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Orderings over Intuitionistic Fuzzy Pairs Generated by the Power Mean and the Weighted Power Mean

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Abstract: In the present work, we prove a result concerning an ordering over intuitionistic fuzzy pairs generated by the power mean (M_p) for $p > 0$. We also introduce a family of orderings over intuitionistic fuzzy pairs generated by the weighted power mean (M_p^{α}) and prove that a similar result holds for them. The considered orderings in a natural way extend the classical partial ordering and allow the comparison of previously incomparable alternatives. In the process of proving these properties, we establish some inequalities involving logarithms which may be of interest by themselves. We also show that there exists $p > 0$ for which a finite set of alternatives, satisfying some reasonable requirements, some of which were not comparable under the classical ordering, has all its elements comparable under the new ordering. Finally, we provide some examples for the possible use of these orderings to a set of alternatives, which are in the form of intuitionistic fuzzy pairs as well as to results from InterCriteria Analysis.

Keywords: intuitionistic fuzzy pair; ordering; power mean; alternatives

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1. Introduction

Decision making under uncertainty, especially when the data are represented by intuitionistic fuzzy values or intuitionistic fuzzy interval values, has gained increasing interest in recent decades [1–9]. Recent research has been concentrated on the reduction in redundant information from the set of considered alternatives without affecting the quality of the decision [10], finding approaches applicable to cases with incomplete or missing evaluations [11], or finding approaches that are suitable for multi-attribute decision making [12,13], while others were focused on investigating the relationships between different possible orderings and their advantages and disadvantages [9,14–16].

Intuitionistic fuzzy pairs (IFPs) may be viewed as single-point intuitionistic fuzzy sets (see [17]), all of which are associated with the same element. Different orderings were defined over IFPs. The two classical ones are due to Atanassov [18] as well as Bustince and Burillo [19] (the latter one was also studied in detail by E. Marinov [20]). In order to select certain IFPs among some which are incomparable under the classical ordering of Atanassov, another ordering has to be used. The approaches usually used either employ a ranking function which produces a real-valued number, thus implying a linear order, or some distance measure to $\langle 1, 0 \rangle$, which also provides a real-valued number as a result. Our approach slightly differs from that, since we consider the families of partial orderings which tend towards linear ordering as certain parameters grow. In a previous investigation, two of the authors of the present work first introduced the component biased power mean-based ordering between IFPs, denoted by $\preceq_{\mu;M_p}$ [21]. It is, in a way, the natural generalization of Atanassov's classical ordering, which in terms of power mean, may be stated as $\preceq_{\mu;M_{-\infty}}$.

The main contributions of the current work are:

- The establishment of the fact that, for $p > 0$, if $u \preceq_{\mu;M_p} v$, then $u \preceq_{\mu;M_{p+\varepsilon}} v$, for any $\varepsilon > 0$,
- The introduction of a new ordering—a first component biased weighted power mean-based ordering $\preceq_{\mu;M_p^\alpha}$—which has the same property both for $p > 0$ and for any $\beta > \alpha$.
- Providing the necessary and sufficient conditions for the possibility to make any two alternatives from a set of IFPs comparable under some of these orderings stated as Proposition 1.

The considered families of orderings ($\preceq_{\mu;M_p}$ and $\preceq_{\mu;M_p^\alpha}$), thus provide an opportunity for a more flexible yet consistent way of comparing a larger selection of IFPs, which are incomparable under the classical ordering. In terms of the orderings between the closed subintervals of the unit interval introduced by Bustince et al. in [8], these families of orderings may be considered admissible successive refinements of the partial order $\leq_2$, as p and/or α grows.

The paper is organized as follows: Section 2 provides the basic definitions and auxiliary results which will be further used. Section 3 provides proofs of the properties of the considered families of orderings. Section 4 establishes the necessary and sufficient condition to make any two alternatives from a set of IFPs comparable under some of these orderings, providing examples for the possible applications of the orderings based on that, as well as a comparison with the results obtained by other existing orderings from the literature. Section 5 provides a brief overview of the results and outlines future directions for research.

2. Preliminaries and Auxiliary Results

Here, we provide a concise description of the notions and auxiliary results that will be required for the formulation and proof of our main results.

Definition 1 (cf. [22])**.** *An intuitionistic fuzzy pair is an ordered couple of real non-negative numbers $\langle a, b \rangle$, such that:*

$$\min(a, b) \leq 1 - \max(a, b). \tag{1}$$

This concept is important in practice since many methods implementing intuitionistic fuzzy techniques generate estimates in the form of IFPs. One such example is the InterCriteria Analysis (ICrA). For these estimates, the first component usually has a sense of validity, similarity, or some form of agreement or parity, while the second component signifies falsity, distance, or some form of disagreement or disparity, etc. When a choice between two IFPs is required, an ordering (or an appropriate ranking method) must be used.

The classical partial ordering introduced by Atanassov (see [18]) is given by

Definition 2 (cf. [18,22])**.** *For two IFPs: $u = \langle u_1, u_2 \rangle$ and $v = \langle v_1, v_2 \rangle$, we say that u is less or equal to v, and we write:*

$$u \leq v,$$

iff

$$\begin{cases} u_1 \leq v_1 \\ u_2 \geq v_2. \end{cases} \tag{2}$$

It is readily obvious that $\leq$ is only partial ordering, since it is transitive, reflexive, and antisymmetric but there exist u and v, for which (2) is not satisfied. For instance, the pairs $\langle 0.35, 0.45 \rangle$ and $\langle 0.4, 0.5 \rangle$ are not comparable under classical ordering.

Other classical ordering worth mentioning is outlined as follows.

Definition 3 (cf. [19,20])**.** *For two IFPs: $u = \langle u_1, u_2\rangle$ and $v = \langle v_1, v_2\rangle$, we say that u precedes v, and we write:*

$$u \preceq v,$$

iff

$$\begin{cases} u_1 \leq v_1 \\ u_2 \leq v_2. \end{cases} \tag{3}$$

This ordering, called π-ordering by E. Marinov, is in some sense counterpart to classical ordering. Indeed, excluding the cases of simultaneous equalities in (3) and (2), we have that, if two elements are comparable under one of these orderings, they are incomparable under the other.

Definition 4 ([23], p. 175)**.** *The power mean of two non-negative numbers x, y is given by:*

$$M_p(x,y) = \left(\frac{x^p + y^p}{2}\right)^{\frac{1}{p}}. \tag{4}$$

Special cases (obtained as a limit) of the power mean worth mentioning are the following: $M_{-\infty}(x,y) = \min(x,y)$, $M_0(x,y) = \sqrt{xy}$, $M_\infty(x,y) = \max(x,y)$.

The power mean has the following nice property [23] (p. 175):

$$M_p(x,y) \leq M_q(x,y) \text{ for } p \leq q.$$

Definition 5 (cf. [23], p. 175)**.** *The power mean with the weight α of two non-negative numbers x, y is given by:*

$$M_p^\alpha(x,y) = (\alpha x^p + (1-\alpha)y^p)^{\frac{1}{p}}, \tag{5}$$

where $\alpha \in (\frac{1}{2}, 1)$.

Definition 6 ([21])**.** *Given two IFPs $u = \langle u_1, u_2\rangle$ and $v = \langle v_1, v_2\rangle$, we say that u is the first component biased power mean-based with a value of p less or equal to v and write $u \preceq_{\mu;M_p} v$ if*

$$\begin{cases} 1 - u_1 \geq 1 - v_1 \\ M_p(1-u_1, u_2) \geq M_p(1-v_1, v_2). \end{cases} \tag{6}$$

Definition 7. *Given two IFPs $u = \langle u_1, u_2\rangle$ and $v = \langle v_1, v_2\rangle$, we say that u is the first component biased power mean with a weight of α based on a value of p, which is less than or equal to v, and write $u \preceq_{\mu;M_p^\alpha} v$ if*

$$\begin{cases} 1 - u_1 \geq 1 - v_1 \\ M_p^\alpha(1-u_1, u_2) \geq M_p^\alpha(1-v_1, v_2). \end{cases} \tag{7}$$

Remark 1. *If we consider the IFP $\langle a, b\rangle$ as corresponding to the closed subinterval $[b, 1-a]$, we can see that the ordering $\leq_2$ introduced in [8] as:*

$$[b, 1-a] \leq_2 [d, 1-c] \Leftrightarrow b \leq d \wedge 1-a \leq 1-c.$$

is equivalent to (see Definition 2):

$$\langle a, b\rangle \geq \langle c, d\rangle$$

Therefore, any refinement on partial order $\geq$ (from Definition 2) would also be a refinement on $\leq_2$. As we shall further prove, $\langle a, b\rangle \geq \langle c, d\rangle$ implies $\langle a, b\rangle \geq_{\mu;p} \langle c, d\rangle$, which in turn implies $\langle a, b\rangle \geq_{\mu;q} \langle c, d\rangle$, for $q \geq p$, we can thus introduce the successive admissible (partial) linear orders

$$[b, 1-a] \leq_2^p [d, 1-c] \Leftrightarrow M_p(1-a, b) \leq M_p(1-c, d) \wedge 1-a \leq 1-c, \tag{8}$$

which, as p grows, refine the previous (partial) order (see Theorem 1 in Section 3). In the same line of reasoning (see Theorem 2 in Section 3), we can introduce the family of admissible orderings which are successive refinements of the $\leq_2^p$ ordering as α approaches 1*:*

$$[b, 1-a] \leq_2^{p;\alpha} [d, 1-c] \Leftrightarrow M_p^\alpha(1-a, b) \leq M_p^\alpha(1-c, d) \wedge 1-a \leq 1-c. \tag{9}$$

Further, we will make use of

Lemma 1. *For any constant $c \in (\frac{1}{2}, 1)$ and for all $t \in (0, 1-c)$, the following inequality is fulfilled:*

$$(2c-1)\ln(2c-1) > (c-t)\ln(c-t) + (c+t)\ln(c+t). \tag{10}$$

Proof. Consider the function

$$h(t) = (c-t)\ln(c-t) + (c+t)\ln(c+t)$$

defined for all $t \in (-c, c)$.

It is increasing in the interval $(0, c)$ since

$$h'(t) = \ln\left(\frac{c+t}{c-t}\right) > 0.$$

and has a minimum for $t = 0$. Since, from the condition of Lemma 1, it follows that $1-c < c$, we have for $t \in [0, 1-c)$

$$h(t) < h(1-c) = (2c-1)\ln(2c-1).$$

□

We will also require the following

Lemma 2. *Let $a \in (\frac{1}{2}, 1)$. Then, for all $k \in \left(1, \frac{a}{1-a}\right)$ and $t \in (0, a-k(1-a)]$, it is true that*

$$ka\ln(a) + (k(1-a)+t)\ln(k(1-a)+t) < t\ln(t). \tag{11}$$

Proof. We rewrite (11) as:

$$ka\ln(a) + k(1-a)\ln(k(1-a)+t) < t\ln\left(\frac{t}{k(1-a)+t}\right). \tag{12}$$

The left-hand side (LHS) of (12) is obviously increasing with t. Let us consider the right-hand side (RHS) and denote it by $g(t)$. We will show that $g(t)$ decreases with t on the interval $(0, a-k(1-a)]$. We have

$$g'(t) = \frac{k(1-a)}{k(1-a)+t} + \ln\left(\frac{t}{k(1-a)+t}\right)$$

Using the fact that, for $t > 0$,

$$\ln\left(\frac{t}{k(1-a)+t}\right) < \frac{t}{k(1-a)+t} - 1,$$

when $\frac{t}{k(1-a)+t} \neq 1$, which is obviously true in our case, we obtain:

$$g'(t) < 0.$$

Hence, g decreases with t on the interval $(0, a-k(1-a)]$. Thus, we conclude that the minimum value of $g(t)$ on the interval $(0, a-k(1-a)]$ is obtained for $t = a-k(1-a)$.

Since the LHS of (12) is increasing with t, it will also reach its maximum value for $t = a - k(1-a)$. Hence, if we establish that

$$\begin{aligned} ka\ln(a) + k(1-a)\ln(k(1-a) + a - k(1-a)) &= \max(\text{LHS}) \\ < \min(\text{RHS}) = (a-k(1-a))\ln&\left(\frac{a-k(1-a)}{k(1-a)+a-k(1-a)}\right) \end{aligned} \tag{13}$$

we will complete the proof. After simplification, this is equivalent to

$$k\ln(a) < (a-k(1-a))\ln\left(\frac{a-k(1-a)}{a}\right), \tag{14}$$

which is equivalent to

$$\ln(a) < a\frac{1-\frac{k(1-a)}{a}}{k}\ln\left(1-\frac{k(1-a)}{a}\right). \tag{15}$$

Let us denote $s = \frac{k(1-a)}{a}$. Then, using the fact that

$$\ln(a) \le a - 1,$$

we see that (15) is true if:

$$a - 1 < \frac{a}{k}(1-s)\ln(1-s).$$

The last can be simplified to (since $1 - a > 0$):

$$-1 < \frac{1-s}{s}\ln(1-s),$$

i.e., we need to establish that

$$\frac{s}{1-s} > \ln\left(\frac{1}{1-s}\right).$$

However, since $1 > s > 0$, we have

$$\frac{s}{1-s} = \frac{1}{1-s} - 1 > \ln\left(\frac{1}{1-s}\right).$$

□

3. Main Results

We are now ready to formulate our main results.

Theorem 1. *Let $u = \langle u_1, u_2\rangle$ and $v = \langle v_1, v_2\rangle$. If*

$$u \preceq_{\mu;M_p} v \tag{16}$$

for some $p > 0$, then

$$u \preceq_{\mu;M_q} v \tag{17}$$

for all $q > p$.

Proof. From (1) and the first inequality of (6), we have $1 - u_1 \ge 1 - v_1 \ge v_2$. If $u_2 \ge v_2$, the statement of Theorem 1 is obviously valid. Furthermore, without loss of generality, we will assume $1 - u_1 > 1 - v_1 > v_2 > u_2$. Let us denote that $1 - u_1 = x, 1 - u_2 = z, v_2 = t, u_2 = y$.

Thus, without loss of generality, we may assume that $1 > x > z > t > y > 0$. The statement of Theorem 1 is equivalent to the following:

If

$$x^p + y^p \geq z^p + t^p, \text{ for } p > 0, \tag{18}$$

we have

$$x^{p+\varepsilon} + y^{p+\varepsilon} \geq z^{p+\varepsilon} + t^{p+\varepsilon} \tag{19}$$

for all $\varepsilon > 0$.

Without loss of generality, we can rewrite (18) as

$$q^p \geq u^p + w^p - 1, \tag{20}$$

where $q = \frac{y}{x}, w = \frac{t}{x}, u = \frac{z}{x}$ and we have $0 < q < w < u < 1$. In the same manner, (19) is equivalent to

$$q^{p+\varepsilon} \geq u^{p+\varepsilon} + w^{p+\varepsilon} - 1, \tag{21}$$

Let us assume that $u^{p+\varepsilon} + w^{p+\varepsilon} - 1 > 0$ (otherwise, the statement is obvious) and it is true that

$$q^{p+\varepsilon_0} < u^{p+\varepsilon_0} + w^{p+\varepsilon_0} - 1 \tag{22}$$

for some $\varepsilon_0 > 0$.

From (20) and (22), it follows:

$$(u^{p+\varepsilon_0} + w^{p+\varepsilon_0} - 1)^{\frac{1}{p+\varepsilon_0}} > (u^p + w^p - 1)^{\frac{1}{p}} \tag{23}$$

However, we will show that (23) is impossible, since

$$f(x) = (u^x + w^x - 1)^{\frac{1}{x}}$$

is monotonously decreasing as x grows.

Let us now consider the first derivative of $f(x)$. We have

$$f'(x) = \frac{(u^x + w^x - 1)^{\frac{1}{x}}}{x^2(u^x + w^x - 1)}(u^x \ln(u^x) + w^x \ln(w^x) - (u^x + w^x - 1)\ln(u^x + w^x - 1)).$$

The sign of $f'(x)$ depends on the sign of

$$u^x \ln(u^x) + w^x \ln(w^x) - (u^x + w^x - 1)\ln(u^x + w^x - 1).$$

Putting $c = \frac{u^x + w^x}{2}$, and from the fact that $\frac{u^x - w^x}{2} < 1 - \frac{u^x + w^x}{2}$ using Lemma 1, we obtain that $f'(x) < 0$, while $u^x + w^x - 1 > 0$. Hence, (23) is impossible as we have

$$f(p) > f(p + \varepsilon).$$

□

This result permits us to consider all ordering generated by $p > 0$ as a transition from a partial to linear ordering, which is obtained (as a limit) for $p = \infty$. Due to (17), we know that this ordering preserves the existing relations, i.e., they increase the number of comparable elements in a consistent manner.

The introduction of the weight α allows us to fine-tune our orderings (while keeping them consistent), shifting them closer to the linear order by providing a higher priority to the first component.

Theorem 2. *Let $u = \langle u_1, u_2 \rangle$ and $v = \langle v_1, v_2 \rangle$. Let $\alpha \in (\frac{1}{2}, 1)$. Then, if*

$$u \preceq_{\mu; M_p^\alpha} v \tag{24}$$

for some $p > 0$, we have

$$u \preceq_{\mu;M_q^\alpha} v \tag{25}$$

for all $q > p$.

Proof. From (1) and the first inequality of (7), we have $1 - u_1 \geq 1 - v_1 \geq v_2$. If $u_2 \geq v_2$, the statement of Theorem 2 is obviously valid. Furthermore, without loss of generality, we will assume $1 - u_1 > 1 - v_1 > v_2 > u_2$. Let us denote $1 - u_1 = x, 1 - u_2 = z, v_2 = t, u_2 = y$.

Thus, $1 > x > z > t > y > 0$. The statement of Theorem 2 (in this case) is equivalent to the following:

If

$$\alpha x^p + (1-\alpha)y^p \geq \alpha z^p + (1-\alpha)t^p, \text{ for } p > 0, \tag{26}$$

then, we have

$$\alpha x^{p+\varepsilon} + (1-\alpha)y^{p+\varepsilon} \geq \alpha z^{p+\varepsilon} + (1-\alpha)t^{p+\varepsilon} \tag{27}$$

for all $\varepsilon > 0$.

Without loss of generality, we can rewrite (26) as

$$q^p \geq \frac{\alpha}{1-\alpha}u^p + w^p - \frac{\alpha}{1-\alpha}, \tag{28}$$

where $q = \frac{y}{x}, w = \frac{t}{x}, u = \frac{z}{x}$ and we have $0 < q < w < u < 1$. In the same manner, (27) is equivalent to

$$q^{p+\varepsilon} \geq \frac{\alpha}{1-\alpha}u^{p+\varepsilon} + w^{p+\varepsilon} - \frac{\alpha}{1-\alpha}, \tag{29}$$

Let us assume that $\frac{\alpha}{1-\alpha}u^{p+\varepsilon} + w^{p+\varepsilon} - \frac{\alpha}{1-\alpha} > 0$ (otherwise the statement is obvious) and that it is true that

$$q^{p+\varepsilon_0} < \frac{\alpha}{1-\alpha}u^{p+\varepsilon_0} + w^{p+\varepsilon_0} - \frac{\alpha}{1-\alpha} \tag{30}$$

for some $\varepsilon_0 > 0$.

From (28) and (30), it follows:

$$\left(\frac{\alpha}{1-\alpha}u^{p+\varepsilon_0} + w^{p+\varepsilon_0} - \frac{\alpha}{1-\alpha}\right)^{\frac{1}{p+\varepsilon_0}} > \left(\frac{\alpha}{1-\alpha}u^{p} + w^{p} - \frac{\alpha}{1-\alpha}\right)^{\frac{1}{p}} \tag{31}$$

However, we will show that (31) is impossible, since

$$\eta(x) = \left(\frac{\alpha}{1-\alpha}u^x + w^x - \frac{\alpha}{1-\alpha}\right)^{\frac{1}{x}}.$$

is monotonously decreasing as x grows.

Let us consider the first derivative of η with respect to x

$$\eta'(x) = \frac{\left(\frac{\alpha}{1-\alpha}u^x + w^x - \frac{\alpha}{1-\alpha}\right)^{\frac{1}{x}}}{x^2\left(\frac{\alpha}{1-\alpha}u^x + w^x - \frac{\alpha}{1-\alpha}\right)}\theta(x),$$

where

$$\theta(x) = \frac{\alpha}{1-\alpha}u^x \ln(u^x) + w^x \ln(w^x) - \left(\frac{\alpha}{1-\alpha}u^x + w^x - \frac{\alpha}{1-\alpha}\right)\ln\left(\frac{\alpha}{1-\alpha}u^x + w^x - \frac{\alpha}{1-\alpha}\right).$$

The sign of $\eta'(x)$, solely depends on $\theta(x)$, if $\frac{\alpha}{1-\alpha}u^x + w^x - \frac{\alpha}{1-\alpha} > 0$.

Putting $k = \frac{\alpha}{1-\alpha}, a = u^x$, and using the fact that there exists $t^* \in (0, u^x - (1-u^x)\frac{\alpha}{1-\alpha}]$ for which $(1-u^x)\frac{\alpha}{1-\alpha} + t^* = w^x$, by Lemma 2, we obtain that $\eta'(x) < 0$.

Hence, (31) can never be true, since we have

$$\eta(p) > \eta(p+\varepsilon).$$

□

Corollary 1. *Let $u = \langle u_1, u_2\rangle$ and $v = \langle v_1, v_2\rangle$. Let $\alpha_1 \in (\frac{1}{2}, 1), \alpha_2 \in (\alpha_1, 1)$. Then, if for some $p > 0$,*

$$u \preceq_{\mu;M_p^{\alpha_1}} v \tag{32}$$

it follows that

$$u \preceq_{\mu;M_p^{\alpha_2}} v. \tag{33}$$

4. Possible Applications of the Obtained Results

Proposition 1. *Given a finite set of alternatives in the forms of IFPs, which does not contain the alternative $\langle 0,0\rangle$ (corresponding to total indeterminacy or complete lack of information), there always exists $p > 0$ and (or) $\alpha \geq \frac{1}{2}$ for which the alternatives are fully ordered under $\preceq_{\mu;M_p}$ and (or) $\preceq_{\mu;M_p^{\alpha}}$.*

Proof. Without loss of generality, we may assume that the set is

$$\{\langle a_1, b_1\rangle, \langle a_2, b_2\rangle, \langle a_3, b_3\rangle, \dots, \langle a_n, b_n\rangle\} \tag{34}$$

such that it is fulfilled $a_i \leq a_j$, for all $i \leq j$.

We will show that we can find $p > 0$ such that all IFPs in the set are comparable under $\preceq_{\mu;M_p}$. The elements for which $a_i = a_j$ or $b_i \geq b_j$, are already ordered under the classical ordering, so without loss of generality, we will assume

$$\begin{cases} a_i < a_j \\ b_i < b_j. \end{cases}$$

In other words, we have $1 - a_i > 1 - a_j \geq b_j > b_i$. If, for a given $i < j$, we have $1 - a_i + b_i \geq 1 - a_j + b_j$, then $\langle a_i, b_i\rangle \preceq_{\mu;M_1} \langle a_j, b_j\rangle$. Therefore, we will further assume that

$$(1 - a_i) - (1 - a_j) = a_j - a_i < b_j - b_i. \tag{35}$$

We will show that, if (35) is true, there exists an integer $n > 1$ for which

$$(1 - a_i)^n - (1 - a_j)^n > b_j^n - b_i^n. \tag{36}$$

To establish that fact, we observe that, from (35), it follows that we must have

$$\frac{1 - a_j}{1 - a_i} > \frac{b_i}{b_j}. \tag{37}$$

The validity of (37) is easily established when we observe that it is equivalent to:

$$(1 - a_j)(b_j - b_i) > b_i((1 - a_i) - (1 - a_j)),$$

which is certainly true in view of $1 - a_j > b_i$ and (35). We note that (36) is equivalent to

$$\frac{(1 - a_i)^n}{b_j^n} \frac{a_j - a_i}{b_j - b_i} > \frac{\sum\limits_{k=0}^{n-1} \frac{b_i}{b_j}}{\sum\limits_{k=0}^{n-1} \frac{1-a_j}{1-a_i}}. \tag{38}$$

In view of (37), we see that the right-hand-side of (38) is decreasing as n grows, while the left-hand side is certainly increasing as n grows. Thus, (38) will be evidently true when

$$\left(\frac{(1-a_i)}{b_j}\right)^n > \frac{b_j - b_i}{a_j - a_i}.$$

Since $\frac{1-a_i}{b_j} > 1$, then for any positive constant c, there must exist a positive integer $n(c)$, such that

$$\left(\frac{(1-a_i)}{b_j}\right)^{n(c)} > c,$$

and therefore, we established that (36) is valid for some $n \geq n\left(\frac{b_j-b_i}{a_j-a_i}\right)$. Thus, for the same n, we have $\langle a_i, b_i\rangle \preceq_{\mu;M_n} \langle a_j, b_j\rangle$, since (36) implies

$$\left(\frac{(1-a_i)^n + b_i^n}{2}\right)^n > \left(\frac{(1-a_j)^n + b_j^n}{2}\right)^n.$$

Therefore, after going through all possible pairs, taking the maximum of all n-s and denoting it by n_{max}, we obtain a set of alternatives that is fully ordered with respect to $\preceq_{\mu;M_{n_{max}}}$.

In view of Corollary 1, it is evident that similar reasoning may be applied with respect to α, as we let it grow. □

Remark 2. *Note that n_{max}, introduced in the above proof, is clearly larger than the smallest value of p, for which the full ordering is obtained. However, since we were only interested in establishing the existence, and not concerned with the practical calculation of the said value, we would postpone the question of finding the smallest value for another time.*

Furthermore, to better illustrate the implications of Proposition 1, we will consider two examples.

Example 1. *Let us be given the set of alternatives:*

$$\{\langle 0.1, 0.8\rangle, \langle 0.5, 0.4\rangle, \langle 0.3, 0.7\rangle, \langle 0.6, 0.3\rangle, \langle 0.2, 0.4\rangle\} \tag{39}$$

The only IFPs here, which are not comparable under the classical ordering, are $\langle 0.2, 0.4\rangle$ and $\langle 0.3, 0.7\rangle$. We can easily see that:

$$\langle 0.2, 0.4\rangle \preceq_{\mu;M_5} \langle 0.3, 0.7\rangle$$

In fact, the value of p for which this ordering becomes true is $p = 4.95292$. Therefore, the set of alternatives for $p \geq 4.95292$ becomes linearly ordered under this power mean ordering and we have

$$\langle 0.1, 0.8\rangle \preceq_{\mu;M_5} \langle 0.2, 0.4\rangle \preceq_{\mu;M_5} \langle 0.3, 0.7\rangle \preceq_{\mu;M_5} \langle 0.5, 0.4\rangle \preceq_{\mu;M_5} \langle 0.6, 0.3\rangle \tag{40}$$

If we instead use the weighted power mean ordering, we can, by using $\alpha = \frac{3}{4}$, achieve the same result for $p = 1$, i.e.,

$$\langle 0.1, 0.8\rangle \preceq_{\mu;M_1^{\frac{3}{4}}} \langle 0.2, 0.4\rangle \preceq_{\mu;M_1^{\frac{3}{4}}} \langle 0.3, 0.7\rangle \preceq_{\mu;M_1^{\frac{3}{4}}} \langle 0.5, 0.4\rangle \preceq_{\mu;M_1^{\frac{3}{4}}} \langle 0.6, 0.3\rangle \tag{41}$$

Example 2. *We consider the data used in [24], which we have processed using VisicrA software v.0.9.1 (developed in Python by N. Ikonomov), as shown in Figure 1.*

The InterCriteria Analysis provides an estimation in the forms of IFPs, regarding the relationships between the different criteria. The factors considered in [24] are the following:

- *Poor work ethic in national labor force (PWE);*
- *Access to financing (ATF);*
- *Corruption (COR);*
- *Crime and theft (CAT);*
- *Foreign currency regulations (FCR);*
- *Government instability/coups (GIC);*
- *Inadequate supply of infrastructure (ISI);*
- *Inadequately educated workforce (IEW);*
- *Inefficient government bureaucracy (IGB);*
- *Inflation (INF);*
- *Insufficient capacity to innovate (ICI);*
- *Policy instability (PIN);*
- *Poor public health (PPH);*
- *Restrictive labor regulations (RLR);*
- *Tax rates (TRA);*
- *Tax regulations (TRE).*

We only concentrate our attention on the couples PWE-ISI, PWE-COR, PWE-ICI, PWE-PIN, PWE-ATF, PWE-IEW, PWE-TRE, PWE-RLR, PWE-IGB, and PWE-TRA, representing the relationship of PWE to the various other factors related to the incentive to do business, which InterCriteria Analysis evaluates as:

$\langle 0.6478, 0.3251\rangle$, $\langle 0.6158, 0.3571\rangle$, $\langle 0.4483, 0.532\rangle$, $\langle 0.3793, 0.5837\rangle$, $\langle 0.4877, 0.4877\rangle$, $\langle 0.7685, 0.2167\rangle$, $\langle 0.367, 0.6034\rangle$, $\langle 0.3276, 0.6527\rangle$, $\langle 0.3744, 0.6084\rangle$, $\langle 0.3399, 0.6453\rangle$.

We can easily see that only two IFPs are incomparable by the classical ordering, namely those corresponding to PWE-TRE and PWE-IGB, with values $\langle 0.367, 0.6034\rangle$ *and* $\langle 0.3744, 0.6084\rangle$. *However, it is not difficult to observe that* $\langle 0.367, 0.6034\rangle \preceq_{\mu;M_1} \langle 0.3744, 0.6084\rangle$; *hence, we have (under* $\preceq_{\mu;M_1}$*) the following order for the strength of the relations between the factors: PWE-RLR* $\preceq_{\mu;M_1}$ *PWE-TRA* $\preceq_{\mu;M_1}$ *PWE-TRE* $\preceq_{\mu;M_1}$ *PWE-IGB* $\preceq_{\mu;M_1}$ *PWE-PIN* $\preceq_{\mu;M_1}$ *PWE-ICI* $\preceq_{\mu;M_1}$ *PWE-ATF* $\preceq_{\mu;M_1}$ *PWE-COR* $\preceq_{\mu;M_1}$ *PWE-ISI* $\preceq_{\mu;M_1}$ *PWE-IEW.*

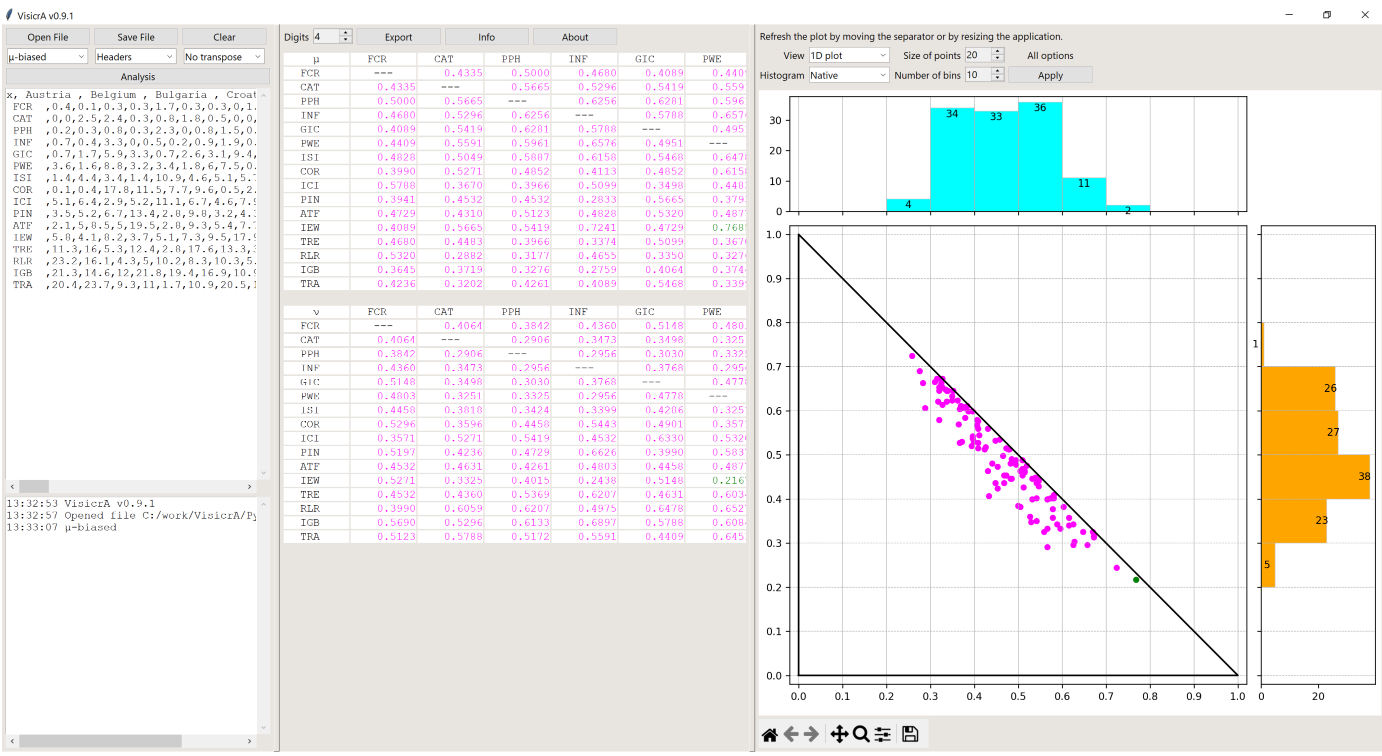

Figure 1. A view of the VisicrA software processing input data from [24].

As can be seen from Example 2, our proposed orderings may be used over results from InterCriteria Analysis to obtain a linear order among them. At any rate, whether it is always appropriate to do so, and whether the considered orderings may infer counter-intuitive or even wrong conclusions, is a matter of a future research. Here, we only established the theoretical framework which permits enables us to implement this possibility in a consistent manner.

Furthermore, we will compare the result of the application of our orderings to previously proposed ones. We require the following definitions:

Definition 8 (cf. [25,26])**.** *For the IFP $u = \langle u_1, u_2 \rangle$, its score S and accuracy H are given by:*

$$S(u) = u_1 - u_2$$
$$H(u) = u_1 + u_2$$

Definition 9 (cf. [2])**.** *For two u, v IFPs, the order $\preceq_{SH}$ is defined as follows:*

$$u \preceq_{SH} v \Leftrightarrow \begin{cases} S(u) < S(v) \\ S(u) = S(v) \wedge H(u) \leq H(v) \end{cases} \tag{42}$$

Definition 10 (cf. [3])**.** *For two u, v IFPs, the order $\preceq_{LH}$ is defined as follows:*

$$u \preceq_{LH} v \Leftrightarrow \begin{cases} L(u) < L(v) \\ L(u) = L(v) \wedge H(u) \leq H(v) \end{cases} \tag{43}$$

where $L(u) = \frac{1-u_2}{1-u_1+1-u_2}$.

Definition 11 (cf. [5])**.** *For two u, v IFPs, the order $\preceq_R$ is defined as follows:*

$$u \preceq_R v \Leftrightarrow R(u) \geq R(v) \tag{44}$$

where $R(u) = \frac{1}{2}(1 - u_1)(1 - u_1 + 1 - u_2)$.

Definition 12 (cf. [6])**.** *For two u, v IFPs, the order $\preceq_Z$ is defined as follows:*

$$u \preceq_Z v \Leftrightarrow Z(u) \leq Z(v) \tag{45}$$

where $Z(u) = \frac{1}{4}(1 + u_1 - u_2)(1 + u_1 + u_2)$.

Definition 13 (cf. [7])**.** *For two u, v IFPs, the order $\preceq_P$ is defined as follows:*

$$u \preceq_P v \Leftrightarrow P(u) \leq P(v) \tag{46}$$

where $P(u) = 1 - M_2(1 - u_1, u_2)$, with M_2 defined by (4).

Definition 14 (cf. [9])**.** *For two u, v IFPs, the order $\preceq_M$ is defined as follows:*

$$u \preceq_M v \Leftrightarrow M(u) \leq M(v) \tag{47}$$

where $M(u) = \sqrt{(u_1)^2 + (1 - u_2)^2} - \sqrt{(1 - u_1)^2 + (u_2)^2} + \sqrt{(u_1)^2 + (u_2)^2}$.

Below, we present a comparison of the orderings over some IFPs.

In Table 1, $\succeq_{\mu;M_{p>i}}$, signifies the smallest integer value for p after which the ordering is valid. It can be seen that $\succeq_{\mu;M_p}$ ordering agrees well with most other orderings, specifically, $\preceq_M$, $\preceq_R$, and $\preceq_P$ (the second example shows the opposite ordering after $p > 5$, but for cases where $i \leq 2$, i.e., the two orderings are actually equivalent). Therefore, it seems to be possible to use the proposed orderings as an auxiliary tool, for instance, when a majority voting regarding the correct ordering between two IFP alternatives is needed.

Table 1. IFPs compared under the considered orderings.

u	v	$\leq$	$\preceq_{SH}$	$\preceq_{LH}$	$\preceq_{R}$	$\preceq_{Z}$	$\preceq_{P}$	$\preceq_{M}$	$\preceq_{\mu;M_p}$
$\langle 0.367, 0.6034\rangle$	$\langle 0.3744, 0.6084\rangle$	N/A	$\preceq_{SH}$	$\succeq_{LH}$	$\preceq_{R}$	$\preceq_{Z}$	$\preceq_{P}$	$\preceq_{M}$	$\preceq_{\mu;M_{p>0}}$
$\langle 0.2, 0.4\rangle$	$\langle 0.3, 0.7\rangle$	N/A	$\succeq_{SH}$	$\succeq_{LH}$	$\preceq_{R}$	$\succeq_{Z}$	$\succeq_{P}$	$\preceq_{M}$	$\preceq_{\mu;M_{p>5}}$
$\langle 0.7, 0.3\rangle$	$\langle 0.4, 0.2\rangle$	N/A	$\succeq_{SH}$	$\succeq_{LH}$	$\succeq_{R}$	$\succeq_{Z}$	$\succeq_{P}$	$\succeq_{M}$	$\succeq_{\mu;M_{p>1}}$
$\langle 0.2, 0.8\rangle$	$\langle 0, 0.74\rangle$	N/A	$\succeq_{SH}$	$\preceq_{LH}$	$\succeq_{R}$	$\succeq_{Z}$	$\succeq_{P}$	$\succeq_{M}$	$\succeq_{\mu;M_{p>1}}$

We conclude this section with a final example showing how our results may be utilized for selection based on IFPs' evaluations.

Example 3. *Further, we consider the selection of traveling by airplane. We suppose there are three feasible ways to travel from the starting destination to the target destination. The criteria that need to be satisfied are as follows c_1—convenience of air travel; c_2—reliability of travel; c_3—total time to reaching final destination; and c_4—total price of travel.*

Convenience of travel signifies whether a flight is direct (as provided by suppliers s_1 and s_2), or whether it is with a transfer flight from the same or allied company (i.e., the luggage is automatically transferred)—as provided by supplier s_3—or the second flight is provided by a different supplier, implying that the passenger will need to claim their luggage and check it in again (s_4). The reliability of travel may be calculated by the number of executed flights and the number of canceled flights divided by all planned flights in the previous month. The total time required to reach the final destination includes an estimation of the expected duration of all flights with the use of some type of affordable public transport to the city center from the final airport. Note that airports located farther from the city may have limited means of transportation (or some may not be available after certain hour), as reflected in the IFPs' values of s_3 and s_4, which although are not provided by the suppliers themselves, are nonetheless affected by the arrival times and the target airport, so it is fair to be attributed to them. The total price of travel includes all possible travel expenses plus the price for some reasonable refreshments (offered as a free service by suppliers s_1, s_2 and s_3) as shown in Table 2.

We consider three types of travelers T_1, T_2, and T_3 whose preferences may be represented by the following weights:

$$T_1(u) = 0.5c_1 + 0.3c_2 + 0.1c_3 + 0.1c_4 \tag{48}$$

$$T_2(u) = 0.3c_1 + 0.4c_2 + 0.1c_3 + 0.2c_4 \tag{49}$$

$$T_3(u) = 0.01c_2 + 0.99c_4 \tag{50}$$

Table 2. Choosing to travel by airplane according to criteria estimated as IFPs.

	c_1	c_2	c_3	c_4
s_1	$\langle 1, 0\rangle$	$\langle 0.8, 0.2\rangle$	$\langle 1, 0\rangle$	$\langle 0.7, 0\rangle$
s_2	$\langle 1, 0\rangle$	$\langle 0.9, 0.1\rangle$	$\langle 1, 0\rangle$	$\langle 0.6, 0\rangle$
s_3	$\langle 0.85, 0.1\rangle$	$\langle 0.95, 0.05\rangle$	$\langle 0.8, 0.1\rangle$	$\langle 0.8, 0\rangle$
s_4	$\langle 0.4, 0.6\rangle$	$\langle 0.7, 0.3\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.9, 0.05\rangle$

As one can readily observe, the orderings of the suppliers by the criteria are as follows:

$$c_1 : s_4 \leq s_3 \leq s_2 = s_1 \tag{51}$$

$$c_2 : s_4 \leq s_1 \leq s_2 \leq s_3 \tag{52}$$

$$c_3 : s_4 \leq s_3 \leq s_2 = s_1 \tag{53}$$

$$c_4 : s_2 \leq_{\mu;M_{0.7}} s_1 \leq_{\mu;M_{0.7}} s_3 \leq_{\mu;M_{0.7}} s_4 \tag{54}$$

In order to provide a meaningful comparison, let us, by analogy with Definition 13, introduce:

$$PM\mu, p(\langle a, b\rangle) = 1 - \left(\frac{(1-a)^p + b^p}{2}\right)^{\frac{1}{p}}$$

fixing $p = 0.7$, we obtain for the maximum theoretical value of $T_1(u)$, $T_2(u)$, $T_3(u)$, from (48)–(50) (by substituting the value of $PM\mu, 0.7$, the best alternative among all suppliers) to be:

$$\begin{aligned} T_1(u_{\max}) &= 0.5PM\mu, 0.7(\langle 1, 0\rangle) + 0.3PM\mu, 0.7(\langle 0.95, 0.05\rangle) \\ &\quad + 0.1PM\mu, 0.7(\langle 1, 0\rangle) + 0.1PM\mu, 0.7(\langle 0.9, 0.05\rangle) = 0.977 \qquad (55) \\ T_2(u_{\max}) &= 0.3PM\mu, 0.7(\langle 1, 0\rangle) + 0.4PM\mu, 0.7(\langle 0.95, 0.05\rangle) \\ &\quad + 0.1PM\mu, 0.7(\langle 1, 0\rangle) + 0.2PM\mu, 0.7(\langle 0.9, 0.05\rangle) = 0.9653 \qquad (56) \\ T_3(u_{\max}) &= 0.01PM\mu, 0.7(\langle 0.95, 0.05\rangle) + 0.99PM\mu, 0.7(\langle 0.9, 0.05\rangle) = 0.9265 \qquad (57) \end{aligned}$$

By calculating the actual values for each supplier, we obtain:

$$T_1(s_1) = 0.9288; T_1(s_2) = 0.9551; T_1(s_3) = 0.9; T_1(s_4) = 0.57,$$

$$T_2(s_1) = 0.897; T_2(s_2) = 0.93028; T_2(s_3) = 0.9131; T_2(s_4) = 0.66041,$$

$$T_3(s_1) = 0.887; T_3(s_2) = 0.8518; T_3(s_3) = 0.9132; T_3(s_4) = 0.924.$$

Thus, according to the score $PM\mu, 0.7$ corresponding to $\geq_{\mu;M_{0.7}}$, we can conclude that the best choice for the T_1 type passenger is a direct flight by s_2, closely followed by a direct flight by s_1, and the best choice is also valid for the T_2 type passenger; however, s_3 is preferred to s_1 in this case. Only T_3—a type 3 passenger—would choose s_4 (based on the best price), a choice extremely closely followed by s_3.

Had we used the $Z(\langle a, b\rangle)$ ordering from Definition 12, we would have obtained similar results to those shown in Table 3.

Table 3. Z-order for the considered example.

	c_1	c_2	c_3	c_4
s_1	1	0.8	1	0.7225
s_2	1	0.9	1	0.64
s_3	0.4143	0.95	0.8075	0.81
s_4	0.4	0.7	0.7125	0.901875

However, for T_2, the Z order asserts that s_1 is better than s_3. But in view of (52) and the fact that the highest weight of T_2 type passenger is focused on the reliability of travel, it would seem that our ordering provides a more reasonable suggestion.

5. Conclusions

In the present work, we constructed new orderings depending on the weighted power mean which allow us to compare a larger number of alternatives in the form of intuitionistic fuzzy pairs in a consistent manner. We proved that these orderings approach the linear ordering as the value of p grows to $+\infty$, thus naturally allowing more intuitionistic fuzzy pairs to become comparable, while preserving the already existing order. We illustrated the possible application to a set of alternatives and to the results obtained from InterCriteria Analysis. We also compared our proposed orderings with the ones mostly used and have provided an example of their use. In future work, we will extend these approaches, if possible, to constructing other orderings depending on different generalized means over the intuitionistic fuzzy pairs, and we shall study their properties.

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