\mathfrak{A}$, then $e = 0$ or $\mathfrak{A}$ is commutative.*

In [52], Lee and Shuie studied that, if a noncommutative prime ring $\mathfrak{A}$ admits a derivation e such that $[e(\ell^m)\ell^n, \ell^r]_k = 0$ for all $\ell \in I$, where I is a nonzero left ideal, then $e = 0$ except when $\mathfrak{A} \cong M_2(GF(2))$. In the year 2000, Carini and De Filippis [27] studied Posner's classical result for power central values. In particular, they discussed this situation for $\mathfrak{A}$ of a characteristic that is not two and proved that, if $([e(\ell), \ell])^n \in \mathfrak{Z}(\mathfrak{A})$ for all $\ell \in L$, a noncentral Lie ideal of $\mathfrak{A}$, then $\mathfrak{A}$ satisfies s_4. In 2006, Wang and You [11] mentioned that the restriction of the characteristic need not be necessary in Theorem 1.1 of [27]. More precisely, they proved the following result.

Theorem 19. *Let $\mathfrak{A}$ be a noncommutative prime ring and L be a noncentral Lie ideal of $\mathfrak{A}$. If $\mathfrak{A}$ admits a derivation e satisfying $([e(\ell^m), \ell^m])^n \in \mathfrak{Z}(\mathfrak{A})$ for all $\ell \in L$, then $\mathfrak{A}$ satisfies s_4, the standard identity in four variables.*

Motivated by these two results, Wang [10] studied the similar condition for $\mathfrak{A}$ of a characteristic that is not two and obtained the same conclusion. In fact, he proved the following results.

Theorem 20. *Let $\mathfrak{A}$ be a noncommutative prime ring of a characteristic that is not two. If $\mathfrak{A}$ admits a nonzero derivation e satisfying $([e(\ell^m), \ell^m]_n)^k \in \mathfrak{Z}(\mathfrak{A})$ for all $\ell \in \mathfrak{A}$, then $\mathfrak{A}$ satisfies s_4, the standard identity in four variables.*

In our main results (Theorems 6, 8, 10, 14 and 15), we investigated the structure of the quotient rings $\mathfrak{A}/\mathfrak{L}$, where $\mathfrak{A}$ is an arbitrary ring and $\mathfrak{L}$ is a prime ideal of $\mathfrak{A}$. Nevertheless, there are various interesting open problems related to our work. In this final section, we will propose a direction for future further research. In view of the above mentioned results and our main theorems, the following problems remain unanswered.

Problem 1. *Let $\mathfrak{A}$ be a ring of a suitable characteristic with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \not\subseteq \mathfrak{L}$. Next, let $f : \mathfrak{A} \to \mathfrak{A}$ be a mapping satisfying $[f(\ell), ((\ell)^*)^m]^n \in \mathfrak{Z}(\mathfrak{A})$ or $\in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$. Then, what can we say about the structure of $\mathfrak{A}$ and f?*

Problem 2. *Let $\mathfrak{A}$ be a ring of a suitable characteristic with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \not\subseteq \mathfrak{L}$. Next, let $e : \mathfrak{A} \to \mathfrak{A}$ be a derivation satisfying $[e(\ell), (\ell)^*]_n \in \mathfrak{Z}(\mathfrak{A})$ or $\in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$. Then, what can we say about the structure of $\mathfrak{A}$ and e?*

Problem 3. *Let $\mathfrak{A}$ be a ring of a suitable characteristic with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \not\subseteq \mathfrak{L}$. Next, let $e : \mathfrak{A} \to \mathfrak{A}$ be a derivation satisfying $([e(x^{k_1}), ((\ell)^*)^{k_2}]_n)^m \in \mathfrak{Z}(\mathfrak{A})$ or $\in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$. Then, what can we say about the structure of $\mathfrak{A}$ and e?*

Problem 4. *Let $\mathfrak{A}$ be a ring of a suitable characteristic with involution $*$ of the second kind and $\mathfrak{L}$ be a prime ideal of $\mathfrak{A}$ such that $S(\mathfrak{A}) \cap \mathfrak{Z}(\mathfrak{A}) \not\subseteq \mathfrak{L}$. Next, let $e : \mathfrak{A} \to \mathfrak{A}$ be a derivation satisfying $(e(\ell^{k_1}) \circ_n ((\ell)^*))^{k_2})^m \in \mathfrak{Z}(\mathfrak{A})$ or $\in \mathfrak{L}$ for all $\ell \in \mathfrak{A}$. Then, what can we say about the structure of $\mathfrak{A}$ and e?*

6. Conclusions

In the present paper, we extended very famous and classical results due to Vukman [2] and Posner [1]. Moreover, we described the structure of quotient rings $\mathfrak{A}/\mathfrak{L}$, where $\mathfrak{A}$ is an arbitrary ring and $\mathfrak{L}$ is a prime ideal of $\mathfrak{A}$. Further, we studied a conventional result due to Bell and Martindale [18]. Finally, we concluded our manuscript with a direction for future further research.

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