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Editorial

Fractional Integrals and Derivatives: "True" versus "False"

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Within the last few years, many of the efforts of the fractional calculus (FC) community have been directed towards clarifying the nature and essential properties of the operators known as fractional integrals and derivatives. Alongside suggestions for some new definitions of the fractional order operators, several authors have tried to formulate axioms or desiderata that the fractional integrals and derivatives should satisfy.

However, until now, there have been no definite answers to questions such as "What are the fractional integrals and derivatives?", "What are their obligatory properties?", and "What fractional operators make sense in applications and why?". However, within the last few years, these and similar questions have been intensively discussed in many FC publications and during scientific conferences, and some important partial answers to the questions formulated above have been suggested.

In particular, the articles published in the Special Issue "Fractional Integrals and Derivatives: "True" versus "False"" and in this Special Issue "Fractional Integrals and Derivatives: "True" versus "False" II" provide serious contributions to the theory and applications of the "true" fractional calculus operators.

One of the most interesting and promising recent approaches to the time-fractional integrals and derivatives is that based on the integral and integro-differential operators with Sonin kernels. In [1], Yuri Luchko investigates in detail an important class of the so-called general fractional integrals and derivatives with Sonin kernels with an integrable singularity of the power law type at point zero. In particular, two fundamental theorems of fractional calculus are formulated and proved for the general fractional integrals and derivatives with Sonin kernels from this class. In addition, the n-fold general fractional integrals and derivatives are constructed and studied. The paper [1] is one of the first FC publications devoted to this promising research topic. It is both well read and well cited, and was awarded the second prize in the *Mathematics* best paper award in 2021.

The line of research started in [1] is continued in the paper [2] by Yuri Luchko, where the general fractional derivatives of arbitrary order are introduced. The main contribution of [2] regards the development of an operational calculus of the Mikusiński type for the general fractional derivatives of arbitrary order. In [2], this calculus is employed for the derivation of an explicit form of solutions to the Cauchy problems for the single- and multi-term linear fractional differential equations with general fractional derivatives of arbitrary order. The solutions are provided in form of convolution series generated by the kernels of the corresponding general fractional integrals.

Vasily E. Tarasov considers in [3] an important particular case of the general fractional operators introduced in [1], namely the Prabhakar fractional integrals and derivatives. Due to the special properties of the kernels of these operators in the form of the three-parameter Mittag–Leffler functions, nowadays often referred to as the Prabhakar functions, they are very useful for many applications. In particular, they allow non-exponential depreciation and fading memory to be taken into account in certain mathematical models in economics. In [3], some integral and integro-differential equations with Prabhakar fractional integrals and derivatives are solved in explicit form and their asymptotic behavior is discussed.

The paper [4] by Manuel D. Ortigueira is devoted to an in-depth investigation of the initial conditions that should be posed while solving the initial-value problems for the

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fractional differential equations. In particular, he finds that the initial conditions provided by the one-sided Laplace transform are not suitable for fractional differential equations in the form of Riemann–Liouville and Caputo derivatives. As an application of the theoretical results deduced in [4], some continuous-time fractional autoregressive–moving average systems are analyzed.

The paper [5] by Masahiro Yamamoto presents a very important contribution to the theory of fractional derivatives and time-fractional partial differential equations based on the operator theory in fractional Sobolev spaces. In the framework of the Sobolev spaces of fractional order, some suitable extensions of the Caputo and the Riemann–Liouville fractional derivatives are suggested. As a result, Masahiro Yamamoto deduces the well-posedness and other important properties of solutions to the initial boundary value problems for a broad class of time-fractional partial differential equations.

The paper [6] by Abdallah El Hamidi, Mokhtar Kirane, and Ali Tfayli is devoted to a class of inverse problems formulated for a generalized time-space-fractional diffusion equation. It is well known that the inverse problems are very important for different applications. Thus, even if the results presented in [6] are of rather theoretical nature, they are potentially useful for analysis of mathematical models in form of the fractional differential equations.

Finally, the papers [7] by Jean-Philippe Aguilar, Jan Korbel, and Nicolas Pescian and [8] by Teodor M. Atanackovic, Cemal Dolicanin, and Enes Kacapor are devoted to applications of FC in two very different areas.

In [7], a discussion of several models for the description of stock return behavior is presented. The models are formulated in terms of the space-fractional pseudo-differential operators. In particular, some formulas for volatility and first- and second-order market sensitivities, as well as for hedging and profit and loss policies in the framework of these models, are provided.

In [8], the authors present a generalization of the classical internal variable approach to viscoelasticity by introducing a fractional derivative into the equation for the time evolution of the internal variables. In particular, some restrictions on the coefficients of the resulting equations are derived from the dissipation inequality. This paper [8] also contains a discussion of numerical solutions to the derived equations.

In summary, these papers [1–8] provide a sound contribution to the theory and applications of FC and thus account for the clarification of some of the basic questions regarding the properties and applicability of the "true" fractional integrals and derivatives.

Conflicts of Interest: The author declares no conflict of interest.

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