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The Whitham Modulation Solution of the Complex Modified KdV Equation

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Abstract: This paper primarily concerns the Whitham modulation equation of the complex modified Korteweg–de Vries (cmKdV) equation with a step-like initial value. By utilizing the Lax pair, we derive the N-genus Whitham equations via the averaging method. The Whitham equation can be integrated using the hodograph transformation. We investigate Krichever's algebro-geometric scheme to propose the averaging method for the cmKdV integrable hierarchy and obtain the Whitham velocities of the integrable hierarchy and the hodograph transformation. The connection between the equations of the Euler–Poisson–Darboux type linear overdetermined system, which determines the solutions of the hodograph transformation, is constructed through Riemann integration, which demonstrates that the Whitham equation can be solved. Finally, a step-like initial value problem is solved and an exotic wave pattern is discovered. The results of direct numerical simulation agree well with the Whitham theory solution, which shows the validity of the theory.

Keywords: the cmKdV equation; Lax pair; averaging method; Whitham theory; algebro-geometric scheme

MSC: 35Q53

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1. Introduction

Nonlinear phenomena play a crucial role in various scientific fields such as fluid mechanics, optical fibers, solid state physics, chemical dynamics and geochemistry, and are modeled using nonlinear partial differential equations. Typical nonlinear dispersion equations include the Korteweg–de Vries (KdV) equation [1,2], the nonlinear Schrödinger (NLS) equation [3,4], the sine-Gordon equation [5,6], the Camassa–Holm equation [7,8], etc. Their precise solutions are of great significance for understanding the mechanisms and dynamic behaviors of complex physical phenomena. Since Gardner et al. [9] proposed the inverse scattering method for solving nonlinear integrable equations, many systematic and effective methods have been developed successively, such as the Hirota bilinear method [10,11], the Darboux transform [12], the Bäcklund transform [13], Painlevé's analysis [14], the inverse scattering method [9], the Riemann–Hilbert Method [15] and the Whitham method [16,17]. The Whitham method is a powerful tool for solving discontinuous initial value problems [18,19], that was developed by G. B. Whitham [20] in the 1960s. The main idea of the Whitham method is to approximate a nonlinear wave by a slowly varying wave envelope that satisfies a set of differential equations known as the Whitham equations.

In [21], Gurevich and Pitaevskii first applied the Whitham theory to the Korteweg–de Vries (KdV) equation to study the self-similar solutions for dispersive shock waves (DSW), whose evolution can be described by the diagonal Whitham equation. Tian [22,23] connected the Whitham equation with the linear overdetermined equations of the Euler–Poison–Darboux equations of the KdV equation and the NLS equation and obtained the solution of the Whitham equation using Krichever's algebro-geometric scheme.

In recent years, the algebro-geometric scheme has been extensively studied and some significant progress has been made [24,25]. Krichever [26,27] showed that when the fast oscillations of periodic solutions are averaged or smoothed, the Whitham equation appears as a modulation equation of the Riemann surface module; they also parameterized the algebro-geometric periodic solutions and constructed an algebro-geometric n-orthogonal curve coordinate system on the plane space. These results are important for understanding the relationship between integrable systems and algebraic geometry. As we all know, the KdV equation is a classical nonlinear wave equation typically used to describe the propagation of shallow water waves. The complex modified Korteweg–de Vries (cmKdV) adds a complex term, making it even more complex, thereby allowing the equation not only to describe traditional shallow water waves but also simulate the nonlinear evolution of plasma waves [28], the propagation of shear waves in molecular chains [29] and generalized elastic solids [30]. Furthermore, the equation can also be used to describe coherent structures in some physical phenomena, such as solitary waves in disordered media such as plasma and liquid crystals, and simulate shock waves and their interactions in the aforementioned media. Therefore, the cmKdV equation has a wide range of applications in modern physics and engineering. We are interested in the modulation theory and Krichever's algebro-geometric scheme of the cmKdV equation,

$$\phi_t + \frac{3}{2}|\phi|^2\phi_x - \frac{\epsilon^2}{4}\phi_{xxx} = 0, \tag{1}$$

where ϕ is the complex wave envelope, ϵ is a small modulation scale, and x and t are independent variables. The cmKdV equation has been studied using the Darboux transformation [31], the Hirota method [32], the inverse scattering transformation [33], the Riemann–Hilbert problem [34,35], and so on. As we all know, the zero dispersion limit of the cmKdV equation can be described using the Whitham equation [36–38]

$$\frac{\partial \tau_i}{\partial t} + v_i(\tau_1, \tau_2, \cdots, \tau_{2g+2})\frac{\partial \tau_i}{\partial x} = 0, \quad i = 1, 2, \ldots, 2g+2, \tag{2}$$

with the ordering $\tau_1 > \tau_2 > \cdots > \tau_{2g+2}$, where v_i depends on the complete hyperelliptic integral of genus g and will be given later. Equation (2) describes the slow modulation of the parameter τ_i on the g-genus solution of integrable nonlinear evolutionary systems. Kodama [36] found that they are neither strictly hyperbolic nor genuinely nonlinear. Wang et al. [37] gave the complete classification of discontinuous initial data. The Whitham Equation (2) in the 1-genus case is transformed into a linear overdetermined system of the Euler–Poisson–Darboux type equations [36],

$$\begin{gathered} 2(\tau_i - \tau_j)\frac{\partial^2 q}{\partial \tau_i \partial \tau_j} = \frac{\partial q}{\partial \tau_i} - \frac{\partial q}{\partial \tau_j} \quad i, j = 2, 3, 4, \\ q(a, \tau, \tau, \tau) = F(\tau) \quad i \neq j, \end{gathered} \tag{3}$$

where a is a constant. The system (3) has a unique solution, which can be expressed explicitly [39]. In this paper, we will connect the g-genus Whitham Equation (2) with a high-dimensional Euler–Poisson–Darboux type linear overdetermined system,

$$\begin{gathered} 2(\tau_i - \tau_j)\frac{\partial^2 q}{\partial \tau_i \partial \tau_j} = \frac{\partial q}{\partial \tau_i} - \frac{\partial q}{\partial \tau_j}, \quad i, j = 1, 2, \ldots, 2g+2, \\ q(\tau, \tau, ..., \tau, b) = F(\tau), \quad i \neq j, \end{gathered} \tag{4}$$

where b is a constant and prove that the solution of Equation (2) can be obtained using Equation (4). The system can provide mathematical models based on the principles of physics to describe the laws of fluid motion. It can be used to study the conservation law, wave propagation, instability and other phenomena in fluid flow, which are of great significance for understanding the properties of fluid mechanics and promoting the scientific progress in related fields.

The organizational structure of this paper is as follows. In Section 2, based on the Lax pair, the Whitham equations are established. In Section 3, Krichever's algebro-geometric scheme is analyzed to find the solution of a Whitham-type Equation (2). In Section 4, the connection between the Whitham Equation (2) and the Euler–Poisson–Darboux type linear overdetermined system is constructed. In Section 5, we solve a step-like initial value problem. Finally, we provide our conclusions in the last section.

2. The Whitham Modulation Equation

In this section, we use an averaging method to establish the Whitham equation of Equation (1) based on the Lax pair. Equation (1) can be obtained from the compatibility condition $\psi_{xt} = \psi_{tx}$ of the Lax pair

$$\psi_x = U(\xi)\psi, \;\; \psi_t = V(\xi)\psi, \tag{5}$$

where

$$\psi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \; U(\xi) = \frac{i}{\epsilon}\begin{pmatrix} -\xi & 0 \\ 0 & \xi \end{pmatrix} + \frac{1}{\epsilon}\begin{pmatrix} 0 & \phi \\ \phi^* & 0 \end{pmatrix}, \; V(\tau) = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{6}$$

and

$$\begin{aligned} A =& \frac{1}{4\epsilon}(-\epsilon\phi^*\phi_x + \epsilon\phi\phi_x^* + 4i\xi(\xi^2 + \frac{|\phi|^2}{2})), \\ B =& \frac{1}{4\epsilon}(-2i\xi\epsilon\phi_x - 2|\phi|^2\phi - 4\xi^2\phi + \epsilon^2\phi_{xx}), \\ C =& \frac{1}{4\epsilon}(2i\xi\epsilon\phi_x^* + \epsilon^2\phi_{xx}^* - 2|\phi|^2\phi^* - 4\xi^2\phi^*). \end{aligned} \tag{7}$$

For the Bloch function φ_1

$$\varphi_1 = e^{i(px-\omega t)+\chi(x,t,\xi)}, \tag{8}$$

which satisfies the identity $\partial_t\partial_x \ln\varphi_1 = \partial_x\partial_t \ln\varphi_1$, where ξ is the spectral parameter, p is the quasimomentum and ω is the quasienergy. From (5)–(7), we have

$$\varphi_1|_{\xi\to\infty} \to e^{i(-\frac{\xi}{\epsilon}x+\frac{\xi^3}{\epsilon}t)}. \tag{9}$$

Consider the Riemann surface of genus $g \geq 0$:

$$S_g = \left\{\Pi = (\xi, \Re), \; \Re^2 = \prod_{j=1}^{2g+2}(\xi - \tau_j)\right\}, \;\; \tau_1 > \tau_2 > \ldots > \tau_{2g+2}, \tag{10}$$

where S_g is a two-sheet cover of $\mathbb{CP}^1$; the upper (or lower) sheet denotes the points $(\xi, \Re) \in S_g$ for which $\Re = \Re(\xi, \vec{\tau})$ ($\Re = -\Re(\xi, \vec{\tau})$).

The averaged equations take the form

$$\partial_T < \partial_x \ln\varphi_1 >= \partial_X < \partial_t \ln\varphi_1 >,$$

where X and T are slowly varying variables and $<>$ denotes averaging over one period. The system of averaged equations for many-genus solutions of the cmKdV equation can be expressed in Riemann invariants, namely the Whitham Equation (2). Since

$\partial_x \ln \varphi_1 = ip + \chi_x$, $\partial_t \ln \varphi_1 = -i\omega + \chi_t$, $\lim_{\tau\to\infty} \chi = 0$, it follows that $\partial p/\partial T = \partial\omega/\partial X$. For p and ω, which are Abelian differentials of the second kind on S_g, we obtain

$$\partial\Omega_1/\partial t = \partial\Omega_3/\partial x. \tag{11}$$

The cmKdV equation is on one sheet of the Riemann surface and the Abelian differential has this form:

$$\Omega_i = \frac{Q_l(\xi)d\xi}{\Re(\xi)}, \quad i = 1, 2, \ldots, g, \tag{12}$$

where

$$\Re(\xi) = \sqrt{(\xi - \tau_1)(\xi - \tau_2)\cdots(\xi - \tau_{2g+2})}, \tag{13}$$

and $Q_l(\xi)$ is a polynomial in ξ of degree l, which is related with the number of genera and the asymptotic behavior.

From (8) and (9), we find the asymptotic behavior:

$$\Omega_1 \to -\frac{1}{\epsilon}d\xi + o\left(\frac{d(\xi)}{\xi}\right) \text{ is the quasimomentum}, \tag{14}$$

$$\Omega_3 \to -\frac{3}{\epsilon}\xi^2 d\xi + o\left(\frac{d(\xi)}{\xi}\right) \text{ is the quasienergy}. \tag{15}$$

For the multi-gap solution of the cmKdV equation, $\mu^2 = \Re_{2g+2}(\xi)$. Therefore, in order to satisfy the requirements of (14) and (15), we need

$$\Omega_1^{(g)} = -\frac{1}{\varepsilon}\frac{Q_{g+1}(\xi)d\xi}{\Re(\xi)}, \quad \Omega_3^{(g)} = -\frac{3}{\varepsilon}\frac{Q_{g+3}(\xi)d\xi}{\Re(\xi)}. \tag{16}$$

The Whitham velocities in (2) are the following [36,40]:

$$v_i = \frac{\Omega_3^{(g)}}{\Omega_1^{(g)}} = 3\frac{Q_{g+3}}{Q_{g+1}} = 3\frac{Q_2}{Q_0}. \tag{17}$$

Now define two polynomials

$$\begin{aligned} Q_0(\xi; \tau_1, \tau_2, \cdots, \tau_{2g+2}) &= \xi^{g+1} + \alpha_1\xi^g + \cdots + \alpha_g, \\ Q_2(\xi; \tau_1, \tau_2, \cdots, \tau_{2g+2}) &= \xi^{g+3} + a_1\xi^{g+2} + a_2\xi^{g+1} + a_3\xi^g + \gamma_1\xi^{g-1} + \cdots + \gamma_g, \end{aligned} \tag{18}$$

with $\alpha_1, \alpha_2, \cdots, \alpha_g, a_1, a_2, a_3$ and $\gamma_1, \gamma_2, \ldots, \gamma_g$ uniquely determined by the conditions

$$\begin{aligned} &\int_{\tau_{2k+2}}^{\tau_{2k+1}} \frac{Q_0(\xi; \tau_1, \tau_2, \cdots, \tau_{2g+2})}{\Re(\xi)} \mathrm{d}\xi = 0, \\ &\int_{\tau_{2k+2}}^{\tau_{2k+1}} \frac{Q_2(\xi; \tau_1, \tau_2, \cdots, \tau_{2g+2})}{\Re(\xi)} \mathrm{d}\xi = 0, \quad k = 1, 2, \ldots, 2g+2, \end{aligned} \tag{19}$$

where

$$\Re(\xi) = \sqrt{(\xi - \tau_1)(\xi - \tau_2)\cdots(\xi - \tau_{2g+2})}, \tag{20}$$

and the sign is given by $\sqrt{1} = 1$.

The motions of $\tau_1, \tau_2, \ldots, \tau_{2g+2}$ are determined by Whitham Equations (2) with velocities

$$v_i(\tau_1, \tau_2, \ldots, \tau_{2g+2}) = 3\frac{Q_2(\tau_i; \tau_1, \tau_2, \ldots, \tau_{2g+2})}{Q_0(\tau_i; \tau_1, \tau_2, \ldots, \tau_{2g+2})}, \quad i = 1, 2, \ldots, 2g+2. \tag{21}$$

We first consider the case in which a single valued function x represents the evolution curve. In this case, $g = 0$ and Equation (18) becomes

$$\begin{aligned} Q_0 &= \xi + \alpha_1, \\ Q_2 &= \xi^3 + a_1\xi^2 + a_2\xi + \gamma_1, \end{aligned} \tag{22}$$

where α_1, a_1, a_2 and γ_1 are determined by (19) and (20). In fact,

$$\alpha_1 = a_1 = -\frac{\tau_1 + \tau_2}{2}, \tag{23}$$

$$a_2 = -\frac{1}{8}(\tau_1^2 + \tau_2^2) + \frac{1}{4}\tau_1\tau_2, \tag{24}$$

$$\gamma_1 = -\frac{1}{16}(\tau_1^3 + \tau_2^3) + \frac{1}{16}(\tau_1^2\tau_2 + \tau_1\tau_2^2). \tag{25}$$

According to Equation (21), Equation (2) becomes

$$\tau_{it} + v_i(\tau_1, \tau_2)\tau_{ix} = 0, \ \ i = 1, 2, \tag{26}$$

where

$$v_1 = \frac{15}{8}\tau_1^2 + \frac{3}{4}\tau_1\tau_2 + \frac{3}{8}\tau_2^2, \quad v_2 = \frac{3}{8}\tau_1^2 + \frac{3}{4}\tau_1\tau_2 + \frac{15}{8}\tau_2^2, \tag{27}$$

whose solution satisfies the characteristic equation

$$x = v_i(\tau_1, \tau_2)t + f_i(\tau_1, \tau_2), \ \ i = 1, 2, \tag{28}$$

where $x = f_i(\tau_1, \tau_2)$ are the initial curves.

The 1-genus case is given by $g = 1$ and Equation (18) becomes

$$\begin{aligned} Q_0 &= \xi^2 + \alpha_1\xi + \alpha_2, \\ Q_2 &= \xi^4 + a_1\xi^3 + a_2\xi^2 + a_3\xi + \gamma_1, \end{aligned} \tag{29}$$

where α_1, a_1, a_2 and γ_1 are

$$\alpha_1 = a_1 = -\frac{1}{2}e_1(\tau), \tag{30}$$

$$\alpha_2 = \frac{1}{2}(\tau_1\tau_2 + \tau_3\tau_4) - \frac{1}{2}(\tau_1 - \tau_3)(\tau_2 - \tau_4)\frac{E(s)}{K(s)}, \tag{31}$$

$$a_2 = -\frac{1}{8}e_1(\tau)^2 + \frac{1}{2}e_2(\tau), \tag{32}$$

$$a_3 = -\frac{1}{16}(\tau_1 - \tau_2 - \tau_3 + \tau_4)(\tau_1 - \tau_2 + \tau_3 - \tau_4)(\tau_1 + \tau_2 - \tau_3 - \tau_4), \tag{33}$$

$$\begin{aligned} \gamma_1 = \frac{1}{48}&(3\tau_1\tau_2(\tau_1^2 + \tau_2^2) + 3\tau_3\tau_4(\tau_3^2 + \tau_4^2) + 2(\tau_1\tau_2 + \tau_3\tau_4)^2 \\ &- 2(\tau_3 + \tau_4)(\tau_1 + \tau_2)(\tau_1\tau_2 + \tau_3\tau_4) - (\tau_1\tau_4 + \tau_2\tau_3)(\tau_1\tau_3 + \tau_2\tau_4)) \\ &- \frac{1}{48}(\tau_1 - \tau_3)(\tau_2 - \tau_4)(3e_1(\tau^2) + 2e_2(\tau))\frac{E(s)}{K(s)}, \end{aligned} \tag{34}$$

with

$$e_1(\tau) = \sum_{i=1}^{4} \tau_i, \quad e_2(\tau) = \sum_{i<j} \tau_i \tau_j, \quad i,j = 1,2,3,4, \tag{35}$$

$$s = \sqrt{\frac{(\tau_1 - \tau_2)(\tau_3 - \tau_4)}{(\tau_1 - \tau_3)(\tau_2 - \tau_4)}}, \tag{36}$$

and $K(s)$ and $E(s)$ are complete elliptic integrals of the first and second kind, respectively.

According to Equation (21), Equation (2) becomes

$$\tau_{it} + v_i(\tau_1, \tau_2, \tau_3, \tau_4)\tau_{ix} = 0, \;\; i = 1,2,3,4, \tag{37}$$

where

$$v_1 = \frac{3}{8}\sum_{i=1}^{4} \tau_i^2 + \frac{1}{4}\sum_{i<j} \tau_i \tau_j + \frac{1}{2}(\sum_{i=1}^{4} \tau_i + 2\tau_1)\frac{(\tau_1 - \tau_2)(\tau_1 - \tau_4)K(s)}{(\tau_1 - \tau_4)K(s) + (\tau_4 - \tau_2)E(s)}, \tag{38}$$

$$v_2 = \frac{3}{8}\sum_{i=1}^{4} \tau_i^2 + \frac{1}{4}\sum_{i<j} \tau_i \tau_j + \frac{1}{2}(\sum_{i=1}^{4} \tau_i + 2\tau_2)\frac{(\tau_1 - \tau_2)(\tau_3 - \tau_2)K(s)}{(\tau_3 - \tau_2)K(s) + (\tau_1 - \tau_3)E(s)}, \tag{39}$$

$$v_3 = \frac{3}{8}\sum_{i=1}^{4} \tau_i^2 + \frac{1}{4}\sum_{i<j} \tau_i \tau_j + \frac{1}{2}(\sum_{i=1}^{4} \tau_i + 2\tau_3)\frac{(\tau_3 - \tau_4)(\tau_3 - \tau_2)K(s)}{(\tau_3 - \tau_2)K(s) + (\tau_2 - \tau_4)E(s)}, \tag{40}$$

$$v_4 = \frac{3}{8}\sum_{i=1}^{4} \tau_i^2 + \frac{1}{4}\sum_{i<j} \tau_i \tau_j + \frac{1}{2}(\sum_{i=1}^{4} \tau_i + 2\tau_4)\frac{(\tau_4 - \tau_3)(\tau_4 - \tau_1)K(s)}{(\tau_4 - \tau_1)K(s) + (\tau_1 - \tau_3)E(s)}. \tag{41}$$

The Whitham system can also be derived using the methods of multiphase averaging [40,41] and multiple scale expansions regardless of the integrability of the systems [42]. Taking the limits $s \to 0$ and $s \to 1$, the velocities of the 1-genus can be reduced to the ones of a 0-genus and the boundaries connecting the 1-genus and 0-genus are continuous. The periodic solution of the cmKdV Equation (1) can be expressed as follows [36],

$$\rho = \frac{1}{4}(\tau_1 - \tau_2 - \tau_3 + \tau_4)^2 + (\tau_1 - \tau_2)(\tau_3 - \tau_4)\mathrm{sn}^2\Big(\frac{\sqrt{(\tau_1 - \tau_3)(\tau_2 - \tau_4)}}{\epsilon}(\xi - \xi_0), s\Big), \tag{42}$$

where sn is the Jacobian elliptic function, $\xi = x - Vt$ is the traveling transformation, $V = \frac{3}{8}(\sum_{i=1}^{4} \tau_i)^2 - \frac{1}{2}\sum_{i<j}^{4} \tau_i \tau_j$ is the traveling velocity and ξ_0 is the phase shift.

Similarly, the velocities of the multi-genus Whitham Equation (2) can be expressed using complete hyperelliptic integrals of genus g. In fact, the Whitham Equation (2) can be integrated using a hodograph transformation and, more precisely, we get the following theorem due to Tsarev [43].

Theorem 1. *If $w_i(\vec{\tau})$, where $\vec{\tau}$ denotes $(\tau_1, \tau_2, \ldots, \tau_{2g+2})$ solves the overdetermined linear system*

$$\frac{\partial w_i}{\partial \tau_i} = a_{ij}(\tau_1, \tau_2, \ldots, \tau_{2g+2})[w_i - w_j], \;\; i,j = 1,2,\ldots,2g+2, \;\; i \neq j, \tag{43}$$

where

$$a_{ij}(\tau_1, \tau_2, \ldots, \tau_{2g+2}) = \frac{1}{v_i - v_j}\frac{\partial v_i}{\partial \tau_j}, \;\; i,j = 1,2,\ldots,2g+2, \;\; i \neq j, \tag{44}$$

then the solution $(\tau_1(x,t), \tau_2(x,t), \ldots, \tau_{2g+2}(x,t))$ of the hodograph transformation

$$x = v_i(\tau_1, \tau_2, \ldots, \tau_{2g+2})t + w_i(\tau_1, \tau_2, \ldots, \tau_{2g+2}), \;\; i = 1,2,\ldots,2g+2, \tag{45}$$

satisfies Equation (2).

It is worth noting that the hodograph method (45) is a generalization of the characteristic method (28). The linear overdetermined system (43) satisfies the compatibility condition and has local solutions. These solutions usually cannot be expanded globally, because a_{ij} in (44) are singular on the phase transition boundaries. Proving the existence of a global solution to the system (43) and using it to construct the solution to the initial value problem of the Whitham Equation (2) are very important problems. In the single-genus case, we will find explicit global solutions to (43) and solve the initial problem of Equation (2).

3. Krichever's Algebro-Geometric Scheme

In this section, we will analyze Krichever's scheme to find the solutions of the Whitham Equation (2). According to Jenkins [44], we can define

$$P_n(\xi, \tau_1, \tau_2, \cdots, \tau_{2g+2}) = \xi^{n+1} + a_{1,n}\xi^n + \cdots + a_{n+1,n}, \tag{46}$$

where the coefficients $a_{1,n}, a_{2,n}, \ldots, a_{n+1,n}$ are uniquely determined by these two conditions

$$\frac{P_n(\xi, \tau_1, \tau_2, \ldots, \tau_{2g+2})}{R(\xi)} = \xi^{n-g-1} + O(\xi^{-2}), \quad \text{for large } |\xi|, \tag{47}$$

and

$$\int_{\tau_{k+1}}^{\tau_k} \frac{P_n(\xi, \tau_1, \tau_2, \cdots, \tau_{2g+2})}{\Re(\xi)} \mathrm{d}\xi = 0, \quad k = 1, 2, \ldots, 2g+2. \tag{48}$$

It is easy to show that these conditions uniquely determine $P_n(\xi)$'s. Obviously, we have $Q_0 = P_g$ and $Q_2 = 3P_{g+2}$ for the cmKdV Equation (1), where Q_0 and Q_2 are given in Equation (18).

We consider a simple initial value with $\tau_2 = b$, then the initial curves are

$$\tau_1(x, 0) = a(x), \;\; \tau_2(x, 0) = b, \tag{49}$$

and the initial function $x = f_1(\tau_1, b) = f(u)$, where $a(x)$ is a function of x.

Suppose the initial function $x = f(u)$ has the form

$$f(u) = \beta_0 + \beta_1 u + \cdots + \beta_k u^k + \cdots, \tag{50}$$

where only a finite number of β_k's are non-zero.

Construct $P(\xi)$ as

$$P(\xi) = \sum_{k=0}^{+\infty} \frac{\beta_k}{A_k} P_{k+g}(\xi), \tag{51}$$

where

$$A_k = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^k k!}. \tag{52}$$

The Whitham-type Equation (2) is equivalent to

$$\left[\frac{Q_0(\xi; \vec{\tau})}{\Re(\xi)}\right]_t + \left[\frac{Q_2(\xi; \vec{\tau})}{\Re(\xi)}\right]_x = 0, \tag{53}$$

which can be generalized to the cmKdV hierarchy by the following theorem.

Theorem 2. *If*

$$\left[\frac{P_g(\xi;\vec{\tau})}{\Re(\xi)}\right]_t + \left[\frac{P(\xi;\vec{\tau})}{\Re(\xi)}\right]_x = 0 \tag{54}$$

holds, then if, and only if,

$$\tau_{it} + W_i(\vec{\tau})\tau_{ix} = 0, \;\; i = 1, 2, \ldots, 2g+2, \tag{55}$$

where

$$W_i(\vec{\tau}) = \frac{P(\tau_i;\vec{\tau})}{P_g(\tau_i;\vec{\tau})}, \;\; i = 1, 2, \ldots, 2g+2. \tag{56}$$

Proof. From (54), we have

$$\begin{aligned}&\left[\frac{P_g(\xi;\vec{\tau})}{\Re(\xi)}\right]_t + \left[\frac{P(\xi;\vec{\tau})}{\Re(\xi)}\right]_x \\ &= \frac{\Re^2(\xi)[(\partial/\partial t)P_g(\xi;\vec{\tau}) + (\partial/\partial x)P(\xi;\vec{\tau})] - (1/2)M}{\Re^3(\xi)},\end{aligned} \tag{57}$$

where

$$M = P_g(\xi;\vec{\tau})(\partial/\partial t)\Re^2(\xi) + P(\xi;\vec{\tau})(\partial/\partial x)\Re^2(\xi)]. \tag{58}$$

According to Equation (20), we get

$$\begin{aligned}&P_g(\xi;\vec{\tau})\frac{\partial}{\partial t}\Re^2(\xi;\vec{\tau}) + P(\xi;\vec{\tau})\frac{\partial}{\partial x}\Re^2(\xi;\vec{\tau}) \\ &= -\sum_{i=1}^{2g+2}[P_g(\xi;\vec{\tau})\tau_{it} + P(\xi;\vec{\tau})\tau_{ix}]\Pi_{j\neq i}(\xi-\tau_j),\end{aligned} \tag{59}$$

which vanish at $\xi = \tau_i$ if, and only if,

$$P_g(\xi;\vec{\tau})\tau_{it} + P(\xi;\vec{\tau})\tau_{ix} = 0. \tag{60}$$

The proof of the second part is as follows. The time evolution, Equation (60), allows us to rewrite Equation (57) as:

$$\left[\frac{P_g(\xi;\vec{\tau})}{\Re(\xi)}\right]_t + \left[\frac{P(\xi;\vec{\tau})}{\Re(\xi)}\right]_x = \frac{N(\xi)}{\Re(\xi)}, \tag{61}$$

where

$$\begin{aligned}N(\xi) = &\frac{\partial P_g(\xi;\vec{\tau})}{\partial t} + \frac{\partial P(\xi;\vec{\tau})}{\partial x} \\ &+ \frac{1}{2}\sum_{i=1}^{2g+2}\left[\frac{P_g(\xi;\vec{\tau}) - P_g(\tau_i;\vec{\tau})}{(\xi-\tau_i)}\tau_{it} + \frac{P(\xi;\vec{\tau}) - P(\tau_i;\vec{\tau})}{(\xi-\tau_i)}\tau_{ix}\right]\end{aligned} \tag{62}$$

is a power series of ξ. By Equation (47) and Equation (51), we get the expansion at $\xi = \infty$

$$\begin{aligned} \frac{P_g(\xi;\vec{\tau})}{\Re(\xi)} &= \xi^0 + O(\xi^{-2}), \\ \frac{P(\xi;\vec{\tau})}{\Re(\xi)} &= \sum_{k=0}^{+\infty} \frac{\beta_k}{A_k}\xi^k + O(\xi^{-2}), \end{aligned} \tag{63}$$

which mean that

$$\left[\frac{P_g(\xi;\vec{\tau})}{\Re(\xi)}\right]_t + \left[\frac{P(\xi;\vec{\tau})}{\Re(\xi)}\right]_x = O(\xi^{-2}), \quad \text{for large } |\xi|. \tag{64}$$

Therefore, from Equation (61), we have

$$N(\xi) = \Re(\xi)O(\xi^{-2}) = O(\xi^{g-1}), \quad \text{for large } |\xi|, \tag{65}$$

which shows that $N(\xi)$ is a polynomial of ξ with a maximum degree of $g-1$.

Now consider the Riemann surface defined by $\Re(\xi)$ and cut along interval $[\tau_{2g+2}, \tau_{2g+1}]$, $[\tau_{2g}, \tau_{2g-1}], \ldots, [\tau_2, \tau_1]$. We define the a_k cycle as a closed contour, which surrounds the notch on the upper Riemann graph clockwise $[\tau_{2k+2}, \tau_{2k+1}]$. The b_k cycle, defined as a closed contour, starts from τ_{2k+2} and goes to τ_{2g+2}, crosses to the lower level and finally returns to τ_{2k+2}.

Integrating (61) along the cycle b_k yields

$$\oint_{b_k} \frac{N(\xi)}{\Re(\xi)}d\xi = \frac{\partial}{\partial t}\left[\oint_{b_k} \frac{P_g(\xi;\vec{\tau})}{\Re(\xi)}d\xi\right] + \frac{\partial}{\partial x}\left[\oint_{b_k} \frac{P(\xi;\vec{\tau})}{\Re(\xi)}d\xi\right] = 0, \quad k = 1, 2, \ldots, g, \tag{66}$$

in which, we used Equation (48) and Equation (51) in the last equation. This is equivalent to

$$\int_{2k+2}^{2k+1} \frac{N(\xi)}{\Re(\xi)}d\xi = 0, \;\; \text{for all } \; k = 1, 2, \ldots, g, \tag{67}$$

which proves that $N(\xi)$ has at least one zero in the interval $[\tau_{2k+2}, \tau_{2k+1}]$, so $N(\xi)$ has at least $g+1$ zeros. This combined with Equation (65) shows that $N = 0$. Equation (54) immediately follows from Equation (61). The proof of Theorem 2 is complete. □

We first state the following classical results about entropy in conservation law theory [45].

Theorem 3. *If Equation (55) implies the existence of an additional conservation law*

$$\frac{\partial}{\partial t}U(\vec{\tau}) + \frac{\partial}{\partial x}G(\vec{\tau}) = 0, \tag{68}$$

then we have

$$W_i(\vec{\tau}) = \frac{(\partial/\partial\tau_i)G(\vec{\tau})}{(\partial/\partial\tau_i)U(\vec{\tau})}, \;\; i = 1, 2, \ldots, 2g+2. \tag{69}$$

The next result shows that the W_i's of Equation (56) solve the system (43) of Theorem 1, which can be proved using the method of Levermore [46].

Theorem 4. *W and v are satisfied for $i, j = 1, 2, \ldots, 2g+2$, and $i \neq j$:*

$$\frac{1}{W_i - W_j}\frac{\partial W_i}{\partial \tau_j} = \frac{1}{v_i - v_j}\frac{\partial v_i}{\partial \tau_j}. \tag{70}$$

Proof. By (20), Theorem 2 and Theorem 3, we have

$$W_j(\vec{\tau}) = \frac{\left(\frac{\partial}{\partial \tau_j}\left[\frac{P(\xi;\vec{\tau})}{\Re(\xi)}\right]\right)}{\left(\frac{\partial}{\partial \tau_j}\left[\frac{P_g(\xi;\vec{\tau})}{\Re(\xi)}\right]\right)}, \ j = 1,2,\ldots,2g+2$$
$$= \frac{(\xi-\tau_j)(\partial/\partial\tau_j)P(\xi;\vec{\tau}) + (1/2)P(\xi;\vec{\tau})}{(\xi-\tau_j)(\partial/\partial\tau_j)P_g(\xi;\vec{\tau}) + (1/2)P_g(\xi;\vec{\tau})},$$

which yields

$$\frac{\partial P}{\partial \tau_j} - W_j \frac{\partial P_g}{\partial \tau_j} = -\frac{1}{2}\frac{P - W_j P_g}{\xi - \tau_j}, \ j = 1,2,\ldots,2g+2. \tag{71}$$

From Equations (56) and (71), we can get

$$\begin{aligned}
\frac{\partial W_i(\vec{\tau})}{\partial \tau_j} &= \frac{\partial}{\partial \tau_j}\left[\frac{P(\tau_i;\vec{\tau})}{P_g(\tau_i;\vec{\tau})}\right], \ i \neq j \\
&= \frac{[\partial_{\tau_j} P(\tau_i;\vec{\tau})]P_g(\tau_i;\vec{\tau}) - [\partial_{\tau_j} P_g(\tau_i;\vec{\tau})]P(\tau_i;\vec{\tau})}{P_g(\tau_i;\vec{\tau})^2} \\
&= \frac{\partial_{\tau_j} P(\tau_i;\vec{\tau}) - W_i \partial_{\tau_j} P_g(\tau_i;\vec{\tau})}{P_g(\tau_i;\vec{\tau})} \\
&= (W_j - W_i)\frac{\partial_{\tau_j} P_g(\tau_i;\vec{\tau})}{P_g(\tau_i;\vec{\tau})} - \frac{1}{2}\frac{P(\tau_i;\vec{\tau}) - W_j P_g(\tau_i;\vec{\tau})}{(\tau_i - \tau_j)P_g(\tau_i;\vec{\tau})} \\
&= (W_j - W_i)\frac{\partial_{\tau_j} P_g(\tau_i;\vec{\tau})}{P_g(\tau_i;\vec{\tau})} - \frac{1}{2}\frac{W_i - W_j}{\tau_i - \tau_j},
\end{aligned} \tag{72}$$

which shows that

$$\frac{1}{W_i - W_j}\frac{\partial W_i}{\partial \tau_j} = -\frac{\partial_{\tau_j} P_g(\tau_i;\vec{\tau})}{P_g(\tau_i;\vec{\tau})} - \frac{1}{2}\frac{1}{\tau_i - \tau_j}. \tag{73}$$

In particular, the above argument also applies to the case of the Whitham-type Equations (1). Therefore, we also have

$$\frac{1}{v_i - v_j}\frac{\partial v_i}{\partial \tau_j} = -\frac{\partial_{\tau_j} P_g(\tau_i;\vec{\tau})}{P_g(\tau_i;\vec{\tau})} - \frac{1}{2}\frac{1}{\tau_i - \tau_j}; \tag{74}$$

when combined with Equation (73), Theorem 4 is proved. □

Finally, we combine Theorem 1 and Theorem 2 to construct the transformation

$$\begin{aligned}
x &= v_i(\vec{\tau})t + W_i(\vec{\tau}), \\
&= \left[3\frac{P_{g+2}(\xi;\vec{\tau})}{P_g(\xi;\vec{\tau})}t + \frac{P(\xi;\vec{\tau})}{P_g(\xi;\vec{\tau})}\right]_{\xi=\tau_i}, \ i = 1,2,\ldots,2g+2.
\end{aligned} \tag{75}$$

4. Linear Overdetermined Systems of the Euler–Poisson–Darboux Type

In this section, we show the relation between the Whitham Equation (2) and linear overdetermined systems of the Euler–Poisson–Darboux type (4).

First we define [44]

$$\phi_i(\xi;\vec{\tau}) = \frac{c_{i1} + c_{i2}\xi + \ldots + c_{ig-1}\xi^{g-2} + c_{ig}\xi^{g-1}}{\Re(\xi)}, \ i = 1,2\ldots,g, \tag{76}$$

where the constants c_{ik} are uniquely determined by the normalization conditions

$$\oint_{a_k} \phi_i(\xi)d\xi = \delta_{ik}, \quad i,k = 1,2,\ldots,g, \tag{77}$$

and the cycles a_k's are as previously given.

Fixing the point A on the Riemann surface, we define $\partial\Pi$ as the standard $4g$-edge path, i.e.,

$$A \xrightarrow{a_1} A \xrightarrow{b_1} A \xrightarrow{a_1^{-1}} A \xrightarrow{b_1^{-1}} A \xrightarrow{a_2} A \xrightarrow{b_2} A \xrightarrow{a_2^{-1}} A \xrightarrow{b_2^{-1}} A \cdots \longrightarrow A. \tag{78}$$

Therefore, we have

$$\begin{aligned}
&\oint_{\partial\Pi} \left[\int_A^{\xi} \phi_i(E)dE \right] \frac{P(\xi)}{\Re(\xi)} d\xi \\
&= \sum_{k=1}^{g} \left\{ \left[\oint_{a_k} + \oint_{b_k} + \oint_{a_k^{-1}} + \oint_{b_k^{-1}} \right] \left[\frac{P(\xi)}{\Re(\xi)} \int_A^{\xi} \phi_i(E)dE \right] d\xi \right\} \\
&= \sum_{k=1}^{g} \left\{ - \left[\oint_{a_k} \frac{P(\xi)}{\Re(\xi)} d\xi \right] \left[\oint_{b_k} \phi_i(\xi)d\xi \right] \right. \\
&\left. + \left[\oint_{a_k} \phi_i(\xi)d\xi \right] \left[\oint_{b_k} \frac{P(\xi)}{\Re(\xi)} d\xi \right] \right\} \\
&= - \sum_{k=1}^{g} \sigma_{ki} \oint_{a_k} \frac{P(\xi)}{\Re(\xi)} d\xi,
\end{aligned}$$

where we have used Equations (48) and (51) in the last equation, and

$$\sigma_{ki} = \oint_{b_k} \phi_i(\xi)d\xi. \tag{79}$$

On the other hand, by Equation (76) we have

$$\oint_{\partial\Pi} \left[\int_A^{\xi} \phi_i(E)dE \right] \frac{P(\xi)}{\Re(\xi)} d\xi = \sum_{j=1}^{g} c_{ij} \oint_{\partial\Pi} \left[\int_A^{\xi} \frac{E^{j-1}}{R(E)} dE \right] \frac{P(\xi)}{\Re(\xi)} d\xi. \tag{80}$$

The integral

$$\left[\int_A^{\xi} \frac{E^{j-1}}{\Re(E)} dE \right] \frac{P(\xi)}{\Re(\xi)} d\xi \tag{81}$$

has only one singular point at $\xi = \infty$, and the residue can be obtained as follows. Expanding $1/\Re(\xi)$ at $\xi = \infty$, we find

$$\frac{1}{\Re(\xi)} = \xi^{-g-1}[\Gamma_0(\vec{\tau}) + \Gamma_1(\vec{\tau})\xi^{-1} + \cdots + \Gamma_m(\vec{\tau})\xi^{-m} + \cdots], \tag{82}$$

which gives

$$\frac{\xi^{j-1}}{\Re(\xi)} = \sum_{m=0}^{\infty} \Gamma_m(\vec{\tau})\xi^{j-g-m-2}. \tag{83}$$

Therefore, we obtain

$$\begin{aligned}\int_A^\xi \frac{E^{j-1}}{\Re(E)}dE &= C_0 + \sum_{m=0}^{\infty} \frac{\Gamma_m(\vec{\tau})}{j-g-m-1}\xi^{j-g-m-1} \\ &= C_0 + \sum_{m=0}^{\infty} \frac{\Gamma_m(\vec{\tau})}{j-g-m-1}\mu^{g+m-j+1},\end{aligned} \tag{84}$$

where we introduce $\xi = \frac{1}{\mu}$, μ is the local coordinate and C_0 is a constant.

By Equations (47) and (51), we have

$$\begin{aligned}\frac{P(\xi)}{\Re(\xi)} &= \sum_{n=0}^{\infty} \frac{\beta_n}{A_n}\frac{P_{n+g}(\xi)}{\Re(\xi)} \\ &= \sum_{n=0}^{\infty} \frac{\beta_n}{A_n}\left[\xi^n + O(\xi^{-2})\right], \text{ for large } |\xi|, \\ &= \sum_{n=0}^{\infty} \left[\frac{\beta_n}{A_n}\mu^{-n} + O(\mu^2)\right], \text{ for small } |\mu|,\end{aligned} \tag{85}$$

which together with (84) gives

$$\begin{aligned}&Residue_{\xi=\infty}\left\{\left[\int_A^\xi \frac{E^{j-1}}{\Re(E)}dE\right]\frac{P(\xi)}{\Re(\xi)}d\xi\right\} \\ &= Residue_{\mu=0}\left\{\left[C_0 + \sum_{m=0}^{\infty}\frac{\Gamma_m(\vec{\tau})}{j-g-m-1}\mu^{g+m-j+1}\right]\right. \\ &\times \left[\sum_{n=0}^{\infty}\frac{\beta_n}{A_n}\mu^{-n} + O(\mu^2)\right](-\mu^{-2})d\mu \\ &= -\sum_{g+m-j-n=0}\frac{\beta_n}{A_n}\frac{\Gamma_m(\vec{\tau})}{j-g-m-1} \\ &= \sum_{m=0}^{\infty}\frac{\beta_{m+g-j}}{(m+g-j+1)A_{m+g-j}}\Gamma_m(\vec{\tau}).\end{aligned} \tag{86}$$

This, together with Equations (79) and (80), yields

$$-\sum_{k=1}^{g}\sigma_{ki}\oint_{a_k}\frac{P(\xi)}{\Re(\xi)}d\xi = 2\pi\sqrt{-1}\sum_{j=1}^{g}c_{ij}\left(\sum_{m=0}^{\infty}\frac{\beta_{m+g-j}}{(m+g-j+1)A_{m+g-j}}\Gamma_m(\vec{\tau})\right), \tag{87}$$

which is the Riemann bilinear relation. Rewriting it in the matrix form, we obtain

$$\begin{pmatrix}\sigma_{11} & \sigma_{21} & \cdots & \sigma_{g1} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{g3} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1g} & \sigma_{2g} & \cdots & \sigma_{gg}\end{pmatrix}\begin{pmatrix}\oint_{a_1}\frac{P(\xi)}{\Re(\xi)}d\xi \\ \oint_{a_2}\frac{P(\xi)}{\Re(\xi)}d\xi \\ \vdots \\ \oint_{a_g}\frac{P(\xi)}{\Re(\xi)}d\xi\end{pmatrix} = -2\pi\sqrt{-1}\begin{pmatrix}c_{11} & c_{12} & \cdots & c_{1g} \\ c_{21} & c_{22} & \cdots & c_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ c_{g1} & c_{g2} & \cdots & c_{gg}\end{pmatrix}\begin{pmatrix}q_g(\vec{\tau}) \\ q_{g-1}(\vec{\tau}) \\ \vdots \\ q_1(\vec{\tau})\end{pmatrix}, \tag{88}$$

where

$$q_n(\vec{\tau}) = \sum_{m=0}^{\infty}\frac{\beta_{m+n-1}}{(m+n)A_{m+n-1}}\Gamma_m(\vec{\tau}), \ \ n = 1,2,\ldots,g. \tag{89}$$

It follows from Equations (76) and (79) that

$$\begin{pmatrix} \sigma_{11} & \sigma_{21} & \cdots & \sigma_{g1} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{g3} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1g} & \sigma_{2g} & \cdots & \sigma_{gg} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1g} \\ c_{21} & c_{22} & \cdots & c_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ c_{g1} & c_{g2} & \cdots & c_{gg} \end{pmatrix} \begin{pmatrix} \oint_{b_1} \frac{1}{\Re(\xi)} d\xi & \oint_{b_2} \frac{1}{\Re(\xi)} d\xi & \cdots & \oint_{b_g} \frac{1}{\Re(\xi)} d\xi \\ \oint_{b_1} \frac{\xi}{\Re(\xi)} d\xi & \oint_{b_2} \frac{\xi}{\Re(\xi)} d\xi & \cdots & \oint_{b_g} \frac{\xi}{\Re(\xi)} d\xi \\ \vdots & \vdots & \ddots & \vdots \\ \oint_{b_1} \frac{\xi^{g-1}}{\Re(\xi)} d\xi & \oint_{b_2} \frac{\xi^{g-1}}{\Re(\xi)} d\xi & \cdots & \oint_{b_g} \frac{\xi^{g-1}}{\Re(\xi)} d\xi \end{pmatrix}. \tag{90}$$

The regularity of matrix (c_{ij}) follows from Equations (76) and (79), while that of matrix $(\oint_{b_k} (\xi^{j-1}/\Re(\xi)) d\xi)$ is also easy to check. This, together with (88) and (90), gives

$$\begin{pmatrix} \oint_{b_1} \frac{1}{\Re(\xi)} d\xi & \oint_{b_2} \frac{1}{\Re(\xi)} d\xi & \cdots & \oint_{b_g} \frac{1}{\Re(\xi)} d\xi \\ \oint_{b_1} \frac{\xi}{\Re(\xi)} d\xi & \oint_{b_2} \frac{\xi}{\Re(\xi)} d\xi & \cdots & \oint_{b_g} \frac{\xi}{\Re(\xi)} d\xi \\ \vdots & \vdots & \ddots & \vdots \\ \oint_{b_1} \frac{\xi^{g-1}}{\Re(\xi)} d\xi & \oint_{b_2} \frac{\xi^{g-1}}{\Re(\xi)} d\xi & \cdots & \oint_{b_g} \frac{\xi^{g-1}}{\Re(\xi)} d\xi \end{pmatrix} \begin{pmatrix} \oint_{a_1} \frac{P(\xi)}{\Re(\xi)} d\xi \\ \oint_{a_2} \frac{P(\xi)}{\Re(\xi)} d\xi \\ \vdots \\ \oint_{a_g} \frac{P(\xi)}{\Re(\xi)} d\xi \end{pmatrix} = -2\pi\sqrt{-1} \begin{pmatrix} q_g(\vec{\tau}) \\ q_{g-1}(\vec{\tau}) \\ \vdots \\ q_1(\vec{\tau}) \end{pmatrix}. \tag{91}$$

Inverting this, we obtain

$$\oint_{a_k} \frac{P(\xi)}{\Re(\xi)} d\xi = \sqrt{-1} \sum_{n=1}^{g} K_n^{(k)} q_n, \quad k = 1, 2, \ldots, g, \tag{92}$$

where

$$\begin{pmatrix} K_g^{(1)} & K_{g-1}^{(1)} & \cdots & K_1^{(1)} \\ K_g^{(2)} & K_{g-1}^{(2)} & \cdots & K_1^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ K_g^{(g)} & K_{g-1}^{(g)} & \cdots & K_1^{(g)} \end{pmatrix} = -2\pi \begin{pmatrix} \oint_{b_1} \frac{1}{\Re(\xi)} d\xi & \oint_{b_2} \frac{1}{\Re(\xi)} d\xi & \cdots & \oint_{b_g} \frac{1}{\Re(\xi)} d\xi \\ \oint_{b_1} \frac{\xi}{\Re(\xi)} d\xi & \oint_{b_2} \frac{\xi}{\Re(\xi)} d\xi & \cdots & \oint_{b_g} \frac{\xi}{\Re(\xi)} d\xi \\ \vdots & \vdots & \ddots & \vdots \\ \oint_{b_1} \frac{\xi^{g-1}}{\Re(\xi)} d\xi & \oint_{b_2} \frac{\xi^{g-1}}{\Re(\xi)} d\xi & \cdots & \oint_{b_g} \frac{\xi^{g-1}}{\Re(\xi)} d\xi \end{pmatrix}^{-1} \tag{93}$$

is the real matrix in view of Equation (20). In particular, by Equations (51), (89) and (91), we get

$$\oint_{a_k} \frac{P_g(\xi)}{\Re(\xi)} d\xi = \sqrt{-1} K_1^{(k)}. \tag{94}$$

Next, integrating Equation (54) yields

$$\frac{\partial}{\partial t}\left[\oint_{a_k} \frac{P_g(\xi;\vec{\tau})}{\Re(\xi)} d\xi\right] + \frac{\partial}{\partial x}\left[\oint_{a_k} \frac{P(\xi;\vec{\tau})}{\Re(\xi)} d\xi\right] = 0, \quad k = 1, 2, \ldots, g. \tag{95}$$

The first and second terms of Equation (95) are proportional to the wave numbers and frequencies of the modulated multiphase waves for the cmKdV hierarchy, respectively.

Applying Theorem 3 to Equation (95) and using Equations (91) and (94), we get that for $i = 1, 2, \ldots, 2g+2$,

$$W_i(\vec{\tau}) = \left[1 + \frac{K_1^{(k)}}{(\partial/\partial\tau_i)K_1^{(k)}} \frac{\partial}{\partial\tau_i}\right] q_1(\vec{\tau}) + \sum_{n=2}^{g} \left[\frac{(\partial/\partial\tau_i)K_n^{(k)}}{(\partial/\partial\tau_i)K_1^{(k)}} + \frac{K_n^{(k)}}{(\partial/\partial\tau_i)K_1^{(k)}} \frac{\partial}{\partial\tau_i}\right] q_n(\vec{\tau}), \tag{96}$$

which establishes the following theorem.

Theorem 5. *There are fixed real functions $H_1(\vec{\tau}), H_2(\vec{\tau}), \ldots, H_g(\vec{\tau})$, so that the system, Equation (43), has solutions for all polynomial initial data Equations (50)*

$$W_i(\vec{\tau}) = \left[1 + \frac{H_1}{(\partial/\partial\tau_i)H_1} \frac{\partial}{\partial\tau_i}\right] q_1(\vec{\tau}) + \sum_{n=2}^{g} \left[\frac{(\partial/\partial\tau_i)H_n}{(\partial/\partial\tau_i)H_1} + \frac{H_n}{(\partial/\partial\tau_i)H_1} \frac{\partial}{\partial\tau_i}\right] q_n(\vec{\tau}) \tag{97}$$

for $i = 1, 2, \ldots, 2g+2$, where the q_n's are given by Equation (89).

It immediately follows from Equation (20) that

$$2(\tau_i - \tau_j)\frac{\partial^2}{\partial\tau_i\partial\tau_j}\left[\frac{1}{\Re(\xi)}\right] = \frac{\partial}{\partial\tau_i}\left[\frac{1}{\Re(\xi)}\right] - \frac{\partial}{\partial\tau_j}\left[\frac{1}{\Re(\xi)}\right], \ i \neq j. \tag{98}$$

Combined with Equation (82), (98) shows that the Γ_m's satisfy Equation (4). From (89), $\Gamma_m(\vec{\tau})$'s of (82) obeys

$$\Gamma_m(\tau, \tau, \ldots, \tau) = \frac{(2g+1)(2g+3)\cdots(2g+2m-1)}{2^m m!}\tau^m, \ m = 0, 1, 2, \ldots, \tag{99}$$

which establishes the following theorem.

Theorem 6. *Each $q_n(\vec{\tau})$ of (89) solves the linear overdetermined system of Euler–Poisson–Darboux Equation (4) with $F(\tau)$ given by*

$$F(\tau) = \frac{2^{g-1}}{1\cdot 3\cdot 5\cdots(2g-1)}\tau^{-n+1/2}\frac{d^{g-n}}{d\tau^{g-n}}\left\{\tau^{g-1/2}\left[\frac{d^{g-1}}{d\tau^{g-1}}f(\tau)\right]\right\} \tag{100}$$

for $n = 1, 2, \ldots, g$.

For the special initial value (49), the Whitham Equation (2) has the property that $\tau_4 = b$, and τ_1, τ_2, τ_3 are nontrivial functions of (x, t). If $g = n = 1$, we have

$$q_1(\tau, \tau, \tau, b) = f(\tau).$$

In particular, when $g = 1$, we have

$$W_i(\vec{\tau}) = \left[1 + \frac{H_1}{(\partial/\partial\tau_i)H_1}\frac{\partial}{\partial\tau_i}\right]q_1(\vec{\tau}), \ i = 1, 2, 3, 4. \tag{101}$$

Choosing $\beta_2 = 3/8$ and the other β_k's to be zero in Equation (56), we obtain $P = P_{g+2}$. Therefore we have

$$\begin{aligned} v_i(\tau_1, \tau_2, \tau_3, \tau_4) &= 3\frac{P_{g+2}(\tau_i; \tau_1, \tau_2, \tau_3, \tau_4)}{P_g(\tau_i; \tau_1, \tau_2, \tau_3, \tau_4)} \\ &= 3\left[1 + \frac{H_1}{(\partial/\partial\tau_i)H_1}\frac{\partial}{\partial\tau_i}\right]\frac{1}{3}\Gamma_2(\tau_1, \tau_2, \tau_3, \tau_4) \\ &= \frac{3}{8}(\tau_1+\tau_2+\tau_3+\tau_4)\left[\tau_1+\tau_2+\tau_3+\tau_4 + 2\frac{H_1}{(\partial/\partial\tau_i)H_1}\right], i = 1, 2, 3, 4. \end{aligned} \tag{102}$$

In there, we used (21) in the first equation, (56), (89) and (21) in the second equation and $\Gamma_2(\tau_1, \tau_2, \tau_3, \tau_4) = (3/8)(\tau_1+\tau_2+\tau_3+\tau_4)^2$ of Equation (82) in the last equation. Combined with Equation (101), we have

$$\begin{aligned} W_i(\vec{\tau}) = \frac{4}{3}&\left[\left(\frac{1}{\tau_1+\tau_2+\tau_3+\tau_4}\right)v_i - \frac{3}{8}(\tau_1+\tau_2+\tau_3+\tau_4)\right]\frac{\partial q_1(\tau_1, \tau_2, \tau_3, \tau_4)}{\partial\tau_i} \\ &+ q_1(\tau_1, \tau_2, \tau_3, \tau_4), \ i = 1, 2, 3, 4, \end{aligned} \tag{103}$$

which is the connection of the solutions between the Whitham Equation (37) of 1-genus and Euler–Poisson–Darboux equations.

5. A Step-like Initial Value Problem

Consider Equation (1) with the initial data

$$\phi(x,0) = \sqrt{\rho(x,0)}\exp(i\frac{S(x,0)}{\epsilon}), \tag{104}$$

where $\rho(x,0)$ and $S(x,0)$ are real functions that are independent of ϵ. The solution of Equation (1) in the form

$$\phi(x,t;\epsilon) = \sqrt{\rho(x,t;\epsilon)}\exp(i\frac{S(x,t;\epsilon)}{\epsilon}), \tag{105}$$

where ρ is the density, and $v = \partial S/\partial x$ is the velocity of the hydrodynamics. Substituting (105) into Equation (1), the conservation form of the cmKdV Equation (1) is obtained. Taking the limit $\epsilon \to 0$ and writing the conservation form in diagonal form, the 0-genus Whitham equation is derived again [36], which is identical to (26).

We consider the step-like initial data

$$\rho(x,0) = \begin{cases} \rho^L, & x < 0, \\ \rho^R, & x > 0, \end{cases} \quad \text{and} \quad v(x,0) = \begin{cases} v^L, & x < 0, \\ v^R, & x > 0, \end{cases} \tag{106}$$

where ρ^L, ρ^R, v^L and v^R are constants. Choosing $\rho^L = 0.1225$, $\rho^R = 1.1025$, $v^L = -1.3$ and $v^R = 0.5$, the distributions of Riemann variables and the structures of density function ρ are shown in Figure 1. Here, we have used a 1-genus Whitham Equation (37). The fundamental wave structures of rarefaction wave (RW) and DSW, along with the relevant mathematical theory, are presented in Ref. [37]. In this case, the Riemann invariants are divided into six regions, which, from left to right, are plateau, DSW, RW, DSW, RW and plateau. We now give the solutions for each region and the boundary velocities.

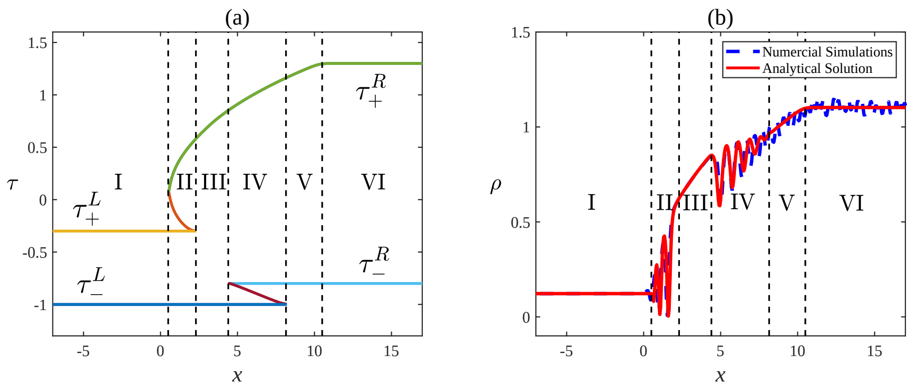

Figure 1. The self-similar solution with initial condition of $\tau_+^R = 1.3$, $\tau_-^R = -0.8$, $\tau_+^L = -0.3$, $\tau_-^L = -1$ with $t = 4$. (**a**) The distribution of Riemann invariants. (**b**) The structure diagram of density function ρ.

(I) For $x/t \leqslant v_2(\tau^*, \tau^*, \tau_+^L, \tau_-^L)$,

$$\tau_+ = \tau_+^L, \quad \tau_- = \tau_-^L,$$

where τ^* satisfies $\partial v_2(\tau^*, \tau^*, \tau_+^L, \tau_-^L)/\partial\tau^* = 0$, is the vertex of the parabola.

(II) For $v_2(\tau^*, \tau^*, \tau_+^L, \tau_-^L) < x/t < v_2(\tau_1, \tau_+^L, \tau_+^L, \tau_-^L)$,

$$\frac{x}{t} = v_1(\tau_1, \tau_2, \tau_+^L, \tau_-^L), \quad \frac{x}{t} = v_2(\tau_1, \tau_2, \tau_+^L, \tau_-^L), \quad \tau_3 = \lambda_+^L, \quad \tau_4 = \lambda_-^L.$$

(III) For $v_2(\tau_1, \tau_+^L, \tau_+^L, \tau_-^L) \leqslant x/t \leqslant v_2(\tau_1, \tau_-^R, \tau_-^R, \tau_-^L)$,

$$\tau_+ = -\frac{1}{5}\tau_-^L + \frac{2}{15}\sqrt{30 \cdot \frac{x}{t} - 9(\tau_-^L)^2}, \ \ \tau_- = \tau_-^L.$$

(IV) For $v_2(\tau_1, \tau_-^R, \tau_-^R, \tau_-^L) < x/t < v_2(\tau_1, \tau_-^R, \tau_-^L, \tau_-^L)$,

$$\frac{x}{t} = v_1(\tau_1, \tau_-^R, \tau_3, \tau_-^L), \ \ \tau_2 = \tau_-^R, \ \ \frac{x}{t} = v_3(\tau_1, \tau_-^R, \tau_3, \tau_-^L), \ \ \tau_4 = \tau_-^L.$$

(V) For $v_2(\tau_1, \tau_-^R, \tau_-^L, \tau_-^L) \leqslant x/t \leqslant v_+(\tau_+^R, \tau_-^R)$

$$\tau_+ = -\frac{1}{5}\tau_-^R + \frac{2}{15}\sqrt{30 \cdot \frac{x}{t} - 9(\tau_-^R)^2}, \ \ \tau_- = \tau_-^R.$$

(VI) For $x/t > v_+(\tau_+^R, \tau_-^R)$,

$$\tau_+ = \tau_+^R, \ \ \tau_- = \tau_-^R.$$

Direct numerical simulation can effectively test the validity of theoretical solutions [47]. Here, we compare the Whitham theoretical solution with the direct numerical simulation solution obtained from finite differences. It turns out that the two results are in good agreement (see Figure 1). However, it is important to note that in this study, we only considered one specific initial value problem. In fact, using the same method, we can obtain solutions for arbitrary step-like initial data.

6. Conclusions

In conclusion, we have studied the cmKdV Equation (1), which is the second member in the cmKdV hierarchy. We have derived the N-genus Whitham equations based on the Lax pair and provided detailed descriptions of the 0-genus (26) and 1-genus (37) Whitham equations as special cases. In order to solve the Whitham equation, Krichever's algebro-geometric scheme was established, and the connection between Whitham equation of single-genus and Euler–Poisson–Darboux equations was constructed. With respect to the cmKdV Equation (1), which involves two initial curves, we have employed a simple example where one curve remained constant while the other one varied. Additionally, a step-like initial value problem as an example was solved, and the Whitham modulation solution agreed well with the numerical solution (seen in Figure 1), which proves the validity of the theoretical solution. The results of this paper also provide an effective mathematical method for solving discontinuous initial value problems.

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