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Some New Properties of Convex Fuzzy-Number-Valued Mappings on Coordinates Using Up and Down Fuzzy Relations and Related Inequalities

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Abstract: The symmetric function class interacts heavily with other types of functions. One of these is the convex function class, which is strongly related to symmetry theory. In this study, we define a novel class of convex mappings on planes using a fuzzy inclusion relation, known as coordinated up and down convex fuzzy-number-valued mapping. Several new definitions are introduced by placing some moderate restrictions on the notion of coordinated up and down convex fuzzy-number-valued mapping. Other uncommon examples are also described using these definitions, which can be viewed as applications of the new outcomes. Moreover, Hermite–Hadamard–Fejér inequalities are acquired via fuzzy double Aumann integrals, and the validation of these outcomes is discussed with the help of nontrivial examples and suitable choices of coordinated up and down convex fuzzy-number-valued mappings.

Keywords: fuzzy-interval-valued function on coordinates; coordinated up and down convex fuzzy-number-valued mapping; fuzzy double integral; Hermite–Hadamard–Fejér-type inequalities

MSC: 26A33; 26A51; 26D10

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1. Introduction

Convex functions are distinguished from other function classes by their widespread application in mathematics, statistics, optimization theory, and applied sciences. This is due to the analytic inequalities, particularly those of the Hermite–Hadamard, Fejér, Hardy, Simpson, and Ostrowski types, that have been established using this concept [1–17]. The concept of a convex function is one of the core theorems of inequality theory, detailed as follows:

Definition 1. *The real-valued mapping $Y : K \to \mathbb{R}$ is called a convex mapping on convex set K if*

$$Y(\tau\sigma + (1-\tau)s) \leq \tau \odot Y(\sigma) + (1-\tau) \odot Y(s), \tag{1}$$

for all $\sigma, s \in K$, $\tau \in [0, 1]$. If Equation (1) is reversed, then Y is called a concave mapping on K. Y is affine if and only if it is both a convex and concave mapping.

The Hermite–Hadamard inequality, which is a key component of the widespread use and geometrical interpretation of convex functions, has piqued the interest of researchers in fundamental mathematics. This inequality has piqued the interest of multiple scholars from around the world due to its numerous applications, particularly in the domains of numerical analysis, engineering, physical science, and chemistry. The idea of inequality has advanced rapidly in recent years. For convex functions, several inequalities can be found; however, Hermite–Hadamard's inequality is one of the most extensively and intensively studied conclusions. It is worthwhile to consider how closely related the theories of inequality and convexity are. As a result of this reality, the concept of inequality becomes more appealing. Many new expansions, generalizations, and definitions of novel convexity have been given in recent years, as have corresponding advancements in the theory of convexity inequality, particularly integral inequality theory. Formally, the Hermite–Hadamard inequality is as follows:

For a convex mapping $Y : K \to \mathbb{R}$ on convex set K, the HH inequality is written as

$$Y\left(\frac{\rho+\mu}{2}\right) \leq \frac{1}{\mu-\rho}\int_{\rho}^{\mu} Y(\sigma)d\sigma \leq \frac{Y(\rho)+Y(\mu)}{2}, \tag{2}$$

for all $\rho,\ \mu \in K$, with $\rho \leq \mu$. If Y is concave, then Equation (2) is reversed.

If it is a concave function, the inequality in Equation (2) holds in both directions. Based on geometry, the Hermite–Hadamard inequality provides an upper and lower estimate for the integral mean of any convex function defined in a closed and limited domain that encompasses the function's ends and midpoint. Because of the importance of this inequality, multiple modifications of it have been studied in the literature for various classes of convexity, including harmonically convex, exponentially convex, s-convex, h-convex, and co-ordinate convex functions [18–33].

Moore [34] was the first to consider interval analysis. Moore [35] researched interval methods for obtaining the upper and lower bounds of accurate values of the integrals of interval-valued functions and studied the integration of interval-valued functions in 1979. Bhurjee and Panda [36] devised a framework for determining effective solutions to a broad multi-objective fractional programming problem whose parameters in the objective functions and constraints are intervals. Zhang et al. [37] expanded the ideas of invexity and pre-invexity to interval-valued functions, resulting in KKT optimality requirements for LU-pre-invex and invex optimization problems with an interval-valued objective function. Zhao et al. [38] defined the interval double integral and provided Chebyshev-type inequalities for interval-valued functions. Interval analysis has practical applications in economics, chemical engineering, beam physics, control circuit design, global optimization, robotics, error analysis, signal processing, and computer graphics (see [39–58]).

Budak et al. [59] defined the interval-valued right-sided Riemann–Liouville fractional integral and derived H-H-type inequalities for such integrals. Sharma et al. [60] proposed interval-valued pre-invex functions and proved fractional H-H-type inequalities for them. Zhao et al. [61,62] recently developed the concept of interval-valued coordinated convex functions on coordinates and proved H-H-type inequalities for these interval-valued coordinated convex functions. Furthermore, Budak et al. [63] introduced a new concept of interval-valued fractional integrals on coordinates and used these fractional integrals to analyze H-H-type inequalities for interval-valued coordinated convex functions. Kara et al. [64] demonstrated that the product of two interval-valued convex functions on coordinates has H-H–Fejér-type inclusions. We refer to [65–76] and the references therein for more information on the links between the various types of coordinated fuzzy-number-valued mappings, interval-valued functions, and integral inequalities. Similarly, most of the authors work in the field of fuzzy calculus as well as fuzzy fractional calculus. Therefore, we refer the readers to [77–97] and the references therein, which will help in understanding fuzzy theory.

Motivated and inspired by the above ongoing research, this manuscript is divided into four sections. In the second section, we recall some classical and preliminary notions and results which will be helpful in discussing the main outcomes. In the third section, some new estimates of integral inequalities via fuzzy double Aumann integrals and a newly defined coordinated class of convex fuzzy-number mappings on up and down fuzzy relations are presented. Some interesting examples are also given to illustrate the main outcomes. In the final section, some conclusions and future plans are discussed.

2. Preliminaries

First, we will review the fundamental notions of fuzzy mathematics. Additional information can be found in the following references: Anastassiou [77]; Anastassiou and Gal [78]; Gal [79]; Goetschel and Voxman [82]; Gal [83]; and Wu and Zengtai [84].

Let $\Lambda \in \mathbb{E}_0$ be a fuzzy number. Then, this fuzzy number is also represented as $\mathfrak{q}$-level sets $[\Lambda]^{\mathfrak{q}}$ defined as

$$\begin{cases} \{\varsigma \in \mathbb{R} | \Lambda(\varsigma) \geq \mathfrak{q}\}, \ \mathfrak{q} \in (0,1] \\ \overline{\{\varsigma \in \mathbb{R} | \Lambda(\varsigma) > \mathfrak{q}\}}, \ \mathfrak{q} = 0, \end{cases} \tag{3}$$

which is a bounded and closed interval of $\mathbb{R}$ and denoted as

$$[\Lambda]^{\mathfrak{q}} = [\Lambda_*(\mathfrak{q}), \ \Lambda^*(\mathfrak{q})].$$

For $\Lambda, \lambda \in \mathbb{E}_0$ and $\varrho \in \mathbb{R}$, the sum $\Lambda \oplus \lambda$, product $\Lambda \otimes \lambda$, scalar product $\varrho \odot \Lambda$, and sum with the scalar are uniquely defined as, for all $\mathfrak{q} \in [0, \ 1]$, we obtain

$$[\Lambda \oplus \lambda]^{\mathfrak{q}} = [\Lambda]^{\mathfrak{q}} + [\lambda]^{\mathfrak{q}}, \tag{4}$$

$$[\Lambda \otimes \lambda]^{\mathfrak{q}} = [\Lambda]^{\mathfrak{q}} \times [\lambda]^{\mathfrak{q}}, \tag{5}$$

$$[\varrho \odot \Lambda]^{\mathfrak{q}} = \varrho \cdot [\Lambda]^{\mathfrak{q}}. \tag{6}$$

$$[\varrho \oplus \Lambda]^{\mathfrak{q}} = \varrho + [\Lambda]^{\mathfrak{q}}. \tag{7}$$

For $\psi \in \mathbb{E}_0$, such that $\Lambda = \lambda \oplus \psi$, via this result, we then determine the existence of Hukuhara difference between Λ and λ, and we can say that ψ is the H-difference between Λ and λ and is denoted as $\Lambda \ominus \lambda$. If H-difference exists, then

$$(\psi)^*(\mathfrak{q}) = (\Lambda \ominus \lambda)^*(\mathfrak{q}) = \Lambda^*(\mathfrak{q}) - \lambda^*(\mathfrak{q}), \ (\psi)_*(\mathfrak{q}) = (\Lambda \ominus \lambda)_*(\mathfrak{q}) = \Lambda_*(\mathfrak{q}) - \lambda_*(\mathfrak{q}). \tag{8}$$

For $[\mathcal{Z}_*, \ \mathcal{Z}^*], \ [\mathcal{Q}_*, \ \mathcal{Q}^*] \in \mathbb{R}_I$, where $\mathbb{R}_I$ is the space of all closed and bounded intervals of real numbers $\mathbb{R}$, the Hausdorff–Pompeiu distance between the intervals $[\mathcal{Z}_*, \ \mathcal{Z}^*]$ and $[\mathcal{Q}_*, \ \mathcal{Q}^*]$ is defined as

$$d_H([\mathcal{Z}_*, \ \mathcal{Z}^*], \ [\mathcal{Q}_*, \ \mathcal{Q}^*]) = max\{|\mathcal{Z}_* - \mathcal{Q}_*|, \ |\mathcal{Z}^* - \mathcal{Q}^*|\}. \tag{9}$$

It is a known fact that $(\mathbb{R}_I, d_H)$ is a complete metric space [82].

Theorem 1 ([82])**.** *The space $\mathbb{E}_0$ dealing with a supremum metric, i.e., for $\widetilde{\psi}, \ \widetilde{\omega} \in \mathbb{E}_0$*

$$d_\infty\left(\widetilde{\psi}, \ \widetilde{\omega}\right) = \sup_{0 \leq \lambda \leq 1} d_H\left(\left[\widetilde{\psi}\right]^{\mathfrak{q}}, \ \left[\widetilde{\omega}\right]^{\mathfrak{q}}\right), \tag{10}$$

is a complete metric space, where H denotes the well-known Hausdorff metric in the space of intervals.

Remark 1 ([86,87])**.** *Let $\mathbb{R}_I$ be the space of all closed and bounded intervals of real numbers $\mathbb{R}$. The relation " $\leq_I$" is defined in $\mathbb{R}_I$ as*

$$[\Lambda_*, \ \Lambda^*] \leq_I [\lambda_*, \ \lambda^*] \text{ if and only if } \Lambda_* \leq \lambda_*, \ \Lambda^* \leq \lambda^*,$$

for all $[\Lambda_*,\ \Lambda^*],\ [\lambda_*,\ \lambda^*] \in \mathbb{R}_I$, *and it is known as the left and right relation.*

The inclusion " $\subseteq$ " means that

$$\Lambda \subseteq_I \lambda \text{ if and only if } [\Lambda_*,\ \Lambda^*] \subseteq_I [\lambda_*,\ \lambda^*], \text{ if and only if } \lambda_* \leq \Lambda_*,\ \Lambda^* \leq \lambda^*.$$

It is known as the up and down relation.

Proposition 1 ([86]). *If* $\Lambda, \lambda \in \mathbb{E}_0$, *then relation* " $\leq_{\mathbb{F}}$ " *is defined in* $\mathbb{E}_0$ *as*

$$\Lambda \leq_{\mathbb{F}} \lambda \text{ if and only if } [\Lambda]^{\mathfrak{q}} \leq_I [\lambda]^{\mathfrak{q}} \text{ for all } \mathfrak{q} \in [0,\ 1],$$

and this relation is known as the left and right fuzzy relation.

Proposition 2 ([80]). *If* $\Lambda, \lambda \in \mathbb{E}_0$, *then relation* " $\supseteq_{\mathbb{F}}$ " *is defined in* $\mathbb{E}_0$ *as*

$$\Lambda \supseteq_{\mathbb{F}} \lambda \text{ if and only if } [\Lambda]^{\mathfrak{q}} \supseteq_I [\lambda]^{\mathfrak{q}} \text{ for all } \mathfrak{q} \in [0,\ 1],$$

and this relation is known as the up and down fuzzy relation.

Definition 2 ([90]). *The IVM* $Y : \Delta = [\mu,\ \sigma] \times [\varsigma,\ \nu] \to \mathbb{R}^+$ *is said to be a coordinated convex function on* Δ *if*

$$\begin{aligned} &Y(\tau\mu + (1-\tau)\sigma,\ s\varsigma + (1-s)\nu) \\ &\leq \tau s Y(\mu,\varsigma) + \tau(1-s)Y(\mu,\nu) + (1-\tau)sY(\sigma,\varsigma) + (1-\tau)(1-s)Y(\sigma,\nu), \end{aligned} \tag{11}$$

for all $(\mu,\ \sigma),\ (\varsigma,\nu) \in \Delta,\ \tau$ *and* $\tau, s \in [0,\ 1]$. *If inequality Equation (11) is reversed, then* Y *is called a coordinated concave IVM on* Δ.

Definition 3 ([87]). *The FN-V-M* $\widetilde{Y} : [\varsigma,\nu] \to \mathbb{E}_0$ *is said to be an up and down convex FN-V-M on* $[\varsigma,\nu]$ *if*

$$\widetilde{Y}(\tau\sigma + (1-\tau)\omega) \supseteq_{\mathbb{F}} \tau \odot \widetilde{Y}(\sigma) \oplus (1-\tau) \odot \widetilde{Y}(\omega), \tag{12}$$

for all $\sigma, \omega \in [\varsigma,\nu],\ \tau \in [0,\ 1]$, *where* $\widetilde{Y}(\sigma) \geq_{\mathbb{F}} \widetilde{0}$. *If* $\widetilde{Y}$ *is an up and down concave FN-V-M on* $[\varsigma,\nu]$, *then inequality Equation (12) is reversed.*

Theorem 2 ([85]). *Let* $\widetilde{Y}, \widetilde{\mathfrak{S}} : [\varsigma,\ \nu] \to \mathbb{E}_0$ *be two up and down convex FN-V-Ms. Then, from the* $\mathfrak{q}$*-levels, we obtain the collection of IVMs* $Y_{\mathfrak{q}},\ \mathfrak{S}_{\mathfrak{q}} : [\varsigma,\ \nu] \subset \mathbb{R} \to \mathbb{R}_I^+$ *given as* $Y_{\mathfrak{q}}(\sigma) = [Y_*(\sigma,\mathfrak{q}),\ Y^*(\sigma,\mathfrak{q})]$ *and* $\mathfrak{S}_{\mathfrak{q}}(\sigma) = [\mathfrak{S}_*(\sigma,\mathfrak{q}),\ \mathfrak{S}^*(\sigma,\mathfrak{q})]$ *for all* $\sigma \in [\varsigma,\ \nu]$ *and for all* $\mathfrak{q} \in [0,\ 1]$. *If* $\widetilde{Y} \otimes \widetilde{\mathfrak{S}}$ *is a fuzzy Riemann integrable, then*

$$\frac{1}{\nu-\varsigma} \odot (FR)\int_{\varsigma}^{\nu} \widetilde{Y}(\sigma) \otimes \widetilde{\mathfrak{S}}(\sigma)d\sigma \supseteq_{\mathbb{F}} \frac{1}{3} \odot \widetilde{\mathcal{M}}(\varsigma,\nu) \oplus \frac{1}{6} \odot \widetilde{\mathcal{N}}(\varsigma,\nu), \tag{13}$$

and

$$2 \odot \widetilde{Y}\left(\frac{\varsigma+\nu}{2}\right) \otimes \widetilde{\mathfrak{S}}\left(\frac{\varsigma+\nu}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\nu-\varsigma} \odot (FR)\int_{\varsigma}^{\nu} \widetilde{Y}(\sigma) \otimes \widetilde{\mathfrak{S}}(\sigma)d\sigma \oplus \frac{1}{6} \odot \widetilde{\mathcal{M}}(\varsigma,\nu) \oplus \frac{1}{3} \odot \widetilde{\mathcal{N}}(\varsigma,\nu). \tag{14}$$

where $\widetilde{\mathcal{M}}(\varsigma,\nu) = \widetilde{Y}(\varsigma) \otimes \widetilde{\mathfrak{S}}(\varsigma) \oplus \widetilde{Y}(\nu) \otimes \widetilde{\mathfrak{S}}(\nu), \widetilde{\mathcal{N}}(\varsigma,\nu) = \widetilde{Y}(\varsigma) \otimes \widetilde{\mathfrak{S}}(\nu) \oplus \widetilde{Y}(\nu) \otimes \widetilde{\mathfrak{S}}(\varsigma)$, *and* $\mathcal{M}_{\mathfrak{q}}(\varsigma,\nu) = [\mathcal{M}_*((\varsigma,\nu),\ \mathfrak{q}),\ \mathcal{M}^*((\varsigma,\nu),\ \mathfrak{q})]$ *and* $\mathcal{N}_{\mathfrak{q}}(\varsigma,\nu) = [\mathcal{N}_*((\varsigma,\nu),\ \mathfrak{q}),\ \mathcal{N}^*((\varsigma,\nu),\ \mathfrak{q})]$.

Theorem 3 ([85]). *Let* $\widetilde{Y} : [\varsigma,\ \nu] \to \mathbb{E}_0$ *be an up and down convex FN-V-M with* $\varsigma < \nu$. *Then, from the* $\mathfrak{q}$*-levels, we obtain the collection of IVMs* $Y_{\mathfrak{q}} : [\varsigma,\ \nu] \subset \mathbb{R} \to \mathbb{R}_I^+$ *given as*

$Y_{\mathfrak{q}}(\sigma) = [Y_*(\sigma,\mathfrak{q}),\ Y^*(\sigma,\mathfrak{q})]$ *for all* $\sigma \in [\varsigma,\ \nu]$ *and for all* $\mathfrak{q} \in [0,\ 1]$. *If* $\widetilde{Y} \in Y\mathcal{R}_{([\varsigma,\ \nu],\mathfrak{q})}$ *and* $\Omega : [\varsigma,\ \nu] \to \mathbb{R}, \Omega(\sigma) \geq 0$, *symmetric with respect to* $\frac{\varsigma+\nu}{2}$, *and* $\int_{\varsigma}^{\nu} \Omega(\sigma)d\sigma > 0$, *then*

$$\widetilde{Y}\left(\frac{\varsigma+\nu}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\int_{\varsigma}^{\nu} \Omega(\sigma)d\sigma} \odot (FR)\int_{\varsigma}^{\nu} \widetilde{Y}(\sigma)\Omega(\sigma)d\sigma \supseteq_{\mathbb{F}} \frac{\widetilde{Y}(\varsigma) \oplus \widetilde{Y}(\nu)}{2}. \tag{15}$$

If $\widetilde{Y}$ *is an up and down concave FN-V-M, then inequality Equation (15) is reversed.*
If $\Omega(\sigma) = 1$, *then via Equation (15) we obtain following inequality:*

$$\widetilde{Y}\left(\frac{\varsigma+\nu}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\nu-\varsigma} \odot (FR)\int_{\varsigma}^{\nu} \widetilde{Y}(\sigma)\Omega(\sigma)d\sigma \supseteq_{\mathbb{F}} \frac{\widetilde{Y}(\varsigma) \oplus \widetilde{Y}(\nu)}{2}. \tag{16}$$

Theorem 4 ([36]). *If* $Y : [\varsigma,\nu] \subset \mathbb{R} \to \mathbb{R}_I$ *is an IVM given as* $(\sigma)\ [Y_*(\sigma), Y^*(\sigma)]$, *then* Y *is Riemann-integrable on* $[\varsigma,\nu]$ *if and only if* Y_* *and* Y^* *are both Riemann-integrable on* $[\varsigma,\nu]$, *such that*

$$(IR)\int_{\varsigma}^{\nu} Y(\sigma)d\sigma = [(R)\int_{\varsigma}^{\nu} Y_*(\sigma)d\sigma,\ (R)\int_{\varsigma}^{\nu} Y^*(\sigma)d\sigma]. \tag{17}$$

The collection of all Riemann-integrable real-valued functions and Riemann-integrable IVMs is denoted as $\mathcal{R}_{[\varsigma,\nu]}$ and $\mathfrak{T}\mathcal{R}_{[\varsigma,\nu]}$, respectively.

Note that Theorem 5 is also true for interval double integrals. The collection of all double-integrable IVMs is denoted as $\mathfrak{T}\mathfrak{O}_{\Delta}$, respectively.

Theorem 5 ([38]). *Let* $\Delta = [\mu,\ \sigma] \times [\varsigma,\ \nu]$. *If* $Y : \Delta \to \mathbb{R}_I$ *is ID-integrable on* Δ, *then we obtain*

$$(ID)\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu} Y(\sigma,\omega)d\omega d\sigma = (IR)\int_{\mu}^{\sigma} (IR)\int_{\varsigma}^{\nu} Y(\sigma,\omega)d\omega d\sigma. \tag{18}$$

Definition 4 ([91]). *A fuzzy-interval-valued map* $\widetilde{Y} : \Delta = [\mu,\ \sigma] \times [\varsigma,\ \nu] \to \mathbb{E}_0$ *is called an FN-V-M on coordinates. Then, from the* $\mathfrak{q}$*-levels, we obtain the collection of IVMs* $Y_{\mathfrak{q}} : \Delta \subset \mathbb{R}^2 \to \mathbb{R}_I$ *on coordinates given as* $Y_{\mathfrak{q}}(\sigma,\omega) = [Y_*((\sigma,\omega),\mathfrak{q}),\ Y^*((\sigma,\omega),\mathfrak{q})]$ *for all* $(\sigma,\omega) \in \Delta$. *Herein, for each* $\mathfrak{q} \in [0,\ 1]$, *the end-point real-valued functions* $Y_*(.,\mathfrak{q}),\ Y^*(.,\mathfrak{q}) : [\mu,\ \sigma] \times [\varsigma,\ \nu] \to \mathbb{R}$ *are called the lower and upper functions of* $Y_{\mathfrak{q}}$.

Definition 5 ([91]). *Let* $\widetilde{Y} : \Delta = [\mu,\ \sigma] \times [\varsigma,\ \nu] \subset \mathbb{R}^2 \to \mathbb{E}_0$ *be a coordinated FN-V-M. Then,* $\widetilde{Y}(\sigma,\omega)$ *is said to be continuous at* $(\sigma,\omega) \in \Delta = [\mu,\ \sigma] \times [\varsigma,\ \nu]$ *if for each* $\mathfrak{q} \in [0,\ 1]$, *both the end-point functions* $Y_*((\sigma,\omega),\mathfrak{q})$ *and* $Y^*((\sigma,\omega),\mathfrak{q})$ *are continuous at* $(\sigma,\omega) \in \Delta$.

Definition 6 ([91]). *Let* $\widetilde{Y} : \Delta = [\mu,\ \sigma] \times [\varsigma,\ \nu] \subset \mathbb{R}^2 \to \mathbb{E}_0$ *be an FN-V-M on coordinates. Then, the fuzzy double integral of* $\widetilde{Y}$ *on* $\Delta = [\mu,\ \sigma] \times [\varsigma,\ \nu]$, *denoted as* $(FD)\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu} \widetilde{Y}(\sigma,\omega)d\omega d\sigma$, *is defined level-wise as*

$$\begin{aligned}\left[(FD)\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu} \widetilde{Y}(\sigma,\omega)d\omega d\sigma\right]^{\mathfrak{q}} &= (ID)\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu} Y_{\mathfrak{q}}(\sigma,\omega)d\omega d\sigma \\ &= (IR)\int_{\mu}^{\sigma} (IR)\int_{\varsigma}^{\nu} Y_{\mathfrak{q}}(\sigma,\omega)d\omega d\sigma,\end{aligned} \tag{19}$$

for all $\mathfrak{q} \in [0,\ 1]$, *and* $\widetilde{Y}$ *is FD-integrable on* Δ *if* $(FD)\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu} \widetilde{Y}(\sigma,\omega)d\omega d\sigma \in \mathbb{E}_0$. *Note that if the end-point functions are Lebesgue-integrable, then* $\widetilde{Y}$ *is a fuzzy double-Aumann-integrable function on* Δ.

Theorem 6 ([91])**.** *Let $\widetilde{Y} : \Delta \subset \mathbb{R}^2 \to \mathbb{E}_0$ be an FN-V- M on coordinates. Then, from the $\mathfrak{q}$-levels, we obtain the collection of IVMs $Y_{\mathfrak{q}} : \Delta \subset \mathbb{R}^2 \to \mathbb{R}_I$ given as $Y_{\mathfrak{q}}(\sigma,\omega) = [Y_*((\sigma,\omega),\mathfrak{q}),\ Y^*((\sigma,\omega),\mathfrak{q})]$ for all $(\sigma,\omega) \in \Delta = [\mu,\ \sigma] \times [\varsigma,\ \nu]$ and for all $\mathfrak{q} \in [0,\ 1]$. Then, $\widetilde{Y}$ is FD-integrable on Δ if and only if $Y_*((\sigma,\omega),\mathfrak{q})$ and $Y^*((\sigma,\omega),\mathfrak{q})$ are both D-integrable on Δ. Moreover, if $\widetilde{Y}$ is FD-integrable on Δ, then*

$$\begin{aligned}\left[(FD)\int_\mu^\sigma \int_\varsigma^\nu \widetilde{Y}(\sigma,\omega)d\omega d\sigma\right]^q &= \left[(FR)\int_\mu^\sigma (FR)\int_\varsigma^\nu \widetilde{Y}(\sigma,\omega)d\omega d\sigma\right]^{\mathfrak{q}} \\ &= (IR)\int_\mu^\sigma (IR)\int_\varsigma^\nu Y_{\mathfrak{q}}(\sigma,\omega)d\omega d\sigma \\ &= (ID)\int_\mu^\sigma \int_\varsigma^\nu Y_{\mathfrak{q}}(\sigma,\omega)d\omega d\sigma\end{aligned} \tag{20}$$

for all $\mathfrak{q} \in [0,\ 1]$.

3. Main Results

In this section, we will first propose the new class of coordinated convex functions with the up and down fuzzy relation, which are known as coordinated UD-convex FN-V-Ms. Secondly, we will present HH–Fejér inequalities with the help of this new class and double fuzzy integrals as well as verify them with the support of some useful examples.

Definition 7. *The FN-V-M $\widetilde{Y} : \Delta \to \mathbb{E}_0$ is said to be a coordinated UD-convex FN-V-M on Δ if*

$$\begin{aligned}&\widetilde{Y}(\tau\mu + (1-\tau)\sigma,\ s\varsigma + (1-s)\nu) \\ \supseteq_{\mathbb{F}}\ \tau s \odot \widetilde{Y}(\mu,\varsigma) \supseteq_{\mathbb{F}}\ \tau(1-s) \odot \widetilde{Y}(\mu,\nu) \oplus (1-\tau)s \odot \widetilde{Y}(\sigma,\varsigma) \oplus (1-\tau)(1-s) \odot \widetilde{Y}(\sigma,\nu),&\end{aligned} \tag{21}$$

for all $(\mu,\ \sigma),\ (\varsigma,\nu) \in \Delta$, and $\tau, s \in [0,\ 1]$, where $\widetilde{Y}(\sigma) \geq_{\mathbb{F}} \widetilde{0}$. If inequality Equation (21) is reversed, then $\widetilde{Y}$ is called a coordinated concave FN-V-M on Δ.

The straightforward proof of Lemma 1 will be omitted herein.

Lemma 1. *Let $\widetilde{Y} : \Delta \to \mathbb{E}_0$ be a coordinated FN-V-M on Δ. Then, $\widetilde{Y}$ is a coordinated UD-convex FN-V-M on Δ if and only if two coordinated UD-convex FN-V-Ms exist, $\widetilde{Y}_{\sigma} : [\varsigma,\nu] \to \mathbb{E}_0$, $\widetilde{Y}_{\sigma}(\omega) = \widetilde{Y}(\sigma,\omega)$ and $\widetilde{Y}_\omega : [\mu,\sigma] \to \mathbb{E}_0$, $\widetilde{Y}_\omega(\varsigma) = \widetilde{Y}(\varsigma,\omega)$*

Proof. From the definition of the coordinated FN-V-M, it can be easily proved. □

From Lemma 1, we can easily note that each UD-convex FN-V-M is a coordinated UD-convex FN-V-M. However, the converse is not true (see Example 1).

Theorem 7. *Let $\widetilde{Y} : \Delta \to \mathbb{E}_0$ be an FN-V-M on Δ. Then, from the $\mathfrak{q}$-levels, we obtain the collection of IVMs $Y_{\mathfrak{q}} : \Delta \to \mathbb{R}_I^+ \subset \mathbb{R}_I$ given as*

$$Y_{\mathfrak{q}}(\sigma,\omega) = [Y_*((\sigma,\omega),\mathfrak{q}),\ Y^*((\sigma,\omega),\mathfrak{q})], \tag{22}$$

for all$(\sigma,\omega) \in \Delta$ and for all $\mathfrak{q} \in [0,\ 1]$. Then, $\widetilde{Y}$ is a coordinated UD-convex FN-V-M on Δ if and only if for all $\mathfrak{q} \in [0,\ 1]$, $Y_*((\sigma,\omega),\ \mathfrak{q})$ and $Y^*((\sigma,\omega),\ \mathfrak{q})$ are coordinated UD-convex and concave functions, respectively.

Proof. Assume that for each $\mathfrak{q} \in [0,\ 1]$, $Y_*(\sigma,\mathfrak{q})$ and $Y^*(\sigma,\mathfrak{q})$ are coordinated UD-convex on Δ. Then, from Equation (21), for all $(\mu,\ \sigma),\ (\varsigma,\nu) \in \Delta,\ \tau$ and $s \in [0,\ 1]$, we obtain

$$\begin{aligned}&Y_*((\tau\mu + (1-\tau)\sigma,\ s\varsigma + (1-s)\nu),\ \mathfrak{q}) \\ \leq\ \tau s Y_*((\mu,\varsigma),\ \mathfrak{q}) + t(1-s)Y_*((\mu,\nu),\ \mathfrak{q}) + s(1-t)Y_*((\mu,\varsigma),\ \mathfrak{q}) + (1-\tau)(1-s)Y_*((\mu,\nu),\ \mathfrak{q}),&\end{aligned}$$

and

$$\begin{gathered} Y^*((\tau\mu + (1-\tau)\sigma,\ s\varsigma + (1-s)\nu),\ \mathfrak{q}) \\ \geq \tau s Y_*((\mu,\varsigma),\ \mathfrak{q}) + t(1-s)Y^*((\mu,\nu),\ \mathfrak{q}) + s(1-t)Y^*((\mu,\varsigma),\ \mathfrak{q}) + (1-\tau)(1-s)Y^*((\mu,\nu),\ \mathfrak{q}), \end{gathered}$$

Then, via Equations (4), (6) and (22), we obtain

$$\begin{gathered} Y_{\mathfrak{q}}((\tau\mu + (1-\tau)\sigma,\ s\varsigma + (1-s)\nu)) \\ = [Y_*((\tau\mu + (1-\tau)\sigma,\ s\varsigma + (1-s)\nu),\ \mathfrak{q}),\ Y^*((\tau\mu + (1-\tau)\sigma,\ s\varsigma + (1-s)\nu),\ \mathfrak{q})], \\ \supseteq_I \tau s[Y_*((\mu,\varsigma),\ \mathfrak{q}),\ Y^*((\mu,\varsigma),\ \mathfrak{q})] + t(1-s)[Y_*((\mu,\nu),\ \mathfrak{q}),\ Y^*((\mu,\nu),\ \mathfrak{q})] \\ +s(1-\tau)[Y_*((\mu,\varsigma),\ \mathfrak{q}),\ Y^*((\mu,\varsigma),\ \mathfrak{q})] + (1-\tau)(1-s)[Y_*((\mu,\nu),\ \mathfrak{q}),\ Y^*((\mu,\nu),\ \mathfrak{q})]. \end{gathered}$$

That is,

$$\begin{gathered} \widetilde{Y}(\tau\mu + (1-\tau)\sigma,\ s\varsigma + (1-s)\nu) \\ \supseteq_{\mathbb{F}} \tau s \odot \widetilde{Y}(\mu,\varsigma) \oplus \tau(1-s) \odot \widetilde{Y}(\mu,\nu) \oplus (1-\tau)s \odot \widetilde{Y}(\sigma,\varsigma) \oplus (1-\tau)(1-s) \odot \widetilde{Y}(\sigma,\nu), \end{gathered}$$

Hence, $\widetilde{Y}$ is a coordinated UD-convex FN-V-M on Δ.

Conversely, let $\widetilde{Y}$ be a coordinated UD-convex FN-V-M on Δ. Then, for all $(\mu,\ \sigma)$, $(\varsigma,\nu) \in \Delta,\ \tau$ and $s \in [0,\ 1]$, we obtain

$$\begin{gathered} \widetilde{Y}(\tau\mu + (1-\tau)\sigma,\ s\varsigma + (1-s)\nu) \\ \supseteq_{\mathbb{F}} \tau s \odot \widetilde{Y}(\mu,\varsigma) \oplus \tau(1-s) \odot \widetilde{Y}(\mu,\nu) \oplus (1-\tau)s \odot \widetilde{Y}(\sigma,\varsigma) \oplus (1-\tau)(1-s) \odot \widetilde{Y}(\sigma,\nu) \end{gathered}$$

Therefore, from Equation (22), for each $\mathfrak{q} \in [0,\ 1]$, we obtain

$$\begin{gathered} Y_{\mathfrak{q}}((\tau\mu + (1-\tau)\sigma,\ s\varsigma + (1-s)\nu)) \\ = [Y_*((\tau\mu + (1-\tau)\sigma,\ s\varsigma + (1-s)\nu),\ \mathfrak{q}),\ Y^*((\tau\mu + (1-\tau)\sigma,\ s\varsigma + (1-s)\nu),\ \mathfrak{q})]. \end{gathered}$$

Again, via Equation (22), we obtain

$$\begin{gathered} \tau s Y_{\mathfrak{q}}(\mu,\varsigma) + \tau(1-s)Y_{\mathfrak{q}}(\mu,\nu) + (1-\tau)sY_{\mathfrak{q}}(\sigma,\varsigma) + (1-\tau)(1-s)Y_{\mathfrak{q}}(\sigma,\nu) \\ = \tau s[Y_*((\mu,\varsigma),\ \mathfrak{q}),\ Y^*((\mu,\varsigma),\ \mathfrak{q})] + t(1-s)[Y_*((\mu,\nu),\ \mathfrak{q}),\ Y^*((\mu,\nu),\ \mathfrak{q})] \\ +s(1-\tau)[Y_*((\mu,\varsigma),\ \mathfrak{q}),\ Y^*((\mu,\varsigma),\ \mathfrak{q})] + (1-\tau)(1-s)[Y_*((\mu,\nu),\ \mathfrak{q}),\ Y^*((\mu,\nu),\ \mathfrak{q})], \end{gathered}$$

for all $\sigma,\omega \in \Delta$ and $\tau \in [0,\ 1]$. Then, via the coordinated UD-convexity of $\widetilde{Y}$, for all $\sigma,\omega \in \Delta$ and $\tau \in [0,\ 1]$, we obtain

$$\begin{gathered} Y_*((\tau\mu + (1-\tau)\sigma,\ s\varsigma + (1-s)\nu),\ \mathfrak{q}) \\ \leq \tau s Y_*(\mu,\varsigma) + \tau(1-s)Y_*(\mu,\nu) + (1-\tau)sY_*(\sigma,\varsigma) + (1-\tau)(1-s)Y_*(\sigma,\nu), \end{gathered}$$

and

$$\begin{gathered} Y^*((\tau\mu + (1-\tau)\sigma,\ s\varsigma + (1-s)\nu),\ \mathfrak{q}) \\ \geq \tau s Y^*(\mu,\varsigma) + \tau(1-s)Y^*(\mu,\nu) + (1-\tau)sY^*(\sigma,\varsigma) + (1-\tau)(1-s)Y^*(\sigma,\nu), \end{gathered}$$

for each $\mathfrak{q} \in [0,\ 1]$. Hence, the result follows. □

Example 1. *We consider the FN-V-Ms* $\widetilde{Y} : [0,\ 1] \times [0,\ 1] \to \mathbb{E}_0$ *defined as*

$$\widetilde{Y}(\sigma)(m) = \begin{cases} \frac{m - o\omega}{5-\sigma\omega}\ m \in [\sigma\omega,\ 5] \\ \frac{\left(6+e^{\sigma}\right)(6+e^{\omega})-m}{\left(6+e^{\sigma}\right)(6+e^{\omega})-5}\ m \in \left(5,\ \left(6+e^{\sigma}\right)(6+e^{\omega})\right] \\ 0 \text{ otherwise}, \end{cases} \tag{23}$$

and then, for each $\mathfrak{q} \in [0,\ 1]$, *we obtain* $Y_{\mathfrak{q}}(\sigma, \omega) = [(1-\mathfrak{q})\sigma\omega + 5\mathfrak{q}, (1-\mathfrak{q})(6+e^{\sigma})(6+e^{\omega}) + 5\mathfrak{q}]$. *The end-point functions* $Y_*((\sigma, \omega), \mathfrak{q})$ *and* $Y^*((\sigma, \omega), \mathfrak{q})$ *are coordinated convex and concave functions for each* $\mathfrak{q} \in [0,\ 1]$, *respectively. Hence,* $\widetilde{Y}(\sigma, \omega)$ *is an up and down coordinated convex FN-V-M.*

From Example 1, it can be easily seen that each coordinated UD-convex FN-V-M is not a UD-convex FN-V-M.

Corollary 1. *Let* $\widetilde{Y} : \Delta \to \mathbb{E}_0$ *be an FN-V-M on* Δ. *Then, from the* $\mathfrak{q}$*-levels, we obtain the collection of IVMs* $Y_{\mathfrak{q}} : \Delta \to \mathbb{R}_I^+ \subset \mathbb{R}_I$ *given as*

$$Y_{\mathfrak{q}}(\sigma, \omega) = [Y_*((\sigma, \omega), \mathfrak{q}),\ Y^*((\sigma, \omega), \mathfrak{q})], \tag{24}$$

for all $(\sigma, \omega) \in \Delta$ *and for all* $\mathfrak{q} \in [0,\ 1]$. *Then,* $\widetilde{Y}$ *is a coordinated left-UD-convex (concave) FN-V-M on* Δ *if and only if for all* $\mathfrak{q} \in [0,\ 1]$, $Y_*((\sigma, \omega),\ \mathfrak{q})$ *and* $Y^*((\sigma, \omega),\ \mathfrak{q})$ *are coordinated convex (concave) and affine functions on* Δ, *respectively.*

Corollary 2. *Let* $\widetilde{Y} : \Delta \to \mathbb{E}_0$ *be an FN-V-M on* Δ. *Then, from the* $\mathfrak{q}$*-levels, we obtain the collection of IVMs* $Y_{\mathfrak{q}} : \Delta \to \mathbb{R}_I^+ \subset \mathbb{R}_I$ *given as*

$$Y_{\mathfrak{q}}(\sigma, \omega) = [Y_*((\sigma, \omega), \mathfrak{q}),\ Y^*((\sigma, \omega), \mathfrak{q})], \tag{25}$$

for all $(\sigma, \omega) \in \Delta$ *and for all* $\mathfrak{q} \in [0,\ 1]$. *Then,* $\widetilde{Y}$ *is a coordinated right-UD-convex (concave) FN-V-M on* Δ *if and only if for all* $\mathfrak{q} \in [0,\ 1]$, $Y_*((\sigma, \omega),\ \mathfrak{q})$ *and* $Y^*((\sigma, \omega),\ \mathfrak{q})$ *are coordinated affine and convex (concave) functions on* Δ, *respectively.*

Theorem 8. *Let* Δ *be a coordinated convex set, and let* $\widetilde{Y} : \Delta \to \mathbb{E}_0$ *be an FN-V- M. Then, from the* $\mathfrak{q}$*-levels, we obtain the collection of IVMs* $Y_{\mathfrak{q}} : \Delta \to \mathbb{R}_I^+ \subset \mathbb{R}_I$ *given as*

$$Y_{\mathfrak{q}}(\sigma, \omega) = [Y_*((\sigma, \omega), \mathfrak{q}),\ Y^*((\sigma, \omega), \mathfrak{q})], \tag{26}$$

for all $(\sigma, \omega) \in \Delta$ *and for all* $\mathfrak{q} \in [0,\ 1]$. *Then,* $\widetilde{Y}$ *is a coordinated UD-concave FN-V-M on* Δ *if and only if for all* $\mathfrak{q} \in [0,\ 1]$, $Y_*((\sigma, \omega),\ \mathfrak{q})$ *and* $Y^*((\sigma, \omega),\ \mathfrak{q})$ *are coordinated concave and convex functions, respectively.*

Proof. The demonstration of the proof of Theorem 8 is similar to the demonstration of the proof of Theorem 7. □

Example 2. *We consider the FN-V-Ms* $\widetilde{Y} : [0,\ 1] \times [0,\ 1] \to \mathbb{E}_0$ *defined as*

$$\widetilde{Y}(\sigma)(m) = \begin{cases} \frac{m - \left(6 - e^{\sigma}\right)(6 - e^{\omega})}{\left(6 - e^{\sigma}\right)(6 - e^{\omega}) - 25},\ m \in \left[\left(6 - e^{\sigma}\right)(6 - e^{\omega}),\ 25\right] \\ \frac{35\sigma\omega - m}{35\sigma\omega - 25},\ m \in (25,\ 35\sigma\omega] \\ 0,\ otherwise. \end{cases} \tag{27}$$

Then, for each $\mathfrak{q} \in [0,1]$, we obtain $Y_{\mathfrak{q}}(\sigma, \omega) = [(1-\mathfrak{q})(6-e^{\sigma})(6-e^{\omega}) + 25\mathfrak{q}, 35(1-\mathfrak{q})\sigma\omega + 25\mathfrak{q}]$. The end-point functions $Y_*((\sigma, \omega), \mathfrak{q})$ and $Y^*((\sigma, \omega), \mathfrak{q})$ are coordinated concave and convex functions for each $\mathfrak{q} \in [0,1]$. Hence, $\widetilde{Y}(\sigma, \omega)$ is a coordinated up and down concave FN-V-M.

In the next results, to avoid confusion, we will not include the symbols (R), (IR), (FR), (ID), and (FD) before the integral sign.

Theorem 9. *Let* $\widetilde{Y} : \Delta \to \mathbb{E}_0$ *be a coordinated UD-convex FN-V-M on* Δ. *Then, from the* $\mathfrak{q}$*-levels, we obtain the collection of IVMs* $Y_{\mathfrak{q}} : \Delta \to \mathbb{R}_I^+$ *given as* $Y_{\mathfrak{q}}(\sigma,\omega) = [Y_*((\sigma,\omega),\mathfrak{q}),\ Y^*((\sigma,\omega),\mathfrak{q})]$ *for all* $(\sigma,\omega) \in \Delta$ *and for all* $\mathfrak{q} \in [0,\ 1]$. *Then, the following inequality holds*:

$$\begin{gathered}\widetilde{Y}\left(\frac{\mu+\sigma}{2},\ \frac{\varsigma+\nu}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{2} \odot \left[\frac{1}{\sigma-\mu} \odot \int_\mu^\sigma \widetilde{Y}\left(\sigma,\ \frac{\varsigma+\nu}{2}\right) d\sigma \oplus \frac{1}{\nu-\varsigma} \odot \int_\varsigma^\nu \widetilde{Y}\left(\frac{\mu+\sigma}{2}, \omega\right) d\omega\right] \\ \supseteq_{\mathbb{F}} \frac{1}{(\sigma-\mu)(\nu-\varsigma)} \odot \int_\mu^\sigma \int_\varsigma^\nu \widetilde{Y}(\sigma,\omega) d\omega d\sigma \\ \supseteq_{\mathbb{F}} \frac{1}{4(\sigma-\mu)} \odot \left[\int_\mu^\sigma \widetilde{Y}(\sigma,\varsigma) d\sigma \oplus \int_\mu^\sigma \widetilde{Y}(\sigma,\nu) d\sigma\right] \oplus \frac{1}{4(\nu-\varsigma)} \odot \left[\int_\varsigma^\nu \widetilde{Y}(\mu,\omega) d\omega \oplus \int_\varsigma^\nu \widetilde{Y}(\sigma,\omega) d\omega\right] \\ \supseteq_{\mathbb{F}} \frac{\widetilde{Y}(\mu,\varsigma) \oplus \widetilde{Y}(\sigma,\varsigma) \oplus \widetilde{Y}(\mu,\nu) \oplus \widetilde{Y}(\sigma,\nu)}{4}.\end{gathered} \tag{28}$$

If $Y(\sigma)$ *is a concave FN-V-M, then*

$$\begin{gathered}\widetilde{Y}\left(\frac{\mu+\sigma}{2},\ \frac{\varsigma+\nu}{2}\right) \subseteq_{\mathbb{F}} \frac{1}{2} \left[\frac{1}{\sigma-\mu} \odot \int_\mu^\sigma \widetilde{Y}\left(\sigma,\ \frac{\varsigma+\nu}{2}\right) d\sigma \oplus \frac{1}{\nu-\varsigma} \odot \int_\varsigma^\nu \widetilde{Y}\left(\frac{\mu+\sigma}{2}, \omega\right) d\omega\right] \\ \subseteq_{\mathbb{F}} \frac{1}{(\sigma-\mu)(\nu-\varsigma)} \odot \int_\mu^\sigma \int_\varsigma^\nu \widetilde{Y}(\sigma,\omega) d\omega d\sigma \\ \subseteq_{\mathbb{F}} \frac{1}{4(\sigma-\mu)} \odot \left[\int_\mu^\sigma \widetilde{Y}(\sigma,\varsigma) d\sigma \oplus \int_\mu^\sigma \widetilde{Y}(\sigma,\nu) d\sigma\right] \oplus \frac{1}{4(\nu-\varsigma)} \odot \left[\int_\varsigma^\nu \widetilde{Y}(\mu,\omega) d\omega \oplus \int_\varsigma^\nu \widetilde{Y}(\sigma,\omega) d\omega\right] \\ \subseteq_{\mathbb{F}} \frac{\widetilde{Y}(\mu,\varsigma) \oplus \widetilde{Y}(\sigma,\varsigma) \oplus \widetilde{Y}(\mu,\nu) \oplus \widetilde{Y}(\sigma,\nu)}{4}\end{gathered} \tag{29}$$

Proof. Let $\widetilde{Y} : [\mu,\ \sigma] \to \mathbb{E}_0$ be a coordinated UD-convex FN-V-M. Then, via hypothesis, we obtain

$$4 \odot \widetilde{Y}\left(\frac{\mu+\sigma}{2}, \frac{\varsigma+\nu}{2}\right) \supseteq_{\mathbb{F}} \widetilde{Y}(\tau\mu + (1-\tau)\sigma,\ \tau\varsigma + (1-\tau)\nu) \oplus \widetilde{Y}((1-\tau)\mu + \tau\sigma,\ (1-\tau)\varsigma + \tau\nu).$$

By using Theorem 7, for every $\mathfrak{q} \in [0,\ 1]$, we obtain

$$\begin{gathered}4Y_*\left(\left(\frac{\mu+\sigma}{2}, \frac{\varsigma+\nu}{2}\right), \mathfrak{q}\right) \\ \leq Y_*((\tau\mu + (1-\tau)\sigma,\ \tau\varsigma + (1-\tau)\nu),\ \mathfrak{q}) + Y_*(((1-\tau)\mu + \tau\sigma,\ (1-\tau)\varsigma + \tau\nu),\ \mathfrak{q}), \\ 4Y^*\left(\left(\frac{\mu+\sigma}{2}, \frac{\varsigma+\nu}{2}\right), \mathfrak{q}\right) \\ \geq Y^*((\tau\mu + (1-\tau)\sigma,\ \tau\varsigma + (1-\tau)\nu),\ \mathfrak{q}) + Y^*(((1-\tau)\mu + \tau\sigma,\ (1-\tau)\varsigma + \tau\nu), \mathfrak{q}).\end{gathered}$$

By using Lemma 1, we obtain

$$\begin{gathered}2Y_*\left(\left(\sigma, \frac{\varsigma+\nu}{2}\right), \mathfrak{q}\right) \leq Y_*((\sigma, \tau\varsigma + (1-\tau)\nu),\ \mathfrak{q}) + Y_*((\sigma, (1-\tau)\varsigma + \tau\nu),\ \mathfrak{q}), \\ 2Y^*\left(\left(\sigma, \frac{\varsigma+\nu}{2}\right), \mathfrak{q}\right) \geq Y^*((\sigma, \tau\varsigma + (1-\tau)\nu),\ \mathfrak{q}) + Y^*((\sigma, (1-\tau)\varsigma + \tau\nu), \mathfrak{q}),\end{gathered} \tag{30}$$

and

$$\begin{gathered}2Y_*\left(\left(\frac{\mu+\sigma}{2}, \omega\right), \mathfrak{q}\right) \leq Y_*((\tau\mu + (1-\tau)\sigma,\ \omega),\ \mathfrak{q}) + Y_*(((1-\tau)\mu + t\sigma,\ \omega),\ \mathfrak{q}), \\ 2Y^*\left(\left(\frac{\mu+\sigma}{2}, \omega\right), \mathfrak{q}\right) \geq Y^*((\tau\mu + (1-\tau)\sigma,\ \omega),\ \mathfrak{q}) + Y^*(((1-\tau)\mu + t\sigma,\ \omega), \mathfrak{q}).\end{gathered} \tag{31}$$

From Equations (30) and (31), we obtain

$$\begin{gathered}2\left[Y_*\left(\left(\sigma, \frac{\varsigma+\nu}{2}\right), \mathfrak{q}\right), Y^*\left(\left(\sigma, \frac{\varsigma+\nu}{2}\right), \mathfrak{q}\right)\right] \\ \supseteq_I [Y_*((\sigma, \tau\varsigma + (1-\tau)\nu),\ \mathfrak{q}), Y^*((\sigma, \tau\varsigma + (1-\tau)\nu),\ \mathfrak{q})] \\ +[Y_*((\sigma, (1-\tau)\varsigma + \tau\nu),\ \mathfrak{q}), Y^*((\sigma, (1-\tau)\varsigma + \tau\nu), \mathfrak{q})],\end{gathered}$$

and

$$\begin{aligned} &2\Big[\Upsilon_*\Big(\Big(\tfrac{\mu+\sigma}{2},\omega\Big),\ \mathfrak{q}\Big),\Upsilon^*\Big(\Big(\tfrac{\mu+\sigma}{2},\omega\Big),\ \mathfrak{q}\Big)\Big] \\ &\supseteq_I [\Upsilon_*((\tau\mu+(1-\tau)\sigma,\ \omega),\ \mathfrak{q}),\Upsilon^*((\tau\mu+(1-\tau)\sigma,\ \omega),\ \mathfrak{q})] \\ &+[\Upsilon_*((\tau\mu+(1-\tau)\sigma,\ \omega),\ \mathfrak{q}),\Upsilon^*((\tau\mu+(1-\tau)\sigma,\ \omega),\mathfrak{q})], \end{aligned}$$

It follows that

$$\Upsilon_{\mathfrak{q}}\left(\sigma,\frac{\varsigma+\nu}{2}\right) \supseteq_I \Upsilon_{\mathfrak{q}}(\sigma,\tau\varsigma+(1-\tau)\nu)+\Upsilon_{\mathfrak{q}}(\sigma,(1-\tau)\varsigma+\tau\nu) \tag{32}$$

and

$$\Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\omega\right) \supseteq_I \Upsilon_{\mathfrak{q}}(\tau\mu+(1-\tau)\sigma,\ \omega)+\Upsilon_{\mathfrak{q}}(\tau\mu+(1-\tau)\sigma,\ \omega) \tag{33}$$

Since $\Upsilon_{\mathfrak{q}}(\sigma,.)$ and $\Upsilon_{\mathfrak{q}}(.,\omega)$ are both coordinated UD-convex-IVMs, from Theorem 7 and inequality Equation (6), for every $\mathfrak{q}\in[0,\ 1]$, and inequality Equations (32) and (33), we then obtain

$$\Upsilon_{\mathfrak{q}}\left(\sigma,\frac{\varsigma+\nu}{2}\right) \supseteq_I \frac{1}{\nu-\varsigma}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\sigma,\omega)d\omega \supseteq_I \frac{\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)+\Upsilon_{\mathfrak{q}}(\sigma,\nu)}{2}. \tag{34}$$

and

$$\Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\omega\right) \supseteq_I \frac{1}{\sigma-\mu}\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}(\sigma,\omega)d\sigma \supseteq_I \frac{\Upsilon_{\mathfrak{q}}(\mu,\omega)+\Upsilon_{\mathfrak{q}}(\sigma,\omega)}{2}. \tag{35}$$

Dividing double inequality Equation (34) by $(\sigma-\mu)$ and integrating with respect to σ on $[\mu,\ \sigma]$, we obtain

$$\frac{1}{\sigma-\mu}\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}\left(\sigma,\frac{\varsigma+\nu}{2}\right)d\sigma \supseteq_I \frac{1}{(\sigma-\mu)(\nu-\varsigma)}\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\sigma,\omega)d\omega d\sigma \supseteq_I \frac{1}{2(\sigma-\mu)}\left[\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)d\sigma+\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}(\sigma,\nu)d\sigma\right] \tag{36}$$

Similarly, dividing double inequality Equation (35) by $(\nu-\varsigma)$ and integrating with respect to σ on $[\varsigma,\ \nu]$, we obtain

$$\frac{1}{\nu-\varsigma}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\omega\right)d\omega \supseteq_I \frac{1}{(\sigma-\mu)(\nu-\varsigma)}\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\sigma,\omega)d\omega d\sigma \supseteq_I \frac{1}{2(\nu-\varsigma)}\left[\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\mu,\omega)d\omega+\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\sigma,\omega)d\omega\right] \tag{37}$$

By adding Equations (36) and (37), we obtain

$$\begin{aligned} &\tfrac{1}{2}\left[\tfrac{1}{\sigma-\mu}\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}\left(\sigma,\tfrac{\varsigma+\nu}{2}\right)d\sigma+\tfrac{1}{\nu-\varsigma}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}\left(\tfrac{\mu+\sigma}{2},\omega\right)d\omega\right] \supseteq_I \tfrac{1}{(\sigma-\mu)(\nu-\varsigma)}\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\sigma,\omega)d\omega d\sigma \\ &\supseteq_I \tfrac{1}{4(\sigma-\mu)}\left[\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)d\sigma+\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}(\sigma,\nu)d\sigma\right]+\tfrac{1}{4(\nu-\varsigma)}\left[\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\mu,\omega)d\omega+\int_{\varsigma}^{d}\Upsilon_{\mathfrak{q}}(\sigma,\omega)d\omega\right] \end{aligned} \tag{38}$$

Since $\widetilde{\Upsilon}$ is an FN-V-M, via inequality Equation (38), we then obtain

$$\begin{aligned} &\tfrac{1}{2}\left[\tfrac{1}{\sigma-\mu}\odot\int_{\mu}^{\sigma}\widetilde{\Upsilon}\left(\sigma,\tfrac{\varsigma+\nu}{2}\right)d\sigma\oplus\tfrac{1}{\nu-\varsigma}\odot\int_{\varsigma}^{\nu}\widetilde{\Upsilon}\left(\tfrac{\mu+\sigma}{2},\omega\right)d\omega\right] \supseteq_{\mathbb{F}} \tfrac{1}{(\sigma-\mu)(\nu-\varsigma)}\odot\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu}\widetilde{\Upsilon}(\sigma,\omega)d\omega d\sigma \\ &\supseteq_{\mathbb{F}} \tfrac{1}{4(\sigma-\mu)}\odot\left[\int_{\mu}^{\sigma}\widetilde{\Upsilon}(\sigma,\varsigma)d\sigma\oplus\int_{\mu}^{\sigma}\widetilde{\Upsilon}(\sigma,\nu)d\sigma\right]\oplus\tfrac{1}{4(\nu-\varsigma)}\odot\left[\int_{\varsigma}^{\nu}\widetilde{\Upsilon}(\mu,\omega)d\omega\oplus\int_{\varsigma}^{\nu}\widetilde{\Upsilon}(\sigma,\omega)d\omega\right] \end{aligned} \tag{39}$$

From Theorem 7 and the left side of inequality Equation (16), for each $\mathfrak{q}\in[0,\ 1]$, we obtain

$$\Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\ \frac{\varsigma+\nu}{2}\right) \supseteq_I \frac{1}{\sigma-\mu}\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}\left(\sigma,\frac{\varsigma+\nu}{2}\right)d\sigma, \tag{40}$$

$$\Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\ \frac{\varsigma+\nu}{2}\right) \supseteq_I \frac{1}{\nu-\varsigma}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\omega\right)d\omega. \tag{41}$$

Adding inequality Equation (40) and inequality Equation (41), we obtain

$$\Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\ \frac{\varsigma+\nu}{2}\right) \supseteq_I \frac{1}{2}\left[\frac{1}{\sigma-\mu}\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}\left(\boldsymbol{\sigma},\frac{\varsigma+\nu}{2}\right)d\boldsymbol{\sigma}+\frac{1}{\nu-\varsigma}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\omega\right)d\omega\right].$$

Since $\widetilde{\Upsilon}$ is an FN-V-M, it follows that

$$\widetilde{\Upsilon}\left(\frac{\mu+\sigma}{2},\ \frac{\varsigma+\nu}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{2}\left[\frac{1}{\sigma-\mu}\odot\int_{\mu}^{\sigma}\widetilde{\Upsilon}\left(\boldsymbol{\sigma},\frac{\varsigma+\nu}{2}\right)d\boldsymbol{\sigma}\oplus\frac{1}{\nu-\varsigma}\odot\int_{\varsigma}^{\nu}\widetilde{\Upsilon}\left(\frac{\mu+\sigma}{2},\omega\right)d\omega\right] \tag{42}$$

Now, from Theorem 7 and the right side of inequality Equation (16), for every $\mathfrak{q}\in[0,\ 1]$, we obtain

$$\frac{1}{\sigma-\mu}\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)d\boldsymbol{\sigma} \supseteq_I \frac{\Upsilon_{\mathfrak{q}}(\mu,\varsigma)+\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)}{2} \tag{43}$$

$$\frac{1}{\sigma-\mu}\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)d\boldsymbol{\sigma} \supseteq_I \frac{\Upsilon_{\mathfrak{q}}(\mu,\nu)+\Upsilon_{\mathfrak{q}}(\sigma,\nu)}{2} \tag{44}$$

$$\frac{1}{\nu-\varsigma}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\mu,\omega)d\omega \supseteq_I \frac{\Upsilon_{\mathfrak{q}}(\mu,\nu)+\Upsilon_{\mathfrak{q}}(\mu,\varsigma)}{2} \tag{45}$$

$$\frac{1}{\nu-\varsigma}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\sigma,\omega)d\omega \supseteq_I \frac{\Upsilon_{\mathfrak{q}}(\sigma,\nu)+\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)}{2} \tag{46}$$

By adding inequalities Equations (43)–(46), we obtain

$$\begin{gathered}\tfrac{1}{4(\sigma-\mu)}\left[\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)d\boldsymbol{\sigma}+\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)d\boldsymbol{\sigma}\right]+\tfrac{1}{4(\nu-\varsigma)}\left[\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\mu,\omega)d\omega+\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\sigma,\omega)d\omega\right]\\ \supseteq_I \tfrac{\Upsilon_{\mathfrak{q}}(\mu,\varsigma)+\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)+\Upsilon_{\mathfrak{q}}(\mu,\nu)+\Upsilon_{\mathfrak{q}}(\sigma,\nu)}{4}\end{gathered}$$

Since Υ is an FN-V-M, it follows that

$$\begin{gathered}\tfrac{1}{4(\sigma-\mu)}\odot\left[\int_{\mu}^{\sigma}\widetilde{\Upsilon}(\boldsymbol{\sigma},\varsigma)d\boldsymbol{\sigma}\oplus\int_{\mu}^{\sigma}\widetilde{\Upsilon}(\boldsymbol{\sigma},\nu)d\boldsymbol{\sigma}\right]\oplus\tfrac{1}{4(\nu-\varsigma)}\odot\left[\int_{\varsigma}^{\nu}\widetilde{\Upsilon}(\mu,\omega)d\omega\oplus\int_{\varsigma}^{\nu}\widetilde{\Upsilon}(\sigma,\omega)d\omega\right]\\ \supseteq_{\mathbb{F}} \tfrac{\widetilde{\Upsilon}(\mu,\varsigma)\oplus\widetilde{\Upsilon}(\sigma,\varsigma)\oplus\widetilde{\Upsilon}(\mu,\nu)\oplus\widetilde{\Upsilon}(\sigma,\nu)}{4}\end{gathered} \tag{47}$$

By combining inequalities Equations (41), (42), and (47), we obtain the desired result. □

Remark 2. *From inequality Equation (28), the following exceptional results can be acquired*:

Let $\Upsilon_*((\boldsymbol{\sigma},\omega),\mathfrak{q})\neq\Upsilon^*((\boldsymbol{\sigma},\omega),\mathfrak{q})$ *with* $\mathfrak{q}=1$. *Then, we can derive the following inclusion (see* [61]):

$$\begin{gathered}\Upsilon\left(\tfrac{\mu+\sigma}{2},\ \tfrac{\varsigma+\nu}{2}\right)\supseteq\tfrac{1}{2}\left[\tfrac{1}{\sigma-\mu}\int_{\mu}^{\sigma}\Upsilon\left(\boldsymbol{\sigma},\tfrac{\varsigma+\nu}{2}\right)d\boldsymbol{\sigma}+\tfrac{1}{\nu-\varsigma}\int_{\varsigma}^{\nu}\Upsilon\left(\tfrac{\mu+\sigma}{2},\omega\right)d\omega\right]\\ \supseteq\tfrac{1}{(\sigma-\mu)(\nu-\varsigma)}\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu}\Upsilon(\boldsymbol{\sigma},\omega)d\omega d\boldsymbol{\sigma}\\ \supseteq\tfrac{1}{4(\sigma-\mu)}\left[\int_{\mu}^{\sigma}\Upsilon(\boldsymbol{\sigma},\varsigma)d\boldsymbol{\sigma}+\int_{\mu}^{\sigma}\Upsilon(\boldsymbol{\sigma},\nu)d\boldsymbol{\sigma}\right]+\tfrac{1}{4(\nu-\varsigma)}\left[\int_{\varsigma}^{\nu}\Upsilon(\mu,\omega)d\omega+\int_{\varsigma}^{\nu}\Upsilon(\sigma,\omega)d\omega\right]\\ \supseteq\tfrac{\Upsilon(\mu,\varsigma)+\Upsilon(\sigma,\varsigma)+\Upsilon(\mu,\nu)+\Upsilon(\sigma,\nu)}{4}.\end{gathered} \tag{48}$$

Let $\Upsilon_*((\boldsymbol{\sigma},\omega),\mathfrak{q})=\Upsilon^*((\boldsymbol{\sigma},\omega),\mathfrak{q})$ *with* $\mathfrak{q}=1$. *Then, we can derive the following inclusion (see* [90]):

$$\begin{gathered}\Upsilon\left(\tfrac{\mu+\sigma}{2},\ \tfrac{\varsigma+\nu}{2}\right)\leq\tfrac{1}{2}\left[\tfrac{1}{\sigma-\mu}\int_{\mu}^{\sigma}\Upsilon\left(\boldsymbol{\sigma},\tfrac{\varsigma+\nu}{2}\right)d\boldsymbol{\sigma}+\tfrac{1}{\nu-\varsigma}\int_{\varsigma}^{\nu}\Upsilon\left(\tfrac{\mu+\sigma}{2},\omega\right)d\omega\right]\\ \leq\tfrac{1}{(\sigma-\mu)(\nu-\varsigma)}\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu}\Upsilon(\boldsymbol{\sigma},\omega)d\omega d\boldsymbol{\sigma}\\ \leq\tfrac{1}{4(\sigma-\mu)}\left[\int_{\mu}^{\sigma}\Upsilon(\boldsymbol{\sigma},\varsigma)d\boldsymbol{\sigma}+\int_{\mu}^{\sigma}\Upsilon(\boldsymbol{\sigma},\nu)d\boldsymbol{\sigma}\right]+\tfrac{1}{4(\nu-\varsigma)}\left[\int_{\varsigma}^{\nu}\Upsilon(\mu,\omega)d\omega+\int_{\varsigma}^{\nu}\Upsilon(\sigma,\omega)d\omega\right]\\ \leq\tfrac{\Upsilon(\mu,\varsigma)+\Upsilon(\sigma,\varsigma)+\Upsilon(\mu,\nu)+\Upsilon(\sigma,\nu)}{4}.\end{gathered} \tag{49}$$

Example 3. *We consider the FN-V-Ms* $\widetilde{Y} : [0,\ 2] \times [0,\ 2] \to \mathbb{E}_0$ *defined as*

$$Y(\sigma, \omega)(m) = \begin{cases} \frac{m - \sigma\omega}{5 - \sigma\omega},\ m \in [\sigma\omega,\ 5] \\ \frac{\left(2+\sqrt{\sigma}\right)\left(2+\sqrt{\omega}\right) - m}{\left(2+\sqrt{\sigma}\right)\left(2+\sqrt{\omega}\right) - 5},\ m \in \left(5,\ \left(2+\sqrt{\sigma}\right)\left(2+\sqrt{\omega}\right)\right] \\ 0,\ otherwise, \end{cases} \tag{50}$$

and then, for each $\mathfrak{q} \in [0,\ 1]$, *we obtain* $Y_{\mathfrak{q}}(\sigma, \omega) = \left[(1-\mathfrak{q})\sigma\omega + 5\mathfrak{q}, (1-\mathfrak{q})\left(2+\sqrt{\sigma}\right)\left(2+\sqrt{\omega}\right) + 5\mathfrak{q}\right]$. *The end-point functions* $Y_*((\sigma, \omega), \mathfrak{q})$, $Y^*((\sigma, \omega), \mathfrak{q})$ *are coordinated concave functions for each* $\mathfrak{q} \in [0,\ 1]$. *Hence,* $\widetilde{Y}(\sigma, \omega)$ *is a coordinated concave FN-V-M.*

$$Y_{\mathfrak{q}}\left(\frac{\mu + \sigma}{2},\ \frac{\varsigma + \nu}{2}\right) = [1 + 4\mathfrak{q}, 9 - 4\mathfrak{q}]$$

$$\begin{aligned} &\tfrac{1}{2}\left[\tfrac{1}{\sigma - \mu}\int_\mu^\sigma Y_{\mathfrak{q}}\left(\sigma, \tfrac{\varsigma+\nu}{2}\right)d\sigma + \tfrac{1}{\nu - \varsigma}\int_\varsigma^\nu Y_{\mathfrak{q}}\left(\tfrac{\mu+\sigma}{2}, \omega\right)d\omega\right] \\ &= \left[1 + 4\mathfrak{q}, \tfrac{1}{3}\left(\left(9 + 2\sqrt{2}\right)\mathfrak{q} - 2\sqrt{2} + 6\right)\right], \end{aligned}$$

$$\begin{aligned} &\tfrac{1}{(\sigma-\mu)(\nu-\varsigma)}\int_\mu^\sigma\int_\varsigma^\nu Y_{\mathfrak{q}}(\sigma, \omega)d\omega d\sigma = \left[1 + 4\mathfrak{q}, \tfrac{1}{9}\left(\left(1 + 24\sqrt{2}\right)\mathfrak{q} - 24\sqrt{2} + 44\right)\right] \\ &\tfrac{1}{4(\sigma-\mu)}\left[\int_\mu^\sigma Y_{\mathfrak{q}}(\sigma, \varsigma)d\sigma + \int_\mu^\sigma Y_{\mathfrak{q}}(\sigma, \nu)d\sigma\right] + \tfrac{1}{4(\nu-\varsigma)}\left[\int_\varsigma^\nu Y_{\mathfrak{q}}(\mu, \omega)d\omega + \int_\varsigma^\nu Y_{\mathfrak{q}}(\sigma, \omega)d\omega\right], \\ &= \left[1 + 4\mathfrak{q}, \tfrac{8 - 5\sqrt{2}}{3}(1-\mathfrak{q}) + \tfrac{9 + 2\sqrt{2}}{3}\mathfrak{q} + \tfrac{6 - 2\sqrt{2}}{3}\right] \end{aligned}$$

$$\frac{Y_{\mathfrak{q}}(\mu, \varsigma) + Y_{\mathfrak{q}}(\sigma, \varsigma) + Y_{\mathfrak{q}}(\mu, \nu) + Y_{\mathfrak{q}}(\sigma, \nu)}{4} = \left[1 + 4\mathfrak{q}, \frac{(1-\mathfrak{q})\left(2 - \sqrt{2}\right)^2 + 4(1-\mathfrak{q})\left(2 - \sqrt{2}\right) + 4(1-\mathfrak{q}) + 20\mathfrak{q}}{4}\right]$$

That is

$$[1 + 4\mathfrak{q}, 9 - 4\mathfrak{q}] \supseteq_I \left[1 + 4\mathfrak{q}, \frac{1}{3}\left(\left(9 + 2\sqrt{2}\right)\mathfrak{q} - 2\sqrt{2} + 6\right)\right]$$

$$\supseteq_I \left[1 + 4\mathfrak{q}, \frac{1}{9}\left(\left(1 + 24\sqrt{2}\right)\mathfrak{q} - 24\sqrt{2} + 44\right)\right] \supseteq_I \left[1 + 4\mathfrak{q}, \frac{8 - 5\sqrt{2}}{3}(1-\mathfrak{q}) + \frac{9 + 2\sqrt{2}}{3}\mathfrak{q} + \frac{6 - 2\sqrt{2}}{3}\right]$$

$$\supseteq_I \left[1 + 4\mathfrak{q}, \frac{(1-\mathfrak{q})\left(2 - \sqrt{2}\right)^2 + 4(1-\mathfrak{q})\left(2 - \sqrt{2}\right) + 4(1-\mathfrak{q}) + 20\mathfrak{q}}{4}\right]$$

Hence, Theorem 9 has been verified.

We will now obtain some HH inequalities to produce coordinated UD-convex FN-V-Ms. These inequalities are refinements of some Pachpatte-type inequalities on coordinates.

Theorem 10. *Let* $\widetilde{Y}, \widetilde{\mathfrak{S}} : \Delta = [\mu,\ \sigma] \times [\varsigma,\ \nu] \subset \mathbb{R}^2 \to \mathbb{E}_0$ *be two coordinated UD-convex FN-V-Ms on* Δ, *whose* $\mathfrak{q}$*-levels* $Y_{\mathfrak{q}}$, $\mathfrak{S}_{\mathfrak{q}} : [\mu,\ \sigma] \times [\varsigma,\ \nu] \to \mathbb{R}_I^+$ *are defined as* $Y_{\mathfrak{q}}(\sigma, \omega) = [Y_*((\sigma, \omega), \mathfrak{q}),\ Y^*((\sigma, \omega), \mathfrak{q})]$ *and* $\mathfrak{S}_{\mathfrak{q}}(\sigma, \omega) = [\mathfrak{S}_*((\sigma, \omega), \mathfrak{q}),\ \mathfrak{S}^*((\sigma, \omega), \mathfrak{q})]$ *for all* $(\sigma, \omega) \in \Delta$ *and for all* $\mathfrak{q} \in [0,\ 1]$. *Then, the following inequality holds*:

$$\begin{aligned} &\tfrac{1}{(\sigma-\mu)(\nu-\varsigma)} \odot \int_\mu^\sigma\int_\varsigma^\nu \widetilde{Y}(\sigma, \omega) \otimes \widetilde{\mathfrak{S}}(\sigma, \omega)d\omega d\sigma \\ &\supseteq_{\mathbb{F}} \tfrac{1}{9} \odot \widetilde{P}(\mu, \sigma, \varsigma, \nu) \oplus \tfrac{1}{18} \odot \widetilde{\mathcal{M}}(\mu, \sigma, \varsigma, \nu) \oplus \tfrac{1}{36} \odot \widetilde{\mathcal{N}}(\mu, \sigma, \varsigma, \nu). \end{aligned} \tag{51}$$

where

$$\widetilde{P}(\mu, \sigma, \varsigma, \nu) = \widetilde{Y}(\mu, \varsigma) \otimes \widetilde{\mathfrak{S}}(\mu, \varsigma) \oplus \widetilde{Y}(\mu, \nu) \otimes \widetilde{\mathfrak{S}}(\mu, \nu) \oplus \widetilde{Y}(\sigma, \varsigma) \otimes \widetilde{\mathfrak{S}}(\sigma, \varsigma) \oplus \widetilde{Y}(\sigma, \nu) \otimes \widetilde{\mathfrak{S}}(\sigma, \nu),$$

$$\widetilde{\mathcal{M}}(\mu, \sigma, \varsigma, \nu) = \widetilde{Y}(\mu, \varsigma) \otimes \widetilde{\mathfrak{S}}(\mu, \nu) \oplus \widetilde{Y}(\mu, \nu) \otimes \widetilde{\mathfrak{S}}(\mu, \varsigma) \oplus \widetilde{Y}(\sigma, \varsigma) \otimes \widetilde{\mathfrak{S}}(\sigma, \nu) \oplus \widetilde{Y}(\sigma, \nu) \otimes \widetilde{\mathfrak{S}}(\sigma, \varsigma),$$

$$\oplus(\mu, \varsigma) \otimes \widetilde{\mathfrak{S}}(\sigma, \varsigma) \oplus \widetilde{Y}(\sigma, \nu) \otimes \widetilde{\mathfrak{S}}(\mu, \nu) \oplus \widetilde{Y}(\sigma, \varsigma) \otimes \widetilde{\mathfrak{S}}(\mu, \varsigma) \oplus \widetilde{Y}(\mu, \nu) \otimes \widetilde{\mathfrak{S}}(\sigma, \nu)$$

$$\widetilde{\mathcal{N}}(\mu,\sigma,\varsigma,\nu) = \widetilde{Y}(\mu,\varsigma) \otimes \widetilde{\mathfrak{S}}(\sigma,\nu) \oplus \widetilde{Y}(\sigma,\varsigma) \otimes \widetilde{\mathfrak{S}}(\mu,\nu) \oplus \widetilde{Y}(\sigma,\nu) \otimes \widetilde{\mathfrak{S}}(\mu,\varsigma) \oplus \widetilde{Y}(\sigma,\varsigma) \otimes \widetilde{\mathfrak{S}}(\mu,\nu)$$

and for each $\mathfrak{q} \in [0,\ 1]$, $\widetilde{P}(\mu,\sigma,\varsigma,\nu)$, $\widetilde{\mathcal{M}}(\mu,\sigma,\varsigma,\nu)$, *and* $\widetilde{\mathcal{N}}(\mu,\sigma,\varsigma,\nu)$ *are defined as follows:*

$$P_{\mathfrak{q}}(\mu,\sigma,\varsigma,\nu) = [P_*((\mu,\sigma,\varsigma,\nu),\ \mathfrak{q}),\ P^*((\mu,\sigma,\varsigma,\nu),\ \mathfrak{q})]$$

$$\mathcal{M}_{\mathfrak{q}}(\mu,\sigma,\varsigma,\nu) = [\mathcal{M}_*((\mu,\sigma,\varsigma,\nu),\ \mathfrak{q}),\ \mathcal{M}^*((\mu,\sigma,\varsigma,\nu),\ \mathfrak{q})]$$

$$\mathcal{N}_{\mathfrak{q}}(\mu,\sigma,\varsigma,\nu) = [\mathcal{N}_*((\mu,\sigma,\varsigma,\nu),\ \mathfrak{q}),\ \mathcal{N}^*((\mu,\sigma,\varsigma,\nu),\ \mathfrak{q})].$$

Proof. Let $\widetilde{Y}$ and $\widetilde{\mathfrak{S}}$ be two coordinated UD-convex FN-V- Ms on $[\mu,\ \sigma] \times [\varsigma,\ \nu]$. Then,

$$\begin{gathered}\widetilde{Y}(\tau\mu + (1-\tau)\sigma,\ s\varsigma + (1-s)\nu) \\ \supseteq_{\mathbb{F}} \tau s \odot \widetilde{Y}(\mu,\varsigma) \oplus \tau(1-s) \odot \widetilde{Y}(\mu,\nu) \oplus (1-\tau)s \odot \widetilde{Y}(\sigma,\varsigma) \oplus (1-\tau)(1-s) \odot \widetilde{Y}(\sigma,\nu),\end{gathered}$$

and

$$\begin{gathered}\widetilde{\mathfrak{S}}(\tau\mu + (1-\tau)\sigma,\ s\varsigma + (1-s)\nu) \\ \supseteq_{\mathbb{F}} \tau s \odot \widetilde{\mathfrak{S}}(\mu,\varsigma) \oplus \tau(1-s) \odot \widetilde{\mathfrak{S}}(\mu,\nu) \oplus (1-\tau)s \odot \widetilde{\mathfrak{S}}(\sigma,\varsigma) \oplus (1-\tau)(1-s) \odot \widetilde{\mathfrak{S}}(\sigma,\nu).\end{gathered}$$

Since $\widetilde{Y}$ and $\widetilde{\mathfrak{S}}$ are both coordinated UD-convex FN-V-Ms, then via Lemma 1, the following exist:

$$\widetilde{Y}_{\boldsymbol{\sigma}} : [\varsigma,\nu] \to \mathbb{E}_0,\ \widetilde{Y}_{\boldsymbol{\sigma}}(\omega) = \widetilde{Y}(\boldsymbol{\sigma},\omega),\ \widetilde{\mathfrak{S}}_{\boldsymbol{\sigma}} : [\varsigma,\nu] \to \mathbb{E}_0,\ \widetilde{\mathfrak{S}}_{\boldsymbol{\sigma}}(\omega) = \widetilde{\mathfrak{S}}(\boldsymbol{\sigma},\omega),$$

and

$$\widetilde{Y}_{\omega} : [\mu,\sigma] \to \mathbb{E}_0,\ \widetilde{Y}_{\omega}(\boldsymbol{\sigma}) = \widetilde{Y}(\boldsymbol{\sigma},\omega),\ \widetilde{\mathfrak{S}}_{\omega} : [\mu,\sigma] \to \mathbb{E}_0,\ \widetilde{\mathfrak{S}}_{\omega}(\boldsymbol{\sigma}) = \widetilde{\mathfrak{S}}(\boldsymbol{\sigma},\omega).$$

Since $\widetilde{Y}_{\boldsymbol{\sigma}}$, $\widetilde{\mathfrak{S}}_{\boldsymbol{\sigma}}$, $\widetilde{Y}_{\omega}$ and $\widetilde{\mathfrak{S}}_{\omega}$ are FN-V-Ms, then via inequality Equation (13), we obtain

$$\begin{gathered}\frac{1}{\sigma-\mu} \odot \int_{\mu}^{\sigma} \widetilde{Y}_{\omega}(\boldsymbol{\sigma}) \otimes \widetilde{\mathfrak{S}}_{\omega}(\boldsymbol{\sigma}) d\boldsymbol{\sigma} \\ \supseteq_{\mathbb{F}} \frac{1}{3} \odot \left[\widetilde{Y}_{\omega}(\mu) \otimes \widetilde{\mathfrak{S}}_{\omega}(\mu) \oplus \widetilde{Y}_{\omega}(\sigma) \otimes \widetilde{\mathfrak{S}}_{\omega}(\sigma)\right] \oplus \frac{1}{6}\left[\widetilde{Y}_{\omega}(\mu) \otimes \widetilde{\mathfrak{S}}_{\omega}(\sigma) \oplus \widetilde{Y}_{\omega}(\sigma) \otimes \widetilde{\mathfrak{S}}_{\omega}(\mu)\right],\end{gathered}$$

and

$$\begin{gathered}\frac{1}{\nu-\varsigma} \int_{\varsigma}^{\nu} \widetilde{Y}_{\boldsymbol{\sigma}}(\omega) \otimes \widetilde{\mathfrak{S}}_{\boldsymbol{\sigma}}(\omega) d\omega \\ \supseteq_{\mathbb{F}} \frac{1}{3} \odot \left[\widetilde{Y}_{\boldsymbol{\sigma}}(\varsigma) \otimes \widetilde{\mathfrak{S}}_{\boldsymbol{\sigma}}(\varsigma) \oplus \widetilde{Y}_{\boldsymbol{\sigma}}(\nu) \otimes \widetilde{\mathfrak{S}}_{\boldsymbol{\sigma}}(\nu)\right] \oplus \frac{1}{6}\left[\widetilde{Y}_{\boldsymbol{\sigma}}(\varsigma) \otimes \widetilde{\mathfrak{S}}_{\boldsymbol{\sigma}}(\nu) \oplus \widetilde{Y}_{\boldsymbol{\sigma}}(\varsigma) \otimes \widetilde{\mathfrak{S}}_{\boldsymbol{\sigma}}(\nu)\right].\end{gathered}$$

For each $\mathfrak{q} \in [0,\ 1]$, we obtain

$$\begin{gathered}\frac{1}{\sigma-\mu} \int_{\mu}^{\sigma} Y_{\mathfrak{q}_{\omega}}(\boldsymbol{\sigma}) \times \mathfrak{S}_{\mathfrak{q}_{\omega}}(\boldsymbol{\sigma}) d\boldsymbol{\sigma} \\ \supseteq_I \frac{1}{3}\left[Y_{\mathfrak{q}_{\omega}}(\mu) \times \mathfrak{S}_{\mathfrak{q}_{\omega}}(\mu) + Y_{\mathfrak{q}_{\omega}}(\sigma) \times \mathfrak{S}_{\mathfrak{q}_{\omega}}(\sigma)\right] + \frac{1}{6}\left[Y_{\mathfrak{q}_{\omega}}(\mu) \times \mathfrak{S}_{\mathfrak{q}_{\omega}}(\sigma) + Y_{\mathfrak{q}_{\omega}}(\sigma) \times \mathfrak{S}_{\mathfrak{q}_{\omega}}(\mu)\right],\end{gathered}$$

and

$$\begin{gathered}\frac{1}{\nu-\varsigma} \int_{\varsigma}^{\nu} Y_{\mathfrak{q}\boldsymbol{\sigma}}(\omega) \times \mathfrak{S}_{\mathfrak{q}\boldsymbol{\sigma}}(\omega) d\omega \\ \supseteq_I \frac{1}{3}\left[Y_{\mathfrak{q}\boldsymbol{\sigma}}(\varsigma) \times \mathfrak{S}_{\mathfrak{q}\boldsymbol{\sigma}}(\varsigma) + Y_{\mathfrak{q}\boldsymbol{\sigma}}(\nu) \times \mathfrak{S}_{\mathfrak{q}\boldsymbol{\sigma}}(\nu)\right] + \frac{1}{6}\left[Y_{\mathfrak{q}\boldsymbol{\sigma}}(\varsigma) \times \mathfrak{S}_{\mathfrak{q}\boldsymbol{\sigma}}(\nu) + Y_{\mathfrak{q}\boldsymbol{\sigma}}(\varsigma) \times \mathfrak{S}_{\mathfrak{q}\boldsymbol{\sigma}}(\nu)\right].\end{gathered}$$

The above inequalities can be written as

$$\begin{aligned}\frac{1}{\sigma-\mu} \int_{\mu}^{\sigma} Y_{\mathfrak{q}}(\boldsymbol{\sigma},\omega) \times \mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\omega) d\boldsymbol{\sigma} \supseteq_I &\ \frac{1}{3}[Y_{\mathfrak{q}}(\mu,\omega) \times \mathfrak{S}_{\mathfrak{q}}(\mu,\omega) + Y_{\mathfrak{q}}(\sigma,\omega) \times \mathfrak{S}_{\mathfrak{q}}(\sigma,\omega)] \\ &+ \frac{1}{6}[Y_{\mathfrak{q}}(\mu,\omega) \times \mathfrak{S}_{\mathfrak{q}}(\sigma,\omega) + Y_{\mathfrak{q}}(\sigma,\omega) \times \mathfrak{S}_{\mathfrak{q}}(\mu,\omega)],\end{aligned} \tag{52}$$

and

$$\begin{aligned}\frac{1}{\nu-\varsigma} \int_{\varsigma}^{\nu} Y_{\mathfrak{q}}(\boldsymbol{\sigma},\omega) \times \mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\omega) d\omega \supseteq_I &\ \frac{1}{3}[Y_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma) \times \mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma) + Y_{\mathfrak{q}}(\boldsymbol{\sigma},\nu) \times \mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)] \\ &+ \frac{1}{6}[Y_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma) \times \mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma) + Y_{\mathfrak{q}}(\boldsymbol{\sigma},\nu) \times \mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)].\end{aligned} \tag{53}$$

Firstly, we will solve inequality Equation (52). Integrating both sides of the inequality with respect to ω on the interval $[\varsigma,\ \nu]$ and dividing both sides by $\nu-\varsigma$, we obtain

$$\begin{aligned} &\frac{1}{(\sigma-\mu)(\nu-\varsigma)}\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu} \Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\omega)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\omega)d\omega d\boldsymbol{\sigma} \\ \supseteq_I\ &\frac{1}{3(\nu-\varsigma)}\int_{\varsigma}^{\nu}[\Upsilon_{\mathfrak{q}}(\mu,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\omega)+\Upsilon_{\mathfrak{q}}(\sigma,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\omega)]d\omega \\ +&\frac{1}{6(\nu-\varsigma)}\int_{\varsigma}^{\nu}[\Upsilon_{\mathfrak{q}}(\mu,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\omega)+\Upsilon_{\mathfrak{q}}(\sigma,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\omega)]d\omega. \end{aligned} \tag{54}$$

Now, via inequality Equation (13), for each $\mathfrak{q}\in[0,\ 1]$, we obtain

$$\begin{aligned} \frac{1}{(\nu-\varsigma)}\int_{\varsigma}^{\nu} \Upsilon_{\mathfrak{q}}(\mu,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\omega)d\omega \supseteq_I \frac{1}{3}\int_{\varsigma}^{\nu}[\Upsilon_{\mathfrak{q}}(\mu,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\varsigma)+\Upsilon_{\mathfrak{q}}(\mu,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\nu)]d\omega \\ +\frac{1}{6}\int_{\varsigma}^{\nu}[\Upsilon_{\mathfrak{q}}(\mu,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\nu)+\Upsilon_{\mathfrak{q}}(\mu,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\nu)]d\omega. \end{aligned} \tag{55}$$

$$\begin{aligned} \frac{1}{(\nu-\varsigma)}\int_{\varsigma}^{\nu} \Upsilon_{\mathfrak{q}}(\sigma,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\omega)d\omega \supseteq_I \frac{1}{3}\int_{\varsigma}^{\nu}[\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\varsigma)+\Upsilon_{\mathfrak{q}}(\sigma,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\nu)]d\omega \\ +\frac{1}{6}\int_{\varsigma}^{\nu}[\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\nu)+\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\nu)]d\omega \end{aligned} \tag{56}$$

$$\begin{aligned} \frac{1}{(\nu-\varsigma)}\int_{\varsigma}^{\nu} \Upsilon_{\mathfrak{q}}(\mu,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\omega)d\omega \supseteq_I \frac{1}{3}\int_{\varsigma}^{\nu}[\Upsilon_{\mathfrak{q}}(\mu,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\varsigma)+\Upsilon_{\mathfrak{q}}(\mu,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\nu)]d\omega \\ +\frac{1}{6}\int_{\varsigma}^{\nu}[\Upsilon_{\mathfrak{q}}(\mu,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\nu)+\Upsilon_{\mathfrak{q}}(\mu,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\varsigma)]d\omega. \end{aligned} \tag{57}$$

$$\begin{aligned} \frac{1}{(\nu-\varsigma)}\int_{\varsigma}^{\nu} \Upsilon_{\mathfrak{q}}(\sigma,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\omega)d\omega \supseteq_I \frac{1}{3}\int_{\varsigma}^{\nu}[\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\varsigma)+\Upsilon_{\mathfrak{q}}(\sigma,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\nu)]d\omega \\ +\frac{1}{6}\int_{\varsigma}^{\nu}[\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\nu)+\Upsilon_{\mathfrak{q}}(\sigma,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\varsigma)]d\omega. \end{aligned} \tag{58}$$

From Equations (55)–(58) and inequality Equation (54), we obtain

$$\frac{1}{(\sigma-\mu)(\nu-\varsigma)}\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu} \Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\omega)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\omega)d\omega d\boldsymbol{\sigma} \supseteq_I \frac{1}{9}P_{\mathfrak{q}}(\mu,\sigma,\varsigma,\nu)+\frac{1}{18}\mathcal{M}_{\mathfrak{q}}(\mu,\sigma,\varsigma,\nu)+\frac{1}{36}\mathcal{N}_{\mathfrak{q}}(\mu,\sigma,\varsigma,\nu).$$

That is,

$$\frac{1}{(\sigma-\mu)(\nu-\varsigma)}\odot\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu} \widetilde{\Upsilon}(\boldsymbol{\sigma},\omega)\otimes\widetilde{\mathfrak{S}}(\boldsymbol{\sigma},\omega)d\omega d\boldsymbol{\sigma} \supseteq_{\mathbb{F}} \frac{1}{9}\odot\widetilde{P}(\mu,\sigma,\varsigma,\nu)\oplus\frac{1}{18}\odot\widetilde{\mathcal{M}}(\mu,\sigma,\varsigma,\nu)\oplus\frac{1}{36}\odot\widetilde{\mathcal{N}}(\mu,\sigma,\varsigma,\nu).$$

Hence, this concludes the proof of the theorem. □

Theorem 11. *Let $\widetilde{\Upsilon},\widetilde{\mathfrak{S}}\ :\Delta=[\mu,\ \sigma]\times[\varsigma,\ \nu]\subset\mathbb{R}^2\to\mathbb{E}_0$ be two UD-convex FN-V-Ms. Then, from the $\mathfrak{q}$-levels, we obtain the collection of IVMs $\Upsilon_{\mathfrak{q}},\ \mathfrak{S}_{\mathfrak{q}}:\Delta\subset\mathbb{R}^2\to\mathbb{R}_I^+$ given as $\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma})=[\Upsilon_*((\boldsymbol{\sigma},\omega),\mathfrak{q}),\ \Upsilon^*((\boldsymbol{\sigma},\omega),\mathfrak{q})]$ and $\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma})=[\mathfrak{S}_*((\boldsymbol{\sigma},\omega),\mathfrak{q}),\ \mathfrak{S}^*((\boldsymbol{\sigma},\omega),\mathfrak{q})]$ for all $(\boldsymbol{\sigma},\omega)\in\Delta$ and for all $\mathfrak{q}\in[0,\ 1]$. Then, the following inequality holds*:

$$\begin{aligned} &4\odot\widetilde{\Upsilon}\left(\frac{\mu+\sigma}{2},\frac{\varsigma+\nu}{2}\right)\otimes\widetilde{\mathfrak{S}}\left(\frac{\mu+\sigma}{2},\frac{\varsigma+\nu}{2}\right) \\ \supseteq_{\mathbb{F}}\ &\frac{1}{(\sigma-\mu)(\nu-\varsigma)}\odot\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu}\widetilde{\Upsilon}(\boldsymbol{\sigma},\omega)\otimes\widetilde{\mathfrak{S}}(\boldsymbol{\sigma},\omega)d\omega d\boldsymbol{\sigma}\oplus\frac{5}{36}\odot\widetilde{P}(\mu,\sigma,\varsigma,\nu)\oplus\frac{7}{36}\odot\widetilde{\mathcal{M}}(\mu,\sigma,\varsigma,\nu)\oplus\frac{2}{9}\odot\widetilde{\mathcal{N}}(\mu,\sigma,\varsigma,\nu). \end{aligned} \tag{59}$$

where $\widetilde{P}(\mu,\sigma,\varsigma,\nu)$, $\widetilde{\mathcal{M}}(\mu,\sigma,\varsigma,\nu)$, and $\widetilde{\mathcal{N}}(\mu,\sigma,\varsigma,\nu)$ are given in Theorem 10.

Proof. Since $\widetilde{\Upsilon},\widetilde{\mathfrak{S}}\ :\Delta\to\mathbb{E}_0$ are two UD-convex FN-V-Ms, then from inequality Equation (14) and for each $\mathfrak{q}\in[0,\ 1]$, we obtain

$$\begin{aligned} &2\Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\frac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\frac{\varsigma+\nu}{2}\right) \\ \supseteq_I\ &\frac{1}{\sigma-\mu}\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}\left(\boldsymbol{\sigma},\frac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathfrak{q}}\left(\boldsymbol{\sigma},\frac{\varsigma+\nu}{2}\right)d\boldsymbol{\sigma}+\frac{1}{6}\left[\Upsilon_{\mathfrak{q}}\left(\mu,\frac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathfrak{q}}\left(\mu,\frac{\varsigma+\nu}{2}\right)+\Upsilon_{\mathfrak{q}}\left(\sigma,\frac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathfrak{q}}\left(\sigma,\frac{\varsigma+\nu}{2}\right)\right] \\ +&\frac{1}{3}\left[\Upsilon_{\mathfrak{q}}\left(\mu,\frac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathfrak{q}}\left(\sigma,\frac{\varsigma+\nu}{2}\right)+\Upsilon_{\mathfrak{q}}\left(\sigma,\frac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathfrak{q}}\left(\mu,\frac{\varsigma+\nu}{2}\right)\right], \end{aligned} \tag{60}$$

and

$$\begin{aligned} 2\Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\frac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\frac{\varsigma+\nu}{2}\right)\supseteq_I \frac{1}{\nu-\varsigma}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\omega\right)\times\mathfrak{S}_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\omega\right)d\omega \\ +\frac{1}{6}\left[\Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\varsigma\right)\times\mathfrak{S}_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\varsigma\right)+\Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\nu\right)\times\mathfrak{S}_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\nu\right)\right] \\ +\frac{1}{3}\left[\Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\varsigma\right)\times\mathfrak{S}_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\nu\right)+\Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\nu\right)\times\mathfrak{S}_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\varsigma\right)\right]. \end{aligned} \tag{61}$$

Summing inequalities Equations (60) and (61) and then multiplying the result by 2, we obtain

$$
\begin{aligned}
&8Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\tfrac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\tfrac{\varsigma+\nu}{2}\right)\\
\supseteq_I\ &\tfrac{2}{\sigma-\mu}\int_\mu^\sigma Y_{\mathrm{q}}\left(\boldsymbol{\sigma},\tfrac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathrm{q}}\left(\boldsymbol{\sigma},\tfrac{\varsigma+\nu}{2}\right)d\boldsymbol{\sigma}+\tfrac{2}{\nu-\varsigma}\int_\varsigma^\nu Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\omega\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\omega\right)d\boldsymbol{\sigma}\\
&+\tfrac{1}{6}\left[2Y_{\mathrm{q}}\left(\mu,\tfrac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathrm{q}}\left(\mu,\tfrac{\varsigma+\nu}{2}\right)+2Y_{\mathrm{q}}\left(\sigma,\tfrac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathrm{q}}\left(\sigma,\tfrac{\varsigma+\nu}{2}\right)\right]\\
&+\tfrac{1}{6}\left[2Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\varsigma\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\varsigma\right)+2Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\nu\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\nu\right)\right]\\
&+\tfrac{1}{3}\left[2Y_{\mathrm{q}}\left(\mu,\tfrac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathrm{q}}\left(\sigma,\tfrac{\varsigma+\nu}{2}\right)+2Y_{\mathrm{q}}\left(\sigma,\tfrac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathrm{q}}\left(\mu,\tfrac{\varsigma+\nu}{2}\right)\right]\\
&+\tfrac{1}{3}\left[2Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\varsigma\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\nu\right)+2Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\nu\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\varsigma\right)\right].
\end{aligned}
\tag{62}
$$

Now, with the help of integral inequality Equation (14), for each integral on the right-hand side of Equation (62), we obtain

$$
\begin{aligned}
&2Y_{\mathrm{q}}\left(\mu,\tfrac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathrm{q}}\left(\mu,\tfrac{\varsigma+\nu}{2}\right)\\
\supseteq_I\ &\tfrac{1}{\nu-\varsigma}\int_\varsigma^\nu Y_{\mathrm{q}}(\mu,\omega)\times\mathfrak{S}_{\mathrm{q}}(\mu,\omega)d\omega+\tfrac{1}{6}[Y_{\mathrm{q}}(\mu,\varsigma)\times\mathfrak{S}_{\mathrm{q}}(\mu,\varsigma)+Y_{\mathrm{q}}(\mu,\nu)\times\mathfrak{S}_{\mathrm{q}}(\mu,\nu)]\\
&+\tfrac{1}{3}[Y_{\mathrm{q}}(\mu,\varsigma)\times\mathfrak{S}_{\mathrm{q}}(\mu,\nu)+Y_{\mathrm{q}}(\mu,\nu)\times\mathfrak{S}_{\mathrm{q}}(\mu,\varsigma)]
\end{aligned}
\tag{63}
$$

$$
\begin{aligned}
&2Y_{\mathrm{q}}\left(\sigma,\tfrac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathrm{q}}\left(\sigma,\tfrac{\varsigma+\nu}{2}\right)\\
\supseteq_I\ &\tfrac{1}{\nu-\varsigma}\int_\varsigma^\nu Y_{\mathrm{q}}(\sigma,\omega)\times\mathfrak{S}_{\mathrm{q}}(\sigma,\omega)d\omega+\tfrac{1}{6}[Y_{\mathrm{q}}(\sigma,\varsigma)\times\mathfrak{S}_{\mathrm{q}}(\sigma,\varsigma)+Y_{\mathrm{q}}(\sigma,\nu)\times\mathfrak{S}_{\mathrm{q}}(\sigma,\nu)]\\
&+\tfrac{1}{3}[Y_{\mathrm{q}}(\sigma,\varsigma)\times\mathfrak{S}_{\mathrm{q}}(\sigma,\nu)+Y_{\mathrm{q}}(\sigma,\nu)\times\mathfrak{S}_{\mathrm{q}}(\sigma,\varsigma)]
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
&2Y_{\mathrm{q}}\left(\mu,\tfrac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathrm{q}}\left(\sigma,\tfrac{\varsigma+\nu}{2}\right)\\
\supseteq_I\ &\tfrac{1}{\nu-\varsigma}\int_\varsigma^\nu Y_{\mathrm{q}}(\mu,\omega)\times\mathfrak{S}_{\mathrm{q}}(\sigma,\omega)d\omega+\tfrac{1}{6}[Y_{\mathrm{q}}(\mu,\varsigma)\times\mathfrak{S}_{\mathrm{q}}(\sigma,\varsigma)+Y_{\mathrm{q}}(\mu,\nu)\times\mathfrak{S}_{\mathrm{q}}(\sigma,\nu)]\\
&+\tfrac{1}{3}[Y_{\mathrm{q}}(\mu,\varsigma)\times\mathfrak{S}_{\mathrm{q}}(\sigma,\nu)+Y_{\mathrm{q}}(\mu,\nu)\times\mathfrak{S}_{\mathrm{q}}(\sigma,\varsigma)].
\end{aligned}
\tag{65}
$$

$$
\begin{aligned}
&2Y_{\mathrm{q}}\left(\sigma,\tfrac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathrm{q}}\left(\mu,\tfrac{\varsigma+\nu}{2}\right)\\
\supseteq_I\ &\tfrac{1}{\nu-\varsigma}\int_\varsigma^\nu Y_{\mathrm{q}}(\sigma,\omega)\times\mathfrak{S}_{\mathrm{q}}(\mu,\omega)d\omega+\tfrac{1}{6}[Y_{\mathrm{q}}(\sigma,\varsigma)\times\mathfrak{S}_{\mathrm{q}}(\mu,\varsigma)+Y_{\mathrm{q}}(\sigma,\nu)\times\mathfrak{S}_{\mathrm{q}}(\mu,\nu)]\\
&+\tfrac{1}{3}[Y_{\mathrm{q}}(\sigma,\varsigma)\times\mathfrak{S}_{\mathrm{q}}(\mu,\nu)+Y_{\mathrm{q}}(\sigma,\nu)\times\mathfrak{S}_{\mathrm{q}}(\mu,\varsigma)].
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
&2Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\varsigma\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\varsigma\right)\\
\supseteq_I\ &\tfrac{1}{\sigma-\mu}\int_\mu^\sigma Y_{\mathrm{q}}(\boldsymbol{\sigma},\varsigma)\times\mathfrak{S}_{\mathrm{q}}(\boldsymbol{\sigma},\varsigma)d\boldsymbol{\sigma}+\tfrac{1}{6}[Y_{\mathrm{q}}(\mu,\varsigma)\times\mathfrak{S}_{\mathrm{q}}(\mu,\varsigma)+Y_{\mathrm{q}}(\sigma,\varsigma)\times\mathfrak{S}_{\mathrm{q}}(\sigma,\varsigma)]\\
&+\tfrac{1}{3}\left[Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\varsigma\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\varsigma\right)+Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\varsigma\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\varsigma\right)\right]
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
&2Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\nu\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\nu\right)\\
\supseteq_I\ &\tfrac{1}{\sigma-\mu}\int_\mu^\sigma Y_{\mathrm{q}}(\boldsymbol{\sigma},\nu)\times\mathfrak{S}_{\mathrm{q}}(\boldsymbol{\sigma},\nu)d\boldsymbol{\sigma}+\tfrac{1}{6}[Y_{\mathrm{q}}(\mu,\nu)\times\mathfrak{S}_{\mathrm{q}}(\mu,\nu)+Y_{\mathrm{q}}(\sigma,\nu)\times\mathfrak{S}_{\mathrm{q}}(\sigma,\nu)]\\
&+\tfrac{1}{3}\left[Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\nu\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\nu\right)+Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\nu\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\nu\right)\right]
\end{aligned}
\tag{68}
$$

$$
\begin{aligned}
&2Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\varsigma\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\nu\right)\\
\supseteq_I\ &\tfrac{1}{\sigma-\mu}\int_\mu^\sigma Y_{\mathrm{q}}(\boldsymbol{\sigma},\varsigma)\times\mathfrak{S}_{\mathrm{q}}(\boldsymbol{\sigma},\nu)d\boldsymbol{\sigma}+\tfrac{1}{6}[Y_{\mathrm{q}}(\mu,\varsigma)\times\mathfrak{S}_{\mathrm{q}}(\mu,\nu)+Y_{\mathrm{q}}(\sigma,\varsigma)\times\mathfrak{S}_{\mathrm{q}}(\sigma,\nu)]\\
&+\tfrac{1}{3}\left[Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\varsigma\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\nu\right)+Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\varsigma\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\nu\right)\right].
\end{aligned}
\tag{69}
$$

$$
\begin{aligned}
&2Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\nu\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\varsigma\right)\\
\supseteq_I\ &\tfrac{1}{\sigma-\mu}\int_\mu^\sigma Y_{\mathrm{q}}(\boldsymbol{\sigma},\nu)\times\mathfrak{S}_{\mathrm{q}}(\boldsymbol{\sigma},\varsigma)d\boldsymbol{\sigma}+\tfrac{1}{6}[Y_{\mathrm{q}}(\mu,\nu)\times\mathfrak{S}_{\mathrm{q}}(\mu,\varsigma)+Y_{\mathrm{q}}(\sigma,\nu)\times\mathfrak{S}_{\mathrm{q}}(\sigma,\varsigma)]\\
&+\tfrac{1}{3}\left[Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\nu\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\varsigma\right)+Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\nu\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\varsigma\right)\right].
\end{aligned}
\tag{70}
$$

From Equations (63)–(70), we obtain

$$
\begin{aligned}
&8Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\tfrac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\tfrac{\varsigma+\nu}{2}\right)\\
\supseteq_I\ &\tfrac{2}{\sigma-\mu}\int_\mu^\sigma Y_{\mathrm{q}}\left(\boldsymbol{\sigma},\tfrac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathrm{q}}\left(\boldsymbol{\sigma},\tfrac{\varsigma+\nu}{2}\right)d\boldsymbol{\sigma}+\tfrac{2}{\nu-\varsigma}\int_\varsigma^\nu Y_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\omega\right)\times\mathfrak{S}_{\mathrm{q}}\left(\tfrac{\mu+\sigma}{2},\omega\right)d\boldsymbol{\sigma}\\
&+\tfrac{1}{6(\nu-\varsigma)}\int_\varsigma^\nu Y_{\mathrm{q}}(\mu,\omega)\times\mathfrak{S}_{\mathrm{q}}(\mu,\omega)d\omega+\tfrac{1}{6(\nu-\varsigma)}\int_\varsigma^\nu Y_{\mathrm{q}}(\sigma,\omega)\times\mathfrak{S}_{\mathrm{q}}(\sigma,\omega)d\omega\\
&+\tfrac{1}{6(\sigma-\mu)}\int_\mu^\sigma Y_{\mathrm{q}}(\boldsymbol{\sigma},\varsigma)\times\mathfrak{S}_{\mathrm{q}}(\boldsymbol{\sigma},\varsigma)d\boldsymbol{\sigma}+\tfrac{1}{6(\sigma-\mu)}\int_\mu^\sigma Y_{\mathrm{q}}(\boldsymbol{\sigma},\nu)\times\mathfrak{S}_{\mathrm{q}}(\boldsymbol{\sigma},\nu)d\boldsymbol{\sigma}\\
&+\tfrac{1}{3(\nu-\varsigma)}\int_\varsigma^\nu Y_{\mathrm{q}}(\mu,\omega)\times\mathfrak{S}_{\mathrm{q}}(\sigma,\omega)d\omega+\tfrac{1}{3(\nu-\varsigma)}\int_\varsigma^\nu Y_{\mathrm{q}}(\sigma,\omega)\times\mathfrak{S}_{\mathrm{q}}(\mu,\omega)d\omega\\
&+\tfrac{1}{3(\sigma-\mu)}\int_\mu^\sigma Y_{\mathrm{q}}(\boldsymbol{\sigma},\varsigma)\times\mathfrak{S}_{\mathrm{q}}(\boldsymbol{\sigma},\nu)d\boldsymbol{\sigma}+\tfrac{1}{3(\sigma-\mu)}\int_\mu^\sigma Y_{\mathrm{q}}(\boldsymbol{\sigma},\nu)\times\mathfrak{S}_{\mathrm{q}}(\boldsymbol{\sigma},\varsigma)d\boldsymbol{\sigma}\\
&+\tfrac{1}{18}P_{\mathrm{q}}(\mu,\sigma,\varsigma,\nu)+\tfrac{1}{9}\mathcal{M}_{\mathrm{q}}(\mu,\sigma,\varsigma,\nu)+\tfrac{2}{9}\mathcal{N}_{\mathrm{q}}(\mu,\sigma,\varsigma,\nu).
\end{aligned}
\tag{71}
$$

Now, with the help of integral inequality Equation (14) for the first two integrals on the right-hand side of Equation (71), we obtain the following relation:

$$\begin{aligned}&\tfrac{2}{\sigma-\mu}\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}\left(\boldsymbol{\sigma},\tfrac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathfrak{q}}\left(\boldsymbol{\sigma},\tfrac{\varsigma+\nu}{2}\right)d\boldsymbol{\sigma}\\ \supseteq_I&\ \tfrac{1}{(\sigma-\mu)(\nu-\varsigma)}\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\omega)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\omega)d\omega d\boldsymbol{\sigma}\\ &+\tfrac{1}{3(\sigma-\mu)}\int_{\mu}^{\sigma}[\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)+\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)]d\boldsymbol{\sigma}\\ &+\tfrac{1}{6(\sigma-\mu)}\int_{\mu}^{\sigma}[\Upsilon_{\mathfrak{q}}(\varsigma,\boldsymbol{\sigma})\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)+\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)]d\boldsymbol{\sigma},\end{aligned}\tag{72}$$

$$\begin{aligned}&\tfrac{2}{\nu-\varsigma}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}\left(\tfrac{\mu+\sigma}{2},\omega\right)\times\mathfrak{S}_{\mathfrak{q}}\left(\tfrac{\mu+\sigma}{2},\omega\right)d\boldsymbol{\sigma}\\ \supseteq_I&\ \tfrac{1}{(\sigma-\mu)(\nu-\varsigma)}\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\omega)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\omega)d\omega d\boldsymbol{\sigma}\\ &+\tfrac{1}{3(\nu-\varsigma)}\int_{\varsigma}^{\nu}[\Upsilon_{\mathfrak{q}}(\mu,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\omega)+\Upsilon_{\mathfrak{q}}(\sigma,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\omega)]d\omega\\ &+\tfrac{1}{6(\nu-\varsigma)}\int_{\varsigma}^{\nu}[\Upsilon_{\mathfrak{q}}(\mu,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\omega)+\Upsilon_{\mathfrak{q}}(\sigma,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\omega)]d\omega.\end{aligned}\tag{73}$$

From Equations (72) and (73), we obtain

$$\begin{aligned}&8\Upsilon_{\mathfrak{q}}\left(\tfrac{\mu+\sigma}{2},\tfrac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathfrak{q}}\left(\tfrac{\mu+\sigma}{2},\tfrac{\varsigma+\nu}{2}\right)\\ \supseteq_I&\ \tfrac{1}{(\sigma-\mu)(\nu-\varsigma)}\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\omega)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\omega)d\omega d\boldsymbol{\sigma}\\ &+\tfrac{1}{3(\sigma-\mu)}\int_{\mu}^{\sigma}[\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)+\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)]d\boldsymbol{\sigma}\\ &+\tfrac{1}{6(\sigma-\mu)}\int_{\mu}^{\sigma}[\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)+\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)]d\boldsymbol{\sigma}\\ &+\tfrac{1}{(\sigma-\mu)(\nu-\varsigma)}\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\omega)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\omega)d\omega d\boldsymbol{\sigma}\\ &+\tfrac{1}{3(\nu-\varsigma)}\int_{\varsigma}^{\nu}[\Upsilon_{\mathfrak{q}}(\mu,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\omega)+\Upsilon_{\mathfrak{q}}(\sigma,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\omega)]d\omega\\ &+\tfrac{1}{6(\nu-\varsigma)}\int_{\varsigma}^{\nu}[\Upsilon_{\mathfrak{q}}(\mu,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\omega)+\Upsilon_{\mathfrak{q}}(\sigma,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\omega)]d\omega\\ &+\tfrac{1}{6(\nu-\varsigma)}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\mu,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\omega)d\omega+\tfrac{1}{6(\nu-\varsigma)}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\sigma,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\omega)d\omega\\ &+\tfrac{1}{6(\sigma-\mu)}\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)d\boldsymbol{\sigma}+\tfrac{1}{6(\sigma-\mu)}\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)d\boldsymbol{\sigma}\\ &+\tfrac{1}{3(\nu-\varsigma)}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\mu,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\omega)d\omega+\tfrac{1}{3(\nu-\varsigma)}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\sigma,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\omega)d\omega\\ &+\tfrac{1}{3(\sigma-\mu)}\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)d\boldsymbol{\sigma}+\tfrac{1}{3(\sigma-\mu)}\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)d\boldsymbol{\sigma}\\ &+\tfrac{1}{18}P_{\mathfrak{q}}(\mu,\sigma,\varsigma,\nu)+\tfrac{1}{9}\mathcal{M}_{\mathfrak{q}}(\mu,\sigma,\varsigma,\nu)+\tfrac{2}{9}\mathcal{N}_{\mathfrak{q}}(\mu,\sigma,\varsigma,\nu).\end{aligned}$$

It follows that

$$\begin{aligned}&8\Upsilon_{\mathfrak{q}}\left(\tfrac{\mu+\sigma}{2},\tfrac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathfrak{q}}\left(\tfrac{\mu+\sigma}{2},\tfrac{\varsigma+\nu}{2}\right)\\ \supseteq_I&\ \tfrac{2}{(\sigma-\mu)(\nu-\varsigma)}\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\omega)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\omega)d\omega d\boldsymbol{\sigma}\\ &+\tfrac{2}{3(\sigma-\mu)}\int_{\mu}^{\sigma}[\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)+\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)]d\boldsymbol{\sigma}\\ &+\tfrac{1}{3(\sigma-\mu)}\int_{\mu}^{\sigma}[\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)+\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)]d\boldsymbol{\sigma}\\ &+\tfrac{2}{3(\nu-\varsigma)}\int_{\varsigma}^{\nu}[\Upsilon_{\mathfrak{q}}(\mu,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\omega)+\Upsilon_{\mathfrak{q}}(\sigma,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\omega)]d\omega\\ &+\tfrac{1}{3(\nu-\varsigma)}\int_{\varsigma}^{\nu}[\Upsilon_{\mathfrak{q}}(\mu,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\omega)+\Upsilon_{\mathfrak{q}}(\sigma,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\omega)]d\omega\\ &+\tfrac{1}{18}P_{\mathfrak{q}}(\mu,\sigma,\varsigma,\nu)+\tfrac{1}{9}\mathcal{M}_{\mathfrak{q}}(\mu,\sigma,\varsigma,\nu)+\tfrac{2}{9}\mathcal{N}_{\mathfrak{q}}(\mu,\sigma,\varsigma,\nu).\end{aligned}\tag{74}$$

Now, using integral inequality Equation (13) for the integrals on the right-hand side of Equation (74), we obtain the following relation

$$\begin{aligned}&\tfrac{1}{\sigma-\mu}\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)d\boldsymbol{\sigma}\\ \supseteq_I&\ \tfrac{1}{3}[\Upsilon_{\mathfrak{q}}(\mu,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\varsigma)+\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\varsigma)]+\tfrac{1}{6}[\Upsilon_{\mathfrak{q}}(\mu,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\varsigma)+\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\varsigma)],\end{aligned}\tag{75}$$

$$\begin{aligned}&\tfrac{1}{\sigma-\mu}\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)d\boldsymbol{\sigma}\\ \supseteq_I&\ \tfrac{1}{3}[\Upsilon_{\mathfrak{q}}(\mu,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\nu)+\Upsilon_{\mathfrak{q}}(\sigma,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\nu)]+\tfrac{1}{6}[\Upsilon_{\mathfrak{q}}(\mu,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\nu)+\Upsilon_{\mathfrak{q}}(\sigma,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\nu)]\end{aligned}\tag{76}$$

$$\begin{aligned}&\tfrac{1}{\sigma-\mu}\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)d\boldsymbol{\sigma}\\ \supseteq_I&\ \tfrac{1}{3}[\Upsilon_{\mathfrak{q}}(\mu,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\nu)+\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\nu)]+\tfrac{1}{6}[\Upsilon_{\mathfrak{q}}(\mu,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\nu)+\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\nu)],\end{aligned}\tag{77}$$

$$\begin{aligned}&\tfrac{1}{\sigma-\mu}\int_{\mu}^{\sigma}\Upsilon_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)d\boldsymbol{\sigma}\\ \supseteq_I&\ \tfrac{1}{3}[\Upsilon_{\mathfrak{q}}(\mu,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\varsigma)+\Upsilon_{\mathfrak{q}}(\sigma,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\varsigma)]+\tfrac{1}{6}[\Upsilon_{\mathfrak{q}}(\mu,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\varsigma)+\Upsilon_{\mathfrak{q}}(\sigma,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\varsigma)],\end{aligned}\tag{78}$$

$$\begin{aligned}&\tfrac{1}{\nu-\varsigma}\int_{\varsigma}^{\nu}\Upsilon_{\mathfrak{q}}(\mu,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\omega)d\omega\\ \supseteq_I&\ \tfrac{1}{3}[\Upsilon_{\mathfrak{q}}(\mu,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\varsigma)+\Upsilon_{\mathfrak{q}}(\mu,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\nu)]+\tfrac{1}{6}[\Upsilon_{\mathfrak{q}}(\mu,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\nu)+\Upsilon_{\mathfrak{q}}(\mu,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\varsigma)],\end{aligned}\tag{79}$$

$$\begin{aligned}&\frac{1}{\nu-\varsigma}\int_{\varsigma}^{\nu} Y_{\mathfrak{q}}(\sigma,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\omega)d\omega\\ \supseteq_I\ &\frac{1}{3}[Y_{\mathfrak{q}}(\sigma,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\varsigma)+Y_{\mathfrak{q}}(\sigma,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\nu)]+\frac{1}{6}[Y_{\mathfrak{q}}(\sigma,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\nu)+Y_{\mathfrak{q}}(\sigma,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\varsigma)],\end{aligned} \tag{80}$$

$$\begin{aligned}&\frac{1}{\nu-\varsigma}\int_{\varsigma}^{\nu} Y_{\mathfrak{q}}(\mu,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\omega)d\omega\\ \supseteq_I\ &\frac{1}{3}[Y_{\mathfrak{q}}(\mu,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\varsigma)+Y_{\mathfrak{q}}(\mu,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\nu)]+\frac{1}{6}[Y_{\mathfrak{q}}(\mu,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\nu)+Y_{\mathfrak{q}}(\mu,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\sigma,\varsigma)],\end{aligned} \tag{81}$$

$$\begin{aligned}&\frac{1}{\nu-\varsigma}\int_{\varsigma}^{\nu} Y_{\mathfrak{q}}(\sigma,\omega)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\omega)d\omega\\ \supseteq_I\ &\frac{1}{3}[Y_{\mathfrak{q}}(\sigma,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\varsigma)+Y_{\mathfrak{q}}(\sigma,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\nu)]+\frac{1}{6}[Y_{\mathfrak{q}}(\sigma,\varsigma)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\nu)+Y_{\mathfrak{q}}(\sigma,\nu)\times\mathfrak{S}_{\mathfrak{q}}(\mu,\varsigma)].\end{aligned} \tag{82}$$

From Equations (75)–(82) and inequality Equation (74), we obtain

$$\begin{aligned}&4Y_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\frac{\varsigma+\nu}{2}\right)\times\mathfrak{S}_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\frac{\varsigma+\nu}{2}\right)\\ \supseteq_I\ &\frac{1}{(\sigma-\mu)(\nu-\varsigma)}\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu} Y_{\mathfrak{q}}(\boldsymbol{\sigma},\omega)\times\mathfrak{S}_{\mathfrak{q}}(\boldsymbol{\sigma},\omega)d\omega d\boldsymbol{\sigma}+\frac{5}{36}P_{\mathfrak{q}}(\mu,\sigma,\varsigma,\nu)+\frac{7}{36}\mathcal{M}_{\mathfrak{q}}(\mu,\sigma,\varsigma,\nu)+\frac{2}{9}\mathcal{N}_{\mathfrak{q}}(\mu,\sigma,\varsigma,\nu)\end{aligned}$$

That is,

$$\begin{aligned}&4\widetilde{Y}\left(\frac{\mu+\sigma}{2},\frac{\varsigma+\nu}{2}\right)\otimes\widetilde{\mathfrak{S}}\left(\frac{\mu+\sigma}{2},\frac{\varsigma+\nu}{2}\right)\\ \supseteq_{\mathbb{F}}\ &\frac{1}{(\sigma-\mu)(\nu-\varsigma)}\odot\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu}\widetilde{Y}(\boldsymbol{\sigma},\omega)\otimes\widetilde{\mathfrak{S}}(\boldsymbol{\sigma},\omega)d\omega d\boldsymbol{\sigma}\oplus\frac{5}{36}\odot\widetilde{P}(\mu,\sigma,\varsigma,\nu)\oplus\frac{7}{36}\odot\widetilde{\mathcal{M}}(\mu,\sigma,\varsigma,\nu)\oplus\frac{2}{9}\odot\widetilde{\mathcal{N}}(\mu,\sigma,\varsigma,\nu).\end{aligned}$$

We will now obtain the HH–Fejér inequality for coordinated UD-convex FN-V-Ms by means of FOR in the following result. □

Theorem 12. *Let $\widetilde{Y}:\Delta=[\mu,\ \sigma]\times[\varsigma,\ \nu]\to\mathbb{E}_0$ be a coordinated UD-convex FN-V-M with $\mu<\sigma$ and $\varsigma<\nu$. Then, from the $\mathfrak{q}$-levels, we obtain the collection of IVMs $Y_{\mathfrak{q}}:\Delta\to\mathbb{R}_I^+$ given as $Y_{\mathfrak{q}}(\boldsymbol{\sigma},\omega)=[Y_*((\boldsymbol{\sigma},\omega),\mathfrak{q}),\ Y^*((\boldsymbol{\sigma},\omega),\mathfrak{q})]$ for all $(\boldsymbol{\sigma},\omega)\in\Delta$ and for all $\mathfrak{q}\in[0,\ 1]$. Let $\Omega:[\mu,\ \sigma]\to\mathbb{R}$ with $\Omega(\boldsymbol{\sigma})\geq 0$, $\int_{\mu}^{\sigma}\Omega(\boldsymbol{\sigma})d\boldsymbol{\sigma}>0$, and $\mathcal{W}:[\varsigma,\ \nu]\to\mathbb{R}$ with $\mathcal{W}(\omega)\geq 0$, $\int_{\varsigma}^{\nu}\mathcal{W}(\omega)d\omega>0$, be two symmetric functions with respect to $\frac{\mu+\sigma}{2}$ and $\frac{\varsigma+\nu}{2}$, respectively. Then, the following inequality holds*:

$$\begin{aligned}\widetilde{Y}\left(\frac{\mu+\sigma}{2},\frac{\varsigma+\nu}{2}\right)&\supseteq_{\mathbb{F}}\frac{1}{2}\left[\frac{1}{\int_{\mu}^{\sigma}\Omega(\boldsymbol{\sigma})d\boldsymbol{\sigma}}\odot\int_{\mu}^{\sigma}\widetilde{Y}\left(\boldsymbol{\sigma},\frac{\varsigma+\nu}{2}\right)\Omega(\boldsymbol{\sigma})d\boldsymbol{\sigma}\oplus\frac{1}{\int_{\mu}^{\sigma}\mathcal{W}(\omega)d\omega}\odot\int_{\mu}^{\sigma}\widetilde{Y}\left(\frac{\mu+\sigma}{2},\omega\right)\mathcal{W}(\omega)d\omega\right]\\ &\supseteq_{\mathbb{F}}\frac{1}{\int_{\mu}^{\sigma}\Omega(\boldsymbol{\sigma})d\boldsymbol{\sigma}\int_{\mu}^{\sigma}\mathcal{W}(\omega)d\omega}\odot\int_{\mu}^{\sigma}\int_{\varsigma}^{\nu}\widetilde{Y}(\boldsymbol{\sigma},\omega)\Omega(\boldsymbol{\sigma})\mathcal{W}(\omega)d\omega d\boldsymbol{\sigma}\\ &\supseteq_{\mathbb{F}}\frac{1}{4\int_{\mu}^{\sigma}\Omega(\boldsymbol{\sigma})d\boldsymbol{\sigma}}\odot\left[\int_{\mu}^{\sigma}\widetilde{Y}(\boldsymbol{\sigma},\varsigma)d\boldsymbol{\sigma}\oplus\int_{\mu}^{\sigma}\widetilde{Y}(\boldsymbol{\sigma},\nu)d\boldsymbol{\sigma}\right]\\ &\oplus\frac{1}{4\int_{\mu}^{\sigma}\mathcal{W}(\omega)d\omega}\odot\left[\int_{\varsigma}^{\nu}\widetilde{Y}(\mu,\omega)d\omega\oplus\int_{\varsigma}^{\nu}\widetilde{Y}(\sigma,\omega)d\omega\right]\\ &\supseteq_{\mathbb{F}}\frac{\widetilde{Y}(\mu,\varsigma)\oplus\widetilde{Y}(\sigma,\varsigma)\oplus\widetilde{Y}(\mu,\nu)\oplus\widetilde{Y}(\sigma,\nu)}{4}\end{aligned} \tag{83}$$

Proof. Since $\widetilde{Y}$ is a coordinated UD-convex FN-V-M on Δ, and it follows those functions, then via Lemma 1, the following exist:

$$\widetilde{Y}_{\boldsymbol{\sigma}}:[\varsigma,\nu]\to\mathbb{E}_0,\ \widetilde{Y}_{\boldsymbol{\sigma}}(\omega)=\widetilde{Y}(\boldsymbol{\sigma},\omega),\ \widetilde{Y}_{\omega}:[\mu,\sigma]\to\mathbb{E}_0,\ \widetilde{Y}_{\omega}(\boldsymbol{\sigma})=\widetilde{Y}(\boldsymbol{\sigma},\omega).$$

Thus, from inequality Equation (15), for each $\mathfrak{q}\in[0,\ 1]$, we obtain

$$Y_{\mathfrak{q}\boldsymbol{\sigma}}\left(\frac{\varsigma+\nu}{2}\right)\supseteq_I\frac{1}{\int_{\varsigma}^{\nu}\mathcal{W}(\omega)d\omega}\int_{\varsigma}^{\nu}Y_{\mathfrak{q}\boldsymbol{\sigma}}(\omega)\mathcal{W}(\omega)d\omega\supseteq_I\frac{Y_{\mathfrak{q}\boldsymbol{\sigma}}(\varsigma)+Y_{\mathfrak{q}\boldsymbol{\sigma}}(\nu)}{2},$$

and

$$Y_{\mathfrak{q}_{\omega}}\left(\frac{\mu+\sigma}{2}\right)\supseteq_I\frac{1}{\int_{\mu}^{\sigma}\Omega(\boldsymbol{\sigma})d\boldsymbol{\sigma}}\int_{\mu}^{\sigma}Y_{\mathfrak{q}_{\omega}}(\boldsymbol{\sigma})\Omega(\boldsymbol{\sigma})d\boldsymbol{\sigma}\supseteq_I\frac{Y_{\mathfrak{q}_{\omega}}(\mu)+Y_{\mathfrak{q}_{\omega}}(\sigma)}{2}$$

The above inequalities can be written as

$$Y_{\mathfrak{q}}\left(\boldsymbol{\sigma},\frac{\varsigma+\nu}{2}\right)\supseteq_I\frac{1}{\int_{\varsigma}^{\nu}\mathcal{W}(\omega)d\omega}\int_{\varsigma}^{\nu}Y_{\mathfrak{q}}(\boldsymbol{\sigma},\omega)\mathcal{W}(\omega)d\omega\supseteq_I\frac{Y_{\mathfrak{q}}(\boldsymbol{\sigma},\varsigma)+Y_{\mathfrak{q}}(\boldsymbol{\sigma},\nu)}{2}, \tag{84}$$

and

$$\Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\omega\right)\supseteq_I \frac{1}{\int_\mu^\sigma \Omega(\sigma)d\sigma}\int_\mu^\sigma \Upsilon_{\mathfrak{q}}(\sigma,\omega)\Omega(\sigma)d\sigma \supseteq_I \frac{\Upsilon_{\mathfrak{q}}(\mu,\omega)+\Upsilon_{\mathfrak{q}}(\sigma,\omega)}{2}. \tag{85}$$

Multiplying Equation (84) by $\Omega(\sigma)$ and then integrating the result with respect to σ on $[\mu,\ \sigma]$, we obtain

$$\int_\mu^\sigma \Upsilon_{\mathfrak{q}}\left(\sigma,\frac{\varsigma+\nu}{2}\right)\Omega(\sigma)d\sigma \supseteq_I \frac{1}{\int_\varsigma^\nu \mathcal{W}(\omega)d\omega}\int_\mu^\sigma\int_\varsigma^\nu \Upsilon_{\mathfrak{q}}(\sigma,\omega)\Omega(\sigma)\mathcal{W}(\omega)d\omega d\sigma \supseteq_I \int_\mu^\sigma \frac{\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)+\Upsilon_{\mathfrak{q}}(\sigma,\nu)}{2}\Omega(\sigma)d\sigma. \tag{86}$$

Now, multiplying Equation (85) by $\mathcal{W}(\omega)$ and then integrating the result with respect to ω on $[\varsigma,\ \nu]$, we obtain

$$\int_\varsigma^\nu \Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\omega\right)\mathcal{W}(\omega)d\omega \supseteq_I \frac{1}{\int_\mu^\sigma \Omega(\sigma)d\sigma}\int_\mu^\sigma\int_\varsigma^\nu \Upsilon_{\mathfrak{q}}(\sigma,\omega)\Omega(\sigma)\mathcal{W}(\omega)d\sigma d\omega \supseteq_I \int_\mu^\sigma \frac{\Upsilon_{\mathfrak{q}}(\mu,\omega)+\Upsilon_{\mathfrak{q}}(\sigma,\omega)}{2}\mathcal{W}(\omega)d\omega \tag{87}$$

Since $\int_\mu^\sigma \Omega(\sigma)d\sigma > 0$ and $\int_\mu^\sigma \mathcal{W}(\omega)d\omega > 0$, then by dividing Equations (86) and (87) by $\int_\mu^\sigma \Omega(\sigma)d\sigma > 0$ and $\int_\mu^\sigma \mathcal{W}(\omega)d\omega > 0$, respectively, we obtain

$$\begin{gathered}\frac{1}{2}\left[\frac{1}{\int_\mu^\sigma \Omega(\sigma)d\sigma}\int_\mu^\sigma \Upsilon_{\mathfrak{q}}\left(\sigma,\frac{\varsigma+\nu}{2}\right)\Omega(\sigma)d\sigma+\frac{1}{\int_\mu^\sigma \mathcal{W}(\omega)d\omega}\int_\mu^\sigma \Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\omega\right)\mathcal{W}(\omega)d\omega\right]\\ \supseteq_I \frac{1}{\int_\mu^\sigma \Omega(\sigma)d\sigma\int_\mu^\sigma \mathcal{W}(\omega)d\omega}\int_\mu^\sigma\int_\varsigma^\nu \Upsilon_{\mathfrak{q}}(\sigma,\omega)\Omega(\sigma)\mathcal{W}(\omega)d\omega d\sigma\\ \supseteq_I \left[\frac{1}{\int_\mu^\sigma \Omega(\sigma)d\sigma}\int_\mu^\sigma \frac{\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)+\Upsilon_{\mathfrak{q}}(\sigma,\nu)}{4}\Omega(\sigma)d\sigma+\frac{1}{\int_\varsigma^\nu \mathcal{W}(\omega)d\omega}\int_\mu^\sigma \frac{\Upsilon_{\mathfrak{q}}(\mu,\omega)+\Upsilon_{\mathfrak{q}}(\sigma,\omega)}{4}\mathcal{W}(\omega)d\omega\right].\end{gathered} \tag{88}$$

Now, from the left part of double inequalities Equations (84) and (85), we obtain

$$\Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\frac{\varsigma+\nu}{2}\right)\supseteq_I \frac{1}{\int_\varsigma^\nu \mathcal{W}(\omega)d\omega}\int_\varsigma^\nu \Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\omega\right)\mathcal{W}(\omega)d\omega, \tag{89}$$

and

$$\Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\frac{\varsigma+\nu}{2}\right)\supseteq_I \frac{1}{\int_\mu^\sigma \Omega(\sigma)d\sigma}\int_\mu^\sigma \Upsilon_{\mathfrak{q}}\left(\sigma,\frac{\varsigma+\nu}{2}\right)\Omega(\sigma)d\sigma \tag{90}$$

Summing inequalities Equations (89) and (90), we obtain

$$\Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\frac{\varsigma+\nu}{2}\right)\supseteq_I \frac{1}{2}\left[\frac{1}{\int_\mu^\sigma \Omega(\sigma)d\sigma}\int_\mu^\sigma \Upsilon_{\mathfrak{q}}\left(\sigma,\frac{\varsigma+\nu}{2}\right)\Omega(\sigma)d\sigma+\frac{1}{\int_\varsigma^\nu \mathcal{W}(\omega)d\omega}\int_\varsigma^\nu \Upsilon_{\mathfrak{q}}\left(\frac{\mu+\sigma}{2},\omega\right)\mathcal{W}(\omega)d\omega\right]. \tag{91}$$

Similarly, from the right part of Equations (84) and (85), we obtain

$$\frac{1}{\int_\varsigma^\nu \mathcal{W}(\omega)d\omega}\int_\varsigma^\nu \Upsilon_{\mathfrak{q}}(\mu,\omega)\mathcal{W}(\omega)d\omega \supseteq_I \frac{\Upsilon_{\mathfrak{q}}(\mu,\varsigma)+\Upsilon_{\mathfrak{q}}(\mu,\ \nu)}{2}, \tag{92}$$

$$\frac{1}{\int_\varsigma^\nu \mathcal{W}(\omega)d\omega}\int_\varsigma^\nu \Upsilon_{\mathfrak{q}}(\sigma,\omega)\mathcal{W}(\omega)d\omega \supseteq_I \frac{\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)+\Upsilon_{\mathfrak{q}}(\sigma,\ \nu)}{2}, \tag{93}$$

and

$$\frac{1}{\int_\mu^\sigma \Omega(\sigma)d\sigma}\int_\mu^\sigma \Upsilon_{\mathfrak{q}}(\sigma,\varsigma)\Omega(\sigma)d\sigma \supseteq_I \frac{\Upsilon_{\mathfrak{q}}(\mu,\varsigma)+\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)}{2}. \tag{94}$$

$$\frac{1}{\int_\mu^\sigma \Omega(\sigma)d\sigma}\int_\mu^\sigma \Upsilon_{\mathfrak{q}}(\sigma,\nu)\Omega(\sigma)d\sigma \supseteq_I \frac{\Upsilon_{\mathfrak{q}}(\mu,\nu)+\Upsilon_{\mathfrak{q}}(\sigma,\nu)}{2}. \tag{95}$$

Adding Equations (92)–(95) and dividing by 4, we obtain

$$\begin{gathered}\frac{1}{4\int_\varsigma^\nu \mathcal{W}(\omega)d\omega}\left[\int_\varsigma^\nu \Upsilon_{\mathfrak{q}}(\mu,\omega)\mathcal{W}(\omega)d\omega+\int_\varsigma^\nu \Upsilon_{\mathfrak{q}}(\sigma,\omega)\mathcal{W}(\omega)d\omega\right]+\frac{1}{4\int_\mu^\sigma \Omega(\sigma)d\sigma}\left[\int_\mu^\sigma \Upsilon_{\mathfrak{q}}(\sigma,\varsigma)\Omega(\sigma)d\sigma+\int_\mu^\sigma \Upsilon_{\mathfrak{q}}(\sigma,\nu)\Omega(\sigma)d\sigma\right]\\ \supseteq_I \frac{\Upsilon_{\mathfrak{q}}(\mu,\varsigma)+\Upsilon_{\mathfrak{q}}(\mu,\ \nu)+\Upsilon_{\mathfrak{q}}(\sigma,\varsigma)+\Upsilon_{\mathfrak{q}}(\sigma,\ \nu)}{4}\end{gathered} \tag{96}$$

Combining inequalities Equations (88), (91), and (96), we obtain

$$\begin{aligned} Y_{\mathfrak{q}}\left(\tfrac{\mu+\sigma}{2}, \tfrac{\varsigma+\nu}{2}\right) &\supseteq_I \tfrac{1}{2}\left[\tfrac{1}{\int_\mu^\sigma \Omega(\sigma)d\sigma}\int_\mu^\sigma Y_{\mathfrak{q}}\left(\sigma, \tfrac{\varsigma+\nu}{2}\right)\Omega(\sigma)d\sigma + \tfrac{1}{\int_\varsigma^\nu \mathcal{W}(\omega)d\omega}\int_\varsigma^\nu Y_{\mathfrak{q}}\left(\tfrac{\mu+\sigma}{2},\omega\right)\mathcal{W}(\omega)d\omega\right]. \\ &\supseteq_I \tfrac{1}{\int_\mu^\sigma \Omega(\sigma)d\sigma\int_\mu^\sigma \mathcal{W}(\omega)d\omega}\int_\mu^\sigma\int_\varsigma^\nu Y(\sigma,\omega)\Omega(\sigma)\mathcal{W}(\omega)d\omega d\sigma \\ &\supseteq_I \tfrac{1}{4\int_\varsigma^\nu \mathcal{W}(\omega)d\omega}\left[\int_\varsigma^\nu Y_{\mathfrak{q}}(\mu,\omega)\mathcal{W}(\omega)d\omega + \int_\varsigma^\nu Y_{\mathfrak{q}}(\sigma,\omega)\mathcal{W}(\omega)d\omega\right] \\ &+\tfrac{1}{4\int_\mu^\sigma \Omega(\sigma)d\sigma}\left[\int_\mu^\sigma Y_{\mathfrak{q}}(\sigma,\varsigma)\Omega(\sigma)d\sigma + \int_\mu^\sigma Y_{\mathfrak{q}}(\sigma,\nu)\Omega(\sigma)d\sigma\right] \\ &\supseteq_I \tfrac{Y_{\mathfrak{q}}(\mu,\varsigma)+Y_{\mathfrak{q}}(\mu,\nu)}{2} + \tfrac{Y_{\mathfrak{q}}(\sigma,\varsigma)+Y_{\mathfrak{q}}(\sigma,\nu)}{2} + \tfrac{Y_{\mathfrak{q}}(\mu,\varsigma)+Y_{\mathfrak{q}}(\sigma,\varsigma)}{2} + \tfrac{Y_{\mathfrak{q}}(\mu,\nu)+Y_{\mathfrak{q}}(\sigma,\nu)}{2}. \end{aligned}$$

That is,

$$\begin{aligned} \widetilde{Y}\left(\tfrac{\mu+\sigma}{2}, \tfrac{\varsigma+\nu}{2}\right) &\supseteq_{\mathbb{F}} \tfrac{1}{2}\left[\tfrac{1}{\int_\mu^\sigma \Omega(\sigma)d\sigma}\odot\int_\mu^\sigma \widetilde{Y}\left(\sigma, \tfrac{\varsigma+\nu}{2}\right)\Omega(\sigma)d\sigma \oplus \tfrac{1}{\int_\mu^\sigma \mathcal{W}(\omega)d\omega}\odot\int_\mu^\sigma \widetilde{Y}\left(\tfrac{\mu+\sigma}{2},\omega\right)\mathcal{W}(\omega)d\omega\right] \\ &\supseteq_{\mathbb{F}} \tfrac{1}{\int_\mu^\sigma \Omega(\sigma)d\sigma\int_\mu^\sigma \mathcal{W}(\omega)d\omega}\odot\int_\mu^\sigma\int_\varsigma^\nu \widetilde{Y}(\sigma,\omega)\Omega(\sigma)\mathcal{W}(\omega)d\omega d\sigma \\ &\supseteq_{\mathbb{F}} \tfrac{1}{4\int_\mu^\sigma \Omega(\sigma)d\sigma}\odot\left[\int_\mu^\sigma \widetilde{Y}(\sigma,\varsigma)d\sigma \oplus \int_\mu^\sigma \widetilde{Y}(\sigma,\nu)d\sigma\right] \\ &\oplus\tfrac{1}{4\int_\mu^\sigma \mathcal{W}(\omega)d\omega}\odot\left[\int_\varsigma^\nu \widetilde{Y}(\mu,\omega)d\omega \oplus \int_\varsigma^\nu \widetilde{Y}(\sigma,\omega)d\omega\right] \\ &\supseteq_{\mathbb{F}} \tfrac{\widetilde{Y}(\mu,\varsigma)\oplus\widetilde{Y}(\sigma,\varsigma)\oplus\widetilde{Y}(\mu,\nu)\oplus\widetilde{Y}(\sigma,\nu)}{4}, \end{aligned}$$

Hence, this concludes the proof. □

Remark 3. *From inequality Equation (56), the following exceptional results can be acquired*:

If $\mathcal{W}(\omega) = 1 = \Omega(\sigma)$, one can then obtain inequality Equation (36).

Let $Y_*((\sigma,\omega),\mathfrak{q}) \neq Y^*((\sigma,\omega),\mathfrak{q})$ with $\mathfrak{q} = 1$. Then, one can derive following inclusion [61]:

$$\begin{aligned} Y\left(\tfrac{\mu+\sigma}{2}, \tfrac{\varsigma+\nu}{2}\right) &\supseteq \tfrac{1}{2}\left[\tfrac{1}{\int_\mu^\sigma \Omega(\sigma)d\sigma}\int_\mu^\sigma Y\left(\sigma, \tfrac{\varsigma+\nu}{2}\right)\Omega(\sigma)d\sigma + \tfrac{1}{\int_\mu^\sigma \mathcal{W}(\omega)d\omega}\int_\mu^\sigma Y\left(\tfrac{\mu+\sigma}{2},\omega\right)\mathcal{W}(\omega)d\omega\right] \\ &\supseteq \tfrac{1}{\int_\mu^\sigma \Omega(\sigma)d\sigma\int_\mu^\sigma \mathcal{W}(\omega)d\omega}\int_\mu^\sigma\int_\varsigma^\nu Y(\sigma,\omega)\Omega(\sigma)\mathcal{W}(\omega)d\omega d\sigma \\ &\supseteq \tfrac{1}{4\int_\mu^\sigma \Omega(\sigma)d\sigma}\left[\int_\mu^\sigma Y(\sigma,\varsigma)d\sigma + \int_\mu^\sigma Y(\sigma,\nu)d\sigma\right] \\ &+\tfrac{1}{4\int_\mu^\sigma \mathcal{W}(\omega)d\omega}\left[\int_\varsigma^\nu Y(\mu,\omega)d\omega + \int_\varsigma^\nu Y(\sigma,\omega)d\omega\right] \\ &\tfrac{Y(\mu,\varsigma)+Y(\sigma,\varsigma)+Y(\mu,\nu)+Y(\sigma,\nu)}{4}. \end{aligned} \tag{97}$$

Let $\widetilde{Y}$ be a left coordinated UD-convex FN-V-M. Then, we can achieve the following outcome (see [91]*):*

$$\begin{aligned} \widetilde{Y}\left(\tfrac{\mu+\sigma}{2}, \tfrac{\varsigma+\nu}{2}\right) &\leq_{\mathbb{F}} \tfrac{1}{2}\left[\tfrac{1}{\int_\mu^\sigma \Omega(\sigma)d\sigma}\odot\int_\mu^\sigma \widetilde{Y}\left(\sigma, \tfrac{\varsigma+\nu}{2}\right)\Omega(\sigma)d\sigma \oplus \tfrac{1}{\int_\mu^\sigma \mathcal{W}(\omega)d\omega}\odot\int_\mu^\sigma \widetilde{Y}\left(\tfrac{\mu+\sigma}{2},\omega\right)\mathcal{W}(\omega)d\omega\right] \\ &\leq_{\mathbb{F}} \tfrac{1}{\int_\mu^\sigma \Omega(\sigma)d\sigma\int_\mu^\sigma \mathcal{W}(\omega)d\omega}\odot\int_\mu^\sigma\int_\varsigma^\nu \widetilde{Y}(\sigma,\omega)\Omega(\sigma)\mathcal{W}(\omega)d\omega d\sigma \\ &\leq_{\mathbb{F}} \tfrac{1}{4\int_\mu^\sigma \Omega(\sigma)d\sigma}\odot\left[\int_\mu^\sigma \widetilde{Y}(\sigma,\varsigma)d\sigma \oplus \int_\mu^\sigma \widetilde{Y}(\sigma,\nu)d\sigma\right] \\ &\oplus\tfrac{1}{4\int_\mu^\sigma \mathcal{W}(\omega)d\omega}\odot\left[\int_\varsigma^\nu \widetilde{Y}(\mu,\omega)d\omega \oplus \int_\varsigma^\nu \widetilde{Y}(\sigma,\omega)d\omega\right] \\ &\leq_{\mathbb{F}} \tfrac{\widetilde{Y}(\mu,\varsigma)\oplus\widetilde{Y}(\sigma,\varsigma)\oplus\widetilde{Y}(\mu,\nu)\oplus\widetilde{Y}(\sigma,\nu)}{4}. \end{aligned} \tag{98}$$

Let $\widetilde{Y}$ be a left coordinated UD-convex FN-V-M and $\mathcal{W}(\omega) = 1 = \Omega(\sigma)$. Then, we can achieve the following outcome (see [91]*):*

$$\begin{aligned} \widetilde{Y}\left(\tfrac{\mu+\sigma}{2}, \tfrac{\varsigma+\nu}{2}\right) &\leq_{\mathbb{F}} \tfrac{1}{2}\odot\left[\tfrac{1}{\sigma-\mu}\odot\int_\mu^\sigma \widetilde{Y}\left(\sigma, \tfrac{\varsigma+\nu}{2}\right)d\sigma \oplus \tfrac{1}{\nu-\varsigma}\odot\int_\varsigma^\nu \widetilde{Y}\left(\tfrac{\mu+\sigma}{2},\omega\right)d\omega\right] \\ &\leq_{\mathbb{F}} \tfrac{1}{(\sigma-\mu)(\nu-\varsigma)}\odot\int_\mu^\sigma\int_\varsigma^\nu \widetilde{Y}(\sigma,\omega)d\omega d\sigma \\ &\leq_{\mathbb{F}} \tfrac{1}{4(\sigma-\mu)}\odot\left[\int_\mu^\sigma \widetilde{Y}(\sigma,\varsigma)d\sigma \oplus \int_\mu^\sigma \widetilde{Y}(\sigma,\nu)d\sigma\right] \oplus \tfrac{1}{4(\nu-\varsigma)}\odot\left[\int_\varsigma^\nu \widetilde{Y}(\mu,\omega)d\omega \oplus \int_\varsigma^\nu \widetilde{Y}(\sigma,\omega)d\omega\right] \\ &\leq_{\mathbb{F}} \tfrac{\widetilde{Y}(\mu,\varsigma)\oplus\widetilde{Y}(\sigma,\varsigma)\oplus\widetilde{Y}(\mu,\nu)\oplus\widetilde{Y}(\sigma,\nu)}{4}. \end{aligned} \tag{99}$$

Let $\Upsilon_*((\sigma,\omega),\mathfrak{q}) \neq \Upsilon^*((\sigma,\omega),\mathfrak{q})$ with $\mathfrak{q}=1$ and $\mathcal{W}(\omega)=1=\Omega(\sigma)$. Then, we acquire following inequality (see [90]):

$$\begin{array}{c} \Upsilon\left(\frac{\mu+\sigma}{2},\ \frac{\varsigma+\nu}{2}\right) \supseteq \frac{1}{2}\left[\frac{1}{\sigma-\mu}\int_\mu^\sigma \Upsilon\left(\sigma,\ \frac{\varsigma+\nu}{2}\right)d\sigma + \frac{1}{\nu-\varsigma}\int_\varsigma^\nu \Upsilon\left(\frac{\mu+\sigma}{2},\omega\right)d\omega\right] \\ \supseteq \frac{1}{(\sigma-\mu)(\nu-\varsigma)}\int_\mu^\sigma\int_\varsigma^\nu \Upsilon(\sigma,\omega)d\omega d\sigma \\ \supseteq \frac{1}{4(\sigma-\mu)}\left[\int_\mu^\sigma \Upsilon(\sigma,\varsigma)d\sigma + \int_\mu^\sigma \Upsilon(\sigma,\nu)d\sigma\right] + \frac{1}{4(\nu-\varsigma)}\left[\int_\varsigma^\nu \Upsilon(\mu,\omega)d\omega + \int_\varsigma^\nu \Upsilon(\sigma,\omega)d\omega\right] \\ \supseteq \frac{\Upsilon(\mu,\varsigma)+\Upsilon(\sigma,\varsigma)+\Upsilon(\mu,\nu)+\Upsilon(\sigma,\nu)}{4}. \end{array} \tag{100}$$

Let $\Upsilon_*((\sigma,\omega),\mathfrak{q}) = \Upsilon^*((\sigma,\omega),\mathfrak{q})$ with $\mathfrak{q}=1$. Then, we can derive the following inclusion:

$$\begin{array}{c} \Upsilon\left(\frac{\mu+\sigma}{2},\frac{\varsigma+\nu}{2}\right) \leq \frac{1}{2}\left[\frac{1}{\int_\mu^\sigma \Omega(\sigma)d\sigma}\int_\mu^\sigma \Upsilon\left(\sigma,\ \frac{\varsigma+\nu}{2}\right)\Omega(\sigma)d\sigma + \frac{1}{\int_\mu^\sigma \mathcal{W}(\omega)d\omega}\int_\mu^\sigma \Upsilon\left(\frac{\mu+\sigma}{2},\omega\right)\mathcal{W}(\omega)d\omega\right] \\ \leq \frac{1}{\int_\mu^\sigma \Omega(\sigma)d\sigma\int_\mu^\sigma \mathcal{W}(\omega)d\omega}\int_\mu^\sigma\int_\varsigma^\nu \Upsilon(\sigma,\omega)\Omega(\sigma)\mathcal{W}(\omega)d\omega d\sigma \\ \leq \frac{1}{4\int_\mu^\sigma \Omega(\sigma)d\sigma}\left[\int_\mu^\sigma \Upsilon(\sigma,\varsigma)d\sigma + \int_\mu^\sigma \Upsilon(\sigma,\nu)d\sigma\right] \\ +\frac{1}{4\int_\mu^\sigma \mathcal{W}(\omega)d\omega}\left[\int_\varsigma^\nu \Upsilon(\mu,\omega)d\omega + \int_\varsigma^\nu \Upsilon(\sigma,\omega)d\omega\right] \\ \leq \frac{\Upsilon(\mu,\varsigma)+\Upsilon(\sigma,\varsigma)+\Upsilon(\mu,\nu)+\Upsilon(\sigma,\nu)}{4}. \end{array} \tag{101}$$

4. Conclusions

In this paper, we introduced and studied a new class of generalized convex fuzzy mappings on coordinates involving the up and down fuzzy relation, which are known as coordinated up and down convex fuzzy mappings. Several new versions of integral inequalities for this class of functions were obtained. It is interesting to note that most of the classes and other results are also exceptional cases of our defined class and main results, and these exceptional cases of our results are discussed as applications. For the validation of our main outcomes in this paper, some examples were also proved. In future, this concept will be explored in the field of quantum calculus.

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