mathematics

Article

Polynomial Recurrence for SDEs with a Gradient-Type Drift, Revisited

Alexander Veretennikov

Kharkevich Institute for Information Transmission Problems, Moscow 127051, Russia; ayv@iitp.ru

Abstract: In this paper, polynomial recurrence bounds for a class of stochastic differential equations with a rotational symmetric gradient type drift and an additive Wiener process are established, as well as certain a priori moment inequalities for solutions. The key feature of this paper is that the approach does not use Lyapunov functions because it is not clear how to construct them. The method based on Dynkin's (nonrandom) chain of equations is applied instead. Another key feature is that the asymptotic conditions on the potential near infinity are assumed as inequalities—which allows for more flexibility compared to a single limit at infinity, making it less restrictive.

Keywords: stochastic differential equations; gradient type drift; polynomial recurrence

MSC: 60H10; 60J60

check for updates

Citation: Veretennikov, A. Polynomial Recurrence for SDEs with a Gradient-Type Drift, Revisited. *Mathematics* **2023**, *11*, 3096. https://doi.org/10.3390/math11143096

Academic Editor: Quanxin Zhu

Received: 29 May 2023
Revised: 3 July 2023
Accepted: 7 July 2023
Published: 13 July 2023

1. Introduction

Let us consider a stochastic differential equation in $\mathbb{R}^d$

$$dX_t = dB_t + \nabla U(X_t)\, dt \tag{1}$$

with initial data

$$X_0 = x. \tag{2}$$

Here ,B_t, $t \geq 0$ is a d-dimensional Brownian motion, X_t takes values in $\mathbb{R}^d$, $U : \mathbb{R}^d \mapsto \mathbb{R}$ is a *symmetric* (i.e., $U(x) = U(x')$ if $|x'| = |x|$), non-positive function with $U(0) = 0$ and $\lim_{|x|\to\infty} U(x) = -\infty$. The function U is assumed to be locally bounded in C^1. The aim of this paper is to establish the recurrence properties of the Markov process X_t. Recurrence properties are the usual preliminary steps to many other features of a process such as ergodicity, existence, and uniqueness of its invariant probability measure, as well as for bounds in the Law of Large Numbers type theorems and bounds for the beta-mixing rate (cf. Ref. [1]). In this paper the goal is to establish some polynomial bound for the hitting time to some compact subset in $\mathbb{R}^d$ by the process X and the moment bounds for the marginal distribution of the process X_t itself. The issues related to the invariant measure are left till further studies except for one auxiliary statement, which is used for the comparison. This hitting time bound will not depend on the first derivatives of the function U, even though the drift in the SDE is of the gradient type. This may look a bit unusual because the drift in the SDE (1) is of the form $\nabla U(x)$. Such a problem—concerning bounds not depending explicitly on ∇U—was posed and in some particular case solved in Ref. [2]. Earlier, other various results in the area of SDEs with a gradient type drift were established by N.I. Portenko [3,4], then by S.Ya. Makhno in Refs. [5–7]. Recently the SDEs of the type (1) with a gradient drift form under the name Ferrari–Spohn diffusion turned out to play an important role in mathematical physics; see Refs. [8,9]. Here we extend, relax, and also correct some of the assumptions from Ref. [2], with the main aim to replace the assumptions of the limit type (see (9) in what follows) by asymptotic inequalities (see (11) in what follows). Results on the solutions for SDEs with an irregular drift of a gradient form were also obtained in Ref. [10], and more recently in Refs. [11,12]. Certain results on strong solutions were established in Ref. [13]. To the best of the author's knowledge, ergodicity issues with slow polynomial convergence rates were not studied, except for in Ref. [2], where the assumptions were more restrictive.

It is common knowledge that the rate of convergence to the invariant distribution as well as the decrease rates for certain mixing coefficients may be derived from the estimates of the type

$$\frac{\mathsf{E}_x \tau^k}{1+|x|^m} \le C \tag{3}$$

along with

$$\frac{\sup_{t\ge 0} \mathsf{E}_x |X_t|^m}{1+|x|^{m'}} \le C, \tag{4}$$

for some $k > 1$, $m, m', C > 0$, where

$$\tau = \inf(t \ge 0 : \ |X_t| \le K) \tag{5}$$

for some $K > 1$, see, e.g., Refs. [14,15]. Naturally, if $|x| \le K$ then $\tau = 0$. The interest lies in the bounds, such as (3), for large values of $|x|$. Here the value of m in (3) and in the left hand side of (4) should be the same; the value m' may equal m, or may be different. In particular, for SDEs (1) it may be derived from (3) and (4) that

$$\|\mu_t^x - \mu^{inv}\|_{TV} \le P(|x|)(1+t)^{-k'}, \tag{6}$$

with some k' and with some polynomial function P, at least, in the case of a *bounded* function ∇U and under certain recurrent condition on ∇U. Here μ_t^x denotes the marginal distribution of the process X_t with the initial value $X_0 = x$, while μ^{inv} stands for the invariant probability measure; the norm is in total variation. Moreover, similar bounds may be established for the beta-mixing coefficient on the basis of the recurrence properties; this is known to be quite useful in various limit theorems (cf. Ref. [16]) as well as in the extreme value theory [17]. However, the pursuit of this goal is not within the scope of this paper; certain applications will be studied in a separate publication.

Bounds such as (3) under various assumptions were obtained for various classes of processes by many authors, see, in particular Refs. [1,14,18–20], and the references therein; yet, for SDEs all assumptions were usually—except for in Ref. [2]—stated in terms of ∇U. See also Refs. [21,22], where stronger sub-exponential bounds were established under another standing assumption.

In [1,15] a recurrence condition

$$-p = \limsup_{|x|\to\infty} (\nabla U(x), x) < 0 \tag{7}$$

was used to get bounds such as (6). Naturally these bounds depend on the asymptotic of ∇U. Here the problem was to find some analogue of the latter condition in terms of the limiting behavior of the function U itself, as in Ref. [2] but under further relaxed assumptions.

In many papers the study of recurrence starts with the assumption about an existence of one or another Lyapunov function, which eventually leads to certain recurrence properties. In other papers the required Lyapunov functions are derived from the assumptions on the coefficients of the equation. The case under investigation in this paper is different: there is no clear way to construct Lyapunov functions here using the assumptions on the potential $U(x)$. It can be considered a rare situation where recurrence bounds are established without the use of Lyapunov functions. *Of course, after the recurrence estimates (14) in Theorem 1 are established, the functions v^q themselves may serve as Lyapunov functions; however, in this case,* *the situation is reversed as Lyapunov functions no longer required after the recurrence bounds have been found.*

The paper consists of six sections, one of them with two subsections. They are as follows: this introduction, the main results divided into two parts (the earlier results and the new results), the proof of the main theorem, examples, a discussion, and an appendix containing a lemma.

2. Main Results

2.1. Earlier Result

Let us recall briefly some earlier results from Ref. [2] where a slightly more general SDE was considered, with the drift of the form $b(x) - \nabla U(x)$. In this paper we assume $b \equiv 0$, and use $+\nabla U$ instead of $-\nabla U(x)$; naturally, the assumptions will be rewritten accordingly, for example, in our case $U(x) \to -\infty$ in place of $U(x) \to +\infty$ in Ref. [2]. This reminder will be useful for comparison, as it allows us to discern what is truly novel in this context. Additionally, the subsequent analysis will be based on similar but less stringent assumptions.

Assume

$$\sup_{x,x':\,|x-x'|\le 1} (U(x) - U(x')) < \infty. \tag{8}$$

Since Equation (1) uses only the gradient of U, the particular value of U at the origin is not important; for example, without loss of generality we may assume $U(0) = 0$.

The function U is also assumed to possess a central symmetry, i.e., it only depends on the value of $|x|$ at each point, and the function $V(u) := U(x)$ for $|x| = u$ here is assumed to be in the class $C^1[0,\infty)$. At the origin it follows that $V'(0) = 0$, otherwise the gradient $\nabla U(0)$ may not exist. In Ref. [2] the bounds were established under the recurrence condition

$$\lim_{\xi\to\infty} \frac{V(\xi)}{\ln \xi} + d = -p < 0, \tag{9}$$

and under certain relationships between the constants. In particular, the bound (3) was established under the condition $p > 1/2$ for any $0 < k < p + 1/2$ and with any $m = 2k + \varepsilon$ ($\varepsilon > 0$); the inequality (4) was shown to be valid with any $m < 2p - 1$ and $m' = m + 2\varepsilon$.

Remark 1. *The following* **corrections** *should be made in Ref. [2]. (1) It was erroneously assumed in Ref. [2] that the function U may have the form $U(x) = U^1(x) + U^2(x)$ with a special requirement on U^2. This is incorrect: U^2 must be identically to zero. Furthermore, all subsequent assumptions regarding U^1 should be understood as assumptions on U itself. (2) Additionally, instead of the reference on the elliptic Harnack inequality in Ref. [23], there should be a reference on the parabolic one, see, for example, Theorem 6.27 in Ref. [24].*

Remark 2. The improvements *in relation to Ref. [2] are as follows.*

It turned out that the dominating process y_t was chosen in Ref. [2] in a non-optimal way, see (18) and (19) in what follows for a better choice. The assumption (11) is weaker than (9) even in the case $p_1 = p_2 = p$ because the value d in the left hand side of (9) can be replaced by a smaller value $(d-1)/2$ in (11). However, a more essential difference is the assumption on U in this paper, which is presented in an interval form (11)—see the following subsection—instead of the limiting form (9) as in Ref. [2]. Thus, the limit (9) may not exist under only (11), but certain recurrence bounds are still available. This extends the class of processes for which such recurrence bounds may be established.

Remark 3. New improvements *in comparison to Ref. [2] are due to the fact that the asymptotic condition on the potential U in the form of a limit (9) is now replaced by a two-sided inequality (11) (see the next subsection), which does not necessarily assume the existence of a limit as in (9).*

Remark 4. **A comparison** *to the paper [15] is as follows. In Theorem 6 in Ref. [15], the recurrence bound was established, in terms of the Equation (1). If condition (7) explicitly related to the gradient* ∇U *is satisfied with some* $p > 1 + \frac{d}{2}$*, then bound (3) holds true for any* $k < p - \frac{d}{2}$ *with a certain value of* m*. After integration and a small elementary calculation, this is equivalent to the (one-sided) asymptotic version of condition (11) with* $p_1 = p_2$ *of Theorem 1 in the next section, also satisfying (10). Clearly, condition (11) with (10) is valid for a much wider class of drifts leading to the same recurrence bound (3).*

2.2. New Result

Theorem 1. *Assume the condition* $U(x) = U(x')$ *for any* x, x' *such that* $|x| = |x'|$*, and let the assumption (8) hold true, and let* ∇U *be locally bounded. Let* $V(\xi) := U(x)$ *for any* x *such that* $|x| = \xi$*, and let there exist two constants* p_1 *and* p_2 *such that*

$$1/2 < p_2 \leq p_1 \tag{10}$$

and

$$p_2 \leq \frac{-V(\xi)}{\ln \xi} - \frac{d-1}{2} \leq p_1, \tag{11}$$

for all $\xi > 0$*, which are large enough,* $\xi \geq K$ *where* $K > 1$ *is arbitrary. (*$K > 1$ *is used to guarantee that* $\ln \xi > 0$ *for any* $\xi \geq K$*.) Then the bound*

$$\sup_{t \geq 0} \mathsf{E}_x |X_t|^m \leq C(1 + |x|^{m'}) \tag{12}$$

holds true with any

$$m < 2p_1 + 1, \tag{13}$$

and with

$$m' = m + 2(p_1 - p_2).$$

Moreover, for any positive **integer** *value of* $k < 1 + \dfrac{2p_2 - 1}{2(1 + p_1 - p_2)} = \dfrac{2p_1 + 1}{2(1 + p_1 - p_2)}$ *and* $m = 2k(1 + p_1 - p_2)$*, the bound holds,*

$$\mathsf{E}_x \tau^k \leq C(1 + |x|^m), \tag{14}$$

where τ *is the stopping time defined in (5).*

Moreover the Markov process (X_t) *possesses an invariant probability measure* μ *which admits the bound*

$$\int |x|^\ell \mu(dx) < \infty, \quad \forall\, \ell < 2p_2 - 1. \tag{15}$$

Remark 5. *The inequality (14) is satisfied with* $k = 1$ *and some appropriate* m *iff* $p_2 > 1/2$*, which is a standing assumption (10). In addition, notice that no exponential rate of convergence is claimed. Even a polynomial rate will be proved only up to some finite power* k*.*

Remark 6. *Under the assumptions of the theorem, this invariant measure is apparently unique. However, we do not need it for the proof of the other statements and, hence, do not claim so in Theorem 1 in order to not overload this presentation. This uniqueness will be rigorously derived in further publications along with the rate of convergence to the invariant regime, for which a polynomial recurrence plays a crucial role. In this paper* some *invariant measure with explicitly estimated finite moments of a certain order is all we need for the proofs.*

Remark 7. *Since* ∇U *is assumed to be locally bounded, the solution of the Equation (1) is strong and pathwise unique, at least, locally in time (e.g., see Ref. [25]). In fact, as it follows from the almost surely finiteness of the hitting time* τ*, this solution does not explode and, hence, it remains strong* *and pathwise unique for all $t \geq 0$. For recent results on weak solutions with a weak uniqueness in a close setting, see Ref. [11].*

3. Proof of Theorem 1

The approach involves comparing the process $|X_t|$ to a one-dimensional process that follows a SDE with reflection. This allows for obtaining an explicit expression for the invariant density of the reflected process. This auxiliary dominating process will be denoted by (y_t). The stopping time τ for the process (X_t) will admit an upper bound by an appropriate stopping time for (y_t), which will be denoted by γ: we will have $\tau \leq \gamma$. Then, for the process (y_t) it will be possible to estimate by induction the moments of γ by solving the chain of ordinary differential equations of the second order, which may be found in Theorem 13.17 in Ref. [26]. In the PDE theory, these equations are known under the name of Duhamel's formula; the chain, or system of these equations—see (21) in what follows—seems to be specific for a probabilistic setting.

1. Comparison to a solution for a 1D equation with reflection. Recall that $K > 1$ and let

$$\bar{V}(y) = V(y) + \frac{d-1}{2}\ln y, \quad y > 0. \tag{16}$$

Then

$$\bar{V}'(y) = V'(y) + (d-1)/(2y), \quad \forall\, y > 0.$$

Notice that condition (11) in terms of the function $\bar{V}$ may be rewritten in the form

$$\xi^{2p_2} \leq \exp(-2\bar{V}(\xi)) \leq \xi^{2p_1}, \quad \xi \geq K. \tag{17}$$

Similarly to Ref. [2], after an application of Itô's formula to $d|X_t|$ and due to the comparison theorems for SDEs with reflection as shown in Lemma A1 in the appendix, one gets,

$$|X_t| \leq y_t, \tag{18}$$

where y_t is a solution of an SDE with reflection on $[K, \infty)$

$$dy_t = d\bar{w}_t + \left(\frac{d-1}{2y_t} + V'(y_t)\right)dt + d\varphi_t \equiv d\bar{w}_t + \bar{V}'(y_t)dt + d\varphi_t, \tag{19}$$

with any initial condition $y_0 > |X_0|$, random or non-random; a formal proof may be found in Lemma A1 in the Appendix A. Here

$$\bar{w}_t = \int_0^t 1(|X_s| > 0)\frac{X_s dB_s}{|X_s|}$$

is a one-dimensional Wiener process due to the Lévy characterization, since $\mathrm{P}(|X_t| = 0) = 0$ for any $t > 0$, which implies that the compensator $\langle \bar{w} \rangle_t = \displaystyle\int_0^t 1(|X_s| > 0)\frac{\sum_{k=1}^d (X_s^k)^2}{|X_s|^2}ds = t$ a.s. The process y_t is a strong and pathwise unique solution of the SDE (19) (see, e.g., Ref. [27]) with a non-sticky boundary condition at $K > 1$, so that $y_t \geq K$ for all t, φ is its local time at K, that is,

$$y_t \geq K \quad \text{a.s.},$$

the function φ_t is non-negative and non-decreasing, with $\varphi_0 = 0$, and

$$\varphi_t = \int_0^t 1(y_s = K)d\varphi_s.$$

As mentioned, the reflection is assumed to be non-sticky, that is,

$$\int_0^t 1(y_s > K)ds = t \text{ (a.s.)},$$

If we want to highlight the initial value y_0, we will denote this solution of Equation (19) by $y_t^{y_0}$.

Indeed, let us show (18): by Itô's formula for $|X_t| \neq 0$, denoting for convenience $Y_t := \sum_k (X_t^k)^2$ we have,

$$\begin{aligned} dY_t &= d\sum_k (X_t^k)^2 = \sum_k d(X_t^k)^2 = \sum_k 2X_t^k dX_t^k + \frac{1}{2}\sum_k 2(dX_t^k)^2 = \sum_k 2X_t^k dX_t^k + \frac{1}{2}\sum_k 2(dB_t^k)^2 \\ &= 2|X_t| \underbrace{\sum_k \frac{X_t^k dB_t^k}{|X_t|}}_{=d\bar{w}_t} + \left(2|X_t|\sum_k \frac{X_t^k}{|X_t|} U'_{x^k}(X_t) + d\right)dt \\ &= 2|X_t|d\bar{w}_t + \left(2|X_t|\langle \frac{X_t}{|X_t|}, \nabla U(X_t)\rangle + d\right)dt = 2|X_t|d\bar{w}_t + \big(2|X_t|V'(|X_t|) + d\big)dt. \end{aligned}$$

Due to the equality $U(x) = V(|x|)$ on the set $|X| > 0$, we have,

$$\begin{aligned} \frac{x}{|x|}\nabla U(x) &= \frac{1}{|x|}\sum_{k=1}^d x_k U'_{x_k}(x) = \frac{1}{|x|}\sum_{k=1}^d x_k \frac{\partial}{\partial x_k} V(|x|) = \frac{1}{|x|}\sum_{k=1}^d x_k V'(|x|)\frac{\partial\sqrt{\sum_{j=1}^d x_j^2}}{\partial x_k} \\ &= \frac{1}{|x|}\sum_{k=1}^d x_k V'(|x|)\frac{x_k}{|x|} = \frac{1}{|x|^2}V'(|x|)\sum_{k=1}^d x_k^2 = V'(|x|). \end{aligned}$$

Therefore, still for $|X_t| \neq 0$,

$$(dY_t)^2 = 4Y_t dt = 4|X_t|^2 dt.$$

So, by Itô's formula on the set $|X_t| \neq 0$,

$$\begin{aligned} d|X_t| = d(Y_t)^{1/2} &= \frac{1}{2}Y_t^{-1/2}dY_t + \frac{1}{2}\times\frac{1}{2}\times(-\frac{1}{2})Y_t^{-3/2}(dY_t)^2 \\ &= \frac{1}{2}\left(\frac{2|X_t|d\bar{w}_t + (2|X_t|V'(|X_t| + d)dt}{|X_t|}\right) - \frac{4|X_t|^2 dt}{8|X_t|^3} \\ &= d\bar{w}_t + \left(V'(|X_t|) + \frac{d}{2|X_t|} - \frac{1}{2|X_t|}\right)dt \\ &= d\bar{w}_t + \left(V'(|X_t|) + \frac{d-1}{2|X_t|}\right)dt = d\bar{w}_t + \bar{V}'(|X_t|)dt. \end{aligned}$$

Thus, where $|X_t| > K$, the stochastic differential of the process $|X_t|$ has the same form as the stochastic differential of y_t, that is, they both satisfy the same SDE on the half line (K, ∞), but unlike y_t, the process $|X_t|$ may take values less than K with a positive probability. By well-known comparison theorems,

$$\mathsf{P}_x(|X_t| \le y_t,\ t \ge 0) = 1. \tag{20}$$

This assertion (20) will be rigorously proved in Lemma A1 in the Appendix A. *It is important here that both solutions are strong and, hence, are well-defined on the same probability space.*

2. Invariant density and its moments. The invariant density $f(x)$ of the process X_t is well-known: it has a form $f(x) = c\exp(2U(x))$. Indeed, we may check the invariance equation $L^* f = 0$:

$$\frac{1}{c}(\frac{1}{2}\Delta f(x) - div(f\nabla U)) = \frac{1}{2}\sum_i \frac{\partial^2}{\partial x_i^2}\exp(2U(x)) - \sum_i \frac{\partial}{\partial x_i}(U_{x_i}(x)\exp(2U(x)))$$
$$= \sum_i \left\{\frac{1}{2}\frac{\partial}{\partial x_i}(2U_{x_i}(x)\exp(2U(x))) - \frac{\partial}{\partial x_i}(U_{x_i}(x)\exp(2U(x)))\right\} = 0.$$

Further, for any finite dimension $d \geq 1$ we have a bound for the moment of the invariant density,

$$\underset{|x|\geq 1}{\int\cdots\int} |x|^m \exp(2U(x))dx = \underset{|x|\geq 1}{\int\cdots\int} |x|^m \exp(2V(|x|))dx$$
$$\leq \underset{|x|\geq 1\vee K}{\int\cdots\int} |x|^m |x|^{-2(p_2+(d-1)/2)}dx = \int_{1\vee K}^{\infty} r^{m-2(p_2+(d-1)/2)} r^{d-1}dr = \int_{1\vee K}^{\infty} r^{m-2p_2}dr.$$

Here $a \vee b = \max(a, b)$. The last integral converges iff $m - 2p_2 < -1$, that is,

$$\boxed{m < 2p_2 - 1.}$$

For any such value of m the moment of the invariant density of (X_t) is finite. Recall that the uniqueness of the invariant measure is not emphasized here, but see Remark 6. The value of m may be integer, or non-integer here.

3. A normalizing constant for the invariant density of (y_t). The invariant density $f(y)$ of the process y_t on the half-line $[|x|, \infty)$ with the reflection barrier $K = |x|$ has a form

$$C(|x|)\exp(2\bar{V}(y)), \quad y > |x|, \tag{21}$$

which can be easily verified by directly computing the stationarity equation $\bar{L}^* f = 0$, following the standard approach using Itô's formula with expectations for any $g(y_t)$ for $g \in C_0^2$ (with a compact support and with $g(|x|) = 0$), where $\bar{L}$ is the generator of (y_t) and $\bar{L}^*$ is its adjoint with respect to Lebesgue's measure.

Notice that at this stage it is irrelevant whether or not this invariant distribution of the process (y_t) is unique. In fact, it is, but we do not use it in what follows and, hence, do not pursue this goal.

The normalizing identity implies the estimation from above under the condition $2p_1 > 1$ (it coincides with $2p_1 > d$ just for $d = 1$),

$$C(|x|) = \left(\int_{|x|}^{\infty} \exp(2\bar{V}(y))\,dy\right)^{-1} \leq \left(\int_{|x|}^{\infty} \xi^{-2p_1}\,dy\right)^{-1} = (2p_1 - 1)|x|^{2p_1-1}, \tag{22}$$

for the values of $|x| \geq K$ where the assumptions (17) are valid. For smaller values of $|x| > 0$ below K, the integral $\int_{|x|}^{\infty} \exp(2\bar{V}(y))\,dy$ may not diverge because in any finite neighborhood of zero the function $\exp(2V(y))$ is bounded; see condition (8) and the definition (16). Naturally, the integral $\int_{|x|}^{\infty} \xi^{-2p_1}\,dy$ increases when $|x|$ decreases, so that for smaller values of $|x|$ we also have smaller values of $C(|x|)$. Additionally, note that the integral $\int_{0+} \exp(2\frac{(d-1)\ln y}{2})dy = \int_{0+} y^{(d-1)}dy$ converges for any dimension $d \geq 1$. Hence, for all values $|x| \geq 0$

$$C(|x|) \leq (2p_1 - 1)|x|^{2p_1-1} \vee C_0 \tag{23}$$

with some finite C_0. Additionally, note for the sequel that due to the assumption (11), the density $f(y)$ admits the bound $f(y) = c\exp(2\bar{V}(y)) \leq cy^{-2p_2}$ for $y \geq K$ and, hence, integrates some power function: namely, under the condition $\boxed{p_2 > 1/2}$ we have,

$$\int |y|^{\ell} f(y)\, dy \leq C(|x|) \int_{|x|}^{\infty} |y|^{\ell-2p_2}\, dy < \infty, \quad \forall \boxed{\ell < 2p_2 - 1.} \tag{24}$$

Note that the range for the possible values of ℓ here coincides with that for ℓ in (15). This prompts that if we had no explicit formula for the invariant distribution of the process (X_t), but only for the dominating one (y_t), then the right order for its finite moments could still have been obtained using the technique based on the Harris–Khasminskii's method.

4. The inequality (12) with any real value $m < 2p_2 - 1$ and with $m' = m + 2(p_1 - p_2)$ (where m' is not necessarily an integer either) follows from a direct calculation. Indeed, since by the comparison theorem the process $y_t^{|x|}$ with $|x|$ large enough does not exceed the stationary version of the Markov process satisfying the same SDE with a non-sticky reflection and with the reflection barrier at $|x|$, then

$$\begin{aligned}
\mathsf{E}_x|X_t|^m \leq \mathsf{E}_{|x|}|y_t|^m \leq C(|x|) \int\limits_{|x|}^{\infty} \xi^m \exp(2\bar{V}(\xi))\, d\xi \\
\leq (C|x|^{2p_1-1} \vee C_0) \int\limits_{|x|}^{\infty} \xi^m \xi^{-2p_2}\, d\xi \\
\leq (C|x|^{m+2(p_1-p_2)+1-1}) \vee C_0 = (C|x|^{m+2(p_1-p_2)}) \vee C_0.
\end{aligned}$$

Here the constants C and C_0 may all be different. The first inequality in this calculus is true for any x large enough, due to comparison theorems for the processes y_t with different initial data y_0, see, e.g., Ref. [27] for the bounded coefficients; this result naturally generalizes to the locally bounded coefficients in the situation where there is no explosion for (y_t). For *any* $|x|$, and not necessarily small, this implies the bound (4), as required. Note that the drift in Ref. [27] was assumed to be bounded, or, at most, satisfying a linear growth condition. However, given that all solutions are strong and that they are defined on the infinite interval of time without explosion, this assumption can be dropped and replaced by a local boundedness of $\bar{V}'$ outside zero. What is important here is that the values of the norms of the drift $\nabla U(x)$ in $\mathbb{R}^d$ do not contribute to the constants in the final bound where only the assumptions on the function U itself will be used.

5. The inequality (14). This is the crucial part of the statement of the theorem. Denote

$$v^q(\xi) = \mathsf{E}_\xi \gamma^q \tag{25}$$

for any integer $q \geq 0$, $\gamma = \inf(t: \ y_t \leq K)$; $v^0(\xi) \equiv 1$. Clearly, $v^q(\xi) = 0$ for $|\xi| \leq K$, except for $q = 0$. Let $\bar{L}$ denote the generator of the process y_t, that is,

$$\bar{L}g(y) = \frac{1}{2}g''(y) + \bar{V}'(y)g'(y), \tag{26}$$

where $\bar{V}(y) = V(y) + \dfrac{(d-1)\ln y}{2}$, $y > 0$ (see (16)). By the first theorem of the calculus (also known as the Newton–Leibniz theorem) we have,

$$\gamma^q = \left(\int_0^\gamma 1\,dt\right)^q = q\int_0^\gamma \left(\int_0^t 1\,ds\right)^{q-1} dt = q\int_0^\gamma \left(\int_t^\gamma 1\,ds\right)^{q-1} dt,$$

Therefore, by taking expectations and considering the Markov property, it follows that,

$$\begin{aligned}
v^q(y_0) &= \mathsf{E}_{y_0}\gamma^q = \mathsf{E}_{y_0} q\int_0^\gamma \left(\int_t^\gamma 1\,ds\right)^{q-1} dt = \mathsf{E}_{y_0} q\int_0^\infty 1(t<\gamma)\left(\int_t^\gamma 1\,ds\right)^{q-1} dt\\
&= q\int_0^\infty \mathsf{E}_{y_0} 1(t<\gamma)\left(\int_t^\gamma 1\,ds\right)^{q-1} dt = q\int_0^\infty \mathsf{E}_{y_0} 1(t<\gamma)\,\mathsf{E}_{y_0}\left(\left(\int_t^\gamma 1\,ds\right)^{q-1} |\mathcal{F}_t^y\right) dt\\
&= q\int_0^\infty \mathsf{E}_{y_0} 1(t<\gamma)\,\mathsf{E}_{y_t}\left(\int_t^\gamma 1\,ds\right)^{q-1} dt = q\int_0^\infty \mathsf{E}_{y_0} 1(t<\gamma) v^{q-1}(y_t)\,dt = q\mathsf{E}_{y_0}\int_0^\gamma v^{q-1}(y_t)\,dt,
\end{aligned}$$

for any q, such that the integral in the right hand side converges. In turn, by Itô's or Dynkin's formula this implies an equation

$$Lv^q(y) = -qv^{q-1}(y),\ y \ge K \quad (q = 1, 2, \ldots) \tag{27}$$

(cf. with Theorem 13.17 in Ref. [26] where this very equation is explained differently and under a stronger assumption, which guarantees some exponential moment of γ). Evidently, one boundary value for the latter equation is $v^q(K) = 0$. The role of the "second boundary value" usual for an ODE, or PDE of the second order is played by the condition of a moderate growth at infinity, that is, not exceeding some power function. The justification of the formula for the solution below can be done by the limiting procedure, as follows. Let $N > K$ be the second boundary (later on N would go to infinity). Let $v_N^q(\xi) = \mathsf{E}_\xi\gamma_N^q$ for any integer $q \ge 0$, $\gamma_N = \inf(t:\ y_t^N \le K)$, where the process y_t^N is a solution of the equation similar to (19) but with another non-sticky reflection at N. Recall that all solutions are strong and, hence, may be constructed on the same probability space; see, e.g., Ref. [27] for SDEs with one boundary, and results from this paper are easily extended for the case with two finite boundaries. Apparently, $y_t^N \le y_t$ for any t and N, and $\gamma_N \uparrow \gamma$ as $N \uparrow \infty$. So, by the monotone convergence, $v_N^q \uparrow v^q$ for all values of q, no matter whether or not the limit v^q is finite. Then the sequence of the functions $v_N^q(\xi)$ satisfies the Equations (27) with boundary conditions

$$v_N^q(K) = 0, \quad (v_N^q)'(N) = 0.$$

The formula for the solution of such an equation reads,

$$v_N^q(\xi) = 2q\int_K^\xi \exp(-2\bar{V}(y_1))\,dy_1 \int_{y_1}^N v_N^{q-1}(y_2)\exp(2\bar{V}(y_2))\,dy_2, \quad K \le \xi \le N,$$

which may be verified by a direct calculation. Indeed, substituting $\xi = K$, we get $v_N^q(K) = 0$, and by taking the derivative, we can see that

$$(v_N^q)'(\xi)|_{\xi=N} = 2q\exp(-2\bar{V}(\xi))\int_\xi^N v_N^{q-1}(y_2)\exp(2\bar{V}(y_2))\,dy_2|_{\xi=N} = 0.$$

The equation itself follows from some calculus, as follows:

$$
\begin{aligned}
&(v_N^q)'(\xi) = 2q\exp(-2\bar{V}(\xi)) \int_\xi^N v_N^{q-1}(y_2)\exp(2\bar{V}(y_2))\,dy_2,\\
&(v_N^q)''(\xi) = -2q\exp(-2\bar{V}(\xi))\,v_N^{q-1}(\xi)\exp(2\bar{V}(\xi))\\
&- 4q\bar{V}'(\xi)\exp(-2\bar{V}(\xi)) \int_\xi^N v_N^{q-1}(y_2)\exp(2\bar{V}(y_2))\,dy_2\\
&= -2qv_N^{q-1}(\xi) - 4q\bar{V}'(\xi)\exp(-2\bar{V}(\xi)) \int_\xi^N v_N^{q-1}(y_2)\exp(2\bar{V}(y_2))\,dy_2,\\
&Lv_N^q(\xi) = \frac{1}{2}(v_N^q)''(\xi) + \bar{V}'(\xi)(v_N^q)'(\xi)\\
&= -qv_N^{q-1}(\xi) - 2q\bar{V}'(\xi)\exp(-2\bar{V}(\xi)) \int_\xi^N v_N^{q-1}(y_2)\exp(2\bar{V}(y_2))\,dy_2\\
&+ \bar{V}'(\xi)\times 2q\exp(-2\bar{V}(\xi)) \int_\xi^N v_N^{q-1}(y_2)\exp(2\bar{V}(y_2))\,dy_2 = -qv_N^{q-1}(\xi),
\end{aligned}
$$

as required. The uniqueness of the solution for a linear ODE system is well-known.

Hence, by induction, the function $v^q(\xi)$ satisfies a representation using the function v^{q-1},

$$
v^q(\xi) = 2q\int_K^\xi \exp(-2\bar{V}(y_1))\,dy_1 \int_{y_1}^\infty v^{q-1}(y_2)\exp(2\bar{V}(y_2))\,dy_2. \tag{28}
$$

By another induction this implies the inequalities (recall that $v^0 \equiv 1$):

$$
\begin{aligned}
&v^1(\xi) = 2\int_K^\xi \exp(-2\bar{V}(y_1))\,dy_1 \int_{y_1}^\infty v^0(y_2)\exp(2\bar{V}(y_2))\,dy_2\\
&= 2\int_K^\xi \exp(-2\bar{V}(y_1))\,dy_1 \int_{y_1}^\infty \exp(2\bar{V}(y_2))\,dy_2 \le 2\int_K^\xi y_1^{2p_1}\,dy_1 \int_{y_1}^\infty y_2^{-2p_2}\,dy_2\\
&= C\int_K^\xi y_1^{2p_1-2p_2+1}\,dy_1 = C(\xi^{2(p_1-p_2)+2} - K^{2(p_1-p_2)+2}) \le C\xi^{2(p_1-p_2)+2},
\end{aligned} \tag{29}
$$

which is finite under the condition that $p_2 > 1/2$ (otherwise the inner integral diverges). Further,

$$
\begin{aligned}
&v^2(\xi) = 4\int_K^\xi \exp(-2\bar{V}(y_1))\,dy_1 \int_{y_1}^\infty v^1(y_2)\exp(2\bar{V}(y_2))\,dy_2\\
&\le C\int_K^\xi \exp(-2\bar{V}(y_1))\,dy_1 \int_{y_1}^\infty y_2^{2(p_1-p_2)+2}\exp(2\bar{V}(y_2))\,dy_2\\
&\le C\int_K^\xi y_1^{2p_1}\,dy_1 \int_{y_1}^\infty y_2^{2(p_1-p_2)+2-2p_2}\,dy_2\\
&= C\int_K^\xi y_1^{2p_1}\,dy_1\, y_1^{2p_1-4p_2+3} = C(\xi^{4(p_1-p_2)+4} - K^{4(p_1-p_2)+4})\\
&\le C\xi^{4(p_1-p_2+1)},
\end{aligned}
$$

where in the calculus it was assumed that $2p_1 - 4p_2 + 2 < -1$, that is, that $p_1 < 2p_2 - 3/2$, otherwise the inner integral in the calculus diverges. Since it was assumed from the beginning that $p_1 \ge p_2$, for the value of p_2, this implies that $p_2 > 3/2$.

It looks plausible that the general bound for a (finite) v^q is provided by the formula

$$
v^q(\xi) \le C_q\xi^{2q(1+p_1-p_2)}. \tag{30}
$$

As the base is already established, let us show the induction step. Assume that for $q = n - 1$, the formula is valid with some constant C_{n-1}, that is,

$$v^{n-1}(\xi) \le C_{n-1}\xi^{2(n-1)(1+p_1-p_2)}.$$

Then for $q = n$, as long as the integrals in the calculus below converge, we have,

$$\begin{aligned}
v^n(\xi) &= 2n\int_K^{\xi} \exp(-2\bar{V}(y_1))\,dy_1 \int_{y_1}^{\infty} v^{n-1}(y_2)\exp(2\bar{V}(y_2))\,dy_2 \\
&\le 2n\int_K^{\xi} {y_1}^{2p_1}\,dy_1 \int_{y_1}^{\infty} C_{n-1}\,y_2^{2(n-1)(p_1-p_2+1)}{y_2}^{-2p_2}\,dy_2 \\
&= C_n\int_K^{\xi} y_1^{2p_1}\,y_1^{2n-1+2(n-1)p_1-2np_2}dy_1 = C_n\int_K^{\xi} y_1^{2n-1+2np_1-2np_2}dy_1 \le C_n\xi^{2n(p_1-p_2+1)}.
\end{aligned}$$

Hence, by induction, Formula (30) is established.

The values of q for which all the integrals in the calculus above converge for each $1 \le n \le q$ must satisfy the bound

$$2(q-1)(1+p_1-p_2) - 2p_2 < -1,$$

that is,

$$q < q_0 := 1 + \frac{2p_2 - 1}{2(1+p_1-p_2)} = \frac{1+2p_1}{2(1+p_1-p_2)}.$$

Recall that in this paper only **integer** values of q are used. However, q_0 introduced above is not necessarily an integer. In any case, the inequality $q_0 > 1$ is equivalent to $p_2 > 1/2$, which is necessary and sufficient for the finiteness of the first v^1. This proves the last statement of the theorem.

4. Examples

In this section, two examples are provided with drift functions that do not satisfy condition (7) or even (9), but for which condition (11) holds. Therefore, Theorem 1 is applicable. The dimension is equal to one. It suffices to construct the potential U for $x \ge 0$; then we extend it to the negative half-line by the formula $U(-x) = -U(x)$.

Example 1. *First of all, let*

$$U(x) \equiv 0, \quad |x| \le 1/2.$$

Further, let $p > 1/2$, and for $x > 1$,

$$U'(x) = -\frac{p}{x} + \sin x,$$

and let U' be smooth between $1/2$ and 1. This function, clearly, does not satisfy condition (7), because the (finite) limit $\limsup_{|x|\to\infty} xU'(x)$ does not exist. So, the results of Refs. [1,15] are not applicable. However, after integration for $x > 1$ we have,

$$U(x) = U(1) - p\ln x + \cos 1 - \cos x.$$

Recall that for negative values of x we pose $U(x) = -U(-x)$. Here $V(\xi) = \bar{V}(\xi) = U(|\xi|)$, the former equality because $d - 1 = 0$. Condition (9) is satisfied, as we have

$$\lim_{\xi\to\infty}\frac{V(\xi)}{\ln\xi} = \lim_{\xi\to\infty}\frac{\bar{V}(\xi)}{\ln\xi} = -p.$$

Theorem 1 is applicable with any $k < p + 1/2$ and with $p_1 = p_2 = p$.

Example 2. *Let $(x_n,\ n \geq 0)$ be a sequence of real numbers satisfying $x_0 = 1$ and*

$$\ln x_{n+1} \gg \ln x_n : \quad \textit{for example,} \quad x_n = \exp(\exp(n^2)),\ n \geq 0.$$

Further, let us choose $1/2 < p_2 < p_1$. Let $U(x)$ be a smooth and even function on the interval $[-1,1]$ such that $U(1) = 0$, $U'(1) = -p_1 + \sin 1$, and $U''(1) = p_1 + \cos 1$.

For $1 = x_0 \leq x \leq x_1$ let

$$U'(x) = -\frac{p_1}{x} + \sin x;$$

then by induction

$$U'(x) = -\frac{p_2}{x} + \sin x, \quad x_{2n+1} < x \leq x_{2n+2},$$

and

$$U'(x) = -\frac{p_1}{x} + \sin x, \quad x_{2n} < x \leq x_{2n+1},\ n \geq 0.$$

It is easily seen that after integration we get,

$$-p_1 \ln x + \cos x \leq U(x) \leq -p_2 \ln x + \cos x, \quad x \geq 1,$$

or, equivalently,

$$-p_1 + \frac{\cos x}{\ln x} \leq \frac{U(x)}{\ln x} \leq -p_2 + \frac{\cos x}{\ln x}, \quad x \geq 1.$$

So, if $0 < \delta < (p_1 - p_2)/2$, then for x large enough we have,

$$-(p_1 + \delta) \leq \frac{U(x)}{\ln x} \leq -(p_2 - \delta).$$

Hence, the assumptions of theorem 1 are met with any couple $(\tilde{p}_1, \tilde{p}_2)$ such that $1/2 < \tilde{p}_2 < \tilde{p}_1$, which is arbitrarily close to (p_1, p_2). At the same time, due to the construction, the function $\dfrac{U(x)}{\ln x}$ has no limit as $x \to \infty$, since $\ln x_{n+1} \gg \ln x_n$. So, no earlier theorem is applicable.

5. Discussion

In most of the earlier papers on the recurrent properties of solutions of SDEs, Lyapunov functions were used to establish such properties. These Lyapunov functions are often assumed "ad hoc", or are derived from some conditions on the coefficients of an SDE. Here the method does not use Lyapunov functions at all because it is totally unclear how to construct them in the case under the consideration. Instead, the approach is based on a comparison idea and on the system of ordinary differential equations, as in Theorem 13.17 in Ref. [26]; the recurrence bounds do not use any norm of the drift itself, but only some of its integral characteristics. To the best of the author's knowledge the only previous result of this sort was established by himself in 2000 (in Russian) and 2001 (English translation) [2]. Some *other properties* related to the convergence of SDE solutions to a limit, when the coefficients converge in a weak integral sense, were established in the works by S.Ya. Makhno [5–7]. Although these works concern different issues from ours, the main idea is similar: certain "standard" or "usual" assumptions for various recurrence and stability properties of SDE solutions in some cases may be replaced by their "integral" versions. An assumption such as (9) may be relaxed to an assumption such as (11), which is, clearly, much more general, while the guaranteed recurrence bound is similar; see Remark 3.

The assumptions of Ref. [2] were considerably extended in this paper; see Remarks 2 and 3. In addition, certain corrections were made concerning the same paper [2]; see Remark 1. In further studies in this direction the author aims to extend or waive the assumptions of the central symmetry on the drift to include a multiplicative Wiener noise, and possibly to work with non-integer values of q.

Funding: This research was funded by the Foundation for the Advancement of Theoretical Physics and Mathematics "BASIS".

Data Availability Statement: Not applicable.

Acknowledgments: The author thanks the anonymous referees for very useful remarks.

Conflicts of Interest: The author declares no conflict of interest. The funders had no role in the design of the study, in the writing of the manuscript, and in the decision to publish the results.

Abbreviations

The following abbreviations are used in this manuscript:

SDE stochastic differential equation

Appendix A. On Comparison Inequality for 1D SDEs with and without Reflection

Here, a rigorous proof of the inequality (18) is presented. The author initially considered it "common knowledge", and provided a brief explanation after the statement of the lemma. However, an anonymous referee suggested that a formal proof would be appropriate.

Lemma A1. *Under the assumptions of Theorem 1*

$$\mathsf{P}(|X_t| \le y_t,\, t \ge 0) = 1. \tag{A1}$$

Firstly, let us present the intuitive idea. As it was assumed, $|X_0| < y_0$. Both trajectories are continuous, so $|X_t| < y_t$, at least, in some neighborhood if $t = 0$. Let $T := \inf(t \ge 0 : |X_t| > y_t)$ (assuming $\inf(\emptyset) = \infty$). Clearly, $|X_T| = y_T$ on $T < \infty$. Note that T is a stopping time. Hence, if $y_T > K$, then in some right neighborhood of T the trajectories of y_t and $|X_t|$ must coincide due to the uniqueness of the solution of the SDE (19) and because of the strong Markov property of both processes X_t and (y_t), at least, until the first hitting of the barrier K after time T, let us denote this new stopping time by T': $T' := \inf(t \ge T : y_t = K)$. After T' the trajectory of y_t will stay in the area $[K, \infty)$, while the trajectory of $|X_t|$ with probability one will visit the interval $(0, K)$ infinitely many times in any right neighborhood of T'. Hence, it *appears* impossible that at any moment after T' it may occur that $|X_t| > y_t$. This will be formally demonstrated in the following lines using the Yamada and Watanabe method.

Proof of Lemma A1. The reasoning above is, in fact, fully strict until T'. However, after T' it becomes intuitive and we need more accurate arguments. As mentioned, such arguments may be based on the method from Ref. [28], which will be combined with a simplified version of Zvonkin's transformation [29] adjusted to the case under the consideration. Notice that the difficulty with the simple reasoning shown above is only around the level K. Indeed, we have seen that the process $|X_t|$ may not exceed the trajectory of y_t for the first time at any time t where $y_t > K$. Therefore, *it suffices* to prove that

$$\mathsf{P}(|X_t| \le y_t,\, 0 \le t \le \hat{T}) = 1, \tag{A2}$$

assuming that $|X_0| = y_0 = K$, where

$$\hat{T} := \inf(t \ge 0 : |X_t| \vee y_t \ge K + 1, \text{ or } |X_t| \wedge y_t \le \delta),$$

with any $0 < \delta < K$. Recall that the function $\bar{V}'$ exists and is bounded on the interval $[\delta, K+1]$; hence, $\bar{V}$ is Lipschitz on this interval. Let

$$u(y) = \int_0^y \exp(-2\bar{V}(r))dr.$$

Then

$$u'(y) = \exp(-2\bar{V}(y)), \quad u''(y) = -2\bar{V}'(y)\exp(-2\bar{V}(y)),$$

and a simple calculation shows that the equation holds true,

$$\bar{L}u(y) = 0, \; y > 0,$$

where $\bar{L}$ is the generator of the process (y_t); see (26).

In addition, the function u is strictly increasing. Hence, its inverse u^{-1} does exist and is also strictly increasing; moreover, the derivative functions u' and $(u^{-1})'$ are bounded and bounded away from zero on $[\delta, K+1]$. Let

$$Y_t := u(y_t), \quad R_t := u(|X_t|), \quad t \geq 0.$$

Recall that the case $y_0 = |X_0| = K$ is considered. By Itô-Krylov's formula (because in general $\bar{V}'$ may not be continuous and, hence, the solution u is reasonable to consider in the Sobolev sense, Chapter 4 in [30]) we have,

$$dY_t = \exp(-2\bar{V}(y_t))d\bar{w}_t + \exp(-2\bar{V}(y_t))d\varphi_t, \quad dR_t = \exp(-2\bar{V}(|X_t|))d\bar{w}_t.$$

There is no drift in both equations here due to the choice of the function u; this is the standing idea of Zvonkin's transformation of the state space.

Using the inverse function u^{-1}, the equations on Y_t and R_t may be rewritten in the form

$$dY_t = \underbrace{\exp(-2\bar{V}(u^{-1}(Y_t)))}_{=:\Sigma(Y_t)} d\bar{w}_t + \exp(-2\bar{V}(u^{-1}(Y_t)))d\varphi_t, \quad dR_t = \exp(-2\bar{V}(u^{-1}(R_t))d\bar{w}_t.$$

where the function $\Sigma(y) := \exp(-2\bar{V}(u^{-1}(Y_t)))$ is non-negative, Lipshcitz on $[\delta, K+1]$, and bounded along with $1/\Sigma(y)$.

Consider a sequence of smooth (C^∞) functions $0 \leq \psi_n(y)$, $y \in \mathbb{R}$ and a sequence of real numbers $a_1 \geq \ldots \geq a_n \geq \ldots \geq 0$ with the following properties:

$$a_n \downarrow 0, \quad n \to \infty,$$

and

$$\psi_n(y) = 0, \; y \leq 0; \quad 0 \leq \psi_n'(y) \leq 1; \quad \psi_n(y) \uparrow y_+ \equiv y \times 1(y > 0), \; n \to \infty,$$
$$\psi_n'(y) = 1, \; y \geq a_{n-1}, \;\; \psi_n'(y) = 0, \; y \leq a_n; \;\; 0 \leq y\psi_n''(y) \leq \frac{2}{n}, \; \forall\, y > 0; \;\; \psi''(y) = 0, \quad y \leq 0.$$

This is a modified sequence of functions and real numbers from Ref. [28] where the modification is just a multiplication of each $\psi_n(y)$ by the indicator $1(y > 0)$. Now let us apply Itô's formula to the expression $\psi_n(R_t - Y_t)$ on the set $t \leq \tilde{T} := \inf(s \geq 0 : y_s \vee |X_s| \geq K+1, \text{ or } |X_s| \leq \delta)$. Note that $\mathsf{P}(\tilde{T} > 0) = 1$. We have,

$$d\psi_n(R_t - Y_t) = \psi_n'(R_t - Y_t)d(R_t - Y_t) + \frac{1}{2}\psi_n''(R_t - Y_t)(d(R_t - Y_t))^2$$
$$= \psi_n'(R_t - Y_t)(\sigma(R_t) - \sigma(Y_t))d\bar{w}_t - \psi_n'(R_t - Y_t)\sigma(Y_t)d\varphi_t + \frac{1}{2}\psi_n''(R_t - Y_t)(\sigma(R_t) - \sigma(Y_t))^2 dt.$$

By integrating and taking expectations, we obtain

$$\mathsf{E}\psi_n(R_{t\wedge\tilde{T}} - Y_{t\wedge\tilde{T}}) = -\mathsf{E}\int_0^{t\wedge\tilde{T}} \psi_n'(R_s - Y_s)\sigma(Y_s)d\varphi_s + \frac{1}{2}\mathsf{E}\int_0^{t\wedge\tilde{T}} \psi_n''(R_s - Y_s)(\sigma(R_s) - \sigma(Y_s))^2 ds$$
$$\leq -\mathsf{E}\int_0^{t\wedge\tilde{T}} \psi_n'(R_s - Y_s)\sigma(Y_s)d\varphi_s + C\mathsf{E}\int_0^{t\wedge\tilde{T}} \psi_n''(R_s - Y_s)(R_s - Y_s)^2 ds$$

Here the first term in the right hand side is non-positive (because of the minus sign), while the second one is bounded and uniformly goes to zero due to the property $0 \le y\psi_n''(y) \rightrightarrows 0$, $n \to \infty$. The left hand side here tends to $\mathsf{E}(R_{t\wedge\tilde T} - Y_{t\wedge\tilde L})1(R_{t\wedge\tilde T} - Y_{t\wedge\tilde T} > 0) = \mathsf{E}(R_{t\wedge\tilde T} - Y_{t\wedge\tilde T})_+$. Thus, in the limit, as $n \to \infty$, we obtain

$$\mathsf{E}(R_{t\wedge\tilde T} - Y_{t\wedge\tilde T})1(R_{t\wedge\tilde T} - Y_{t\wedge\tilde T} > 0) = 0.$$

This straightforwardly implies that

$$\mathsf{P}(R_{t\wedge\tilde T} - Y_{t\wedge\tilde T} > 0) = 0 \quad \Longleftrightarrow \quad P(R_{t\wedge\tilde T} \le Y_{t\wedge\tilde T}) = 1 \quad \Longleftrightarrow \quad P(|X_{t\wedge\tilde T}| \le y_{t\wedge\tilde T}) = 1.$$

This establishes (A2). So, Lemma A1 is proved. This justifies (20), as promised. □

References

1. Veretennikov, A.Y. On polynomial mixing and convergence rate for stochastic difference and differential equations. *Theory Probab. Appl.* **2000**, *44*, 361–374. [CrossRef]
2. Veretennikov, A.Y. On polynomial mixing for SDEs with a gradient-type drift. *Theory Probab. Appl.* **2001**, *45*, 160–164. [CrossRef]
3. Portenko, N.I. Stochastic differential equations with generalized drift vector. *Theory Probab. Appl.* **1979**, *24*, 338–353. [CrossRef]
4. Portenko, N.I. *Generalized Diffusion Processes, Kiev, Naukova Dumka, 1982 (In Russian)*; Eng. Transl.; American Mathematical Society: Providence, RI, USA, 1990. [CrossRef]
5. Makhno, S.Y. Convergence of solutions of one-dimensional stochastic equations. *Theory Probab. Appl.* **2000**, *44*, 495–510. [CrossRef]
6. Makhno, S.Y. Limit theorem for one-dimensional stochastic equations. *Theory Probab. Appl.* **2004**, *48*, 164–169. [CrossRef]
7. Makhno, S.Y. *Stochastic Equations. Limit Theorems*; Naukova Dumka: Kyiv, Ukraine, 2012. (In Russian)
8. Ferrari, P.L.; Spohn, H. Constrained Brownian motion: Fluctuations away from circular and parabolic barriers. *Ann. Probab.* **2005**, *33*, 1302–1325. [CrossRef]
9. Ferrari, P.L.; Shlosman, S. The Airy$_2$ process and the 3D Ising model. *Phys. Math. Theor.* **2023**, *56*, 014003. [CrossRef]
10. Flandoli, F.; Russo, F.; Wolf, J. Some SDEs with distributional drift. I. General calculus. *Osaka J. Math.* **2003**, *40*, 493–542.
11. Delarue, F.; Diel, R. Rough paths and 1d SDE with a time dependent distributional drift: Application to polymers. *Probab. Theory Relat. Fields* **2016**, *165*, 1–63. [CrossRef]
12. Flandoli, F.; Issoglio, E.; Russo, F. Multidimensional stochastic differential equations with distributional drift. *Trans. Am. Math. Soc.* **2017**, *369*, 1665–1688. [CrossRef]
13. Russo, F.; Trutnau, G. Some parabolic PDEs whose drift is an irregular random noise in space. *Ann. Probab.* **2007**, *35*, 2213–2262. [CrossRef]
14. Kalashnikov, V.V. The property of γ-reflexivity for Markov sequences. *Sov. Math. Dokl.* **1973**, *14*, 1869–1873.
15. Veretennikov, A.Y. On polynomial mixing bounds for stochastic differential equations. *Stoch. Process. Appl.* **1997**, *70*, 115–127. [CrossRef]
16. Ibragimov, I.A.; Linnik, Y.V. *Independent and Stationary Sequences of Random Variables*; Wolters-Noordhoff Publishing: Groningen, The Netherlands, 1971.
17. Leadbetter, M.R.; Lindgren, G.; Rootzen, H. *Extremes and Related Properties of Random Sequences and Processes*; Springer: New York, NY, USA, 1982.
18. Aspandiiarov, S.; Iasnogorodski, R.; Menshikov, M. Passage-time moments for nonnegative stochastic processes and an application to reflected random walks in a quadrant. *Ann. Probab.* **1996**, *24*, 932–960. [CrossRef]
19. Lamperti, J. Criteria for stochastic processes II: Passage-time moments. *Math. Anal. Appl.* **1963**, *7*, 127–145. [CrossRef]
20. Menshikov, M.; Williams, R.J. Passage-time moments for continuous non-negative stochastic processes and applications. *Adv. Appl. Probab.* **1996**, *28*, 747–762. [CrossRef]
21. Ganidis, H.; Roynette, B.; Simonot, F. Convergence rate of some semi-groups to their invariant probability. *Stoch. Process. Appl.* **1999**, *79*, 243–263. [CrossRef]
22. Malyshkin, M.N. Subexponential estimates of the rate of convergence to the invariant measure for stochastic differential equations. *Theory Probab. Appl.* **2001**, *45*, 466–479. [CrossRef]
23. Gilbarg, D.; Trudinger, N.S. *Elliptic Partial Differential Equations of Second Order*, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 1983.
24. Lieberman, G.M. *Second Order Parabolic Differential Equations*; World Scientific: Singapore, 1996.
25. Veretennikov, A.J. On strong solutions and explicit formulas for solutions of stochastic integral equations. *Math. USSR Sb.* **1981**, *39*, 387–403. [CrossRef]
26. Dynkin, E.B. *Markov Processes*; Academic Press: New York, NY, USA, 1965.
27. Veretennikov, A.Y. On strong and weak solutions of one–dimensional stochastic equations with boundary conditions. *Theory Probab. Appl.* **1981**, *26*, 670–686. [CrossRef]
28. Yamada, T.S. Watanabe: On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ.* **1971**, *11*, 155–167.

29. Zvonkin, A.K. A transformation of the phase space of a diffusion process that removes the drift. *Math. USSR Sb.* **1974**, *22*, 129–149. [CrossRef]
30. Ladyzhenskaya, O.A.; Solonnikov, V.A.; Ural'tseva, N.N. *Linear and Quasi-Linear Equations of Parabolic Type*; AMS: Providence, RI, USA, 1968.