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On the Stabilization of the Solution to the Initial Boundary Value Problem for One-Dimensional Isothermal Equations of Viscous Compressible Multicomponent Media Dynamics

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Abstract: An initial boundary value problem is considered for one-dimensional isothermal equations of the dynamics of viscous compressible multicomponent media, which are a generalization of the Navier–Stokes equations. The stabilization of the solution to the initial boundary value problem is proven while the time tends to infinity, without simplifying assumptions for the structure of the viscosity matrix, except for the standard physical requirements of symmetry and positive definiteness. It is shown that the stabilization of the solution is exponential.

Keywords: viscous compressible medium; multicomponent flows; stabilization of solution; stabilization rate

MSC: 76N10; 76T99

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1. Introduction

Both physics and mathematics are concerned with constructing models for the motion of multicomponent media and solving mathematical problems that arise. Nowadays, this is a relevant area of research, yet the field is little studied. No unified approach has been developed so far, and there is no advanced mathematical theory about the existence, uniqueness, and properties of solutions to initial boundary value problems that arise in constructing models. A detailed review of this problem can be found in monographs [1,2] and in papers [3,4]. This paper is aimed at a mathematical study, but in view of the physical grounds of the problem it is necessary to give some brief explanations of the mechanical meaning of certain assertions.

In this work, one of the numerous options for constructing models of movement of multicomponent media, namely a multi-velocity model, is chosen. This means that at each point of space there are all components of the medium that are in the same phase, but each has its own local velocity. The interaction between the components can occur through viscous friction and exchange of momentum, as well as through heat transfer, in the heat conductive version. The model mentioned represents a certain generalization of the well-known system of Navier–Stokes equations of the dynamics of a one-component viscous compressible medium and includes the equations of continuity and momentum, as well as the energy equation in the heat conductive case. The characteristic feature of these equations, in addition to their nonlinearity, is the presence of higher order derivatives of the velocities of all components in the momentum (and energy) equations, due to the composite structure of the viscous stress tensors [1,2,5–9]. This specificity of multicomponent media can be described by means of the concept of viscosity matrices. Unlike the one-component case in which the viscosities are scalars, in the multicomponent case they form matrices whose entries describe viscous friction. Diagonal entries describe viscous friction within each component, and non-diagonal entries describe friction between the components. In the case of diagonal viscosity matrices, the momentum equations are probably connected via the lower order terms only. In this paper, the more complicated case of off-diagonal (filled) viscosity matrices is under consideration.

There are several important technological problems where multicomponent media with off-diagonal viscosity matrices are used. For example, one can mention a variety of problems involving lubrication in machining operations such as cold rolling, milling, cutting and grinding. Mixture theory has been used to study a variety of lubrication problems: the problem of cold rolling by Wang et al. [10], lubrication with bubbly oils (i.e., oil–gas mixtures) by Chamniprasart et al. [11], lubrication of an elastohydrodynamic conjunction by Wang et al. [12] and the flow of oil–water mixtures in a journal bearing by Al-Sharif et al. [13]. A detailed discussion of lubrication with liquid–liquid and liquid–gas mixtures can be found in the book by Szeri [14]. The goal of this research is to conduct a mathematical study of the problem for the equations of the multicomponent media dynamics with off-diagonal viscosity matrices, that is, to investigate the asymptotic behavior of a non-stationary solution as $t \to +\infty$ to the initial boundary value problem for isothermal (non heat conductive) equations of viscous compressible multicomponent media in the case of one-dimensional movement in a bounded domain with impenetrable boundaries. Therefore, known results in this area should be discussed.

As mentioned above, the multi-velocity model of the viscous compressible multicomponent media dynamics under consideration is a certain generalization of the well-known Navier–Stokes system of equations, which describes flows of viscous compressible one-component media and, of course, mathematical results for the viscous compressible multicomponent media appeared after a certain progress for the Navier–Stokes equations had been made. The stabilization of solutions to one-dimensional Navier–Stokes equations is studied in [15–22]. The unique solvability and asymptotic behavior (as $t \to +\infty$) of the solution to the considered one-dimensional equations of viscous compressible multicomponent media in the polytropic case are studied in [23–25]. Similar questions for related one-dimensional models of multicomponent media are discussed in [26–36]. Thus, the novelty of the paper consists of the immediate consideration of such diverse factors as multicomponent medium, viscosity, compressibility and intercomponent viscous friction with off-diagonal viscosity matrices. Additional information and prospects can be taken from [37–39].

Notations of functional spaces [40,41] are standard; $L_p(a,b)$ is the Banach space which consists of all functions measurable on (a,b) which are integrable on (a,b) with the power p normed as

$$\|u\|_{L_p(a,b)} = \left(\int\limits_a^b |u|^p \, dx \right)^{\frac{1}{p}};$$

$W_p^l(a,b)$ is the Banach space which consists of all elements of $L_p(a,b)$ which possess distributional derivatives up to order l integrable on (a,b) with the power p normed as

$$\|u\|_{W_p^l(a,b)} = \left(\sum_{i=0}^{l} \int\limits_a^b \left| \frac{\partial^i u}{\partial x^i} \right|^p dx \right)^{\frac{1}{p}};$$

$\overset{\circ}{W}{}_p^l(a,b)$ is the set of elements of $W_p^l(a,b)$ supported in (a,b); $C^k[a,b]$ is the Banach space which consists of all functions continuous in $[a,b]$ and which possess derivatives up to order k, continuous in $[a,b]$ and normed as

$$\|u\|_{C^k[a,b]} = \sum_{i=0}^{k} \sup_{[a,b]} \left| \frac{\partial^i u}{\partial x^i} \right|,$$

and, $C^0[a,b]$ is denoted as $C[a,b]$.

The structure of the work is as follows. Section 2 contains the statement of the initial boundary value problem and the formulation of the result of the article, namely, Theorem 1 on the stabilization of a non-stationary solution. In Section 3, a priori estimates that are not dependent on the time are constructed, on the basis of which, in Section 4, the stabilization of the solution is proved and the proof of Theorem 1 is completed. In Section 5, estimates of the rate of stabilization of the solution to the non-stationary solution are made and Theorem 2 is formulated, which states that the rate of stabilization is exponential. Section 6 presents a summary of the results of the work.

2. Statement of the Initial Boundary Value Problem and Formulation of the Result

The paper is aimed at a mathematical study of the asymptotic behavior of the solution at $t \to +\infty$ of the following initial boundary value problem for isothermal equations of viscous compressible multicomponent media in the case of one-dimensional movement in a bounded domain with impenetrable boundaries [1–4]:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \tag{1}$$

$$\rho \frac{\partial u_i}{\partial t} + \rho v \frac{\partial u_i}{\partial x} + K \frac{\partial \rho}{\partial x} = \sum_{j=1}^{N} \nu_{ij} \frac{\partial^2 u_j}{\partial x^2}, \quad i = 1, \ldots, N, \tag{2}$$

$$\rho|_{t=0} = \rho_0(x), \quad u_i|_{t=0} = u_{0i}(x), \quad i = 1, \ldots, N, \tag{3}$$

$$u_i|_{x=0} = u_i|_{x=1} = 0, \quad i = 1, \ldots, N, \tag{4}$$

where ρ, u_i, $i = 1, \ldots, N$ are the density and velocity of the medium components, respectively, and they are the sought functions of time $t \in [0, T]$, $0 < T < +\infty$, and of the point x of the flow domain $\Omega = \{x \in \mathbb{R} \mid 0 < x < 1\}$, $v = \frac{1}{N} \sum_{i=1}^{N} u_i$ is the average velocity of the medium, $N \geqslant 2$ is the number of components, $K = \text{const} > 0$, and constant viscosity coefficients ν_{ij}, $i, j = 1, \ldots, N$ form a symmetric matrix $\mathbf{N} > 0$.

The proof of the existence and uniqueness of the regular generalized solution to problem (1)–(4) in the domain $(0, T) \times (0, 1)$ with any final $T > 0$, for $\rho_0 \in W_2^1(0, 1)$, $\rho_0 \geqslant \text{const} > 0$, $u_{0i} \in \overset{\circ}{W}{}_2^1(0, 1)$, $i = 1, \ldots, N$, repeats, step by step, the proof of the unique solvability of the initial boundary value problem for the one-dimensional equations of viscous compressible multicomponent media in a polytropic case, presented in the works [23,24], taking into account the further a priori estimates.

It will be useful to resort to mass Lagrangian coordinates in the study of Equations (1) and (2). Let us consider t and $y(x, t) = \int_0^x \rho(s, t)\, ds$ as new independent variables. Then, Equations (1) and (2) take the form

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial v}{\partial y} = 0, \quad v = \frac{1}{N} \sum_{j=1}^{N} u_j, \tag{5}$$

$$\frac{\partial u_i}{\partial t} + K \frac{\partial \rho}{\partial y} = \sum_{j=1}^{N} \nu_{ij} \frac{\partial}{\partial y} \Big(\rho \frac{\partial u_j}{\partial y} \Big), \quad i = 1, \ldots, N. \tag{6}$$

After that, the flow domain Ω transforms to the domain $\widetilde{\Omega} = \{y \in \mathbb{R} \mid 0 < y < d\}$, where $d = \int_0^1 \rho_0(x)\, dx > 0$, and the initial and boundary conditions arrive at the form

$$\rho|_{t=0} = \widetilde{\rho}_0(y), \quad u_i|_{t=0} = \widetilde{u}_{0i}(y), \quad i = 1, \ldots, N, \tag{7}$$

$$u_i|_{y=0} = u_i|_{y=d} = 0, \quad i = 1, \ldots, N. \tag{8}$$

The result of the article is the following assertion.

Theorem 1. *Assume that* $\rho_0 \in W_2^1(0,1)$, $\rho_0 \geqslant \text{const} > 0$, $u_{0i} \in \overset{\circ}{W}{}_2^1(0,1)$, $i = 1, \ldots, N$. *Then,* $\rho \to d$, $u_i \to 0$, $i = 1, \ldots, N$, *as* $t \to +\infty$, *in the norm of the space* $W_2^1(0,1)$.

Proof. Let us obtain a priori estimates for the solution to the problem (1)–(4), which would be uniform in t, and from which the stabilization of the solution follows.

3. A Priori Estimates

Let us multiply Equation (2) by u_i, summarize in i and integrate in y; thus, using (1), the equality

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{N}\int_0^1 \rho u_i^2\,dx + \sum_{i,j=1}^{N}\nu_{ij}\int_0^1\left(\frac{\partial u_i}{\partial x}\right)\left(\frac{\partial u_j}{\partial x}\right)dx = -K\sum_{i=1}^{N}\int_0^1 u_i\left(\frac{\partial \rho}{\partial x}\right)dx \tag{9}$$

is obtained. Since $\mathbf{N} > 0$, the second summand in the left-hand side of (9) satisfies the inequality (everywhere, C_k, $k \in \mathbb{N}$ stand for various positive constants which depend on the values given in brackets, but do not depend on t)

$$\sum_{i,j=1}^{N}\nu_{ij}\int_0^1\left(\frac{\partial u_i}{\partial x}\right)\left(\frac{\partial u_j}{\partial x}\right)dx \geqslant C_1(\mathbf{N})\sum_{i=1}^{N}\int_0^1\left(\frac{\partial u_i}{\partial x}\right)^2 dx. \tag{10}$$

Multiplying (1) by $\ln\rho - \ln d$ and integrating in x, we obtain for the right-hand side of (9)

$$-K\sum_{i=1}^{N}\int_0^1 u_i\left(\frac{\partial \rho}{\partial x}\right)dx = KN\sum_{i=1}^{N}\int_0^1 \rho\left(\frac{\partial v}{\partial x}\right)dx = -KN\frac{d}{dt}\int_0^1(\rho\ln\rho - (\ln d + 1)\rho + d)\,dx. \tag{11}$$

Thus, using (10) and (11), from (9) we derive

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{N}\int_0^1 \rho u_i^2\,dx + KN\frac{d}{dt}\int_0^1(\rho\ln\rho - (\ln d + 1)\rho + d)\,dx + C_1\sum_{i=1}^{N}\int_0^1\left(\frac{\partial u_i}{\partial x}\right)^2 dx \leqslant 0. \tag{12}$$

In the Lagrangian coordinates (t, y), the formula (12) looks like

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{N}\int_0^d u_i^2\,dy + KN\frac{d}{dt}\int_0^d \frac{1}{\rho}(\rho\ln\rho - (\ln d + 1)\rho + d)\,dy + C_1\sum_{i=1}^{N}\int_0^d \rho\left(\frac{\partial u_i}{\partial y}\right)^2 dy \leqslant 0. \tag{13}$$

From (13), after integration in t, we obtain the estimate

$$\sum_{i=1}^{N}\int_0^d u_i^2 dy + \int_0^d \frac{\rho\ln\rho - (\ln d + 1)\rho + d}{\rho}\,dy + \sum_{i=1}^{N}\int_0^t\int_0^d \rho\left(\frac{\partial u_i}{\partial y}\right)^2 dy d\tau \leqslant C_2, \tag{14}$$

where $C_2 = C_2\big(C_1, K, N, d, \inf\limits_{(0,d)}\widetilde{\rho}_0, \sup\limits_{(0,d)}\widetilde{\rho}_0, \{\|\widetilde{u}_{0i}\|_{L_2(0,d)}\}\big)$. Note that all terms in the left-hand side of (14) are nonnegative.

Let us rewrite Equation (6) as

$$\frac{1}{N}\sum_{j=1}^{N}\widetilde{\nu}_{ij}\frac{\partial u_j}{\partial t} + \frac{K}{N}\left(\sum_{j=1}^{N}\widetilde{\nu}_{ij}\right)\frac{\partial \rho}{\partial y} = \frac{1}{N}\frac{\partial}{\partial y}\left(\rho\frac{\partial u_i}{\partial y}\right), \quad i = 1, \ldots, N, \tag{15}$$

where $\widetilde{\nu}_{ij}$, $i, j = 1, \ldots, N$ are the entries of the matrix $\widetilde{\mathbf{N}} = \mathbf{N}^{-1}$. Summing (15) in i, we obtain

$$\frac{1}{N}\sum_{i,j=1}^{N}\widetilde{\nu}_{ij}\frac{\partial u_j}{\partial t} + \widetilde{K}\frac{\partial \rho}{\partial y} = \frac{\partial}{\partial y}\left(\rho\frac{\partial v}{\partial y}\right), \tag{16}$$

where $\widetilde{K} = \dfrac{K}{N}\left(\sum\limits_{i,j=1}^{N}\widetilde{\nu}_{ij}\right) > 0$. Equation (5) implies that

$$\rho\frac{\partial v}{\partial y} = -\frac{\partial \ln\rho}{\partial t}. \tag{17}$$

Substituting $\rho\dfrac{\partial v}{\partial y}$ from (17) into the equality (16), we arrive at the relation

$$\frac{\partial^2 \ln\rho}{\partial t\partial y} + \widetilde{K}\frac{\partial \rho}{\partial y} = -\frac{1}{N}\sum_{i,j=1}^{N}\widetilde{\nu}_{ij}\frac{\partial u_j}{\partial t}. \tag{18}$$

Let us multiply (18) by $\dfrac{\partial \ln\rho}{\partial y}$ and integrate in y, then we obtain the equality

$$\frac{1}{2}\frac{d}{dt}\int_0^d\left(\frac{\partial \ln\rho}{\partial y}\right)^2 dy + \widetilde{K}\int_0^d \rho\left(\frac{\partial \ln\rho}{\partial y}\right)^2 dy = -\frac{1}{N}\sum_{i,j=1}^{N}\widetilde{\nu}_{ij}\int_0^d\left(\frac{\partial u_j}{\partial t}\right)\left(\frac{\partial \ln\rho}{\partial y}\right) dy. \tag{19}$$

The right-hand side of (19) can be converted via integration by parts and using (8) and (17):

$$\begin{aligned}-\frac{1}{N}\sum_{i,j=1}^{N}\widetilde{\nu}_{ij}\int_0^d\left(\frac{\partial u_j}{\partial t}\right)\left(\frac{\partial \ln\rho}{\partial y}\right) dy = &-\frac{d}{dt}\left(\frac{1}{N}\sum_{i,j=1}^{N}\widetilde{\nu}_{ij}\int_0^d u_j\left(\frac{\partial \ln\rho}{\partial y}\right) dy\right)\\ &+\frac{1}{N}\sum_{i,j=1}^{N}\widetilde{\nu}_{ij}\int_0^d \rho\left(\frac{\partial u_j}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right) dy.\end{aligned} \tag{20}$$

Hence, from (19), it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^d\left(\frac{\partial \ln\rho}{\partial y}\right)^2 dy + \widetilde{K}\int_0^d \rho\left(\frac{\partial \ln\rho}{\partial y}\right)^2 dy = &-\frac{d}{dt}\left(\frac{1}{N}\sum_{i,j=1}^{N}\widetilde{\nu}_{ij}\int_0^d u_j\left(\frac{\partial \ln\rho}{\partial y}\right) dy\right)\\ &+\frac{1}{N}\sum_{i,j=1}^{N}\widetilde{\nu}_{ij}\int_0^d \rho\left(\frac{\partial u_j}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right) dy.\end{aligned} \tag{21}$$

Let us integrate (21) in t:

$$\begin{aligned}&\frac{1}{2}\int_0^d\left(\frac{\partial \ln\rho}{\partial y}\right)^2 dy + \widetilde{K}\int_0^t\int_0^d \rho\left(\frac{\partial \ln\rho}{\partial y}\right)^2 dy d\tau\\ &= \frac{1}{2}\int_0^d\frac{1}{\widetilde{\rho}_0^2}\left(\frac{\partial \widetilde{\rho}_0}{\partial y}\right)^2 dy - \frac{1}{N}\sum_{i,j=1}^{N}\widetilde{\nu}_{ij}\int_0^d u_j\left(\frac{\partial \ln\rho}{\partial y}\right) dy\\ &\quad+\frac{1}{N}\sum_{i,j=1}^{N}\widetilde{\nu}_{ij}\int_0^d\frac{\widetilde{u}_{0j}}{\widetilde{\rho}_0}\left(\frac{\partial \widetilde{\rho}_0}{\partial y}\right) dy + \frac{1}{N}\sum_{i,j=1}^{N}\widetilde{\nu}_{ij}\int_0^t\int_0^d \rho\left(\frac{\partial u_j}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right) dy d\tau.\end{aligned}$$

The last equality and (14) lead to the estimate

$$\int_0^d \left(\frac{\partial \ln\rho}{\partial y}\right)^2 dy + \int_0^t\int_0^d \rho\left(\frac{\partial \ln\rho}{\partial y}\right)^2 dyd\tau \leqslant C_3, \tag{22}$$

where $C_3 = C_3\Big(C_2, \widetilde{\mathbf{N}}, \widetilde{K}, N, \inf\limits_{(0,d)}\widetilde{\rho}_0, \|\widetilde{\rho}_0\|_{W_2^1(0,d)}, \{\|\widetilde{u}_{0i}\|_{L_2(0,d)}\}\Big)$.

Let us note that Equation (5) implies, for every $t \in [0,T]$, the existence of a point $\widetilde{y}(t) \in [0,d]$ such that

$$\rho(t, \widetilde{y}(t)) = d. \tag{23}$$

Since

$$\ln\rho(t,y) = \ln\rho(t,\widetilde{y}(t)) + \int_{\widetilde{y}(t)}^{y} \frac{\partial(\ln\rho(t,s))}{\partial s}\, ds,$$

then by the Hölder inequality, taking into account (22) and (23), we deduce that

$$|\ln\rho(y,t)| \leqslant |\ln d| + \sqrt{d}\left\|\frac{\partial \ln\rho}{\partial y}\right\|_{L_2(0,d)} \leqslant C_4(C_3, d),$$

and we immediately conclude that

$$0 < \frac{1}{C_5} \leqslant \rho(t,y) \leqslant C_5 < +\infty, \quad C_5 = C_5(C_4). \tag{24}$$

Thus, from (14), (22) and (24), we obtain the estimate

$$\sum_{i=1}^N \int_0^t\int_0^d \left(\frac{\partial u_i}{\partial y}\right)^2 dyd\tau + \int_0^d \left(\frac{\partial \rho}{\partial y}\right)^2 dy + \int_0^t\int_0^d \left(\frac{\partial \rho}{\partial y}\right)^2 dyd\tau \leqslant C_6(C_2, C_3, C_5). \tag{25}$$

In order to derive the next group of estimates, we multiply (6) by $\dfrac{\partial^2 u_i}{\partial y^2}$ and integrate in y; thus, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^d \left(\frac{\partial u_i}{\partial y}\right)^2 dy + \sum_{j=1}^N \nu_{ij}\int_0^d \rho\left(\frac{\partial^2 u_i}{\partial y^2}\right)\left(\frac{\partial^2 u_j}{\partial y^2}\right) dy = -\sum_{j=1}^N \nu_{ij}\int_0^d \left(\frac{\partial \rho}{\partial y}\right)\left(\frac{\partial^2 u_i}{\partial y^2}\right)\left(\frac{\partial u_j}{\partial y}\right) dy + K\int_0^d \left(\frac{\partial \rho}{\partial y}\right)\left(\frac{\partial^2 u_i}{\partial y^2}\right) dy. \tag{26}$$

Let us sum Equation (26) over i and integrate over t, then we obtain

$$\frac{1}{2}\sum_{i=1}^N \int_0^d \left(\frac{\partial u_i}{\partial y}\right)^2 dy + \sum_{i,j=1}^N \nu_{ij}\int_0^t\int_0^d \rho\left(\frac{\partial^2 u_i}{\partial y^2}\right)\left(\frac{\partial^2 u_j}{\partial y^2}\right) dyd\tau = \frac{1}{2}\sum_{i=1}^N \int_0^d \left(\frac{\partial \widetilde{u}_{0i}}{\partial y}\right)^2 dy - \sum_{i,j=1}^N \nu_{ij}\int_0^t\int_0^d \left(\frac{\partial \rho}{\partial y}\right)\left(\frac{\partial^2 u_i}{\partial y^2}\right)\left(\frac{\partial u_j}{\partial y}\right) dyd\tau + K\sum_{i=1}^N \int_0^t\int_0^d \left(\frac{\partial \rho}{\partial y}\right)\left(\frac{\partial^2 u_i}{\partial y^2}\right) dyd\tau. \tag{27}$$

The left-hand side of (27) satisfies the estimate

$$\frac{1}{2}\sum_{i=1}^N \int_0^d \left(\frac{\partial u_i}{\partial y}\right)^2 dy + \sum_{i,j=1}^N \nu_{ij}\int_0^t\int_0^d \rho\left(\frac{\partial^2 u_i}{\partial y^2}\right)\left(\frac{\partial^2 u_j}{\partial y^2}\right) dyd\tau \geqslant C_7(C_1, C_5)\left(\sum_{i=1}^N \int_0^d \left(\frac{\partial u_i}{\partial y}\right)^2 dy + \sum_{i=1}^N \int_0^t\int_0^d \left(\frac{\partial^2 u_i}{\partial y^2}\right)^2 dyd\tau\right). \tag{28}$$

Due to (25) and the inequalities ($i = 1, \ldots, N$)

$$\left\| \frac{\partial u_i}{\partial y} \right\|^2_{C[0,d]} \leqslant 2 \left\| \frac{\partial u_i}{\partial y} \right\|_{L_2(0,d)} \left\| \frac{\partial^2 u_i}{\partial y^2} \right\|_{L_2(0,d)}, \tag{29}$$

the second summand in the right-hand side of (27) satisfies the estimate

$$-\sum_{i,j=1}^{N} \nu_{ij} \int_0^t \int_0^d \left(\frac{\partial \rho}{\partial y}\right)\left(\frac{\partial^2 u_i}{\partial y^2}\right)\left(\frac{\partial u_j}{\partial y}\right) dy d\tau \leqslant \frac{C_7}{4} \sum_{i=1}^{N} \int_0^t \int_0^d \left(\frac{\partial^2 u_i}{\partial y^2}\right)^2 dy d\tau + C_8, \tag{30}$$

where $C_8 = C_8(C_6, C_7, \mathbf{N}, N)$. The third summand in the right-hand side of (27) can be estimated as follows:

$$K \sum_{i=1}^{N} \int_0^t \int_0^d \left(\frac{\partial \rho}{\partial y}\right)\left(\frac{\partial^2 u_i}{\partial y^2}\right) dy d\tau \leqslant \frac{C_7}{4} \sum_{i=1}^{N} \int_0^t \int_0^d \left(\frac{\partial^2 u_i}{\partial y^2}\right)^2 dy d\tau + C_9(C_6, C_7, K, N). \tag{31}$$

Thus, from (27), due to (28), (30) and (31), the estimate

$$\sum_{i=1}^{N} \int_0^d \left(\frac{\partial u_i}{\partial y}\right)^2 dy + \sum_{i=1}^{N} \int_0^t \int_0^d \left(\frac{\partial^2 u_i}{\partial y^2}\right)^2 dy d\tau \leqslant C_{10}\left(C_7, C_8, C_9, \left\{ \|\widetilde{u}_{0i}\|_{W_2^1(0,d)} \right\}\right) \tag{32}$$

follows. Finally, integrating (26) in t, we arrive at the inequalities

$$\int_0^t \left| \frac{d}{d\tau} \int_0^d \left(\frac{\partial u_i}{\partial y}\right)^2 dy \right| d\tau \leqslant C_{11}(C_5, C_6, C_{10}, \mathbf{N}, K, N), \quad i = 1, \ldots, N. \tag{33}$$

4. Stabilization of the Solution with an Unlimited Increase in Time

From (25) and (33), the following convergence follows

$$\left\| \frac{\partial u_i}{\partial y} \right\|_{L_2(0,d)} \to 0, \quad i = 1, \ldots, N \tag{34}$$

as $t \to +\infty$. Differentiating (5) with respect to y and multiplying by $\dfrac{\partial \rho}{\partial y}$, using (29), we arrive at the estimate

$$\int_0^t \left| \frac{d}{d\tau} \int_0^d \left(\frac{\partial \rho}{\partial y}\right)^2 dy \right| d\tau \leqslant C_{12}(C_5, C_6, C_{10}, N). \tag{35}$$

Hence, as $t \to +\infty$,

$$\left\| \frac{\partial \rho}{\partial y} \right\|_{L_2(0,d)} \to 0. \tag{36}$$

Thus, it is proved that (see (23)) in the norm of $W_2^1(0, d)$

$$u_i \to 0, \quad i = 1, \ldots, N, \quad \rho \to d \tag{37}$$

as $t \to +\infty$. Now, it can be easily verified that the same convergence takes place in the Euler variables in the norm of the space $W_2^1(0, 1)$. Theorem 1 is proved. □

5. Estimation of the Stabilization Rate

The estimates obtained above prove the fact of stabilization of the solution to problem (1)–(4), but do not give information about the rate of convergence. Since Equations (1) and (2) are a generalization of one-dimensional Navier–Stokes equations, for which the exponential stabilization rate is well known [18,19,22], it is reasonable to expect the same stabilization rate for the solution to the problem (1)–(4), i.e., in the multicomponent case.

In order to verify this hypothesis, Equation (2) is multiplied by $-\frac{\sigma}{\rho}\int\limits_0^x (\rho - d)\, ds$, $\sigma = \text{const} > 0$, summed in i and integrated by x, and after that, using the relations

$$-\sigma \sum_{i=1}^{N} \int\limits_0^1 \left(\frac{\partial u_i}{\partial t}\right)\left(\int\limits_0^x (\rho - d)\, ds\right) dx = -\sigma \frac{d}{dt} \sum_{i=1}^{N} \int\limits_0^1 u_i \left(\int\limits_0^x (\rho - d)\, ds\right) dx - \sigma N \int\limits_0^1 \rho v^2\, dx,$$

$$-\sigma \sum_{i=1}^{N} \int\limits_0^1 v \left(\frac{\partial u_i}{\partial x}\right)\left(\int\limits_0^x (\rho - d)\, ds\right) dx = \frac{\sigma N}{2} \int\limits_0^1 (\rho - d) v^2\, dx,$$

$$-\sigma \sum_{i=1}^{N} \int\limits_0^1 K\left(\frac{1}{\rho}\right)\left(\frac{\partial \rho}{\partial x}\right)\left(\int\limits_0^x (\rho - d)\, ds\right) dx = \sigma K N \int\limits_0^1 (\ln\rho - \ln d)(\rho - d)\, dx,$$

$$\begin{aligned} -\sigma \int\limits_0^1 \left(\frac{1}{\rho}\right)\left(\sum_{i,j=1}^{N} \nu_{ij} \frac{\partial^2 u_j}{\partial x^2}\right)\left(\int\limits_0^x (\rho - d)\, ds\right) dx = \sigma \sum_{i,j=1}^{N} \nu_{ij} \int\limits_0^1 \left(\frac{1}{\rho}\right)\left(\frac{\partial u_j}{\partial x}\right)(\rho - d)\, dx \\ + \sigma \sum_{i,j=1}^{N} \nu_{ij} \int\limits_0^1 \left(\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\right)\left(\frac{\partial u_j}{\partial x}\right)\left(\int\limits_0^x (\rho - d)\, ds\right) dx, \end{aligned}$$

the equality

$$\begin{aligned} -\sigma \frac{d}{dt} \sum_{i=1}^{N} \int\limits_0^1 u_i \left(\int\limits_0^x (\rho - d)\, ds\right) dx - \frac{\sigma N}{2} \int\limits_0^1 (\rho + d) v^2\, dx \\ + \sigma K N \int\limits_0^1 (\ln\rho - \ln d)(\rho - d)\, dx - \sigma \sum_{i,j=1}^{N} \nu_{ij} \int\limits_0^1 \left(\frac{1}{\rho}\right)\left(\frac{\partial u_j}{\partial x}\right)(\rho - d)\, dx \\ - \sigma \sum_{i,j=1}^{N} \nu_{ij} \int\limits_0^1 \left(\frac{\partial}{\partial x}\left(\frac{1}{-}\rho\right)\right)\left(\frac{\partial u_j}{\partial x}\right)\left(\int\limits_0^x (\rho - d)\, ds\right) dx = 0 \end{aligned} \quad (38)$$

is obtained. Summing (12) and (38), the inequality

$$\frac{dF(t)}{dt} + G(t) \leqslant 0 \quad (39)$$

is obtained, where

$$\begin{aligned} F(t) = -\sigma \sum_{i=1}^{N} \int\limits_0^1 u_i \left(\int\limits_0^x (\rho - d)\, ds\right) dx + \frac{1}{2} \sum_{i=1}^{N} \int\limits_0^1 \rho u_i^2\, dx \\ + K N \int\limits_0^1 (\rho \ln\rho - (\ln d + 1)\rho + d)\, dx, \end{aligned} \quad (40)$$

$$G(t) = -\frac{\sigma N}{2}\int_0^1 (\rho+d)v^2\,dx + \sigma K N\int_0^1 (\ln\rho - \ln d)(\rho-d)\,dx - \sigma\sum_{i,j=1}^{N} \nu_{ij}\int_0^1 \left(\frac{1}{\rho}\right)\left(\frac{\partial u_j}{\partial x}\right)(\rho-d)\,dx - \sigma\sum_{i,j=1}^{N}\nu_{ij}\int_0^1 \left(\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\right)\left(\frac{\partial u_j}{\partial x}\right)\left(\int_0^x (\rho-d)\,ds\right)dx + C_1\sum_{i=1}^{N}\int_0^1 \left(\frac{\partial u_i}{\partial x}\right)^2 dx. \quad (41)$$

In view of the estimates (see (24) and (25))

$$\frac{1}{C_{13}}(\rho-d)^2 \leqslant \ln\rho - \ln d + \frac{d}{\rho} - 1 \leqslant C_{13}(\rho-d)^2, \quad C_{13} = C_{13}(C_5, d), \quad (42)$$

$$\frac{1}{C_{14}}(\rho-d)^2 \leqslant (\ln\rho - \ln d)(\rho-d) \leqslant C_{14}(\rho-d)^2, \quad C_{14} = C_{14}(C_5, d), \quad (43)$$

$$\sigma\left|\sum_{i=1}^{N}\int_0^1 u_i\left(\int_0^x (\rho-d)\,ds\right)dx\right| \leqslant \frac{1}{4}\sum_{i=1}^{N}\int_0^1 \rho u_i^2\,dx + \sigma^2 C_{15}(C_5, N)\int_0^1 (\rho-d)^2\,dx, \quad (44)$$

$$\frac{\sigma N}{2}\int_0^1 (\rho+d)v^2\,dx \leqslant \sigma C_{16}(C_5, d)\sum_{i=1}^{N}\int_0^1 \left(\frac{\partial u_i}{\partial x}\right)^2 dx, \quad (45)$$

$$\sigma\left|\sum_{i,j=1}^{N}\nu_{ij}\int_0^1 \left(\frac{1}{\rho}\right)\left(\frac{\partial u_j}{\partial x}\right)(\rho-d)\,dx\right| \leqslant \frac{C_1}{8}\sum_{i=1}^{N}\int_0^1 \left(\frac{\partial u_i}{\partial x}\right)^2 dx + \sigma^2 C_{17}(C_1, C_5, \mathbf{N}, N)\int_0^1 (\rho-d)^2\,dx, \quad (46)$$

$$\sigma\left|\sum_{i,j=1}^{N}\nu_{ij}\int_0^1 \left(\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)\right)\left(\frac{\partial u_j}{\partial x}\right)\left(\int_0^x (\rho-d)\,ds\right)dx\right| \leqslant \frac{C_1}{8}\sum_{i=1}^{N}\int_0^1 \left(\frac{\partial u_i}{\partial x}\right)^2 dx + \sigma^2 C_{18}(C_1, C_5, C_6, \mathbf{N}, N)\int_0^1 (\rho-d)^2\,dx, \quad (47)$$

it follows that, for all $\sigma \in \left(0, \min\left(\sqrt{\frac{KN}{C_{13}C_{15}}}, \frac{C_1}{4C_{16}}, \frac{KN}{2C_{14}(C_{17}+C_{18})}\right)\right)$, the following information for $F(t)$ and $G(t)$ is valid:

$$\frac{1}{C_{19}}\left(\int_0^1 (\rho-d)^2\,dx + \sum_{i=1}^{N}\int_0^1 u_i^2\,dx\right) \leqslant F(t) \leqslant C_{19}\left(\int_0^1 (\rho-d)^2\,dx + \sum_{i=1}^{N}\int_0^1 u_i^2\,dx\right), \quad (48)$$

$$G(t) \geqslant C_{20}\left(\int_0^1 (\rho-d)^2\,dx + \sum_{i=1}^{N}\int_0^1 u_i^2\,dx\right), \quad (49)$$

where $C_{19} = C_{19}(C_5, C_{13}, C_{15})$, $C_{20} = C_{20}(C_1, C_{14}, C_{16}, C_{17}, C_{18}, \sigma, K, N)$. Then, from (39), taking into account (48) and (49), we have

$$\frac{dF(t)}{dt} + \kappa F(t) \leqslant 0, \quad \kappa = \kappa(C_{19}, C_{20}) = \text{const} > 0, \quad (50)$$

since $G(t) \geqslant \kappa F(t)$. From (50), the estimate

$$\int_0^1 (\rho - d)^2\,dx + \sum_{i=1}^N \int_0^1 u_i^2\,dx \leqslant e^{-\kappa t} C_{21}(C_{19}) \left(\int_0^1 (\rho_0 - d)^2\,dx + \sum_{i=1}^N \int_0^1 u_{0i}^2\,dx \right) \ \forall\, t \geqslant 0, \quad (51)$$

immediately follows, which confirms the validity of our hypothesis about the exponential rate of stabilization. Hence, the following assertion has been proven.

Theorem 2. *Let the conditions of Theorem 1 be valid. Then, the solution to problem* (1)–(4) *satisfies the estimate* (51) *for the rate of convergence.*

6. Conclusions

For the system of differential equations of isothermal viscous compressible multicomponent media with a non-diagonal, symmetric and positive-definite viscosity matrix, the asymptotic behavior (as $t \to +\infty$) of the solution to the initial boundary value problem has been analyzed in the case of one spatial variable. The main difficulty was to obtain a priori estimates. To overcome this, the mass Lagrangian coordinates were used. As a result, new a priori estimates are obtained and the stabilization of the solution to the initial boundary value problem is proven. In addition, the rate of convergence of the solution was estimated, which allows us to argue that the stabilization rate is exponential.

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References

1. Rajagopal, K.L.; Tao, L. *Mechanics of Mixtures*; World Scientific Publishing: Hackensack, NJ, USA, 1995.
2. Nigmatulin, R.I. *Dynamics of Multiphase Media: V. 1–2*; Hemisphere: New York, NY, USA, 1990.
3. Mamontov, A.E.; Prokudin, D.A. Viscous compressible homogeneous multi–fluids with multiple velocities: Barotropic existence theory. *Sib. Electron. Math. Rep.* **2017**, *14*, 388–397.
4. Mamontov, A.E.; Prokudin, D.A. Viscous compressible multi–fluids: Modeling and multi–D existence. *Methods Appl. Anal.* **2013**, *20*, 179–195. [CrossRef]
5. Kirwan, A.D.; Massoudi, M. The heat flux vector(s) in a two component fluid mixture. *Fluids* **2020**, *5*, 77. [CrossRef]
6. Lee, R.T.; Yang, K.T.; Chiou, Y.C. A novel model for a mixed-film lubrication with oil-in-water emulsions. *Tribol. Int.* **2013**, *66*, 241–248. [CrossRef]
7. Malek, J.; Rajagopal, K.R. A thermodynamic framework for a mixture of two liquids. *Nonlinear Anal. Real World Appl.* **2008**, *9*, 1649–1660. [CrossRef]
8. Atkin, R.J.; Craine, R.E. Continuum theories of mixtures: Applications. *IMA J. Appl. Math.* **1976**, *17*, 153–207. [CrossRef]
9. Muller, I. A thermodynamic theory of mixtures of fluids. *Arch. Ration. Mech. Anal.* **1968**, *28*, 1–39. [CrossRef]
10. Wang, S.H.; Szeri, A.Z.; Rajagopal, K.R. Lubrication with emulsion in cold-rolling. *J. Tribol. Trans. ASME* **1993**, *115*, 523–531. [CrossRef]
11. Chamniprasart, K.; Al-Sharif, A.; Rajagopal, K.R.; Szeri, A.Z. Lubrication with binary mixtures—Bubbly oil. *J. Tribol. Trans. ASME* **1993**, *115*, 253–260. [CrossRef]
12. Wang, S.H.; Al-Sharif, A.; Rajagopal, K.R.; Szeri, A.Z. Lubrication with binary mixtures—Liquid-liquid emulsion in an EHL conjunction. *J. Tribol. Trans. ASME* **1993**, *115*, 515–522. [CrossRef]
13. Al-Sharif, A.; Chamniprasart, K.; Rajagopal, K.R.; Szeri, A.Z. Lubrication with binary mixtures—Liquid-liquid emulsion. *J. Tribol. Trans. ASME* **1993**, *115*, 45–66. [CrossRef]
14. Szeri, A.Z. *Fluid Film Lubrication: Theory and Design*; Cambridge University Press: Cambridge, UK, 1998.
15. Zlotnik, A.A.; Ducomet, B. Stabilization rate and stability for viscous compressible barotropic symmetric flows with free boundary for a general mass force. *Sb. Math.* **2005**, *196*, 1745–1799. [CrossRef]
16. Ducomet, B.; Zlotnik, A. Lyapunov functional method for 1D radiative and reactive viscous gas dynamics. *Arch. Ration. Mech. Anal.* **2005**, *177*, 185–229. [CrossRef]
17. Straskraba, I.; Zlotnik, A. Global properties of solutions to 1D–viscous compressible barotropic fluid equations with density dependent viscosity. *Z. Für Angew. Math. Phys.* **2003**, *54*, 593–607. [CrossRef]
18. Straskraba, I.; Zlotnik, A. On a decay rate for 1D-viscous compressible barotropic fluid equations. *J. Evol. Equ.* **2002**, *2*, 69–96. [CrossRef]

19. Straskraba, I. Recent progress in the mathematical theory of 1D barotropic flow. *Ann. Dell'Universita Ferrara* **2009**, *55*, 395–405. [CrossRef]
20. Zlotnik, A.A.; Nguen, Z.B. Properties and asymptotic behavior of solutions of some problems of one-dimensional motion of a viscous barotropic gas. *Math. Notes* **1994**, *55*, 471–482. [CrossRef]
21. Antontsev, S.N.; Kazhikhov, A.V.; Monakhov, V.N. *Boundary Value Problems in Mechanics of Nonhomogeneous Fluids*; North–Holland Publishing Co.: Amsterdam, The Netherlands, 1990.
22. Kazhikhov, A.V. Stabilization of solutions of the initial–boundary value problem for barotropic viscous fluid equations. *Differ. Uravn.* **1979**, *15*, 662–667.
23. Prokudin, D.A. Unique solvability of initial–boundary value problem for a model system of equations for the polytropic motion of a mixture of viscous compressible fluids. *Sib. Electron. Math. Rep.* **2017**, *14*, 568–585.
24. Prokudin, D.A. Global solvability of the initial boundary value problem for a model system of one-dimensional equations of polytropic flows of viscous compressible fluid mixtures. *J. Phys. Conf. Ser.* **2017**, *894*, 012076. [CrossRef]
25. Prokudin, D.A. On the stabilization of solutions to the initial–boundary value problem for the equations of dynamics of viscous compressible multicomponent media. *Sib. Electron. Math. Rep.* **2021**, *18*, 1278–1285. [CrossRef]
26. Li, S. On one–dimensional compressible Navier–Stokes equations for a reacting mixture in unbounded domains. *Z. Für Angew. Math. Phys.* **2017**, *68*, 106. [CrossRef]
27. Bresh, D.; Huang, X.; Li, J. Global weak solutions to one–dimensional non–conservative viscous compressible two–phase system. *Commun. Math. Phys.* **2012**, *309*, 737–755. [CrossRef]
28. Papin, A.A. On the uniqueness of the solutions of an initial boundary–value problem for the system of a heat–conducting two–phase mixture. *Math. Notes* **2010**, *309*, 594–598. [CrossRef]
29. Papin, A.A.; Akhmerova, I.G. Solvability of the System of Equations of One-Dimensional Motion of a Heat-Conducting Two-Phase Mixture. *Math. Notes* **2010**, *87*, 230–243. [CrossRef]
30. Zlotnik, A.A. Weak solutions to the equations of motion of viscous compressible reacting binary mixtures: Uniqueness and Lipschitz-continuous dependence on data. *Math. Notes* **2004**, *75*, 278–283. [CrossRef]
31. Chen, G.Q.; Hof, D.; Trivisa, K. Global solutions to a model for exothermically reacting, compressible flows with large discontinuous initial data. *Arch. Ration. Mech. Anal.* **2003**, *166*, 321–358. [CrossRef]
32. Zlotnik, A.A. Uniform estimates and stabilization of solutions to equations of one-dimensional motion of a multicomponent barotropic mixture. *Math. Notes* **1995**, *58*, 885–889. [CrossRef]
33. Tabata, M. Local solutions to the Cauchy problem for the system of equations describing the motion of a mixture of compressible viscous and heat-conductive fluids. *Kobe J. Math.* **1992**, *9*, 195–205.
34. Petrov, A.N. Well-posedness of initial-boundary value problems for one-dimensional equations of mutually penetrating flows of ideal gases. *Din. Sploshnoĭ Sredy* **1982**, *56*, 105–121.
35. Antontsev, S.N.; Papin, A.A. Approximation methods of solving problems of two–phase filtration. *Dokl. Akad. Nauk. SSSR* **1979**, *247*, 521–525.
36. Kazhikov, A.V.; Petrov, A.N. Well-posedness of the initial-boundary value problem for a model system of equations of a multicomponent mixture. *Din. Sploshnoĭ Sredy* **1978**, *35*, 61–73.
37. Zhang, J.X.; Yang, G.H. Low-complexity tracking control of strict-feedback systems with unknown control directions. *IEEE Trans. Autom. Control* **2019**, *64*, 5175–5182. [CrossRef]
38. Zhang, X.; Huang, W. Adaptive neural network sliding mode control for nonlinear singular fractional order systems with mismatched uncertainties. *Fractal Fract.* **2020**, *4*, 50. [CrossRef]
39. Zhang, J.X.; Yang, G.H. Fault-tolerant output-constrained control of unknown Euler–Lagrange systems with prescribed tracking accuracy. *Automatica* **2020**, *111*, 108606. [CrossRef]
40. Maz'ya, V. *Sobolev Spaces*; Springer: Berlin/Heidelberg, Germany, 2011.
41. Ladyzhenskaya, O.A.; Ural'tseva, N.N. *Linear and Quasi-Linear Elliptic Equations*; Academic Press: New York, NY, USA, 1968.