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Backward Stackelberg Games with Delay and Related Forward–Backward Stochastic Differential Equations

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Abstract: In this paper, we study a kind of Stackelberg game where the controlled systems are described by backward stochastic differential delayed equations (BSDDEs). By introducing a new kind of adjoint equation, we establish the sufficient verification theorem for the optimal strategies of the leader and the follower in a general case. Then, we focus on the linear–quadratic (LQ) backward Stackelberg game with delay. The backward Stackelberg equilibrium is presented by the generalized fully coupled anticipated forward–backward stochastic differential delayed Equation (AFBSDDE), which is composed of anticipated stochastic differential equations (ASDEs) and BSDDEs. Moreover, we obtain the unique solvability of the AFBSDDE using the continuation method. As an application of the theoretical results, the pension fund problem with delay effect is considered.

Keywords: Stackelberg game; state delay; forward–backward stochastic differential equation; linear–quadratic problem

MSC: 60H10; 49N70; 93E20

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1. Induction

Stackelberg games, also called leader–follower games, were introduced in [1] in order to investigate the optimal strategies in competitive systems in which there exist two asymmetric players. A Stackelberg game is a non-zero-sum game that assumes that one of the players (the follower) waits until the other player (the leader) announces their strategy, i.e., the follower chooses a strategy to optimize their own cost functional with respect to a given leader's policy. Meanwhile, with the knowledge of the follower's response, the leader will make a decision to optimize their own cost functional.

In recent years, considerable attention has been given to Stackelberg games. Under the deterministic framework, Simaan and Cruz [2] studied the Nash equilibrium and Tolwinski [3] obtained the closed-loop solution to a multistage LQ Stackelberg game. For the stochastic Stackelberg games, Øksendal et al. [4] established the maximum principle for a general stochastic Stackelberg differential game and applied it to news vendor problems. Bensoussan et al. [5] derived the maximum principle for the global solutions. They introduced several information structures, and obtained the optimal strategies of the follower and the leader under the adapted open-loop and adapted closed-loop memoryless information structures. Ref. [6] also studied the Stackelberg games under tow stochastic settings taken from [5], however, the diffusion term in their system allowed them to depend on the control variable. Shi et al. [7,8] studied the Stackelberg games with asymmetric information. Yong [9] illustrated the existence and uniqueness of the LQ Stackelberg game by the unique solvability of a stochastic Riccati equation. Wang and Zhang [10] considered a mean-field LQ Stackelberg differential game with one leader and two followers. Li et al. [11] proposed a kind of LQ Stackelberg game with multilevel hierarchy. They presented the Stackelberg equilibrium by forward–backward stochastic differential equations (FBSDEs) in a closed form.

On the other hand, controlled systems with delay have also received significant attention due to the path-dependence of the state and the time lag between the input and output (see [12–16]). Compared with the system driven by the Wiener process or the non-Gaussian process such as Rosenblatt process, the references [17–21] discussed Stackelberg games with different delays which are based on the system driven by a standard Brownian motion. In [17], the open-loop Stackelberg strategies were obtained for a deterministic LQ game with state delay. In [18,20], the authors were concerned with the games with delay appearing in the leader's control. The open-loop strategies were explicitly given by some symmetric Riccati equations. In [19], the authors considered a linear–quadratic mean-field game between a leader and a group of followers, which may arise from an advertising model and an interbank lending and borrowing model with delay. Finally, in [21], the authors considered a general model in which the system involves both state delay and control delay. Under some assumptions, the state feedback representation of the Stackelberg strategy was derived.

The aforementioned papers are all based on the forward system. In addition to forward systems, controlled backward systems can also be encountered in optimal control problems and have a wide range of applications in finance and related fields. In particular, we will have a controlled BSDDE when considering delayed recursive utility or the pricing of derivative security for which the underlying goods are described by stochastic differential delayed equations (SDDEs). One can refer to [22–26]. Zheng and Shi [27] introduced a Stackelberg game described by a backward stochastic differential Equation (BSDE) and applied the theoretical results to solve a pension fund problem.

In this paper, we shall study the backward Stackelberg game with state delay. To the best of our knowledge, it is the first attempt to address this problem. The problem with delay remains challenging since the state feedback representation cannot be easily obtained in general cases. In order to obtain the optimal feedback of a LQ optimal control problem of BSDDEs, Meng and Shi [26] introduced a new class of delayed Riccati equations, which are very complicated to compute. In the Stackelberg game, the controlled system of the leader is more complex, which is described by coupled BSDDE and ASDE. Therefore, this paper focuses more on the solvability of the problem, rather than the feedback form of the control. The sufficient conditions of the backward equilibrium are given by the corresponding generalized Hamilton system composed of ASDE and BSDDE. For the LQ backward Stackelberg game with delay, we obtain that the solvability of the game is equivalent to that of a related AFBSDDE. Furthermore, we discuss the unique solvability of the AFBSDDE.

The rest of this paper is organized as follows. Section 2 is devoted to presenting the sufficient conditions of the follower and the leader in the general backward Stackelberg game with delay. In Section 3, the LQ backward Stackelberg game with delay is investigated. A new kind of AFBSDDE is used to characterize the unique equilibrium of the game. In Section 4, as an application, a pension fund problem in markets with delayed effects is discussed. Some conclusions and unsolved issues for future research are displayed in Section 5.

To conclude this section, we introduce some notations which will be used throughout this paper. Let $\mathbb{R}^k$ be the k-dimensional Euclidean space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. Let $\mathbb{R}^{k\times d}$ be the Hilbert space consisting of all $k \times d$ matrices with the inner product $\langle A, B\rangle := \mathrm{tr}\{AB^\top\}$, for any $A, B \in \mathbb{R}^{k\times d}$, where $\top$ appears in the superscript and denotes the transposition of a matrix. In particular, $\mathcal{S}^n$ denotes the set of all $n \times n$ symmetric matrices.

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete probability space equipped with natural filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ generated by a d-dimensional standard Brownian motion $W(\cdot)$. Assume that $\mathcal{F}_t = \sigma\{W(s), 0 \leq s \leq t\} \vee \mathcal{N}$, where $\mathcal{N}$ denotes the totality of P-null sets. $T > 0$ is a finite time duration, and $0 \leq \delta < T$ is a time delay. We define $\mathcal{F}_t \equiv \mathcal{F}_0$ for all $t \in [-\delta, 0)$ and $\mathcal{F} = \mathcal{F}_{T+\delta}$. We introduce some spaces which will be used throughout this paper as follows:

$$L^p(\mathcal{F}_T, \mathbb{R}^k) = \Big\{\xi : \xi \text{ is } \mathbb{R}^k \text{ valued-}\mathcal{F}_T\text{-measurable random variable such that } \mathbb{E}|\xi|^p < +\infty\Big\},$$
$$L^p_{\mathbb{F}}(a,b;\mathbb{R}^k) = \Big\{\xi(t) : \xi(t) \text{ is } \mathbb{R}^k\text{-valued } \mathbb{F}\text{-adapted process such that } \mathbb{E}\int_a^b |\xi(t)|^p dt < +\infty\Big\},$$
$$\mathcal{S}^p_{\mathbb{F}}(a,b;\mathbb{R}^k) = \Big\{\xi(t) : \xi(t) \text{ is } \mathbb{R}^k\text{-valued } \mathbb{F}\text{-adapted process such that } \mathbb{E}\sup_{a\leq t\leq b} |\xi(t)|^p < +\infty\Big\},$$
$$L^\infty_{\mathbb{F}}(a,b;\mathbb{R}^k) = \Big\{\xi(t) : \xi(t) \text{ is } \mathbb{R}^k\text{-valued } \mathbb{F}\text{-adapted bounded process}\Big\},$$

where $p \geqslant 1$ is a real number.

2. Backward Stackelberg Game with Delay

Let us consider a backward Stackelberg game with a time delay involving two players. For the sake of simplicity, we only consider the following BSDDE driven by a standard Brownian motion $W(\cdot)$:

$$\begin{cases} -dy(t) = & g(t, y(t), y(t-\delta), z(t), z(t-\delta), u(t), w(t))dt - z(t)dW(t),\ t \in [0,T], \\ \quad y(T) = & \gamma,\ y(t) = \eta(t),\ z(t) = \zeta(t),\ t \in [-\delta, 0), \end{cases} \tag{1}$$

where $(y(\cdot), z(\cdot)) \in \mathbb{R}^n \times \mathbb{R}^{n\times d}$ is the state process pair with terminal condition γ and initial path $(\eta(\cdot), \zeta(\cdot))$. We assume that $g : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^{n\times d} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}^n$ is a given function and $\eta(\cdot), \zeta(\cdot)$ are continuous functions on $[-\delta, 0]$. $u(\cdot)$ and $w(\cdot)$ constitutes the control decisions of player 1 (the follower) and player 2 (the leader) with values in U_1 and U_2. Additionally, U_1 and U_2 are non-empty convex sub-sets of $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, respectively.

The admissible strategy sets for the follower and the leader are denoted by

$$\mathcal{U}_1 := \{u(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{m_1}) | u(\cdot) \in U_1,\ a.e.\ a.s.\},$$
$$\mathcal{U}_2 := \{w(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{m_2}) | w(\cdot) \in U_2,\ a.e.\ a.s.\}.$$

We suppose that the coefficients satisfy the following assumptions **(H1)** :

(H1.1) $\eta(\cdot), \zeta(\cdot) \in L^2_{\mathbb{F}}(-\delta, 0; \mathbb{R}^n)$, $\gamma \in L^2(\mathcal{F}_T; \mathbb{R}^n)$;

(H1.2) For any $y, y_\delta \in \mathbb{R}^n, z, z_\delta \in \mathbb{R}^{n\times d}$ and $u \in \mathbb{R}^{m_1}, w \in \mathbb{R}^{m_2}$, the mapping $g(\cdot, y, y_\delta, z, z_\delta, u, w)$ is $\mathbb{F}$-adapted;

(H1.3) $g(\cdot, 0, 0, 0, 0, 0, 0) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$;

(H1.4) g is twice continuously differentiable with respect to $(y, y_\delta, z, z_\delta, u, w)$ and the derivatives up to two orders of g are uniformly bounded.

The cost functionals of the follower and the leader to minimize, respectively, are described as follows:

$$\begin{cases} J_1(\gamma; u(\cdot), w(\cdot)) = \mathbb{E}[\int_0^T l_1(t, y(t), z(t), u(t), w(t))dt + G_1(y(0))], \\ J_2(\gamma; u(\cdot), w(\cdot)) = \mathbb{E}[\int_0^T l_2(t, y(t), z(t), u(t), w(t))dt + G_2(y(0))]. \end{cases} \tag{2}$$

where $l_i : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ and $G_i : \Omega \times \mathbb{R}^n \to \mathbb{R} (i = 1,2)$ are also given continuous functions.

Now, we give the assumptions **(H2)** of the cost functionals for $i = 1, 2$.

(H2.1) For any $y \in \mathbb{R}^n$, $z \in \mathbb{R}^{n\times d}$, $u \in \mathbb{R}^{m_1}, w \in \mathbb{R}^{m_2}$, the mapping $l_i(\cdot, y, z, u, w)$ is $\mathbb{F}$-adapted and $G_i(y)$ is $\mathcal{F}_0$-measurable;

(H2.2) $l_i(\cdot, 0, 0, 0, 0) \in L^1_{\mathbb{F}}(0,T;\mathbb{R})$ and $G_i(0) \in L^1(\mathcal{F}_0;\mathbb{R})$;

(H2.3) l_i is twice continuously differentiable with respect to (y, z, u, w) and G_i is twice continuously differentiable with respect to y. The derivatives up to two orders of l_i and G_i are uniformly bounded;

(H2.4) $l_{iy}, l_{iz}, l_{iu}, l_{iw}$ are bounded by $C(1 + |y| + |z| + |u| + |w|)$ and G_{iy} is bounded by $C(1 + |y|)$.

From Chen and Huang [24], it follows that the BSDDE (1) admits a unique pair of solutions $(y(\cdot), z(\cdot)) \in L^2_{\mathbb{F}}(-\delta, T; \mathbb{R}^n) \times \mathcal{S}^2_{\mathbb{F}}(-\delta, T; \mathbb{R}^{n\times d})$ for a sufficiently small time delay δ and any $(u(\cdot), w(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$. The functional $J_i (i = 1, 2)$ in (2) is also well defined.

The main feature of the Stackelberg game is that the follower makes their decision according to the leader's strategy. In our backward framework, for any $w(\cdot) \in \mathcal{U}_2$ of the leader and given terminal condition $\gamma \in L^2(\mathcal{F}_T; \mathbb{R}^n)$, the follower would like to choose $u^*(\cdot) \in \mathcal{U}_1$ such that $J_1(\gamma; u^*(\cdot), w(\cdot))$ is the minimum of $J_1(\gamma; u(\cdot), w(\cdot))$ over $u(\cdot) \in \mathcal{U}_1$. The rational choice of the leader is determined by knowing the follower's optimal control $u^*(\cdot)$. The leader would like to choose $w^*(\cdot) \in \mathcal{U}_2$ to minimize $J_2(\gamma; u^*(\cdot), w(\cdot))$ over $w(\cdot) \in \mathcal{U}_2$. Then, we propose the definition of the backward Stackelberg equilibrium similarly to [27].

Definition 1. *The pair $(u^*(\cdot), w^*(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ is called the Stackelberg equilibrium of the backward Stackelberg game with delay, if it satisfies the following conditions. First, for a given $\gamma \in L^2(\mathcal{F}_T, \mathbb{R}^n)$ and any $w(\cdot) \in \mathcal{U}_2$, there exists a map $\Gamma : \mathcal{U}_2 \times L^2(\mathcal{F}_T, \mathbb{R}^n) \to \mathcal{U}_1$ such that*

$$J_1(\gamma; \Gamma(w(\cdot), \gamma), w(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}_1} J_1(\gamma; u(\cdot), w(\cdot)).$$

Secondly , there exists $w^*(\cdot) \in \mathcal{U}_2$, such that

$$J_2(\gamma; \Gamma(w^*(\cdot), \gamma), w^*(\cdot)) = \inf_{w(\cdot)\in\mathcal{U}_2} J_2(\gamma; \Gamma(w(\cdot), \gamma), w(\cdot)),$$

and the optimal strategy of the follower is $u^*(\cdot) = \Gamma(w^*(\cdot), \gamma)$.

2.1. Optimization for the Follower

For a given $\gamma \in L^2(\mathcal{F}_T, \mathbb{R}^n)$ and $w(\cdot) \in \mathcal{U}_2$, the follower faces an optimal control problem as follows:

Problem $(BSG)_f$

$$\inf_{u\in\mathcal{U}_1} J_1(\gamma; u(\cdot), w(\cdot)),$$

$$s.t.\begin{cases} -dy(t) = & g(t, y(t), y(t-\delta), z(t), z(t-\delta), u(t), w(t))dt - z(t)dW(t),\ t \in [0, T], \\ \quad y(T) = & \gamma,\ y(t) = \eta(t),\ z(t) = \zeta(t),\ t \in [-\delta, 0). \end{cases}$$

Due to the adjoint relationship between BSDDE and ASDE, we introduce the following adjoint ASDE:

$$\begin{cases} dx(t) = & \{g_y^\top(t, y(t), y(t-\delta), z(t), z(t-\delta), u^*(t), w(t))x(t) \\ & + \mathbb{E}^{\mathcal{F}_t}[g_{y_\delta}^\top(t, y(t), y(t-\delta), z(t), z(t-\delta), u^*(t), w(t))|_{t+\delta}x(t+\delta)] \\ & - l_{1y}(t, y(t), z(t), u^*(t), w(t))\}dt \\ & + \{g_z^\top(t, y(t), y(t-\delta), z(t), z(t-\delta), u^*(t), w(t))x(t) \\ & + \mathbb{E}^{\mathcal{F}_t}[g_{z_\delta}^\top(t, y(t), y(t-\delta), z(t), z(t-\delta), u^*(t), w(t))|_{t+\delta}x(t+\delta)] \\ & - l_{1z}(t, y(t), z(t), u^*(t), w(t))\}dW(t),\ t \in [0, T], \\ x(0) = & -G_{1y}(y(0)),\ x(t) = 0,\ t \in (T, T+\delta]. \end{cases} \tag{3}$$

Here, g_{y_δ} and g_{z_δ} denote the partial derivatives of g with respect to $y(t-\delta)$ and $z(t-\delta)$, respectively. $g_{y_\delta}(t, y(t), y(t-\delta), z(t), z(t-\delta), u^*(t), w(t))|_{t+\delta}$ denotes the value of

g_{y_δ} when time t takes value $t+\delta$, i.e., $g_{y_\delta}(t, y(t), y(t-\delta), z(t), z(t-\delta), u^*(t), w(t))|_{t+\delta} = g_{y_\delta}(t+\delta, y(t+\delta), y(t), z(t+\delta), z(t), u^*(t+\delta), w(t+\delta))$.

Remark 1. *The assumptions (H1)–(H2) imply that BSDDE (1) admits a unique solution for $(u(\cdot), w(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, the cost functional (2) is well defined, and ASDE (3) admits a unique solution for $(u^*(\cdot), w(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ and the sufficiently small time delay δ by Theorem 2.2 in [24].*

We define the Hamiltonian function $H_1 : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^{n\times d} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^n \to \mathbb{R}$ by

$$H_1(t, y, y_\delta, z, z_\delta, u, w, x) = l_1(t, y, z, u, w) - \langle g(t, y, y_\delta, z, z_\delta, u, w), x\rangle. \tag{4}$$

Then, the adjoint Equation (3) can be rewritten as the following Hamiltonian type:

$$\begin{cases} dx(t) =\{-H_{1y}(t) - \mathbb{E}^{\mathcal{F}_t}[H_{1y_\delta}(t)|_{t+\delta}]\}dt \\ \qquad\qquad + \{-H_{1z}(t) - \mathbb{E}^{\mathcal{F}_t}[H_{1z_\delta}(t)|_{t+\delta}]\}dW(t),\ t \in [0, T], \\ x(0) = - G_{1y}(y(0)),\ x(t) = 0,\ t \in (T, T+\delta], \end{cases} \tag{5}$$

where $H_{1k}(t) = H_{1k}(t, y(t), y(t-\delta), z(t), z(t-\delta), u^*(t), w(t), x(t)), k = y, y_\delta, z, z_\delta, u$.

Now, we can give the sufficient conditions to ensure the optimality of the follower's optimal control problem using Theorem 3.2 in [24].

Theorem 1. *Let (H1)–(H2) hold and $x(\cdot)$ be the solution of adjoint Equation (3) associated with $(u^*(\cdot), w(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$. Suppose that $H_1(t, \cdot, \cdot, \cdot, \cdot, \cdot, w(t), x(t))$ is convex in $(y, y_\delta, z, z_\delta, u)$ for any $t \in [0, T]$ and $G_1(\cdot)$ is convex in y. If*

$$\langle H_{1u}(t), u - u^*(t)\rangle \geqslant 0,\ \forall u \in U_1,\ a.e., a.s. \tag{6}$$

holds, then $u^*(\cdot)$ is an optimal strategy of the follower's Problem $(BSG)_f$.

Remark 2. *What we need to clarify is that $(y(\cdot), z(\cdot))$ and $x(\cdot)$ in Equation (3) and inequality (6) should be the corresponding solutions with respect to $(u^*(\cdot), w(\cdot))$.*

2.2. Optimization for the Leader

Since the follower has adopted strategy $u^*(t)$, then we can obtain the following control problem of the leader.

Problem $(BSG)_l$**:**

$$\inf_{w\in\mathcal{U}_2} J_2(\gamma; u^*(\cdot), w(\cdot)),$$

$$s.t. \begin{cases} -dy(t) = & g(t, y(t), y(t-\delta), z(t), z(t-\delta), u^*(t), w(t))dt - z(t)dW(t),\ t \in [0, T], \\ dx(t) = & \{-H_{1y}(t) - \mathbb{E}^{\mathcal{F}_t}[H_{1y_\delta}(t)|_{t+\delta}]\}dt \\ & + \{-H_{1z}(t) - \mathbb{E}^{\mathcal{F}_t}[H_{1z_\delta}(t)|_{t+\delta}]\}dW(t),\ t \in [0, T], \\ y(T) = & \gamma,\ y(t) = \eta(t),\ z(t) = \zeta(t),\ t \in [-\delta, 0), \\ x(0) = & - G_{1y}(y(0)),\ x(t) = 0,\ t \in (T, T+\delta]. \end{cases} \tag{7}$$

Remark 3. *We should stress that, in Equation (7), the backward equation is a BSDDE and the forward equation is an ASDE. This is different from the results in the existing research literature, which are focused on forward–backward controlled systems with forward SDDE and anticipated backward stochastic differential Equation (ABSDE) (see [28–31]).*

We define the Hamiltonian function of the leader by

$$\begin{aligned} &H_2(t, y, y_\delta, z, z_\delta, x, x_\delta, u^*, w, \xi, p, q) \\ &= l_2(t, y, z, u^*, w) + \langle p, b(t, \Sigma, x, x_\delta, u^*, w)\rangle + \langle q, \sigma(t, \Sigma, x, x_\delta, u^*, w)\rangle - \langle \xi, g(t, \Sigma, u^*, w)\rangle. \end{aligned} \tag{8}$$

Here,

$$
\begin{aligned}
& b(t,\Sigma(t),x(t),\mathbb{E}^{\mathcal{F}_t}[x(t+\delta)],u^*(t),w(t)) \\
= \; & g_y^\top(t,y(t),y(t-\delta),z(t),z(t-\delta),u^*(t),w(t))x(t) \\
& +\mathbb{E}^{\mathcal{F}_t}[g_{y_\delta}^\top(t,y(t),y(t-\delta),z(t),z(t-\delta),u^*(t),w(t))|_{t+\delta}x(t+\delta)] \\
& -l_{1y}(t,y(t),z(t),u^*(t),w(t)),
\end{aligned}
$$

$$
\begin{aligned}
& \sigma(t,\Sigma(t),x(t),\mathbb{E}^{\mathcal{F}_t}[x(t+\delta)],u^*(t),w(t)) \\
= \; & g_z^\top(t,y(t),y(t-\delta),z(t),z(t-\delta),u^*(t),w(t))x(t) \\
& +\mathbb{E}^{\mathcal{F}_t}[g_{z_\delta}^\top(t,y(t),y(t-\delta),z(t),z(t-\delta),u^*(t),w(t))|_{t+\delta}x(t+\delta)] \\
& -l_{1z}(t,y(t),z(t),u^*(t),w(t)),
\end{aligned}
$$

with $\Sigma(t)=(y(t),y(t-\delta),z(t),z(t-\delta))$.

In order to give the sufficient conditions for the optimality of the leader's strategy, we need the following additional assumptions.

(H3) $G_1(y)=\frac{1}{2}Ky^\top y$, for $K\in\mathbb{R}^{n\times n}$.

(H4) $H_2(t,\cdot,\cdot,\cdot,\cdot,\cdot,u^*(t),\cdot,\xi(t),p(t),q(t))$ is convex in $(y,y_\delta,z,z_\delta,x,x_\delta,w)$ and $G_2(\cdot)$ is convex in y for any $t\in[0,T]$.

Theorem 2. *Let (H1)–(H4) hold. Assume that the follower adopted their optimal strategy $u^*(t)\in\mathcal{U}_1$, and $w^*(\cdot)$ is an admissible control of the leader. $(y^*(\cdot),z^*(\cdot),x^*(\cdot))$ is the corresponding solution with respect to $(u^*(\cdot),w^*(\cdot))$ with $y^*(T)=\gamma$. We also assume that $(p(t),q(t),\xi(t))$ is the unique solution of the following adjoint equation:*

$$
\begin{cases}
-dp(t)= & \{H_{2x}(t)+H_{2x_\delta}(t)|_{t-\delta}\}dt-q(t)dW(t),\ t\in[0,T], \\
d\xi(t)= & \{-H_{2y}(t)-\mathbb{E}^{\mathcal{F}_t}[H_{2y_\delta}(t)|_{t+\delta}]\}dt \\
& +\{-H_{2z}(t)-\mathbb{E}^{\mathcal{F}_t}[H_{2z_\delta}(t)|_{t+\delta}]\}dW(t),\ t\in[0,T], \\
p(T)= & 0,\ p(t)=0,\ q(t)=0,\ t\in[-\delta,0), \\
\xi(0)= & G_{1yy}^\top(y^*(0))p(0)-G_{2y}(y^*(0)),\ \xi(t)=0,\ t\in(T,T+\delta],
\end{cases}
\tag{9}
$$

where

$$H_{2k}(t)=H_{2k}(t,\Sigma^*(t),x^*(t),\mathbb{E}^{\mathcal{F}_t}[x^*(t+\delta)],u^*(t),w^*(t),p(t),q(t),\xi(t)),$$

with $k=y,y_\delta,z,z_\delta,x,x_\delta,w$. Then, $w^*(\cdot)$ is an optimal strategy of the leader if it satisfies

$$\langle H_{2w}(t),w-w^*(t)\rangle\geqslant 0,\ \forall w\in U_2, a.e., a.s.. \tag{10}$$

Proof. Let $w(\cdot)$ be an arbitrary admissible control of the leader and $\Sigma(t)$ be the corresponding trajectory.

We consider

$$J_2(\gamma,;u^*(\cdot),w^*(t))-J_2(\gamma;u^*(\cdot),w(\cdot))=I+II,$$

where

$$
\begin{aligned}
I= & \ \mathbb{E}\int_0^T[l_2(t,y^*(t),z^*(t),u^*(t),w^*(t))-l_2(t,y(t),z(t),u^*(t),w(t))]dt, \\
II= & \ \mathbb{E}[G_2(y^*(0)-G_2(y(0))].
\end{aligned}
$$

By the convexity of H_2 and G_2, we have

$$\begin{aligned}
I = & \ \mathbb{E}\int_0^T [H_2(t,\Sigma^*(t),x^*(t),\mathbb{E}^{\mathcal{F}_t}[x^*(t+\delta)],u^*(t),w^*(t),\xi(t),p(t),q(t)) \\
& - H_2(t,\Sigma(t),x(t),\mathbb{E}^{\mathcal{F}_t}[x(t+\delta)],u^*(t),w(t),\xi(t),p(t),q(t))]dt \\
& - \mathbb{E}\int_0^T \Big\{ \langle p(t), b(t,\Sigma^*(t),x^*(t),\mathbb{E}^{\mathcal{F}_t}[x^*(t+\delta)],u^*(t),w^*(t)) \\
& - b(t,\Sigma(t),x(t),\mathbb{E}^{\mathcal{F}_t}[x(t+\delta)],u^*(t),w(t))\rangle \\
& + \mathrm{tr}\{q^T(t)\big(\sigma(t,\Sigma^*(t),x^*(t),\mathbb{E}^{\mathcal{F}_t}[x^*(t+\delta)],u^*(t),w^*(t)) \\
& - \sigma(t,\Sigma(t),x(t),\mathbb{E}^{\mathcal{F}_t}[x(t+\delta)]u^*(t),w(t))\big)\} \\
& - \langle \xi(t), g(t,\Sigma^*(t),u^*(t),w^*(t)) - g(t,\Sigma(t),u^*(t),w(t))\rangle \Big\}dt \\
\leqslant & \ \mathbb{E}\int_0^T \Big\{ \langle y^*(t)-y(t), H_{2y}(t)\rangle + \langle y^*(t-\delta)-y(t-\delta), H_{2y_\delta}(t)\rangle \\
& + \langle z^*(t)-z(t), H_{2z}(t)\rangle + \langle z^*(t-\delta)-z(t-\delta), H_{2z_\delta}(t)\rangle \\
& + \langle w^*(t)-w(t), H_{2w}(t)\rangle + \langle x^*(t)-x(t), H_{2x}(t)\rangle \\
& + \langle \mathbb{E}^{\mathcal{F}_t}[x^*(t+\delta)-x(t+\delta)], H_{2x_\delta}(t)\rangle \Big\}dt \\
& - \mathbb{E}\int_0^T \Big\{ \langle p(t), b(t,\Sigma^*(t),x^*(t),\mathbb{E}^{\mathcal{F}_t}[x^*(t+\delta)],u^*(t),w^*(t)) \\
& - b(t,\Sigma(t),x(t),\mathbb{E}^{\mathcal{F}_t}[x(t+\delta)],u^*(t),w(t))\rangle \\
& + \mathrm{tr}\{q^\top(t)\big(\sigma(t,\Sigma^*(t),x^*(t),\mathbb{E}^{\mathcal{F}_t}[x^*(t+\delta)],u^*(t),w^*(t)) \\
& - \sigma(t,\Sigma(t),x(t),\mathbb{E}^{\mathcal{F}_t}[x(t+\delta)],u^*(t),w(t))\big)\} \\
& - \langle \xi(t), g(t,\Sigma^*(t),u^*(t),w^*(t)) - g(t,\Sigma(t),u^*(t),w(t))\rangle \Big\}dt,
\end{aligned}$$

$$\begin{aligned}
II \leqslant & \ \mathbb{E}[G_{2y}^\top(y^*(0))(y^*(0)-y(0))] \\
= & \ \mathbb{E}[\big(-\xi^\top(0)+p^\top(0)G_{1yy}(y^*(0))\big)(y^*(0)-y(0))].
\end{aligned}$$

Applying Itô's formula to $\langle \xi(t), y^*(t)-y(t)\rangle + \langle p(t), x^*(t)-x(t)\rangle$, we derive that

$$\begin{aligned}
& \mathbb{E}[-\xi(0)^\top(y^*(0)-y(0)) + p^\top(0)G_{1y}(y^*(0)) - G_{1y}(y(0))] \\
= & -\mathbb{E}\int_0^T \Big\{ \langle \xi(t), g(t,\Sigma^*(t),u^*(t),w^*(t)) - g(t,\Sigma(t),u^*(t),w(t))\rangle \\
& + \langle y^*(t)-y(t), H_{2y}(t)+\mathbb{E}^{\mathcal{F}_t}[H_{2y_\delta}(t)|_{t+\delta}]\rangle \\
& + \langle z^*(t)-z(t), H_{2z}(t)+\mathbb{E}^{\mathcal{F}_t}[H_{2z_\delta}(t)|_{t+\delta}]\rangle \Big\}dt \\
& - \mathbb{E}\int_0^T \Big\{ \langle x^*(t)-x(t), H_{2x}(t)+H_{2x_\delta}(t)|_{t-\delta}\rangle \\
& - \langle p(t), b(t,\Sigma^*(t),x^*(t),\mathbb{E}^{\mathcal{F}_t}[x^*(t+\delta)],u^*(t),w^*(t)) \\
& - b(t,\Sigma(t),x(t),\mathbb{E}^{\mathcal{F}_t}[x(t+\delta)],u^*(t),w(t))\rangle \\
& - \mathrm{tr}\{q^\top(t)\big(\sigma(t,\Sigma^*(t),x^*(t),\mathbb{E}^{\mathcal{F}_t}[x^*(t+\delta)],u^*(t),w^*(t)) \\
& - \sigma(t,\Sigma(t),x(t),\mathbb{E}^{\mathcal{F}_t}[x(t+\delta)],u^*(t),w(t))\big)\} \Big\}dt.
\end{aligned}$$

Therefore, we can obtain the following result using the optimal condition (10).

$$
\begin{aligned}
& J_2(\gamma,;u^*(\cdot),w^*(t)) - J_2(\gamma;u^*(\cdot),w(\cdot)) \\
\leqslant\ & \mathbb{E}\int_0^T \Big\{\langle y^*(t-\delta)-y(t-\delta), H_{2y_\delta}(t)\rangle - \langle y^*(t)-y(t), \mathbb{E}^{\mathcal{F}_t}[H_{2y_\delta}(t)|_{t+\delta}]\rangle\Big\}dt \\
& +\mathbb{E}\int_0^T \Big\{\langle z^*(t-\delta)-z(t-\delta), H_{2z_\delta}(t)\rangle - \langle z^*(t)-z(t), \mathbb{E}^{\mathcal{F}_t}[H_{2z_\delta}(t)|_{t+\delta}]\rangle\Big\}dt \\
& +\mathbb{E}\int_0^T \Big\{\langle \mathbb{E}^{\mathcal{F}_t}[x^*(t+\delta)-x(t+\delta)], H_{2x_\delta}(t)\rangle - \langle x^*(t)-x(t), H_{2x_\delta}(t)|_{t-\delta}\rangle\Big\}dt
\end{aligned}
\tag{11}
$$

Now, we need to analyze the right-hand side of inequality (11). In fact,

$$
\begin{aligned}
& \mathbb{E}\int_0^T \Big\{\langle y^*(t-\delta)-y(t-\delta), H_{2y_\delta}(t)\rangle - \langle y^*(t)-y(t), \mathbb{E}^{\mathcal{F}_t}[H_{2y_\delta}(t)|_{t+\delta}]\rangle\Big\}dt \\
=\ & \mathbb{E}\int_{-\delta}^{T-\delta} \langle y^*(t)-y(t), \mathbb{E}^{\mathcal{F}_t}[H_{2y_\delta}(t)|_{t+\delta}]\rangle dt - \mathbb{E}\int_0^T \langle y^*(t)-y(t), \mathbb{E}^{\mathcal{F}_t}[H_{2y_\delta}(t)|_{t+\delta}]\rangle\, dt \\
=\ & \mathbb{E}\int_{-\delta}^{0} \langle y^*(t)-y(t), \mathbb{E}^{\mathcal{F}_t}[H_{2y_\delta}(t)|_{t+\delta}]\rangle dt - \mathbb{E}\int_{T-\delta}^T \langle y^*(t)-y(t), \mathbb{E}^{\mathcal{F}_t}[H_{2y_\delta}(t)|_{t+\delta}]\rangle dt \\
=\ & 0,
\end{aligned}
$$

since $y^*(t) = y(t)$ for any $t \in [-\delta, 0]$ and $H_{2y_\delta}(t) = 0$ for any $t \in [T, T+\delta]$. Similarly, we have

$$
\mathbb{E}\int_0^T \Big\{\langle z^*(t-\delta)-z(t-\delta), H_{2z_\delta}(t)\rangle - \langle z^*(t)-z(t), \mathbb{E}^{\mathcal{F}_t}[H_{2z_\delta}(t)|_{t+\delta}]\rangle\Big\}dt = 0,
$$

$$
\mathbb{E}\int_0^T \Big\{\langle \mathbb{E}^{\mathcal{F}_t}[x^*(t+\delta)-x(t+\delta)], H_{2x_\delta}(t)\rangle - \langle x^*(t)-x(t), H_{2x_\delta}(t)|_{t-\delta}\rangle\Big\}dt = 0,
$$

Then, we deduce $J_2(\gamma,;u^*(\cdot),w^*(t)) - J_2(\gamma;u^*(\cdot),w(\cdot)) \leqslant 0$ and complete the proof. □

Remark 4. *The adjoint Equation (9) is a new kind of FBSDE. Due to the complex form of the Hamiltonian function H_2, the conditions (H1)–(H2) may not guarantee the solvability of this equation. In addition, in Theorem 2, we look for the leader's optimal strategy under the assumption that $u^*(\cdot)$ is known. However, the follower's optimal strategy $u^*(t)$ depends on the current value of the leader's strategy $w(\cdot)$. So, we can set*

$$
\begin{aligned}
u^*(t) &= u^*(t, y(t), z(t), y(t-\delta), z(t-\delta), w(t), x(t), \mathbb{E}^{\mathcal{F}_t}[x(t+\delta)]) \\
&:= u^*(t, \Sigma(t), w(t), x(t), \mathbb{E}^{\mathcal{F}_t}[x(t+\delta)]),
\end{aligned}
$$

for its dependence on the leader's strategy $w(\cdot)$, the state $(y(\cdot), z(\cdot))$ and the adjoint variable $x(t)$.

Then, the state equation of the leader can be rewritten as

$$
\begin{cases}
-dy(t) = & g(t,\Sigma(t),u^*(t,\Sigma(t),w(t),x(t),\mathbb{E}^{\mathcal{F}_t}[x(t+\delta)]),w(t))dt - z(t)dW(t),\ t\in[0,T], \\
dx(t) = & \{g_y^\top(t,\Sigma(t),u^*(t,\Sigma(t),w(t),x(t),\mathbb{E}^{\mathcal{F}_t}[x(t+\delta)]),w(t))x(t) \\
& + \mathbb{E}^{\mathcal{F}_t}[g_{y_\delta}^\top(t,\Sigma(t),u^*(t,\Sigma(t),w(t),x(t),\mathbb{E}^{\mathcal{F}_t}[x(t+\delta)]),w(t))|_{t+\delta}x(t+\delta)] \\
& - l_{1y}(t,y(t),z(t),u^*(t,\Sigma(t),w(t),x(t),\mathbb{E}^{\mathcal{F}_t}[x(t+\delta)]),w(t))\}dt \\
& + \{g_z^\top(t,\Sigma(t),u^*(t,\Sigma(t),w(t),x(t),\mathbb{E}^{\mathcal{F}_t}[x(t+\delta)]),w(t))x(t) \\
& + \mathbb{E}^{\mathcal{F}_t}[g_{z_\delta}^\top(t,\Sigma(t)),u^*(t,\Sigma(t),w(t),x(t),\mathbb{E}^{\mathcal{F}_t}[x(t+\delta)]),w(t))|_{t+\delta}x(t+\delta)] \\
& - l_{1z}(t,y(t),z(t),u^*(t,\Sigma(t)),w(t),x(t),\mathbb{E}^{\mathcal{F}_t}[x(t+\delta)]),w(t))\}dW(t),\ t\in[0,T], \\
y(T) = & \gamma,\ y(t)=\eta(t),\ z(t)=\zeta(t),\ t\in[-\delta,0), \\
x(0) = & -G_{1y}(y(0)),\ x(t)=0,\ t\in(T,T+\delta].
\end{cases}
$$

If we want to obtain the sufficient conditions of the optimal strategy in this case, we need the function $u^*(t, \Sigma(t), w(t), x(t), \mathbb{E}^{\mathcal{F}_t}[x(t+\delta)])$ to satisfy some conditions. In Bensoussan et al. [5], they assumed that u^* is uniquely defined and is uniformly Lipschitz-continuous with respect to the state variable, the leader's control variable and the adjoint variable.

In our framework, the backward Stackelberg equilibrium relies on a fully coupled AFBSDDE consisting of the above equation and adjoint Equation (9). Since it is difficult to obtain the solvability of AFBSDDE in the general case, we will discuss the LQ backward Stackelberg game in the next section.

3. Linear–Quadratic Backward Stackelberg Game

In this section, we apply the theory studied in Section 2 to deal with an LQ backward Stackelberg game. We assume that the Brownian motion is one-dimensional for simplicity. The controlled system is given by

$$\begin{cases} -dy(t) = & [A(t)y(t) + B(t)y(t-\delta) + C(t)u(t) + D(t)w(t) \\ & + E(t)z(t) + F(t)z(t-\delta)]dt - z(t)dW(t),\ t \in [0,T], \\ y(T) = & \gamma,\ y(t) = \eta(t),\ z(t) = \zeta(t),\ t \in [-\delta, 0), \end{cases} \tag{12}$$

where $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $E(\cdot)$, $D(\cdot)$, and $F(\cdot)$ are deterministic continuous matrix-valued functions with suitable dimensions.

The corresponding cost functionals of the follower and the leader in system (12) are

$$\begin{cases} J_1(\gamma; u(\cdot), w(\cdot)) = \dfrac{1}{2}\mathbb{E}\{\displaystyle\int_0^T [\langle M_1(t)y(t), y(t)\rangle + \langle R_1(t)z(t), z(t)\rangle \\ \qquad\qquad + \langle N_1(t)u(t), u(t)\rangle]dt + \langle G_1 y(0), y(0)\rangle\}, \\ J_2(\gamma; u(\cdot), w(\cdot)) = \dfrac{1}{2}\mathbb{E}\{\displaystyle\int_0^T [\langle M_2(t)y(t), y(t)\rangle + \langle R_2(t)z(t), z(t)\rangle \\ \qquad\qquad + \langle N_2(t)w(t), w(t)\rangle]dt + \langle G_2 y(0), y(0)\rangle\}. \end{cases} \tag{13}$$

Then, we give the assumption (H5) which is a special case of (H1)–(H4).

(H5) $A(\cdot), B(\cdot), E(\cdot), F(\cdot) \in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{n\times n})$, $C(\cdot) \in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{n\times m_1})$, $D(\cdot) \in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{n\times m_2})$. For $i = 1,2$, we suppose $M_i(\cdot), R_i(\cdot) \in L^\infty_{\mathbb{F}}(0,T;\mathcal{S}^n)$, $N_i(\cdot) \in L^\infty_{\mathbb{F}}(0,T;\mathcal{S}^{m_i})$, $G_i \in \mathcal{S}^n$, and $M_i(\cdot) \geqslant 0, R_i(\cdot) \geqslant 0, N_i(\cdot) \gg 0, G_i \geqslant 0$.

We still consider the follower's optimal control problem first.

Problem $(LQBSG)_f$**:** For $\gamma \in L^2(\mathcal{F}_T;\mathbb{R}^n)$ and any given leader's control $w(\cdot) \in \mathcal{U}_2$, we find $u^*(\cdot) \in \mathcal{U}_1$ such that $J_1(\gamma; u^*(\cdot), w(\cdot)) = \inf\limits_{u(\cdot)\in\mathcal{U}_1} J_1(\gamma; u(\cdot), w(\cdot))$.

Applying Theorem 1 to Problem $(LQBSG)_f$, we have the following result:

Theorem 3. *Suppose that (H5) holds. For $\gamma \in L^2(\mathcal{F}_T;\mathbb{R}^n)$ and any given leader's control $w(\cdot) \in \mathcal{U}_2$, the follower's optimal control problem $(LQBSG)_f$ can be uniquely solvable if and only if there exists a unique four-tuple $(y(t), z(t), u^*(t), x(t))$ satisfying*

$$\begin{cases} -dy(t) = & [A(t)y(t) + B(t)y(t-\delta) + C(t)u^*(t) + D(t)w(t) \\ & + E(t)z(t) + F(t)z(t-\delta)]dt - z(t)dW(t),\ t \in [0,T], \\ dx(t) = & \{A^\top(t)x(t) + \mathbb{E}^{\mathcal{F}_t}[B^\top(t+\delta)x(t+\delta)] - M_1(t)y(t)\}dt \\ & + \{E^\top(t)x(t) + \mathbb{E}^{\mathcal{F}_t}[F^\top(t+\delta)x(t+\delta)] - R_1(t)z(t)\}dW(t),\ t \in [0,T], \\ y(T) = & \gamma,\ y(t) = \eta(t),\ z(t) = \zeta(t),\ t \in [-\delta, 0), \\ x(0) = & -G_1 y(0),\ x(t) = 0,\ t \in (T, T+\delta], \end{cases} \tag{14}$$

such that

$$u^*(t) = N_1^{-1}(t)C^\top(t)x(t). \tag{15}$$

The second equation in (14) is a linear ASDE and it admits a unique solution under assumption (H5) for a sufficiently small δ. Since the leader can choose their optimal strategy with the knowledge of the follower's reaction, then the state equation of the leader's optimal control is

$$\begin{cases} -dy(t) = & [A(t)y(t) + B(t)y(t-\delta) + C(t)N_1^{-1}(t)C^\top(t)x(t) \\ & + D(t)w(t) + E(t)z(t) + F(t)z(t-\delta)]dt - z(t)dW(t),\ t \in [0,T], \\ dx(t) = & \{A^\top(t)x(t) + \mathbb{E}^{\mathcal{F}_t}[B^\top(t+\delta)x(t+\delta)] - M_1(t)y(t)\}dt \\ & + \{E^\top(t)x(t) - R_1(t)z(t) + \mathbb{E}^{\mathcal{F}_t}[F^\top(t+\delta)x(t+\delta)]\}dW(t),\ t \in [0,T], \\ y(T) = & \gamma,\ y(t) = \eta(t),\ z(t) = \zeta(t),\ t \in [-\delta, 0), \\ x(0) = & -G_1 y(0),\ x(t) = 0,\ t \in (T, T+\delta]. \end{cases} \tag{16}$$

Problem $(LQBSG)_l$**:** Find the optimal strategy $w^*(\cdot) \in \mathcal{U}_2$ such that $J_2(\gamma; u^*(\cdot), w^*(\cdot)) = \inf\limits_{w(\cdot)\in\mathcal{U}_2} J_2(\gamma; u^*(\cdot), w(\cdot))$.

The adjoint equation of the leader is the following FBSDE:

$$\begin{cases} -dp(t) = & \{A(t)p(t) + B(t)p(t-\delta) - C(t)N_1^{-1}(t)C^\top(t)\xi(t) \\ & + E(t)q(t) + F(t)q(t-\delta)\}dt - q(t)dW(t),\ t \in [0,T], \\ d\xi(t) = & \{A^\top(t)\xi(t) + \mathbb{E}^{\mathcal{F}_t}[B^\top(t+\delta)\xi(t+\delta)] + M_1^\top(t)p(t) \\ & - M_2(t)y(t)\}dt + \{E^\top(t)\xi(t) + \mathbb{E}^{\mathcal{F}_t}[F^\top(t+\delta)\xi(t+\delta)] \\ & + R_1^\top(t)q(t) - R_2(t)z(t)\}dW(t),\ t \in [0,T], \\ p(T) = & 0,\ p(t) = 0,\ q(t) = 0,\ t \in [-\delta, 0), \\ \xi(0) = & G_1^\top p(0) - G_2 y(0),\ \xi(t) = 0,\ t \in (T, T+\delta]. \end{cases} \tag{17}$$

Theorem 4. *Under assumption $(H5)$, the leader's optimal control problem $(LQBSG)_l$ can be uniquely solvable if and only if there exists a unique $(y^*(t), z^*(t), x^*(t), p^*(t), q^*(t), \xi^*(t), w^*(t))$ satisfying Equation (16) and Equation (17) such that*

$$w^*(t) = N_2^{-1}(t)D^\top(t)\xi^*(t). \tag{18}$$

We declare that, if FBSDE (16) and (17) admit a unique solution, then $w^*(t)$ in the form of (18) is the unique optimal control of the leader by Theorem 2. Meanwhile, we can also prove that $w^*(t)$ in (18) is optimal using the classical completion of the squares method. Thus, we omit the detailed proof of Theorem 4.

Theorems 3 and 4 show us an equivalence between the solvability of the LQ backward Stackelberg game and that of the coupled AFBSDDE. In order to achieve the existence of the optimal control (u^*, w^*), we study the solvability of the following AFBSDDE.

$$\begin{cases} -dy(t) = & [A(t)y(t) + B(t)y(t-\delta) + I(t)C^\top(t)x(t) + \bar{I}(t)D^\top(t)\xi(t) + E(t)z(t) \\ & + F(t)z(t-\delta)]dt - z(t)dW(t),\ t \in [0,T], \\ dx(t) = & \{A^\top(t)x(t) + \mathbb{E}^{\mathcal{F}_t}[B^\top(t+\delta)x(t+\delta)] - M_1(t)y(t)\}dt \\ & + \{E^\top(t)x(t) + \mathbb{E}^{\mathcal{F}_t}[F^\top(t+\delta)x(t+\delta)] - R_1(t)z(t)\}dW(t),\ t \in [0,T], \\ -dp(t) = & \{A(t)p(t) + B(t)p(t-\delta) - I(t)C^\top(t)\xi(t) + E(t)q(t) \\ & + F(t)q(t-\delta)\}dt - q(t)dW(t), t \in [0,T], \\ d\xi(t) = & \{A^\top(t)\xi(t) + \mathbb{E}^{\mathcal{F}_t}[B^\top(t+\delta)\xi(t+\delta)] + M_1^\top(t)p(t) - M_2(t)y(t)\}dt \\ & + \{E^\top(t)\xi(t) + \mathbb{E}^{\mathcal{F}_t}[F^\top(t+\delta)\xi(t+\delta)] + R_1^\top(t)q(t) - R_2(t)z(t)\}dW_t, t \in [0,T], \\ y(T) = & \gamma,\ y(t) = \eta(t),\ z(t) = \zeta(t),\ t \in [-\delta, 0), \\ x(0) = & -G_1 y(0),\ x(t) = 0,\ t \in (T, T+\delta], \\ p(T) = & 0,\ p(t) = 0,\ q(t) = 0,\ t \in [-\delta, 0), \\ \xi(0) = & G_1^\top p(0) - G_2 y(0),\ \xi(t) = 0,\ t \in (T, T+\delta], \end{cases} \tag{19}$$

where $I_t = C(t)N_1^{-1}(t)$, $\bar{I}_t = D(t)N_2^{-1}(t)$.

Set

$$\lambda(t) = (y(t), z(t), x(t)),\ \theta(t) = (p(t), q(t), \xi(t)),$$

and define

$$\begin{aligned} L^2(-\delta, T+\delta) =& \mathcal{S}_{\mathbb{F}}^2(-\delta, T; \mathbb{R}^n) \times L_{\mathbb{F}}^2(-\delta, T; \mathbb{R}^{n\times d}) \times \mathcal{S}_{\mathbb{F}}^2(0, T+\delta; \mathbb{R}^n) \\ & \times \mathcal{S}_{\mathbb{F}}^2(-\delta, T; \mathbb{R}^n) \times L_{\mathbb{F}}^2(-\delta, T; \mathbb{R}^{n\times d}) \times \mathcal{S}_{\mathbb{F}}^2(0, T+\delta; \mathbb{R}^n) \end{aligned}$$

with the norm

$$\begin{aligned} \| (\lambda(\cdot), \theta(\cdot)) \|^2 =& \ \mathbb{E} \sup_{t\in[-\delta,T]} |y(t)|^2 + \mathbb{E}\int_{-\delta}^T |z(t)|^2 dt + \mathbb{E} \sup_{t\in[0,T+\delta]} |x(t)|^2 \\ & + \mathbb{E} \sup_{t\in[-\delta,T]} |p(t)|^2 + \mathbb{E}\int_{-\delta}^T |q(t)|^2 dt + \mathbb{E} \sup_{t\in[0,T+\delta]} |\xi(t)|^2. \end{aligned}$$

Remark 5. *As mentioned before, Equation (19) is a new kind of double FBSDE composed of ASDEs and BSDDEs. This kind of FBSDE has not been studied in the previous literature. Using the method of continuation introduced in [32], we can obtain the unique solvability of AFBSDDE (19). We should note that the delay time δ must be sufficiently small for ASDEs and BSDDEs. We will give the detailed procedure below.*

Theorem 5. *For sufficiently small δ, AFBSDDE (19) admits a unique solution*

$$(\lambda(\cdot), \theta(\cdot)) = (y(\cdot), z(\cdot), x(\cdot), p(\cdot), q(\cdot), \xi(\cdot)) \in L^2(-\delta, T+\delta).$$

In order to prove Theorem 5, we consider a family of AFBSDDEs with parameter $\rho \in [0,1]$ as follows.

$$
\begin{cases}
-dy^\rho(t) = & \{\rho(A(t)y^\rho(t) + B(t)y^\rho(t-\delta) + I(t)C^\top(t)x^\rho(t) + \bar{I}(t)D^\top(t)\xi^\rho(t) \\
& + E(t)z^\rho(t) + F(t)z^\rho(t-\delta)) + \vartheta_1(t) + (1-\rho)(D(t)D^\top(t) \\
& + C(t)C^\top(t))\xi^\rho(t)\}dt - z^\rho(t)dW(t),\ t \in [0,T], \\
dx^\rho(t) = & \{\rho\big(A^\top(t)x^\rho(t) + \mathbb{E}^{\mathcal{F}_t}[B^\top(t+\delta)x^\rho(t+\delta)] - M_1(t)y^\rho(t)\big) + \kappa_1(t)\}dt \\
& + \{\rho\big(E^\top(t)x^\rho(t) + \mathbb{E}^{\mathcal{F}_t}[F^\top(t+\delta)x^\rho(t+\delta)] \\
& - R_1(t)z^\rho(t)\big) + \varpi_1(t)\}dW(t),\ t \in [0,T], \\
-dp^\rho(t) = & \{\rho\big(A(t)p^\rho(t) + B(t)p^\rho(t-\delta) - I(t)C^\top(t)\xi^\rho(t) + E(t)q^\rho(t) \\
& + F(t)q^\rho(t-\delta)\big) + \vartheta_2(t) + (1-\rho)C(t)C^\top(t)x^\rho(t)\}dt \\
& - q^\rho(t)dW(t), t \in [0,T], \\
d\xi^\rho(t) = & \{\rho\big(A^\top(t)\xi^\rho(t) + \mathbb{E}^{\mathcal{F}_t}[B^\top(t+\delta)\xi^\rho(t+\delta)] + M_1^\top(t)p^\rho(t) - M_2(t)y^\rho(t)\big) \\
& + \kappa_2(t)\}dt + \{\rho\big(E^\top(t)\xi^\rho(t) + \mathbb{E}^{\mathcal{F}_t}[F^\top(t+\delta)\xi^\rho(t+\delta)] + R_1^\top(t)q^\rho(t) \\
& - R_2(t)z^\rho(t)\big) + \varpi_2(t)\}dW(t), t \in [0,T], \\
y^\rho(T) = & \gamma,\ y^\rho(t) = \eta(t),\ z^\rho(t) = \zeta(t),\ t \in [-\delta, 0), \\
x^\rho(0) = & -\rho G_1 y^\rho(0) + \phi_1,\ x^\rho(t) = 0,\ t \in (T, T+\delta], \\
\xi^\rho(0) = & \rho\big(G_1^\top p^\rho(0) - G_2 y^\rho(0)\big) + \phi_2,\ \xi^\rho(t) = 0,\ t \in (T, T+\delta], \\
p^\rho(T) = & \psi,\ p^\rho(t) = 0,\ q^\rho(t) = 0,\ t \in [-\delta, 0),
\end{cases}
\tag{20}
$$

where $\vartheta_i(\cdot), \kappa_i(\cdot), \varpi_i(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$, $\psi \in L^2(\mathcal{F}_T;\mathbb{R}^n)$, $\phi_i \in L^2(\mathcal{F}_0;\mathbb{R}^n)$ $(i = 1,2)$.

When $\rho = 0$, FBSDE (20) is decoupled. Equation (20) has a unique solution by the properties of SDE and BSDE. When $\rho = 1$, if Equation (20) has a unique solution, then the AFBSDDE (19) also has a unique solution. The following lemma gives a prior estimate for the existence of an interval of Equation (20) with respect to parameter $\rho \in [0,1]$.

Lemma 1. *We assume that (H5) holds. There exists a constant $\tau_0 > 0$ such that, if AFBSDDE (20) admits a unique solution $(\lambda_t^{\rho_0}, \theta_t^{\rho_0})$ for some $\rho_0 \in [0,1)$, then AFBSDDE (20) admits a unique solution $(\lambda_t^{\rho_0+\tau}, \theta_t^{\rho_0+\tau})$ for sufficiently small δ and $\rho_0 + \tau \leqslant 1$ with $\tau \in [0, \tau_0]$.*

Proof. We will use the notation

$$\Lambda = (Y, Z, X),\ \Theta = (P, Q, \Xi).$$

Since AFBSDDE (20) admits a unique solution when $\rho = \rho_0 \in [0,1)$ for each $\vartheta_i(\cdot), \kappa_i(\cdot)$, $\varpi_i(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$, $\psi \in L^2(\mathcal{F}_T;\mathbb{R}^n)$, $\phi_i \in L^2(\mathcal{F}_0;\mathbb{R}^n)$ $(i = 1,2)$, there thus exists a unique $(\Lambda, \Theta) \in L^2(-\delta, T+\delta)$ satisfying the following AFBSDDE for each $(\lambda, \theta) \in L^2(-\delta, T+\delta)$.

$$\begin{cases} -dY(t) = & \{\rho_0\big(A(t)Y(t)+B(t)Y(t-\delta)+I(t)C^\top(t)X(t)+\bar{I}(t)D^\top(t)\Xi(t) \\ & +E(t)Z(t)+F(t)Z(t-\delta)\big)+\tau\big(A(t)y(t)+B(t)y(t-\delta) \\ & +I(t)C^\top(t)x(t)+\bar{I}(t)D^\top(t)\xi(t)+E(t)z(t)+F(t)z(t-\delta)\big) \\ & +(1-\rho_0)\big(D(t)D^\top(t)+C(t)C^\top(t)\big)\Xi(t)+\tau\big(D(t)D^\top(t) \\ & +C(t)C^\top(t)\big)\xi(t)+\vartheta_1(t)\}dt-Z(t)dW(t),\ t\in[0,T], \\ dX(t) = & \{\rho_0\big(A^\top(t)X(t)+\mathbb{E}^{\mathcal{F}_t}[B^\top(t+\delta)X(t+\delta)]-M_1(t)Y(t)\big) \\ & +\kappa_1(t)+\tau\big(A^\top(t)x(t)+\mathbb{E}^{\mathcal{F}_t}[B^\top(t+\delta)x(t+\delta)] \\ & -M_1(t)y(t)\big)\}dt+\{\rho_0\big(E^\top(t)X(t)+\mathbb{E}^{\mathcal{F}_t}[F^\top(t+\delta)X(t+\delta)] \\ & -R_1(t)Z(t)\big)+\varpi_1(t)+\tau\big(E^\top(t)x(t) \\ & +\mathbb{E}^{\mathcal{F}_t}[F^\top(t+\delta)x(t+\delta)]-R_1(t)z(t)\big)\}dW(t),\ t\in[0,T], \\ -dP(t) = & \{\rho_0\big(A(t)P(t)+B(t)P(t-\delta)-I(t)C^\top(t)\Xi(t)+E(t)Q(t) \\ & +F(t)Q(t-\delta)\big)+\vartheta_2(t)+\tau\big(A(t)p(t)+B(t)p(t-\delta) \\ & -I(t)C^\top(t)\xi(t)+E(t)q(t)+F(t)q(t-\delta)\big) \\ & +(1-\rho_0)C(t)C^\top(t)X(t)+\tau C(t)C^\top(t)x(t)\}dt-Q(t)dW(t),t\in[0,T], \\ d\Xi(t) = & \{\rho_0\big(A^\top(t)\Xi(t)+\mathbb{E}^{\mathcal{F}_t}[B^\top(t+\delta)\Xi(t+\delta)]+M_1^\top(t)P(t) \\ & -M_2(t)Y(t)\big)+\kappa_2(t)+\tau\big(A^\top(t)\xi(t)+\mathbb{E}^{\mathcal{F}_t}[B^\top(t+\delta)\xi(t+\delta)] \\ & +M_1^\top(t)p(t)-M_2(t)y(t)\big)\}dt \\ & +\{\rho_0\big(E^\top(t)\Xi(t)+\mathbb{E}^{\mathcal{F}_t}[F^\top(t+\delta)\Xi(t+\delta)]+R_1^\top(t)Q(t) \\ & -R_2(t)Z(t)\big)+\varpi_2(t)+\tau\big(E^\top(t)\xi(t)+\mathbb{E}^{\mathcal{F}_t}[F^\top(t+\delta)\xi(t+\delta)] \\ & +R_1^\top(t)q(t)-R_2(t)z(t)\big)\}dW(t),t\in[0,T], \\ Y(T) = & \gamma,\ Y(t)=\eta(t),\ Z(t)=\zeta(t),\ t\in[-\delta,0), \\ X(0) = & -\rho(0)G_1Y(0)-\tau G_1y(0)+\phi_1,\ X(t)=\ 0,\ t\in(T,T+\delta], \\ P(T) = & \psi,\ P(t)=0,\ Q(t)=0,\ t\in[-\delta,0), \\ \Xi(0) = & \rho_0(G_1^\top P(0)-G_2Y(0))+\tau(G_1^\top p(0)-G_2y(0))+\phi_2, \\ \Xi(t) = & 0,\ t\in(T,T+\delta], \end{cases} \tag{21}$$

where $0\leqslant\rho_0+\tau\leqslant 1$.

We define the following mapping:

$$(\Lambda,\Theta)=I_{\rho_0+\tau}(\lambda,\theta):L^2(-\delta,T+\delta)\to L^2(-\delta,T+\delta).$$

We proceed to prove that the mapping $I_{\rho_0+\tau}$ is a contraction.

Suppose $(\bar{\lambda},\bar{\theta})\in L^2(-\delta,T+\delta)$, $(\bar{\Lambda},\bar{\Theta})=I_{\rho_0+\tau}(\bar{\lambda},\bar{\theta})$, and set

$$\begin{aligned} (\hat{\lambda},\hat{\theta}) =&\ (\lambda-\bar{\lambda},\theta-\bar{\theta})=(y-\bar{y},z-\bar{z},x-\bar{x},p-\bar{p},q-\bar{q},\xi-\bar{\xi}), \\ (\hat{\Lambda},\hat{\Theta}) =&\ (\Lambda-\bar{\Lambda},\Theta-\bar{\Theta})=(Y-\bar{Y},Z-\bar{Z},X-\bar{X},P-\bar{P},Q-\bar{Q},\Xi-\bar{\Xi}). \end{aligned}$$

Applying *Itô*'s formula to $\langle\hat{X}(t),\hat{P}(t)\rangle+\langle\hat{\Xi}(t),\hat{Y}(t)\rangle$, we obtain

$$\begin{aligned} & \rho_0\mathbb{E}\langle G_2\hat{Y}(0),\hat{Y}(0)\rangle+\tau\mathbb{E}\langle G_1\hat{y}(0),\hat{P}(0)\rangle+\tau\mathbb{E}\langle G_2\hat{y}(0),\hat{Y}(0)\rangle-\tau\mathbb{E}\langle G_1^\top\hat{p}(0),\hat{Y}(0)\rangle \\ =&\ \mathbb{E}\int_0^T\{-(1-\rho_0)\langle C^\top(t)\hat{X}(t),C^\top(t)\hat{X}(t)\rangle \\ & -(1-\rho_0)\langle D^\top(t)\hat{\Xi}(t),D^\top(t)\hat{\Xi}(t)\rangle \\ & -(1-\rho_0)\langle C^\top(t)\hat{\Xi}(t),C^\top(t)\hat{\Xi}(t)\rangle-\rho_0\langle M_2(t)\hat{Y}(t),\hat{Y}(t)\rangle \\ & -\rho_0\langle\bar{I}(t)D^\top(t)\hat{\Xi}(t),\hat{\Xi}(t)\rangle-\rho_0\langle R_2(t)\hat{Z}(t),\hat{Z}(t)\rangle+\tau\langle A^\top(t)\hat{x}(t) \end{aligned}$$

$$+\mathbb{E}^{\mathcal{F}_t}[B^\top(t+\delta)\hat{x}(t+\delta)]-M_1(t)\hat{y}(t),\hat{P}(t)\rangle-\tau\langle A(t)\hat{p}(t)+B(t)\hat{p}(t-\delta)$$
$$-I(t)C^\top(t)\hat{\xi}(t)+E(t)\hat{q}(t)+F(t)\hat{q}(t-\delta),\hat{X}(t)\rangle-\tau\langle C(t)C^\top(t)\hat{x}(t),\hat{X}(t)\rangle$$
$$+\tau\langle E^\top(t)\hat{x}(t)+\mathbb{E}^{\mathcal{F}_t}[F^\top(t+\delta)\hat{x}(t+\delta)]-R_1(t)\hat{z}(t),\hat{Q}(t)\rangle$$
$$+\tau\langle A^\top(t)\hat{\xi}(t)+\mathbb{E}^{\mathcal{F}_t}[B^\top(t+\delta)\hat{\xi}(t+\delta)]+M_1^\top(t)\hat{p}(t)-M_2(t)\hat{y}(t),\hat{Y}(t)\rangle$$
$$-\tau\langle A(t)\hat{y}(t)+B(t)\hat{y}(t-\delta)+I(t)C^\top(t)\hat{x}(t)+\bar{I}(t)D^\top(t)\hat{\xi}(t)+E(t)\hat{z}(t)$$
$$+F(t)\hat{z}(t-\delta),\hat{\Xi}(t)\rangle+\tau\langle E^\top(t)\hat{\xi}(t)+\mathbb{E}^{\mathcal{F}_t}[F^\top(t+\delta)\hat{\xi}(t+\delta)]+R_1^\top(t)\hat{q}(t)$$
$$-R_2(t)\hat{z}(t),\hat{Z}(t)\rangle-\tau\langle(D(t)D^\top(t)+C(t)C^\top(t))\hat{\xi}(t),\hat{\Xi}(t)\rangle\}dt.$$

Since $\rho_0 \geqslant 0, 1-\rho_0>0$ and $M_2(\cdot)$, $\bar{I}(\cdot)D^\top(\cdot)$, $R_2(\cdot)$ are non-negative, we have

$$\begin{aligned}&\mathbb{E}\int_0^T[\langle C^\top(t)\hat{X}(t),C^\top(t)\hat{X}(t)\rangle+\langle D^\top(t)\hat{\Xi}(t),D^\top(t)\hat{\Xi}(t)\rangle\\&+\langle C^\top(t)\hat{\Xi}(t),C^\top(t)\hat{\Xi}(t)\rangle]dt\\&\leqslant\ \varepsilon C_1\parallel(\hat{\Lambda},\hat{\Theta})\parallel^2+\frac{\tau^2C_1}{\varepsilon}\parallel(\hat{\lambda},\hat{\theta})\parallel^2,\ \forall\varepsilon>0.\end{aligned}\tag{22}$$

Applying $It\hat{o}$'s formula to $|\hat{Y}(t)|^2$, for sufficiently small $\delta \geqslant 0$, we have the following estimate by Gronwall's inequality:

$$\begin{aligned}&\mathbb{E}\sup_{t\in[-\delta,T]}|\hat{Y}(t)|^2+\mathbb{E}\int_{-\delta}^T|\hat{Z}(t)|^2dt\\&\leqslant\ \tau C_1\parallel(\hat{\lambda},\hat{\theta})\parallel^2+C_1\mathbb{E}\int_0^{T+\delta}[|C^\top(t)\hat{X}(t)|^2+|D^\top(t)\hat{\Xi}(t)|^2+|C^\top(t)\hat{\Xi}(t)|^2]dt.\end{aligned}\tag{23}$$

Similarly, applying $It\hat{o}$'s formula to $|\hat{X}(t)|^2$, $|\hat{P}(t)|^2$ and $|\hat{\Xi}(t)|^2$, we can obtain

$$\mathbb{E}\sup_{t\in[0,T+\delta]}|\hat{X}(t)|^2\leqslant\ \tau C_1\parallel(\hat{\lambda},\hat{\theta})\parallel^2+C_1\mathbb{E}\int_{-\delta}^T[|\hat{Y}(t)|^2+|\hat{Z}(t)|^2]dt,\tag{24}$$

$$\begin{aligned}&\mathbb{E}\sup_{t\in[-\delta,T]}|\hat{P}(t)|^2+\mathbb{E}\int_{-\delta}^T|\hat{Q}(t)|^2dt\\&\leqslant\ \tau C_1\parallel(\hat{\lambda},\hat{\theta})\parallel^2+C_1\mathbb{E}\int_0^{T+\delta}[|C^\top(t)\hat{X}(t)|^2+|C^\top(t)\hat{\Xi}(t)|^2]dt,\end{aligned}\tag{25}$$

$$\mathbb{E}\sup_{t\in[0,T+\delta]}|\hat{\Xi}(t)|^2\leqslant\ \tau C_1\parallel(\hat{\lambda},\hat{\theta})\parallel^2+C_1\mathbb{E}\int_{-\delta}^T[|\hat{Y}(t)|^2+|\hat{Z}(t)|^2+|\hat{P}(t)|^2+|\hat{Q}(t)|^2]dt,\tag{26}$$

for sufficiently small $\delta \geqslant 0$. Here, C_1 is a positive constant. Combing the above estimates (22)–(26), we conclude that

$$\parallel(\hat{\Lambda},\hat{\Theta})\parallel^2\leq C\tau\parallel(\hat{\lambda},\hat{\theta})\parallel^2,$$

where C is a positive constant independent of ρ_0 and τ. Taking $\tau=\frac{1}{2C}$, it follows that

$$\parallel(\hat{\Lambda},\hat{\Theta})\parallel^2\leq\frac{1}{2}\parallel(\hat{\lambda},\hat{\theta})\parallel^2.$$

Then, the contract mapping $I_{\rho_0+\tau}$ has a unique fixed point $(\lambda^{\rho_0+\tau}(t),\theta^{\rho_0+\tau}(t))$, which is the unique solution of AFBSDDE (20) for $\rho=\rho_0+\tau$. The proof is completed. □

Lemma 1 shows that, if (20) can be uniquely solved for $\rho=0$, then there exists a unique solution of Equation (20) for $\rho=1$.

Proof of Theorem 5. When $\rho = 0$, AFBSDDE (20) is decoupled and can be uniquely solved. From Lemma 1, there exists a positive constant τ_0 such that for each $\tau \in [0, \tau_0]$, AFBSDDE (20) admits a unique solution for $\rho = 0 + \tau$ and sufficiently small δ. By an inductive argument, we can increase the parameter ρ step by step from $\rho = 0$ to $\rho = 1$. Therefore, AFBSDDE (20) has a unique solution for $\rho = 1$. Especially when we let $\vartheta_i(\cdot)$, $\kappa_i(\cdot)$, $\varpi_i(\cdot)$, ϕ_i, and ψ be zero ($i = 1, 2$) in AFBSDDE (20), we conclude that AFBSDDE (19) admits a unique solution and completes the proof. □

The following result is a direct sequence of Theorems 3–5.

Theorem 6. *Under assumption (H5), for any given $\gamma \in L^2(\mathcal{F}_T, \mathbb{R}^n)$ and sufficiently small time delay δ,*

$$(u^*(t), w^*(t)) = (N_1^{-1}(t)C^\top(t)x(t), N_2^{-1}(t)D^\top(t)\xi(t))$$

provides a unique backward Stackelberg equilibrium for the problems $(LQBSG)_f - (LQBSG)_l$, where $(y(t), z(t), x(t), p(t), q(t), \xi(t))$ is the unique solution of the AFBSDDE (19).

4. Application to Pension Fund Problem

There are many areas where BSDDEs may arise, such as portfolio management, anticipating contract and variable annuities, and unit-linked products problems, as mentioned in [23]. An investor's wealth process may depend on its own past value or the past values of the underlying investment portfolio. We can introduce different delay effects into the financial market, which will make the model more practical.

This section is devoted to the study of a pension fund problem arising from financial markets. Suppose that there are two players that continuously make contributions to a pension fund (see [27]). One of the players is usually the supervisor, government, or company (called the leader), who pays a premium proportion $w(\cdot)$ as their contribution. The other is the individual investor (called the follower) with a premium proportion $u(\cdot)$ as their contribution. Suppose the pension fund could be invested in both a bond and a stock in the market. The price of the bond (risk-free asset) satisfies

$$dB(t) = r(t)B(t)dt,$$

where $r(\cdot)$ is the interest rate of the bond. In a securities market with delayed responses, the price of the stock can be described by an SDDE

$$\begin{cases} dS(t) = \mu(t)S(t-\delta)dt + \sigma S(t)dW(t),\ t \in [0, T], \\ S(t) = \nu(t),\ t \in [-\delta, 0], \end{cases}$$

where $\mu(\cdot)$ is the appreciation rate of a return and σ is the volatility coefficient of the stock. We assume that $r(\cdot)$, $\mu(\cdot)$, and $\nu(\cdot)$ are deterministic $\mathbb{R}$-valued bounded functions, and $\sigma > 0$ is a constant.

Let $\pi(\cdot)$ be the amount invested in the stock. Then, the dynamic of the pension fund is modeled by

$$\begin{cases} dy(t) = [r(t)y(t) - r(t)\pi(t) + \mu(t)\pi(t-\delta) + u(t) + w(t)]dt + \sigma\pi(t)dW(t),\ t \in [0, T], \\ y(T) = \gamma,\ \pi(t) = 0,\ t \in [-\delta, 0), \end{cases} \tag{27}$$

where γ is the terminal wealth goal. If we set $z(t) = \sigma\pi(t)$, the BSDDE (27) can be written as

$$\begin{cases} -dy(t) = [-r(t)y(t) + \sigma^{-1}r(t)z(t) - \sigma^{-1}\mu(t)z(t-\delta) - u(t) - w(t)]dt \\ \qquad\qquad - z(t)dW(t),\ t \in [0, T], \\ y(T) = \gamma,\ z(t) = 0,\ t \in [-\delta, 0). \end{cases} \tag{28}$$

We assume that the time delay $\delta \geqslant 0$ is sufficiently small to guarantee that BSDDE (28) admits a unique solution.

Let

$$\mathcal{U}_1 := \{u(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}) | u(\cdot) \geq 0, \ a.e.\ a.s.\},$$
$$\mathcal{U}_2 := \{w(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}) | w(\cdot) \geq 0, \ a.e.\ a.s.\}.$$

The associated cost functionals of the follower and leader have the form

$$\begin{cases} J_1(\gamma; u(\cdot), w(\cdot)) = \dfrac{1}{2}\mathbb{E}[\displaystyle\int_0^T L_1 e^{-\beta t} u^2(t)dt + G_1 y^2(0)], \\ J_2(\gamma; u(\cdot), w(\cdot)) = \dfrac{1}{2}\mathbb{E}[\displaystyle\int_0^T L_2 e^{-\beta t} w^2(t)dt + G_2 y^2(0)]. \end{cases} \tag{29}$$

where L_i and G_i $(i = 1,2)$ are positive constants, and $\beta > 0$ is a discount factor. Under the precondition that the terminal goal γ is attained, it is natural to want to minimize the cost functionals (29).

By Theorem 6, we can give an explicit characterization of the equilibrium strategy for the pension fund problem.

Proposition 1. *The pension fund problem (28) with (29) admits a unique equilibrium strategy $(u^*(\cdot), w^*(\cdot))$ as follows:*

$$u^*(t) = -L_1^{-1} e^{\beta t} x(t), \ w^*(t) = -L_2^{-1} e^{\beta t} \xi(t), \tag{30}$$

where $x(\cdot)$ and $\xi(\cdot)$ satisfy

$$\begin{cases} -dy(t) = & [-r(t)y(t) + L_1^{-1}e^{\beta t}x(t) + L_2^{-1}e^{\beta t}\xi(t) + \sigma^{-1}r(t)z(t) \\ & -\sigma^{-1}\mu(t)z(t-\delta)]dt - z(t)dW(t), \ t \in [0,T], \\ dx(t) = & -r(t)x(t)dt + \{\sigma^{-1}r(t)x(t) - \sigma^{-1}\mathbb{E}^{\mathcal{F}_t}[\mu(t+\delta)x(t+\delta)]\}dW(t), \ t \in [0,T], \\ -dp(t) = & [-r(t)p(t) - L_1^{-1}e^{\beta t}\xi(t) + \sigma^{-1}r(t)q(t) - \sigma^{-1}\mu(t)q(t-\delta)]dt \\ & -q(t)dW(t), t \in [0,T], \\ d\xi(t) = & -r(t)\xi(t)dt + \{\sigma^{-1}r(t)\xi(t) - \sigma^{-1}\mathbb{E}^{\mathcal{F}_t}[\mu(t+\delta)\xi(t+\delta)]\}dW_t, t \in [0,T], \\ y(T) = & \gamma, \ z(t) = 0, \ t \in [-\delta, 0), \\ x(0) = & -G_1 y(0), \ x(t) = 0, \ t \in (T, T+\delta], \\ p(T) = & 0, \ p(t) = 0, \ q(t) = 0, \ t \in [-\delta, 0), \\ \xi(0) = & G_1 p(0) - G_2 y(0), \ \xi(t) = 0, \ t \in (T, T+\delta]. \end{cases} \tag{31}$$

5. Concluding Remarks

In this paper, we discuss the stochastic backward Stackelberg games with delay motivated by some interesting economic and financial problems. Since the corresponding Hamilton system is particularly complicated, we focus on the linear systems. The optimal strategies of the follower and the leader are expressed by an AFBSDDE, which is a new kind of double FBSDE. Furthermore, the AFBSDDE is proved to be uniquely solvable, and we then obtain the unique equilibrium of the LQ backward Stackelberg game. Although this paper deals with systems with state delays, our approach is still valid for the systems with control delays. In addition, it is worthwhile to consider the state feedback representation of the equilibrium. Another possible research direction is to investigate the maximum principle for the leader's optimal control problem under a structure that describes the follower's control as a function of the leader's control and the adjoint variable. We will study these challenging topics in the future.

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