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On Characterization of Balance and Consistency Preserving d-Antipodal Signed Graphs

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Abstract: A signed graph is an ordered pair $\Sigma = (G, \sigma)$, where G is a graph and $\sigma\colon E(G) \longrightarrow \{+1, -1\}$ is a mapping. For $e \in E(G)$, $\sigma(e)$ is called the sign of e and for any sub-graph H of G, $\sigma(H) = \prod_{e \in E(H)} \sigma(e)$ is called the sign of H. A signed graph having a sign of each cycle $+1$ is called balanced. Two vertices in a graph G are called antipodal if $d_G(u, v) = diam(G)$. The antipodal graph $A(G)$ of a graph G is the graph with a vertex set that is the same as that of G, and two vertices u, v in $A(G)$ are adjacent if u, v are antipodal. By the d-antipodal graph G^A_d of a graph G, we refer to the union of G and $A(G)$. Given a signed graph $\Sigma = (G, \sigma)$, the signed graph $\Sigma^A_d = (G^A_d, \sigma_d)$ is called the d-antipodal signed graph of G, where σ_d is defined as follows: $\sigma_d(e) = \sigma(e)$ if $e \in E(G)$ and otherwise, $\sigma_d(e) = \prod_{P \in \mathcal{P}_e} \sigma(P)$, where $\mathcal{P}_e$ is the collection of all diametric paths in Σ connecting the end vertices of an antipodal edge e in Σ^A_d. In this article, the balance property and canonical consistency of d-antipodal signed graphs of Smith signed graphs (connected graphs having a highest eigenvalue of 2) are studied.

Keywords: signed graphs; balanced and consistent signed graphs; marked graphs; Smith graphs; antipodal signed graphs

MSC: 05C10

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1. Introduction

The book [1] by Harary may be referred for basic terminologies in graph theory. A graph G is an ordered pair (V, E), where V is a non-empty set whose elements are called vertices and E is a collection of unordered pair of distinct vertices whose elements are called edges of the graph G. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of the graph G. By a signature on a graph G, we mean a function $\sigma : E(G) \to \{+1, -1\}$. A graph G together with a signature σ is called a signed graph and will be denoted by $\Sigma = (G, \sigma)$. The graph G is referred as the underlying graph of the signed graph $\Sigma = (G, \sigma)$. The signed graph with all positive (negative) edges having the underlying graph G is denoted by G^+(G^-). By the vertex set of Σ, we refer to the vertex set of the underlying graph G. For any edge $e \in E(G)$, $\sigma(e)$ is referred to as the sign of the edge e in Σ.

By the sign of a sub-graph H of a signed graph $\Sigma = (G, \sigma)$, we mean the product of the sign of the edges in H, and it is denoted by $\sigma(H)$. The concept of a signed graph was first introduced by Harary in [2] to model social problems. A signed graph is said to be balanced if every cycle in it has the sign $+1$. In [2], Harary characterized balanced signed graphs as the signed graph whose vertex set can be partitioned into V_1, V_2 such that any negative edge connects a vertex from V_1 to a vertex in V_2 and a positive edge connects a pair of vertices either from V_1 or from V_2. We refer to such a partition of a balanced signed graph as Harary's partition.

The term "antipodal" was first coined by R. Singleton [3] to refer to a pair of vertices, the distance between which is equal to the diameter of the graph. By antipodes of a vertex in a graph, we refer to the vertices that are antipodal to it. Given a graph $G(V,E)$, by its antipodal graph $A(G)$, we refer to the graph with vertex set $V(A(G)) = V(G)$ such that for $u, v \in V(A(G)), \{u, v\} \in E(A(G))$ if u and v are antipodes in G. The concept of an antipode was first used to establish regularity in Moore graphs, i.e., finite undirected graphs which are regular, connected with diameter $k \geq 1$ and girth $2k+1$. D H Smith [4] had used a version of antipodal graphs while studying distance transitive graphs to characterize primitive and imprimitive graphs. R Aravamudhan [5] studied the behavior of antipodal graphs with regard to completeness, connectedness, etc., and they derived necessary and sufficient conditions for a graph to be an antipodal graph of a graph in terms of its compliment. Acharya and Acharya [6] have studied self antipodal graphs and given important characterizations. Further works can be found in [7]. S-antipodal graphs have been introduced by Nair and Vijaykumar [8] as an induced sub-graph of antipodal graphs whose vertex set comprises those vertices having maximum eccentricity in which two vertices are joined by an edge if they are at a diametrical distance. P S K Reddy et al. [9] introduced Smarandachely antipodal signed digraphs and obtained some structural characterizations. In [10], Reddy and Prashanth have introduced the S-antipodal signed graph $A^*(\Sigma)$ of a signed graph $\Sigma = (G, \sigma)$ inspired by the complement of a graph in which the sign of an edge uv is the product of canonical marking of u and v and reported several characterizations of the balanced S-antipodal signed graph of a signed graph.

In this article, the concept of a d-antipodal graph of a graph has been introduced. This has been inspired by real-life situations where we want to study the changes brought in the network by introducing antipodal edges while retaining the original edges of the network. Such a graph has many real-life applications in networking, defense, diplomatic relationships, etc. For example, consider a graph representing the diplomatic relationship among various countries. It might be interesting to investigate how the network will behave if diplomatic relationship develops between its antipodes.

2. Preliminaries

Given a graph G, its antipodal graph, $A(G)$, is a graph that has a vertex set that is the same as that of G, and two vertices in $A(G)$ are adjacent if they are at a distance of $diam(G)$ in G. By the d-antipodal graph G_d^A of a graph G, we refer to the union of the graphs G and $A(G)$.

Remark 1. *The following are true for d-antipodal graphs:*

- *The d-antipodal graph of an even cycle is* 3*-regular because each vertex in an even cycle has exactly one antipodal vertex.*
- *The d-antipodal graph of an odd cycle is* 4*-regular because each vertex in an even cycle has exactly two antipodal vertices.*
- *The d-antipodal graph of* $K_{1,n}$ *for* $n \geq 2$ *is n-regular because every pair of pendant vertices in* $K_{1,n}$ *consists of antipodes.*
- C_n *and* $K_{1,4}$ *are the only Smith graphs whose d-antipodal graphs are regular.*

Given a signed graph $\Sigma = (G, \sigma)$, the d-antipodal signed graph of Σ, denoted by Σ_d^A, is the signed graph (G_d^A, σ_d) with sign function σ_d defined as follows:

$$\sigma_d(e) = \begin{cases} \sigma(e), & \text{if } e \in E(G) \\ \prod\limits_{P \in \mathcal{P}_e} \sigma(P), & \text{otherwise} \end{cases}$$

and $\mathcal{P}_e$ is the collection of all diametric paths in Σ connecting the end vertices of an antipodal edge e in Σ_d^A.

Remark 2. *Restriction of* σ_d *to* $\Sigma = (G, \sigma)$ *is* σ.

A marking on a graph G is a function $\mu : V(G) \rightarrow \{+1, -1\}$. A graph G provided with a marking μ is called a marked graph. A signed graph $\Sigma = (G, \sigma)$ provided with a marking μ is called a marked signed graph, and it is denoted by Σ_μ.

Let μ be a marking on a graph G. By the mark of a sub-graph H of G, we mean the product of the marks of the vertices in H, and it is denoted by $\mu(H)$. The concept of a marked graph was first introduced by Harary and Cartwright in [11] to model a social problem. A marked graph is said to be *consistent* if every cycle in it has the mark $+1$. The following theorem characterizes marked graphs.

Theorem 1 ([12])**.** *Any marked graph with a mark of each vertex $+1$ is consistent.*

Proof. Let Σ be a marked graph with a mark of each vertex $+1$. Since in any cycle of Σ, each vertex will have a mark $+1$, it will be consistent. Hence, Σ is consistent. □

Theorem 2 ([12])**.** *Any marked graph with a mark of each vertex -1 is consistent if and only if its underlying graph is bipartite.*

Proof. Let Σ be a marked graph with a mark of each vertex -1. Then, each cycle in Σ will be consistent if and only if it has an even number of vertices. Since a graph is bipartite if and only if each of its cycles is even, so the result follows. □

The following corollary is immediate from Theorem 2.

Corollary 1. *If a marked graph is consistent, then the sub-graph induced by its vertices with a mark -1 is bipartite.*

Furthermore, in [13], it was noted that a marked graph is consistent if and only if for any spanning tree T of it, all its fundamental cycles are positive and all common paths shared by a pair of fundamental cycles have end points with the same marking. For more literature on consistency, we refer to [13–18].

Given a signed graph $\sum = (G, \sigma)$, we can associate a natural marking

$$\mu : V(\Sigma) \rightarrow \{-1, +1\}$$

as follows: For any vertex $v \in V(\Sigma)$

$$\mu(v) = \begin{cases} +1, & \text{if } v \text{ is isolated;} \\ \prod\limits_{u \in N(v)} \sigma(uv), & \text{otherwise;} \end{cases}$$

where $N(v)$ is the set of all vertices adjacent to v in Σ. This marking μ is known as the *canonical marking* of the signed graph Σ, and we use Σ_μ to denote the corresponding marked signed graph. A signed graph Σ is said to be *canonically consistent* if it is consistent with respect to the canonical marking.

Remark 3. *All signed cycles are canonically consistent.*

Unless otherwise stated, for a given signed graph Σ, we use μ to represent the canonical marking on Σ and μ_d to represent the canonical marking on Σ_d^A.

Throughout this article, edges of the underlying graph are represented by bold lines, antipodal edges are represented by dotted lines and paths are represented by dashed lines. A blue color used to represent edges with the $+1$ sign, and a red color is used to represent edges with the -1 sign of a signed graph.

3. *d*-Antipodal Signed Graphs

In the theory of graph spectra, an important role is played by graphs with the largest eigenvalue of 2. In the literature, these graphs are known as Smith graphs [19]. There are six different classes of Smith graphs, as shown in Figure 1.

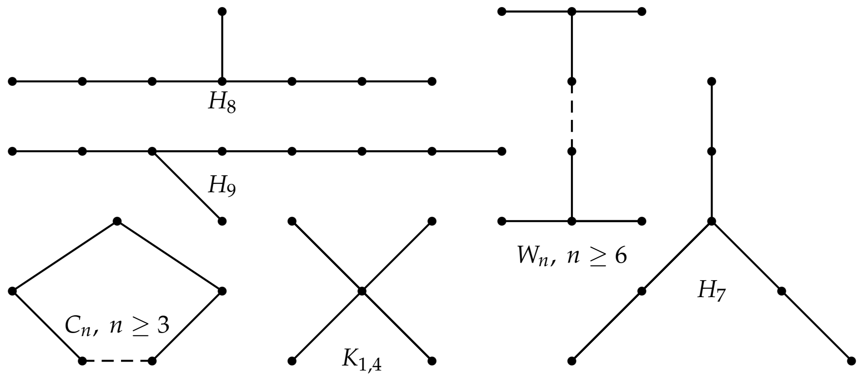

Figure 1. All classes of Smith graphs.

Remark 4. *A Smith signed graph is balanced and consistent if and only if either it is acyclic or it has an even number of negative edges.*

The Smith signed graph (C_n^+, σ) is balanced and consistent, and so is its d-antipodal Σ_d^A. The Smith signed graph Σ shown in Figure 2 is balanced and consistent, but its d-antipodal Σ_d^A is neither balanced nor consistent. In this article, the balanced and consistent Smith signed graphs are characterized that have balanced and consistent d-antipodal values.

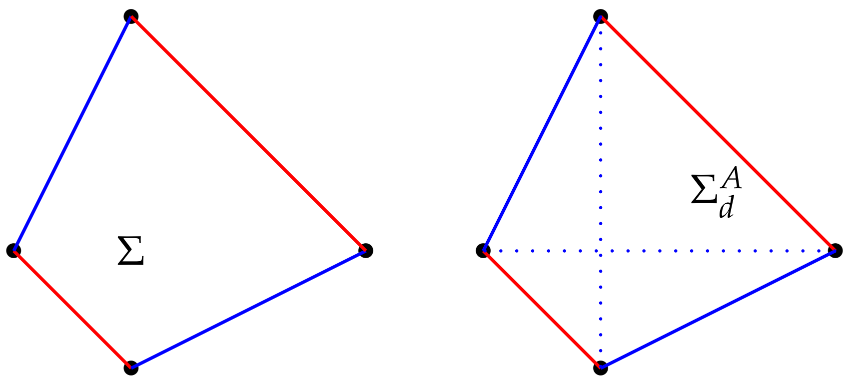

Figure 2. A Smith signed graph Σ and its d-antipodal Σ_d^A.

Theorem 3. *Let $\Sigma = (G, \sigma)$ be a balanced signed graph and V_1, V_2 be Harary's partition of its vertex set. Then, Σ_d^A is balanced if and only if for each pair of antipodes $u \in V_1$, $v \in V_2$, the number of distinct diametric paths in Σ joining u, v with a sign of -1 is odd.*

Proof. Suppose Σ_d^A is balanced. Let $u, v \in V$ be two antipodes. On the contrary, let us assume that there is even number of distinct negative $u - v$ diametric paths $P_1, P_2, ..., P_n$ in Σ. Let a be the antipodal edge joining u and v in Σ_d^A. Clearly, $\sigma_d(a) = +1$ is the product of the signs of an even number of diametric paths with a sign of -1. Consider a cycle $P_i a, 1 \leq i \leq n$, then $\sigma_d(P_i a) = -1$. This contradicts that Σ_d^A is balanced. Hence, the number of distinct negative paths must be odd.

Conversely, suppose that for each pair of antipodes $u \in V_1$, $v \in V_2$, the number of distinct $u - v$ diametric paths in Σ with a sign of -1 is odd. We want to show that Σ_d^A is balanced. We complete the proof by showing that V_1 and V_2 is Harary's partition for Σ_d^A.

Let $e = \{u, v\}$ be any edge in Σ_d^A. If $e \in E(\Sigma)$, then there is nothing to prove. So, let e be an antipodal edge.

Case 1: $\sigma_d(e) = -1$. In this case, there is a $u - v$ diametric path in Σ with a negative sign and hence with an odd number of negative edges. Since each negative edge in Σ connects a vertex from V_1 to a vertex from V_2, so u, v cannot be from the same set of the partition V_1, V_2.

Case 2: $\sigma_d(e) = +1$. In this case, either all the diametric $u - v$ paths in Σ has a sign of $+1$ or the number of distinct $u - v$ diametric paths in Σ with a sign of -1 is even. The existence of $u - v$ diametric paths in Σ with a sign of -1 demands that u, v should be from different parts of the partition. So, by assumption, the second possibility is ruled out. Hence, either $u, v \in V_1$ or $u, v \in V_2$.

Thus every edge in Σ_d^A with sign $+1$ connects two vertices from the same part of the partition V_1, V_2 and every edge in Σ_d^A with sign -1 connects a vertex from V_1 to a vertex in V_2. Therefore, V_1, V_2 is a Harary's partition of the vertex set of Σ_d^A and so it is balanced. □

Remark 5. *For any tree T, its antipodal edges are the chords of T_d^A with respect to the spanning tree T.*

Remark 6. *For any tree T, the fundamental cycles of T_d^A with respect to the spanning tree T are balanced.*

4. Smith Signed Graphs

Let $\Sigma = (C_n, \sigma)$. The d-antipodal graph of a cycle does not have any new edge for $n = 3$. However, $diam(C_n) = \lfloor n/2 \rfloor$, where $\lfloor n/2 \rfloor$ stands for the greatest integer less than or equal to $n/2$. However

$$|E(\Sigma_d^A)| = \begin{cases} 3n/2, & \text{if } n \text{ is even} \\ 2n, & \text{if } n \text{ is odd.} \end{cases}$$

An immediate question is whether the d-antipodal signed graph of a balanced cycle is balanced or not? Here, we have characterized the d-antipodal signed graph of cycles and obtained some results on balancedness, consistency and regularity.

Proposition 1. *Let $\Sigma = C_{2n}^-$. Then, Σ_d^A is balanced if and only if $2|n$.*

Proof. Let the vertices of Σ be labeled by $v_1, v_2, ..., v_{2n}$ in cyclic order. Then, $V_1 = \{v_i \mid i = 1, 3, \cdots, 2n-1\}$ and $V_2 = \{v_i \mid i = 2, 4, \cdots, 2n\}$ gives a partition of $V(\Sigma)$ in which each edge of G connects a vertex from V_1 to a vertex from V_2. Since Σ is balanced, this partition also serves as Harary's partition of the balanced graph Σ. We note that if e is an antipodal edge in Σ_d^A, then $\sigma_d(e) = +1$.

First, assume that Σ_d^A is balanced. Then, the cycle $v_1 v_2 \cdots v_{n+1} v_1$ in Σ_d^A must be balanced. As this cycle contains n negative edges, so n must be even.

Conversely, suppose that n is even. Then, each antipodal edge in Σ_d^A is positive and both the end vertices of such an edge is either from V_1 or from V_2. So, V_1, V_2 forms a partition of $V(\Sigma_d^A)$ such that each negative edge connects a vertex from V_1 to a vertex from V_2 and positive edges connect vertices within the same set. Hence, Σ_d^A is balanced. □

Proposition 2. *Let $\Sigma = (C_{2n}, \sigma)$ be a balanced cycle with Harary's partition V_1, V_2 of $V(\Sigma)$. Then, Σ_d^A is balanced if and only if v_i and v_{i+n} are in the same partition for each $1 \le i \le n$.*

Proof. First, suppose Σ_d^A is balanced and let V_1, V_2 be Harary's partition of $V(\Sigma_d^A)$. Then, each cycle in Σ_d^A is balanced and hence Σ is also balanced. Furthermore, since $E(\Sigma) \subset E(\Sigma_d^A)$, so V_1, V_2 is also Harary's partition of $V(\Sigma)$. Since Σ has an even number of negative edges, so each antipodal edge in Σ_d^A must have a sign of $+1$. As for each $i = 1, 2, \cdots, n$, the vertices v_i, v_{i+n} are antipodes, so both must be either in V_1 or in V_2.

Conversely, since Σ is balanced, so it has even number of negative edges. Hence each antipodal edges in Σ_d^A must have sign $+1$. Since the antipodal vertices of Σ are v_i,

v_{i+n} for each $i = 1, 2, \cdots, n$, so V_1, V_2 also forms Harary's partition of $V(\Sigma_d^A)$. So, Σ_d^A is balanced. □

We have the following characterization for odd cycles.

Proposition 3. *Let $\Sigma = (C_{2n+1}, \sigma)$; then, Σ_d^A is balanced if and only if Σ is balanced.*

Proof. First, suppose that Σ is balanced with Harary's partition of $V(\Sigma)$ as V_1, V_2. In Σ, each diametric path has a length of n and there is exactly one diametric path between each pair of antipodes. Let us label the vertices of Σ using $v_1, v_2, \cdots, v_{2n+1}$ in cyclic order.

Each vertex $v_i \in V, 1 \leq i \leq n$ has two antipodes v_{i+n}, v_{i+n+1}. Without a loss of generality, suppose that $v_i \in V_1$ and let $e = \{v_i, v_{i+n}\}, e' = \{v_i, v_{i+n+1}\}$ be the two associated antipodal edges.

Case 1: Suppose, $\sigma_d(e) = +1$. Then, the number of negative edges in the diametric path joining the vertices v_i and v_{i+n} in Σ must be even. Since each negative edge in Σ joins a vertex from V_1 to a vertex from V_2, so both the end vertices of e must be in the same part, i.e., $v_{i+n} \in V_1$.

Case 2: Suppose, $\sigma_d(e) = -1$. Then, the number of negative edges in the diametric path joining the vertices v_i and v_{i+n} in Σ must be odd. Since each negative edge in Σ joins a vertex from V_1 to a vertex from V_2, so the two end vertices of e must be from different parts, i.e., $v_{i+n} \in V_2$.

A similar argument holds for the end vertices of the antipodal edge e'.

Thus, each antipodal edge with a sign of $+1$ connects vertices from the same part and an antipodal edge with a sign of -1 connects a vertex from V_1 to a vertex from V_2. Hence, V_1 and V_2 gives a partition of $V(\Sigma_d^A)$ such that each negative edge of Σ_d^A connects vertices from different sets and each positive edge connects vertices from the same set. That is, V_1, V_2 serves as Harary's partition of the vertices in Σ_d^A. Hence, Σ_d^A is balanced. □

Remark 7. *If the sign of each edge in a signed graph Σ is $+1$, then Σ and Σ_d^A both are canonically consistent.*

However, (Σ, μ) being canonically consistent need not imply that its d-antipodal marked graph is canonically consistent . For example, the graph in Figure 2 is canonically consistent but its d-antipodal is not canonically consistent . Hence, the conditions under which the canonical consistency of signed graphs is invariant under the d-antipodal operation of canonically consistent signed graphs is essential. The following results give some characterization for signed cycles.

Proposition 4. *If $\Sigma = (C_n^-, \sigma)$, then Σ_d^A is canonically consistent for any $n \in \mathbb{N}$.*

Proof. We observe that $\mu(v_i) = +1, v_i \in V(\Sigma)$ for each i.

Case 1: n is even. Suppose $n = 2l$ for some $l \in N$. Then, $\sigma_d(a_{il}) = +1$; hence, $\mu_d(v_i) = +1$ remains positive in (Σ_d^A, μ_d). Thus, it is canonically consistent.

Case 2: n is odd. In this case, each antipodal edge in Σ_d^A will have the same sign. Since each vertex is incident with exactly two antipodal edges, so each vertex in Σ_d^A has a canonical marking of $+1$. Hence, the result follows. □

Corollary 2. *Let $\Sigma = C_{2n}^-$. Then, Σ_d^A is balanced and canonically consistent if and only if $2|n$.*

Lemma 1. *Let $\Sigma = (C_{2n}, \sigma)$. If there exist antipodes $u, v \in V(\Sigma)$ such that $\mu(u)\mu(v) = -1$, then Σ_d^A is not canonically consistent .*

Proof. Let $\Sigma = (C_{2n}, \sigma)$ have antipodes $u, v \in V(\Sigma)$ with $\mu(u)\mu(v) = -1$. Since in Σ_d^A, $\{u, v\}$ is the only edge that is incident with u or v apart from the edges that were present in Σ, so the marks of u and v remain opposite in Σ_d^A. At most, their marks may become interchanged depending on whether Σ is balanced or not. Therefore, $\mu_d(u)\mu_d(v) = -1$. Let P_{uv} and P'_{uv} be the two diametric paths in Σ joining u to v.

If possible, suppose that Σ_d^A is canonically consistent. Let C_1, C_2 be the cycles in Σ_d^A consisting of the edge $\{u, v\}$ and the paths P_{uv}, P'_{uv}, respectively. Then, by the consistency of Σ_d^A, each of the cycles C_1, C_2 has an even number of vertices with a mark of -1. However, these two cycles have exactly one vertex with the mark of -1 in common, namely u or v. So, the cycle Σ in Σ_d^A which the symmetric difference of C_1 and C_2 has an odd number of vertices with a mark of -1. This contradicts our assumption that Σ_d^A is consistent. Hence, the result follows. $\square$

Proposition 5. *Let $\Sigma = (C_{2n}, \sigma)$ be balanced. Then, Σ_d^A is canonically consistent if and only if $\Sigma = C_{2n}^-$ or $\Sigma = C_{2n}^+$.*

Proof. First, assume that $\Sigma = C_{2n}^-$ or $\Sigma = C_{2n}^+$. In each case, the number of negative edges in Σ is even. Since the two diametric paths connecting a pair of antipodes contain each edge of Σ exactly once, so the sign of an antipodal edge in Σ_d^A is $+1$. Hence,

$$\mu_d(v) = \mu(v) = +1 \text{ for all } v \in V(\Sigma_d^A).$$

So, Σ_d^A is canonically consistent.

Conversely, suppose that Σ has edges with a sign of $+1$ as well as with a sign of -1. Since Σ is balanced, so the total number of negative edges in both the diametric paths connecting a pair of antipodes is even. Hence, the sign of each antipodal edge in Σ_d^A is positive. If we have antipodes $u, v \in V(\Sigma)$ such that $\mu(u)\mu(v) = -1$, then by Lemma 1, Σ_d^A is not canonically consistent.

So, let for each pair of antipodes $u, v \in V(\Sigma)$, $\mu(u) = \mu(v)$. Then, Σ must have a pair of antipodes with negative marking.

If for all $u \in V(\Sigma)$, $\mu(u) = -1$, then the signs of the edges of Σ are alternately $+1$ or -1. Since Σ is balanced and half of the edges in it are negative, so n must be even. Thus, both the cycles in Σ_d^A that are generated by an antipodal edge consist of an odd number of vertices each with mark -1 and so, Σ_d^A is not canonically consistent.

Otherwise, let $x, y \in V(\Sigma)$ be antipodes such that $\mu(x) = \mu(y) = +1$. Then, both the edges incident with x are of the same sign and those with y are of same sign. We consider the following cases:

Case 1. The edges incident with x and those incident with y are opposite in sign, as shown in Figure 3. In this case, each of the cycles in Σ_d^A generated by the antipodal edge $\{x, y\}$ contains an odd number of vertices with a mark of -1, and so, both are not consistent. Hence, Σ_d^A is not canonically consistent.

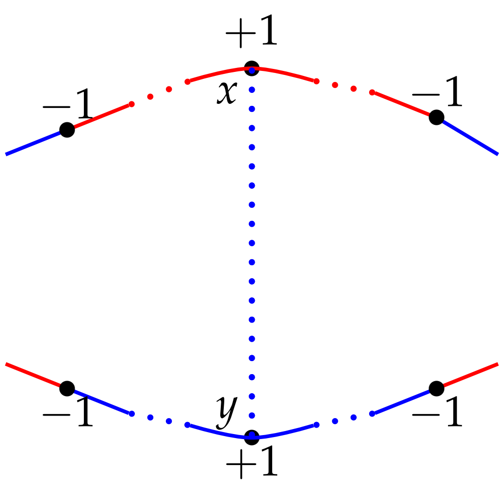

Figure 3. Mark distribution after introducing antipodal edge $\{x, y\}$ in Σ.

Case 2. The edges incident with x and those incident with y are of the same sign. As we move along the cycle clockwise, starting from x, let x' be the first vertex with $\mu(x') = -1$. The existence of x' is guaranteed, because it has at least two edges with different signs. Furthermore, we must obtain such a vertex before reaching y, because antipodes have the same sign. Notice that the signs of all the edges in the x–x' path in Σ not containing y are the same.

Let s, t be the vertices adjacent to x' in Σ, as shown in Figure 4. Let s', y', t' be the antipodes of s, x', t in Σ, respectively. Since antipodal vertices have the same marking, the signs of all the edges in the $y - y'$ path in Σ not containing x are the same sign as that of the edges incident with x. So, the pair of edges $\{s, x'\}, \{s'y'\}$ has the same sign and the pair of edges $\{x', t\}, \{y', t'\}$ has the same sign. Since $\mu(x') = \mu(y') = -1$, each of the cycles generated by introducing the antipodal edge $\{x', y'\}$ in Σ will have an odd number of vertices with a mark of -1 in Σ_d^A, and hence, these cycles are not consistent in Σ_d^A. So, Σ_d^A is not canonically consistent. Hence, the proof is complete. □

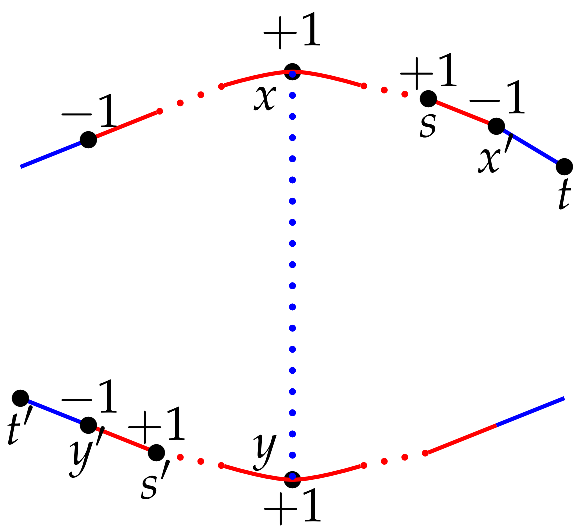

Figure 4. Mark distribution after introducing antipodal edge $\{x, y\}$ in Σ.

Corollary 3. *Let $\Sigma = (C_{2n}, \sigma)$ be balanced. Then, Σ_d^A is balanced and canonically consistent if and only if either of the following holds:*

1. $\Sigma = C_{2n}^-$ *and n is even.*
2. $\Sigma = C_{2n}^+$.

Theorem 4. *Let $\Sigma = (C_{2n+1}, \sigma)$ and $n > 1$. Then, Σ_d^A is canonically consistent if and only if $\mu_d(u) = +1$ for every vertex u in Σ_d^A.*

Proof. Let $\Sigma = (C_{2n+1}, \sigma)$. If $\mu_d(u) = +1$ for every vertex u in Σ_d^A, then obviously Σ_d^A is consistent.

Conversely, let Σ_d^A have a vertex u with $\mu_d(u) = -1$. We need to show that Σ_d^A is not consistent. We shall prove this by the method of contradiction. If possible, suppose that Σ_d^A is consistent. Let v, w be the antipodes of u. Let x be the vertex adjacent to v other than w and let y be the vertex adjacent to w other than v in Σ.

Let C_1 be the cycle in Σ_d^A consisting of the antipodal edge $\{u, v\}$ and let $u - v$ be the sub-path of Σ containing x. Let C_2 be the cycle in Σ_d^A consisting of the antipodal edge $\{u, w\}$ and let $u - w$ be the sub-path of Σ containing y. Let C_3 be the triangle $uvwu$. As Σ_d^A is consistent, each C_1, C_2, C_3 and Σ are consistent. Now, C_3 is consistent, which implies $\mu_d(v)\mu_d(w) = -1$. Without a loss of generality, let $\mu_d(v) = -1$. Then, $\mu_d(w) = +1$ and the $u - v$ sub-path in Σ containing x should have an even number of vertices with a mark of -1 in Σ_d^A. In addition, the $u - w$ sub-path in Σ containing y should have an even number of vertices with a mark of -1 in Σ_d^A. Therefore, the number of vertices in Σ_d^A with a mark of -1 is odd and hence $\mu_d(\Sigma) = -1$, which is a contradiction. Hence, Σ_d^A cannot be consistent. □

Proposition 6. *Let $\Sigma = (C_{2n+1}, \sigma)$ be balanced and $n > 1$. If Σ has three consecutive edges with signs in the order (i) $-1, +1, +1$ or (ii) $+1, -1, +1$ or (iii) $-1, -1, -1$, then Σ_d^A is not consistent.*

Proof. Consider four vertices v_1, v_2, v_3, v_4 in Σ such that

$$\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\} \in E(\Sigma).$$

Let the antipodes of v_1, v_2, v_3, v_4 be $a, b;\ b, c;\ c, d;\ d, e$, respectively.
(i) Let $\sigma(\{v_1, v_2\}) = -1$, $\sigma(\{v_2, v_3\}) = +1$, $\sigma(\{v_3, v_4\}) = +1$. Then, by Theorem 4

$$\mu_d(v) = +1, \forall v \in V(\Sigma_d^A)$$

and so $\sigma_d(\{v_3, c\}) = \sigma_d(\{v_3, d\})$. If possible, let Σ_d^A be consistent.

Case 1. $\sigma_d(\{v_3, c\}) = \sigma_d(\{v_3, d\}) = +1$. Since Σ is balanced, so $\sigma_d(\{d, c\}) = +1$. Now, $\sigma_d(\{v_3, c\}) = \sigma_d(\{d, c\}) = \sigma_d(\{v_2, v_3\}) = +1$ implies that $\sigma_d(\{v_2, c\}) = +1$ and using the consistency of Σ_d^A, we conclude $\sigma_d(\{v_2, b\}) = -1$. So, $\sigma_d(\{b, c\}) = -1$. Hence, $\mu_d(c) = -1$, which is a contradiction to Theorem 4. Thus, Σ_d^A is not consistent. (Figure 5 is a representation of this case.)

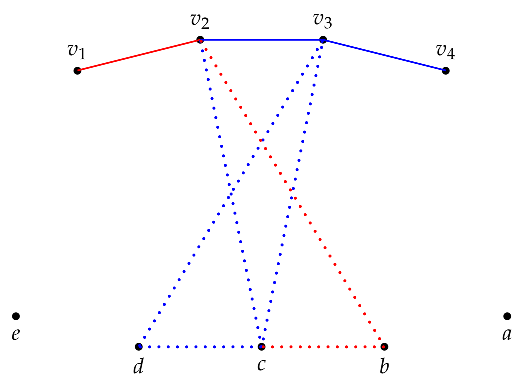

Figure 5. Representative diagram: Case 1.

Case 2. $\sigma_d(\{v_3, c\}) = \sigma_d(\{v_3, d\}) = -1$. Since Σ is balanced, so $\sigma_d(\{d, c\}) = +1$. Now, $\sigma_d(\{v_3, v_2\}) = \sigma_d(\{d, c\}) = +1$ and $\sigma_d(\{v_3, d\}) = -1$ implies that $\sigma_d(\{v_2, c\}) = -1$ and using the consistency of Σ_d^A, we conclude $\sigma_d(\{v_2, b\}) = +1$. So, $\sigma_d(\{b, c\}) = -1$ (Refer to the Figure 6). Hence, $\mu_d(c) = -1$, which is a contradiction to Theorem 4. Thus, Σ_d^A is not consistent.

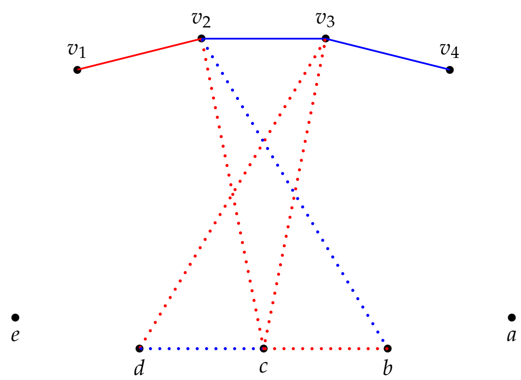

Figure 6. Representative diagram: Case 2.

(ii) Let $\sigma(\{v_1, v_2\}) = +1,\ \sigma(\{v_2, v_3\}) = -1,\ \sigma(\{v_3, v_4\}) = +1$. Then, by Theorem 4

$$\mu_d(v) = +1, \forall v \in V(\Sigma_d^A)$$

and so $\sigma_d(\{v_2, c\})\sigma_d(\{v_2, b\}) = -1$. If possible, let Σ_d^A be consistent.

Case 1: $\sigma_d(\{v_2, c\}) = -1$ and $\sigma_d(\{v_2, b\}) = +1$. Since Σ is balanced, so $\sigma_d(\{b, c\}) = -1$. Therefore, $\sigma_d(\{v_3, c\}) = +1$ and a consistency of Σ_d^A implies $\sigma_d(\{v_3, d\}) = -1$. Hence, $\sigma_d(\{d, c\}) = +1$, which in turn implies that $\sigma_d(\{v_3, d\}) = +1$. Therefore, $\mu_d(v_3) = -1$.

Case 2: $\sigma_d(\{v_2, c\}) = +1$ and $\sigma_d(\{v_2, b\}) = -1$. In this case also, applying an argument similar to Case 1, we can show that $\mu_d(v_3) = -1$.

Hence, in either case, Σ_d^A has a vertex with a sign of -1, which is a contradiction. So, Σ_d^A is not consistent.

(iii) Let $\sigma(\{v_1, v_2\}) = -1,\ \sigma(\{v_2, v_3\}) = -1,\ \sigma(\{v_3, v_4\}) = -1$. Then, by Theorem 4

$$\mu_d(v) = +1, \forall v \in V(\Sigma_d^A)$$

and so $\sigma_d(\{v_2, c\}) = \sigma_d(\{v_2, b\})$. If possible, let Σ_d^A be consistent. Since Σ is balanced, it has at least one edge with sign $+1$.

Case 1: $\sigma_d(\{v_2, c\}) = \sigma_d(\{v_2, b\}) = -1$. In this case, since Σ is balanced and each of the antipodal paths connecting v_2, c and v_2, b has an odd number of negative edges, so $\sigma_d(\{b, c\}) = +1$. Now, $\sigma_d(\{v_2, b\}) = -1$ implies that the antipodal path connecting v_2 and c has an odd number of negative edges and hence the antipodal path joining v_3 and c has an even number of edges with a sign of -1. So, $\sigma_d(\{v_3, c\}) = +1$. Now, the consistency of Σ_d^A implies that $\sigma_d(\{v_3, d\}) = +1$, which in turn implies that $\sigma_d(\{c, d\}) = -1$. Finally, $\sigma_d(\{v_3, c\}) = \sigma_d(\{v_3, d\}) = +1$ implies that each of the antipodal paths connecting v_3, c and v_3, d has an aeven number of edges with a sign of -1. Since the union of the antipodal paths connecting v_3, c and v_3, d together with the edge $\{c, d\}$ is Σ, so $\sigma_d(\Sigma) = -1$, which is a contradiction.

Case 2: $\sigma_d(\{v_2, c\}) = \sigma_d(\{v_2, b\}) = +1$. Proceeding in the way as we have taken in case 1, we can show that Σ is not consistent, which is a contradiction.

Hence, Σ_d^A is not consistent. □

Theorem 5. *Let $\Sigma = (C_{2n+1}, \sigma)$ be balanced and $\Sigma \neq C_{2n+1}^+$. Then, Σ_d^A is canonically consistent if and only if the signs of any three consecutive edges has the pattern either $-1, -1, +1$ or $-1, +1, -1$ or $+1, -1, -1$.*

Proof. First, suppose that Σ has three consecutive edges with a sign pattern different from $-1, -1, +1$; $-1, +1, -1$ and $+1, -1, -1$. Since Σ is balanced, so it must have an edge with sign $+1$. So, Σ must have three consecutive edges with a sign pattern of either $-1, +1, +1$ or $+1, -1, +1$ or $-1, -1, -1$. In each of these cases, Σ_d^A is not consistent.

Conversely, let the sign of the edges follow the given patterns. Then, for every single edge with a sign of $+1$ in the graph, there will be two exclusive edges with a sign of -1. This implies that $2n + 1$ is a multiple of 3. Let $2n + 1 = 3k$. Then, k must be odd and the number of edges in Σ with a sign of -1 is $2k$. We claim that $\sigma_d(v) = +1, \forall v \in \Sigma_d^A$. On the contrary, assume that $\mu_d(v) = -1$ for some $v \in \Sigma_d^A$. Let u, w be the vertices adjacent to v in Σ. Let x, y be the antipodes of v, as shown in Figure 7. We consider the following cases:

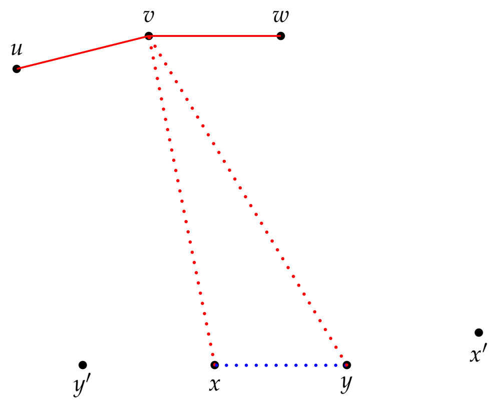

Figure 7. Representation diagram: Case 1.

Case 1: $\sigma_d(\{u,v\}) = \sigma_d(\{v,w\}) = -1$. Then, by the assumption of sign pattern, $\sigma_d(\{x,y\}) = +1$. So, each of the diametric paths connecting v, x and v, y should have an odd number of edges with a sign of -1. Therefore, $\sigma_d(\{v,x\}) = \sigma_d(\{v,y\}) = -1$, which is a contradiction to our assumption that $\mu_d(v) = -1$.

Case 2: $\sigma_d(\{u,v\}) = +1$ and $\sigma_d(\{v,w\}) = -1$. Then, by the assumption of a sign pattern, $\sigma_d(\{x,y\}) = -1$. So, if the diametric path connecting v to x has an odd number of negative edges, then the diametric path connecting v to y should have an even number of negative edges and vice versa. Thus, either $\sigma_d(\{v,x\}) = -1, \sigma_d(\{v,y\}) = +1$ or $\sigma_d(\{v,x\}) = +1, \sigma_d(\{v,y\}) = -1$. In either case, $\mu_d(v) = +1$, which is a contradiction.

Thus, in each case, we have arrived at a contradiction. Hence, Σ_d^A is canonically consistent. $\square$

5. The Smith Graph H_7

Consider the labeling of the vertices of H_7 using $v_1, v_2, v_3, v_4, v_5, v_6, v_7$, as shown in Figure 8. Let $\Sigma = (H_7, \sigma)$. We shall investigate the conditions under which Σ is balanced and canonical consistent.

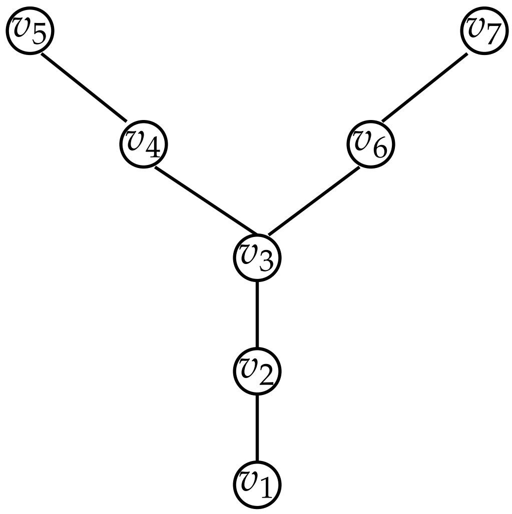

Figure 8. The Smith graph H_7.

Proposition 7. *If $\Sigma = (H_7, \sigma)$, then Σ_d^A is balanced.*

Proof. Since Σ is acyclic, so it is balanced. Let V_1 and V_2 be Harary's partition of vertices. We claim that V_1 and V_2 also serve as Harary's partition of vertices for Σ_d^A. Let a_{15}, a_{17} and a_{57} be the antipodal edges joining the pair of antipodal vertices $v_1, v_5; v_1, v_7$ and v_5, v_7, respectively. Now, $\sigma_d(a_{ij}) = -1$ implies that the path in Σ joining v_i and v_j has an odd number of edges with a sign of -1 and so v_i and v_j belongs to different partitions. In addition, $\sigma_d(a_{ij}) = +1$ implies that the path in Σ joining v_i and v_j has an even number of edges with a sign of -1 and so v_i and v_j belong to the same partition. Hence, V_1, V_2 serves as the desired Harary's partition of Σ_d^A and so it is balanced. $\square$

Lemma 2. *If $\Sigma = (H_7, \sigma)$ has an edge with a sign of -1, then there exists a vertex in Σ_d^A with a mark of -1.*

Proof. Consider the labeling of H_7, as shown in Figure 8. Let e be any edge in Σ with $\sigma(e) = -1$. If possible, let $\mu_d(v) = +1$ for all $v \in V(\Sigma_d^A)$. We consider the following cases:

Case 1: e is a pendant edge. Without loss of generality, let $e = \{v_1, v_2\}$. Then, $\sigma_d(\{v_2, v_3\}) = -1$, otherwise $\mu_d(v_2) = -1$, and hence, exactly one of the edges $\{v_4, v_3\}$, $\{v_6, v_3\}$ has a sign of -1; otherwise, $\mu_d(v_3) = -1$. Without a loss of generality, assume that $\sigma_d(\{v_4, v_3\}) = -1$ and $\sigma_d(\{v_6, v_3\}) = +1$. Then, $\sigma_d(\{v_6, v_7\}) = +1$ and $\sigma_d(\{v_4, v_5\}) = -1$; otherwise, at least one of v_4, v_6 will have a mark of -1. In this case, both the antipodal edges incident with v_1 has a sign of $+1$ and so $\mu_d(v_1) = -1$, which is a contradiction.

Case 2: e is a non-pendant edge. Without a loss of generality, let $e = \{v_3, v_2\}$. Then, $\sigma_d(\{v_2, v_1\}) = -1$; otherwise, $\mu_d(v_2) = -1$. Since $\{v_2, v_1\}$ is a pendant edge, so as in case 1, it will lead to a contradiction.

Hence, the result follows. $\square$

Theorem 6. *If $\Sigma = (H_7, \sigma)$, then Σ_d^A is balanced and canonically consistent if and only if $\Sigma = H_7^+$.*

Proof. By Proposition 7, Σ_d^A is always balanced. So, first suppose that Σ_d^A is canonically consistent. First, we show that $\mu_d(v_3) = +1$. On the contrary, let us assume that $\mu_d(v_3) = -1$. Since Σ_d^A is canonically consistent, so $\mu_d(v_1v_2v_3v_4v_5v_1) = +1$, and this in turn implies that $\mu_d(v_1)\mu_d(v_2) \neq \mu_d(v_4)\mu_d(v_5)$. Similarly,

$$\mu_d(v_1v_2v_3v_6v_7v_1) = +1 \implies \mu_d(v_1)\mu_d(v_2) \neq \mu_d(v_6)\mu_d(v_7).$$

Consequently, $\mu_d(v_4)\mu_d(v_5) = \mu_d(v_6)\mu_d(v_7)$. Then

$$\mu_d(v_5v_4v_3v_6v_7v_5) = \mu_d(v_5)\mu_d(v_4)\mu_d(v_3)\mu_d(v_6)\mu_d(v_7) = -1$$

This implies that Σ_d^A is not consistent, which is a contradiction. So, the only possibility is $\mu_d(v_3) = +1$.

We now claim that the canonical marking of each of the vertices v_1, v_5, v_7 is $+1$. Without a loss of generality, let $\mu_d(v_1) = -1$. Since Σ_d^A is canonically consistent, so the mark of the cycle $v_1v_5v_7v_1$ must be $+1$, and hence $\mu_d(v_5)\mu_d(v_7) = -1$. Then, at least one of the pairs of cycles $v_1v_2v_3v_6v_7v_1$, $v_1v_2v_3v_6v_7v_5v_1$ or $v_1v_2v_3v_4v_5v_1$, $v_1v_2v_3v_4v_5v_7v_1$ has an opposite canonical marking. So, Σ_d^A is not consistent, which is a contradiction. This proves our claim.

We now claim that the canonical marking of each of the vertices v_2, v_4, v_6 is $+1$. Without a loss of generality, let $\mu_d(v_2) = -1$. The canonical consistency of Σ_d^A implies that $\mu_d(v_4) = \mu_d(v_6) = -1$. Now, $\mu_d(v_2) = -1$ implies that $\sigma_d(\{v_1, v_2\}) = -1$ or $\sigma_d(\{v_3, v_2\}) = -1$. If $\sigma_d(\{v_1, v_2\}) = -1$, then $\sigma_d(\{v_5, v_7\}) = -1$. Since $\mu_d(v_5) = \mu_d(v_7) = +1$, so $\sigma_d(\{v_6, v_7\}) = -1$ and $\sigma_d(\{v_4, v_5\}) = +1$. This in turn implies that $\sigma_d(\{v_3, v_4\}) = -1$ and $\sigma_d(\{v_3, v_6\}) = +1$. However, $\sigma_d(\{v_3, v_2\}) = +1$. Therefore, $\mu_d(v_3) = -1$, a contradiction. If $\sigma_d(\{v_3, v_2\}) = -1$, then $\sigma_d(\{v_4, v_3\})\sigma_d(\{v_6, v_3\}) = -1$ because $\sigma_d(v_3) = +1$. Without a loss of generality, let $\sigma_d(\{v_4, v_3\}) = +1$ and $\sigma_d(\{v_6, v_3\}) = -1$. Then, the cycles $v_1v_2v_3v_6v_7v_1$, $v_1v_2v_3v_4v_5v_1$ will have opposite canonical markings, which is a contradiction. So, the canonical marking of each of the vertices v_2, v_4, v_6 must be $+1$.

Thus, each of the vertices in Σ_d^A has a canonical marking of $+1$ and so, by the Lemma 2, $\Sigma = H_7^+$. $\square$

Remark 8. *If $\Sigma = (H_i, \sigma)$, $i = 8, 9$ then Σ_d^A will have exactly one cycle, and so it is always balanced. However, Σ_d^A will be canonically consistent if and only if the non-diametric edge in Σ has a sign of $+1$.*

Double-Headed Snake $(W_n), n \geq 6$

In this section, we consider the d-antipodal graph of the Smith graph W_n, $n \geq 6$. We label the vertices of W_n using $v_1, v_2, \cdots, v_n$, as shown in Figure 9. The antipodal pairs in W_n are $v_1, v_3; v_1, v_4; v_2, v_3; v_2, v_4$ and the antipodal edges are represented by dotted lines. Let $a_{i,j}$ be the antipodal edge joining a pair of antipodes v_i, v_j.

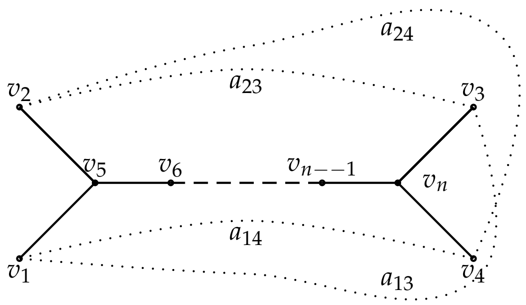

Figure 9. d-antipodal graph of the Smith graph W_n, $n \geq 6$.

The following remark about the d-antipodal graph of the Smith graph W_n, $n \geq 6$ follows from Theorem 3.

Remark 9. *Let $\Sigma = (W_n, \sigma)$, for $n \geq 6$. Then, Σ_d^A is balanced.*

Proposition 8. *If $\Sigma = (W_n, \sigma)$, then Σ_d^A is canonically consistent if and only if all the six edges incident with both the vertices of degree 3 in Σ are of the same sign.*

Proof. First, assume that Σ_d^A is canonically consistent . If possible, let there be a pair of edges e, e' among the six edges incident with the two vertices of degree 3 in Σ that have opposite signs. We consider the following two cases:

Case 1: e and e' are pendant edges with a vertex in common. Without a loss of generality, let $e = \{v_1, v_5\}$ and $e' = \{v_2, v_5\}$. In this case, $\sigma_d(a_{13})\sigma_d(a_{23}) = -1$ and $\sigma_d(a_{14})\sigma_d(a_{24}) = -1$. Therefore,

$$\begin{aligned} \mu_d(v_1) &= \sigma_d(e)\sigma_d(a_{13})\sigma_d(a_{14}) \\ &= [-\sigma_d(e')][-\sigma_d(a_{23})][-\sigma_d(a_{24})] \\ &= -\mu_d(v_2) \end{aligned}$$

Since Σ_d^A is consistent, both the cycles $v_1v_5v_2v_3v_1$ and $v_1v_5v_2v_4v_1$ should have a mark of $+1$, and this is possible only if $\mu_d(v_3) = \mu_d(v_4) \neq \mu_d(v_5)$. Hence,

$$\mu_d(v_1v_3v_2v_4v_1) = \mu_d(v_1)\mu_d(v_3)\mu_d(v_2)\mu_d(v_4)) = \mu_d(v_1)\mu_d(v_2) = -1,$$

which is a contradiction. Therefore, $\sigma(e) = \sigma(e')$—that is, pendant edges in Σ sharing a common vertex should have the same sign.

Case 2: e and e' are pendant edges with no vertex in common. Without a loss of generality, let $e = \{v_1, v_5\}$ and $e' = \{v_n, v_3\}$. By case 1, $\sigma_d(\{v_1, v_5\}) = \sigma_d(\{v_2, v_5\})$ and

$\sigma_d(\{v_n, v_3\}) = \sigma_d(\{v_n, v_4\})$, which in turn implies that all the antipodal edges have the same sign. Hence, $\mu_d(v_1) = \mu_d(v_2)$ and $\mu_d(v_3) = \mu_d(v_4)$. However,

$$\begin{aligned} \mu_d(v_1) &= \sigma_d(e)\sigma_d(a_{13})\sigma_d(a_{14}) \\ &= -\sigma_d(e')\sigma_d(a_{13})\sigma_d(a_{23}) \\ &= -\mu_d(v_3) \end{aligned}$$

Since the mark of the cycle $v_1v_5v_2v_3v_1$ is $+1$, so $\mu_d(v_5) = +1$. In addition, the mark of the cycle $v_3v_nv_4v_2v_3$ is $+1$, so $\mu_d(v_n) = -1$. Therefore, the mark of the cycle $v_1v_5v_2v_4v_nv_3v_1$ is -1, which is a contradiction. Hence, all the four pendant edges should have the same sign in Σ.

Case 3: e and e' have a vertex in common, and only one of these edges is a pendant. Without a loss of generality, let $e = \{v_1, v_5\}$ and $e' = \{v_5, v_6\}$. By case 2, all pendant edges have the same sign and all the four vertices v_1, v_2, v_3, v_4 have the same marking. Therefore, $\mu_d(v_1) \neq \mu_d(v_5)$ and so the mark of the cycle $v_1v_5v_2v_3v_1$ is -1, which is a contradiction. Hence, the result follows.

Conversely, let all the edges in Σ incident with both the vertices of degree 3 have the same sign. Then,

$$\mu_d(v_i) = +1, \text{ for } i = 1, 2, 3, 4, 5, n.$$

Since each cycle in Σ_d^A consists of either an even number of vertices entirely from the set $\{v_i : i = 1, 2, 3, 4, 5, n\}$ or an even number of vertices from the set $\{v_i : i = 1, 2, 3, 4, 5, n\}$ and the vertices on the path connecting v_6 to v_{n-1}, so the mark of each cycle in Σ_d^A must be positive. Hence, Σ_d^A is canonically consistent. $\square$

6. Conclusions

In this paper, the concept of d-antipodal signed graphs has been introduced. The underlying graph of the d-antipodal signed graph of a signed graph $\Sigma = (G, \sigma)$ is the union of G and its antipodal graph $A(G)$. The sign assignment to the antipodal edges is inspired by similar works available in the literature. We have characterized balanced and canonically consistent Smith signed graphs whose d-antipodal signed graphs are balanced and canonically consistent. In particular, it is shown that

1. For any sign assignment to the cycle C_{2n+1}, its d-antipodal is canonically consistent and each vertex in the d-antipodal has a sign of $+1$.
2. For any sign assignment to the double-headed snake W_n, its d-antipodal is canonically consistent if and only if both the degree three vertices are of the same sign.
3. For any sign assignment to the Smith graph H_7, its d-antipodal is canonically consistent if and only if all its edges have a sign of $+1$.

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