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Approximate Subdifferential of the Difference of Two Vector Convex Mappings

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Abstract: This paper deals with the strong approximate subdifferential formula for the difference of two vector convex mappings in terms of the star difference. This formula is obtained via a scalarization process by using the approximate subdifferential of the difference of two real convex functions established by Martinez-Legaz and Seeger, and the concept of regular subdifferentiability. This formula allows us to establish approximate optimality conditions characterizing the approximate strong efficient solution for a general DC problem and for a multiobjective fractional programming problem.

Keywords: vector optimization; optimality condition; approximate subdifferential; DC programming

MSC: 90C46; 58C20; 90C32

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1. Introduction

It is well known that the theory of DC mathematical programming, dealing with functions expressed as a difference of two convex functions, is now very well developed due to its theoretical aspects and extensive range of practical applications in optimal control, mechanics, operations research, and others (see [1–8] and references therein). This theory constitutes an important approach to nonconvex optimization problems. In machine learning, a lot of important learning problems such as Boltzmann machines can be formulated as DC programming (see [9]).

The overview paper [4] presents essential results on theory, applications, and solution methods for DC programming in the sense of global optimization. Significant advances have been made in the study of duality theory associated with constrained DC optimization problems (see [10–15]).

The motivation for this paper stems from the significant contributions of Martinez-Legaz and Seeger [16], who established a formula for the approximate subdifferential of the difference of two convex functions over a locally convex topological vector space. This formula is expressed in terms of the star difference of two subsets, and the authors provided an application for DC programming.

The aim of this work is to show how the formula established by Martinez-Legaz and Seeger can be used to obtain the approximate subdifferential of the difference of two vector convex mappings by using the vector strong subdifferential, the concept of subdifferential regularity [17], and a scalarization process. Two illustrations are given: the first deals with a constrained DC vector programming problem, and the second deals with a constrained multiobjective fractional programming problem. The rest of the work is organized as follows. In Section 2, we present some basic definitions and preliminary material. In

Section 3, we recall the formula established by Martinez-Legaz and Seeger [16] and show how this formula can be used to obtain the approximate subdifferential of the difference of two vector convex mappings. In Sections 4 and 5, we derive, from the obtained formula, optimality conditions for two vector cone-constrained programming problems. Finally, the paper ends with a conclusion and future work.

2. Preliminaries

In this paper, let E, F, and G be tree real Hausdorff locally convex topological vector spaces. The space F (respectively, G) is endowed with a nonempty convex cone $F_+ \subset F$ (respectively, $G_+ \subset G$) introducing a partial preorder in F (respectively, in G) defined by: for $y,\ y' \in F_+$

$$y \leq_{F_+} y' \Longleftrightarrow y' - y \in F_+.$$

We adjoin to F (respectively to G) two abstract elements $+\infty_F$ and $-\infty_F$, such that

$$\begin{cases} -\infty_F = -(+\infty_F), \\ y - \infty_F \leq_{F_+} y',\ \forall y, y' \in F \\ (+\infty_F) - (+\infty_F) = +\infty_F, \\ y \leq_{F_+} y' + \infty_F = +\infty_F,\ \forall y, y' \in F \cup \{+\infty_F\} \\ \beta.(+\infty_F) = +\infty_F,\ \forall \beta \geq 0 \end{cases}.$$

The dual topological spaces of E and G are denoted respectively by E^* and G^*, and the duality pairing in G is denoted by $\langle g^*, z\rangle$, with $g^* \in G^*$ and $z \in G$. The positive dual cone of G_+ is defined by

$$G_+^* := \{g^* \in G^* : \langle g^*, z\rangle \geq 0,\ \forall z \in G_+\}.$$

Let $\emptyset \neq S \subset F$. The point $m \in F$ is said to be a lower bound of S if $m \leq_{F_+} y$, for all $y \in S$. We denote by $\inf S$, if it exists, the greatest lower bound of S.

Let B and C be two nonempty subsets of F, and $\alpha \geq 0$. The following operations will be used:

$$\begin{aligned} B + C &:= \{x + y : x \in B,\ y \in C\} \\ \alpha B &:= \{\alpha x,\ x \in B\} \\ \emptyset + B = B + \emptyset &:= \emptyset. \end{aligned}$$

Let $H : E \longrightarrow F \cup \{+\infty_F\}$ be a given mapping. The effective domain of H is denoted by

$$domH := \{x \in E :\ H(x) \in F\}.$$

We say that H is proper when $domH \neq \emptyset$. The epigraph of the mapping H is denoted by $EpiH$, which is defined as follows:

$$EpiH := \{(x, y) : H(x) \leq_{F_+} y\}.$$

H is called F_+-convex if

$$H(\alpha x + (1-\alpha)\tilde{x}) \leq_{F_+} \alpha H(x) + (1-\alpha)H(\tilde{x}),\ \forall \alpha \in [0,1],\ \forall x, \tilde{x} \in E.$$

A mapping $K : F \longrightarrow G \cup \{+\infty_G\}$ is said to be (F_+, G_+)-increasing if for all $y, y' \in F$,

$$y \leq_{F_+} y' \Longrightarrow K(y) \leq_{G_+} K(y').$$

The composed mapping $K \circ H : E \longrightarrow G \cup \{+\infty_G\}$ is defined as follows:

$$(K \circ H)(x) := \begin{cases} K(H(x)), \text{ if } x \in domH \\ \\ +\infty_G, \qquad \text{else.} \end{cases}$$

Let us note that if K is (F_+, G_+)-increasing and G_+-convex and if H is F_+-convex, then $K \circ H$ is G_+-convex.

Following [18], whenever $\widetilde{x} \in domH$ and $\epsilon \in F_+$, the strong ϵ-subdifferential of H at $\widetilde{x}$ is defined by

$$\partial^s_\epsilon H(\widetilde{x}) := \left\{T \in L(E,F) : T(x - \widetilde{x}) - \epsilon \leq_{F_+} H(x) - H(\widetilde{x}),\ \forall x \in E\right\},$$

where $L(E,F)$ denotes the vector space of continuous linear mappings from E to F. For $\epsilon = 0$, we have the usual strong vector subdifferential

$$\partial^s H(\widetilde{x}) := \left\{T \in L(E,F) : T(x - \widetilde{x}) \leq_{F_+} H(x) - H(\widetilde{x}), \forall x \in E\right\}.$$

If $\widetilde{x} \notin domH$, we set $\partial^s_\epsilon H(\widetilde{x}) = \partial^s H(\widetilde{x}) := \emptyset$. Let us note that when $F = \mathbb{R}$, $\partial^s_\epsilon H(\widetilde{x})$ reduces to the usual subdifferential of convex analysis, denoted by

$$\partial_\epsilon H(\widetilde{x}) := \{e^* \in E^* : \langle e^*, x - \widetilde{x}\rangle - \epsilon \leq H(x) - H(\widetilde{x}), \forall x \in E\}.$$

3. Approximate Subdifferential of the Difference of Two Vector Convex Mappings

In this section, we attempt to extend the formula of [16] for the difference of two vector-valued mappings. Let us recall this scalar formula [16] expressed by means of the star difference of two subsets of E^*.

Definition 1 ([19])**.** *The star difference between two subsets B and C of E^* is given by*

$$B \boxminus C = \{e^* \in E^* : e^* + C \subset B\}.$$

Theorem 1 ([16])**.** *Let H, $K : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be two proper functions, $x \in domH \cap domK$, and $\epsilon \geq 0$. If H and K are lower semicontinuous and convex, then*

$$\partial_\epsilon (H - K)(x) = \bigcap_{\eta \geq 0} \left\{\partial_{\eta+\epsilon} H(x) \boxminus \partial_\eta K(x)\right\}.$$

Let $H : E \longrightarrow G \cup \{+\infty_G\}$ and $g^* \in G^*_+ \backslash \{0\}$. The scalar function $g^* \circ H : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$(g^* \circ H)(x) = \begin{cases} \langle g^*, H(x)\rangle, \text{ if } x \in domH \\ \\ +\infty, \qquad \text{else.} \end{cases}$$

Let us note that, for any $g^* \in G^*_+ \backslash \{0\}$, $dom(g^* \circ H) = domH$ and, if H is G_+-convex, then $g^* \circ H$ is convex. In order to state our main result, we will need the following lemma.

Lemma 1. *1. If G_+ is closed and if there exists $z \in G$ such that $\langle g^*, z\rangle \geq 0$, for all $g^* \in G^*_+ \backslash \{0\}$, then $z \in G_+$;*

2. Let $H : E \longrightarrow G \cup \{+\infty_G\}$ be a given G_+-convex mapping, $\epsilon \in G_+$, and $\widetilde{x} \in E$. If G_+ is closed, then

$$\partial^s_\epsilon H(\widetilde{x}) = \bigcap_{g^* \in G^*_+ \backslash \{0\}} \left\{A \in L(E,G),\ g^* \circ A \in \partial_{\langle g^*, \epsilon\rangle}(g^* \circ H)(\widetilde{x})\right\};$$

3. *If the topological interior of G_+ is nonempty and $intG_+ \neq \emptyset$, then for any $g^* \in G_+^* \backslash \{0\}$, we have*
$$\mathbb{R}^+ = \{\langle g^*, \theta\rangle,\ \theta \in (intG_+) \cup \{0_G\}\}, \tag{1}$$
where $\mathbb{R}^+ = [0, +\infty[$.

Proof. 1. See ([20] Proposition 2.1).
2. See ([17] Theorem 3.2).
3. We have $\{\langle g^*, \theta\rangle,\ \theta \in (intG_+) \cup \{0_G\}\} \subset \mathbb{R}^+$ for any $g^* \in G_+^* \backslash \{0\}$. For the reverse inclusion, let $g^* \in G_+^* \backslash \{0\}$ and $\alpha \in \mathbb{R}^+$. If $\alpha = 0$, we obviously have $0 = \langle g^*, 0_G\rangle$. Following ([20] Proposition 2.1), there exists some $z_0 \in intG_+$ such that $\langle g^*, z_0\rangle = 1$, and hence we write $\alpha = \langle g^*, \alpha z_0\rangle$ for any $\alpha > 0$. Let us note that $\alpha z_0 \in intG_+$ since $\alpha > 0$, $z_0 \in intG_+$, and $intG_+$ is a cone.

□

We say that a vector valued mapping $K : E \longrightarrow G \cup \{+\infty_G\}$ is star G_+-lower semicontinuous at $\tilde{x}$ if the function $g^* \circ K$ is lower semicontinuous at $\tilde{x}$ for any $g^* \in G_+^*$ (see [21]), and K is called weak regular γ-subdifferentiable at $\tilde{x} \in domK$, where $\gamma \geq 0$ (see [17]), if
$$\partial_\gamma(g^* \circ K)(\tilde{x}) = \bigcup_{\substack{\eta \in G_+^\gamma \\ \langle g^*, \eta\rangle = \gamma}} g^* \circ \partial_\eta^s K(\tilde{x}),\ \forall g^* \in G_+^* \backslash \{0\}$$
where $G_+^0 = \{0_G\}$ and $G_+^\gamma = G_+$ if $\gamma > 0$.

If $\gamma = 0$, we say K is weak regular subdifferentiable at $\tilde{x}$.

Theorem 2. *Let $H,\ K : E \longrightarrow G \cup \{+\infty_G\}$ be two given mappings, $\tilde{x} \in domH \cap domK$ and $\epsilon \in G_+$. Then*
$$\partial_\epsilon^s(H - K)(\tilde{x}) \subseteq \bigcap_{\eta \in G_+} \left\{\partial_{\eta+\epsilon}^s H(x) \boxminus \partial_\eta^s K(\tilde{x})\right\}$$
with equality if H and K are proper, G_+-convex, and star G_+-lower semicontinuous; K is weak regular γ-subdifferentiable at $\tilde{x}$ for all $\gamma \geq 0$ and the cone G_+ is closed; and $intG_+ \neq \emptyset$.

Proof. Let $T \in \partial_\epsilon^s(H - K)(\tilde{x})$, i.e.,
$$H(x) - K(x) - H(\tilde{x}) + K(\tilde{x}) - T(x - \tilde{x}) + \epsilon \in G_+,\ \forall x \in E. \tag{2}$$

Let $\eta \in G_+$. Then, for all $T' \in \partial_\eta^s K(x)$, we have
$$K(x) - K(\tilde{x}) - T'(x - \tilde{x}) + \eta \in G_+,\ \forall x \in E. \tag{3}$$

Adding (2) and (3) term by term, and since G_+ is a convex cone, we obtain
$$H(x) - H(\tilde{x}) - \big(T + T'\big)(x - \tilde{x}) + (\eta + \epsilon) \in G_+,\ \forall x \in E,$$
i.e.,
$$T + T' \in \partial_{\epsilon+\eta}^s H(\tilde{x}),\ \forall T' \in \partial_\eta^s K(\tilde{x}),$$
which yields that, for any $\eta \in G_+$,
$$T \in \partial_{\eta+\epsilon}^s H(\tilde{x}) \boxminus \partial_\eta^s K(\tilde{x}),$$
i.e.,
$$T \in \bigcap_{\eta \in G_+} \left\{\partial_{\eta+\epsilon}^s H(\tilde{x}) \boxminus \partial_\eta^s K(\tilde{x})\right\},$$

and the direct inclusion is proved. For the reverse inclusion, let

$$T \in \bigcap_{\eta \in G_+} \left\{ \partial^s_{\eta+\epsilon} H(\tilde{x}) \boxminus \partial^s_\eta K(\tilde{x}) \right\};$$

then, for every $\eta \in G_+$, we have

$$T \in \partial^s_{\eta+\epsilon} H(\tilde{x}) \boxminus \partial^s_\eta K(\tilde{x}),$$

i.e.,

$$T + T' \in \partial^s_{\epsilon+\eta} H(\tilde{x}),\ \forall T' \in \partial^s_\eta K(\tilde{x}).$$

Since H is G_+-convex, it follows according to property (2) of Lemma 1 that

$$g^* \circ T + g^* \circ T' \in \partial_{\langle g^*, \eta+\epsilon\rangle}(g^* \circ H)(\tilde{x}),\ \forall T' \in \partial^s_\eta K(\tilde{x}),\ \forall g^* \in G^*_+ \backslash \{0\},$$

and then

$$g^* \circ T + g^* \circ \partial^s_\eta K(\tilde{x}) \subset \partial_{\langle g^*, \epsilon\rangle + \langle g^*, \eta\rangle}(g^* \circ H)(\tilde{x}),\ \forall \eta \in G_+,\ \forall g^* \in G^*_+ \backslash \{0\}. \quad (4)$$

Let $\theta \in (intG_+) \cup \{0_G\}$. Since K is weak regular γ-subdifferentiable at $\tilde{x}$ for all $\gamma \geq 0$, then K is weak regular $\langle g^*, \theta\rangle$-subdifferentiable at $\tilde{x}$ for all $g^* \in G^*_+ \backslash \{0\}$, i.e.,

$$\partial_{\langle g^*, \theta\rangle}(g^* \circ K)(\tilde{x}) = \bigcup_{\substack{\eta \in G_+^{\langle g^*, \theta\rangle} \\ \langle g^*, \eta\rangle = \langle g^*, \theta\rangle}} g^* \circ \partial^s_\eta K(\tilde{x}),\ \forall g^* \in G^*_+ \backslash \{0\}, \quad (5)$$

with

$$G_+^{\langle g^*, \theta\rangle} := \begin{cases} 0, & \text{if } \theta = 0_G \\ G_+, & \text{if } \theta \in intG_+. \end{cases}$$

From (4), we deduce that, for any $\theta \in (intG_+) \cup \{0_G\}$ and $g^* \in G^*_+ \backslash \{0\}$, we have

$$g^* \circ T + \bigcup_{\substack{\eta \in G_+^{\langle g^*, \theta\rangle} \\ \langle g^*, \eta\rangle = \langle g^*, \theta\rangle}} g^* \circ \partial^s_\eta K(\tilde{x}) \subset \bigcup_{\substack{\eta \in G_+^{\langle z^*, \theta\rangle} \\ \langle g^*, \eta\rangle = \langle g^*, \theta\rangle}} \partial_{\langle g^*, \eta\rangle + \langle g^*, \epsilon\rangle}(g^* \circ H)(\tilde{x}), \quad (6)$$

i.e.,

$$g^* \circ T + \bigcup_{\substack{\eta \in G_+^{\langle g^*, \theta\rangle} \\ \langle g^*, \eta\rangle = \langle g^*, \theta\rangle}} g^* \circ \partial^s_\eta K(\tilde{x}) \subset \partial_{\langle g^*, \theta\rangle + \langle g^*, \epsilon\rangle}(g^* \circ H)(\tilde{x}). \quad (7)$$

From (5) and (7), it follows that

$$g^* \circ T + \partial_{\langle g^*, \theta\rangle}(g^* \circ K)(\tilde{x}) \subset \partial_{\langle g^*, \theta\rangle + \langle g^*, \epsilon\rangle}(g^* \circ H)(\tilde{x}),\ \forall \theta \in (intG_+) \cup \{0_G\},\ \forall g^* \in G^*_+ \backslash \{0\},$$

i.e.,

$$g^* \circ T \in \partial_{\langle g^*, \epsilon\rangle + \langle g^*, \theta\rangle}(g^* \circ H)(\tilde{x}) \boxminus \partial_{\langle g^*, \theta\rangle}(g^* \circ K)(\tilde{x}), \forall g^* \in G^*_+ \backslash \{0\}, \forall \theta \in (intG_+) \cup \{0_G\}.$$

Again, by applying the property (3) of Lemma 1, we can write

$$g^* \circ T \in \partial_{\langle g^*, \epsilon\rangle + \beta}(g^* \circ H)(\tilde{x}) \boxminus \partial_\beta(g^* \circ K)(\tilde{x}),\ \forall g^* \in G^*_+ \backslash \{0\},\ \forall \beta \geq 0,$$

which yields

$$g^* \circ T \in \bigcap_{\beta \geq 0} \left\{ \partial_{\langle g^*, \epsilon\rangle + \beta}(g^* \circ H)(\tilde{x}) \boxminus \partial_\beta(g^* \circ K)(\tilde{x}) \right\},\ \forall g^* \in G^*_+ \backslash \{0\}.$$

Since H and K are proper G_+-convex, star G_+-lower semicontinuous at $\widetilde{x} \in domH \cap domK$, then $g^* \circ H$ and $g^* \circ K$ are proper convex, lower semicontinuous functions and finite at $\widetilde{x}$; hence, by applying Theorem 1, we obtain

$$g^* \circ T \in \partial_{\langle g^*, \epsilon \rangle}(g^* \circ H - g^* \circ K)(\widetilde{x}),\ \forall g^* \in G_+^* \backslash \{0\},$$

i.e.,

$$g^* \circ T \in \partial_{\langle g^*, \epsilon \rangle} g^* \circ (H - K)(\widetilde{x}),\ \forall g^* \in G_+^* \backslash \{0\}.$$

By using the scalarization process of the strong subdifferential given by property (2) of Lemma 1, we obtain

$$T \in \partial_\epsilon^s (H - K)(\widetilde{x}).$$

This completes the proof. □

By taking $\epsilon = 0_G$ in Theorem 2, we obtain the formula of the exact subdifferential of the difference of two vector convex mappings.

Corollary 1. *Let $H,\ K : E \longrightarrow G \cup \{+\infty_G\}$ be two given mappings and $\widetilde{x} \in domH \cap domK$. Then*

$$\partial^s (H - K)(\widetilde{x}) \subseteq \bigcap_{\eta \in G_+} \left\{ \partial_\eta^s H(\widetilde{x}) \boxminus \partial_\eta^s K(\widetilde{x}) \right\}$$

with equality if H and K are proper, G_+-convex, and star G_+-lower semicontinuous; K is weak regular γ-subdifferentiable at $\widetilde{x}$ for all $\gamma \geq 0$; and the positive cone G_+ is closed and $intG_+ \neq \emptyset$.

4. Application to DC Vector Programming Problems

Let $H : E \longrightarrow G \cup \{+\infty_G\}$ be a mapping and $\epsilon \in G_+$. A point $\widetilde{x} \in domH$ is called an ϵ-minimizer of H on C if

$$H(\widetilde{x}) - \epsilon \leq_{G_+} H(x), \quad \forall x \in C,$$

where $\emptyset \neq C \subset E$. If $C = E$, we have that $\widetilde{x}$ is an ϵ-minimizer of H if and only if $0 \in \partial_\epsilon^s H(\widetilde{x})$. The vector indicator mapping $\delta_C^v : E \longrightarrow G \cup \{+\infty_G\}$ is defined by

$$x \longmapsto \delta_C^v(x) := \begin{cases} 0, & \text{if } x \in C \\ +\infty_G, & \text{else.} \end{cases}$$

The ϵ-normal set of C at $\widetilde{x} \in C$ in a vector sense is defined by

$$N_\epsilon^v(C, \widetilde{x}) := \partial_\epsilon^s \delta_C^v(\widetilde{x}) = \left\{ T \in L(E, G) : T(x - \widetilde{x}) \leq_{G_+} \epsilon,\ \forall x \in C \right\}.$$

It is clear that if $T \in L_+(F, G) := \{T \in L(F, G) : T(F_+) \subset G_+\}$, then T is (F_+, G_+)-increasing. By taking a G_+-convex mapping $L : E \longrightarrow F \cup \{+\infty_F\}$, it follows that the composed mapping $T \circ L : E \longrightarrow G \cup \{+\infty_G\}$ is G_+-convex.

We will need the following result later.

Lemma 2 ([22]). *We suppose that the convex cone G_+ is closed. For every $\epsilon \in G_+$, we have*

1. *If $y' \in F_+$, then*

$$T \in N_\epsilon^v(F_+, y') \Longleftrightarrow \begin{cases} -\ T \in L_+(F, G) \\ -\ \epsilon \leq_{G_+} T(y'); \end{cases}$$

2. *If $y' \in -F_+$, then*

$$T \in N_\epsilon^v(-F_+, y') \Longleftrightarrow \begin{cases} T \in L_+(F, G) \\ -\ \epsilon \leq_{G_+} T(y'). \end{cases}$$

In [18], Théra developed the calculus formula for the strong ϵ-subdifferential of the addition of two convex vector mappings. We need to recall that (G, G_+) is said to be order complete if $\inf A$ exists, for each nonempty subset $A \subset G$ order-bounded from below. We say that G is normal if there exists a basis of neighborhoods N of 0_G such that

$$N = (N + G_+) \cap (N - G_+).$$

Theorem 3 ([18]). *Let $H, K : E \longrightarrow G \cup \{+\infty_G\}$ be two G_+-convex mappings. If H is continuous at some point of $domH \cap domK$ and (G, G_+) is normal order-complete, then for every $x \in E$ and $\epsilon \in G_+$, we have*

$$\partial^s_\epsilon (H + K)(x) = \bigcup_{\substack{\epsilon_1 + \epsilon_2 = \epsilon \\ \epsilon_1,\, \epsilon_2 \in G_+}} \left\{ \partial^s_{\epsilon_1} H(x) + \partial^s_{\epsilon_2} K(x) \right\}.$$

Consider the following constrained DC vector minimization problem,

$$(P_1) \quad \begin{cases} \min(H(x) - K(x)) \\ \\ \quad x \in C \end{cases},$$

where $\emptyset \neq C \subset E$ is convex and $H, K : E \longrightarrow G \cup \{+\infty_G\}$ are two proper G_+-convex mappings. By using the vector indicator mapping, the problem (P_1) is equivalent to the following unconstrained problem:

$$\begin{cases} \min(H(x) + \delta^v_C(x) - K(x)) \\ \\ \quad x \in E \end{cases}.$$

Now, we establish necessary and sufficient optimality conditions for the minimization problem (P_1) characterizing an ϵ-minimizer.

Theorem 4. *Let $H, K : E \longrightarrow G \cup \{+\infty_G\}$ be two proper, G_+-convex and star G_+-lower semicontinuous mappings, and $\emptyset \neq C \subset E$ be convex and closed. If H is continuous at some point of $domH \cap C$, (G, G_+) is normal order-complete, K is weak regular γ-subdifferentiable at $\widetilde{x} \in domH \cap domK$ for all $\gamma \geq 0$, and the cone G_+ is closed and $intG_+ \neq \emptyset$. Then $\widetilde{x}$ is an ϵ-minimizer of the problem (P_1) if and only if, for all $\eta \in G_+$,*

$$\partial^s_\eta K(\widetilde{x}) \subset \bigcup_{\substack{\epsilon_1 + \epsilon_2 = \eta + \epsilon \\ \epsilon_1,\, \epsilon_2 \in G_+}} \left\{ \partial^s_{\epsilon_1} H(\widetilde{x}) + N^v_{\epsilon_2}(C, \widetilde{x}) \right\}.$$

Proof. We have that $\widetilde{x}$ is an ϵ-minimizer of the problem (P_1) if and only if

$$0 \in \partial^s_\epsilon ((H + \delta^v_C) - K)(\widetilde{x}). \tag{8}$$

Since the subset C is convex and nonempty, the vector indicator mapping δ^v_C is G_+-convex and proper. It is simple to observe that $g^* \circ \delta^v_C = \delta_C$, for any $g^* \in G^*_+$, where δ_C is the scalar indicator function of the subset C. Given the fact that C is closed, it follows that δ_C is lower semicontinuous, and we deduce that δ^v_C is star G_+-lower semicontinuous.

Since H and δ_C^v are G_+-convex, star G_+-lower semicontinuous, and $domH \cap C \neq \emptyset$, then $\left(H + \delta_C^v\right)$ is G_+-convex, proper, and star G_+-lower semicontinuous. As K is weak regular γ-subdifferentiable at $\tilde{x}$ for any $\gamma \geq 0$, and the positive cone G_+ is closed and $intG_+ \neq \emptyset$, then by virtue of Theorem 2, (8) becomes

$$0 \in \partial^s_{\eta+\epsilon}(H + \delta_C^v)(\tilde{x}) \boxminus \partial^s_\eta K(\tilde{x}), \quad \forall \eta \in G_+,$$

i.e.,

$$\partial^s_\eta K(\tilde{x}) \subset \partial^s_{\eta+\epsilon}(H + \delta_C^v)(\tilde{x}), \quad \forall \eta \in G_+. \tag{9}$$

As H and δ_C^v are G_+-convex, H is continuous at some point of $domH \cap C$, and (G, G_+) is normal order-complete; then, according to Theorem 3, (9) becomes

$$\partial^s_\eta K(x) \subset \bigcup_{\substack{\epsilon_1 + \epsilon_1 = \epsilon + \eta \\ \epsilon_1,\, \epsilon_2 \in G_+}} \left\{\partial^s_{\epsilon_1} H(\tilde{x}) + N^v_{\epsilon_2}(C, \tilde{x})\right\}, \quad \forall \eta \in G_+.$$

The proof is complete. $\square$

By taking $C = E$ in (9) of the above proof, we deduce the following proposition.

Proposition 1. *Let $H,\ K : E \longrightarrow G \cup \{+\infty_G\}$ be two proper, G_+-convex, and star G_+-lower semicontinuous mappings. If K is weak regular γ-subdifferentiable at $\tilde{x} \in domH \cap domK$ for all $\gamma \geq 0$, and the cone G_+ is closed and $intG_+ \neq \emptyset$, then $\tilde{x}$ is an ϵ-minimizer of $H - K$ if and only if*

$$\partial^s_\eta K(\tilde{x}) \subset \partial^s_{\eta+\epsilon} H(\tilde{x}),\ \forall \eta \in G_+.$$

In particular, $\tilde{x}$ is a minimizer of $H - K$ if and only if

$$\partial^s_\eta K(\tilde{x}) \subset \partial^s_\eta H(\tilde{x}),\ \forall \eta \in G_+.$$

Remark 1. *The above proposition generalizes a result due to Hiriart-Urruty's [3] characterizing a global minimum for a scalar DC programming problem.*

A point $\tilde{x} \in domH$ is said to be an ϵ-maximizer of H on C if

$$H(x) \leq_{G_+} H(\tilde{x}) + \epsilon,\ \forall x \in C.$$

Consider the following constraint convex vector maximization problem:

$$(P_2)\begin{cases} \max K(x) \\ x \in C \end{cases}.$$

The problem (P_2) becomes equivalent to

$$\begin{cases} \min(\delta_C^v(x) - K(x)) \\ x \in E \end{cases}.$$

Corollary 2. *Let $K : E \longrightarrow G \cup \{+\infty_G\}$ be a proper, G_+-convex, and star G_+-lower semicontinuous mapping and $\emptyset \neq C \subset E$ be convex and closed. If K is weak regular γ-subdifferentiable at $\tilde{x} \in domK \cap C$ for all $\gamma \geq 0$, and the cone G_+ is closed and $intG_+ \neq \emptyset$, then $\tilde{x}$ is an ϵ-maximizer of the problem (P_2) if and only if*

$$\partial^s_\eta K(\tilde{x}) \subset N^v_{\eta+\epsilon}(C, \tilde{x}),\ \forall \eta \in G_+.$$

Proof. It suffices to take $H = \delta^v_C$ in Proposition 1. $\square$

Let us now consider the following constrained vector minimization problem,

$$(P_3)\begin{cases}\min H(x) - K(x)\\ \\ L(x) \in -F_+,\end{cases}$$

where $H,\ K : E \longrightarrow G \cup \{+\infty_G\}$ are two G_+-convex mappings and $L : E \longrightarrow F \cup \{+\infty_F\}$ is a proper F_+-convex mapping. By using the vector indicator mapping $\delta^v_{-F_+}$, the unconstrained minimization problem below is equivalent to the problem (P_3):

$$\begin{cases}\min(H(x) + \delta^v_{-F_+} \circ L(x) - K(x))\\ \\ x \in E\end{cases}.$$

The following result will be required to state the necessary and sufficient approximate optimality conditions that characterize an ϵ-minimizer of problem (P_3).

Theorem 5 ([23])**.** *Let $H : E \longrightarrow G \cup \{+\infty_G\}$ be a proper G_+-convex mapping, $K : F \longrightarrow G \cup \{+\infty_G\}$ be a proper, G_+-convex, and (F_+, G_+)-increasing mapping and $L : E \longrightarrow F \cup \{+\infty_F\}$ be a proper and F_+-convex mapping. If there exists $a \in domH \cap domL \cap L^{-1}(domK)$ such that K is continuous at the point $L(a)$, then*

$$\partial^s_\epsilon(H + K \circ L)(\widetilde{x}) = \bigcup_{\substack{\eta+\eta'=\epsilon\\ \eta,\ \eta'\in G_+}} \Big\{\partial^s_\eta(H + T \circ L)(\widetilde{x}), T \in \partial^s_{\eta'}K(L(\widetilde{x}))\Big\},$$

for any $\widetilde{x} \in E$ and $\epsilon \in G_+$.

Now, we are prepared to announce the approximate optimality conditions related to problem (P_3).

Theorem 6. *Let $H,\ K : E \longrightarrow G \cup \{+\infty_G\}$ be two proper, G_+-convex, and star G_+-lower semicontinuous mappings, and $L : E \longrightarrow F \cup \{+\infty_F\}$ be a proper and F_+-convex mapping. If there exists some point $a \in domH \cap L^{-1}(-intF_+)$, $L^{-1}(-F_+)$ is closed, K is weak regular γ-subdifferentiable at $\widetilde{x} \in domH \cap domK \cap L^{-1}(-F_+)$ for all $\gamma \geq 0$, and the cone G_+ is closed and $intG_+ \neq \emptyset$, then $\widetilde{x}$ is an ϵ-minimizer of the problem (P_3) if and only if for any $\eta \in G_+$ and for any $A \in \partial^s_\eta K(\widetilde{x})$, there exist $\epsilon_1,\ \epsilon_2 \in G_+$ and $T \in L_+(F, G)$, satisfying $\epsilon_1 + \epsilon_2 = \eta + \epsilon$, $A \in \partial^s_{\epsilon_1}(H + T \circ L)(\widetilde{x})$, and $-\epsilon_2 \leq_{G_+} T(L(\widetilde{x}))$.*

Proof. The point $\widetilde{x}$ is an ϵ-minimizer of the problem (P_3) if and only if

$$0 \in \partial^s_\epsilon(H + \delta^v_{-F_+} \circ L - K)(\widetilde{x}). \tag{10}$$

Let us recall that the vector indicator mapping $\delta^v_{-F_+} : F \longrightarrow G \cup \{+\infty_G\}$ is (F_+, G_+)-increasing (see [20]) and G_+-convex. Since L is F_+-convex, then $\delta^v_{-F_+} \circ L$ is G_+-convex. The fact that $g^* \circ \delta^v_{-F_+} \circ L = \delta_{-F_+} \circ L$ for any $g^* \in G^*_+ \backslash \{0\}$, it follows that

$$\begin{aligned} Epi\Big(g^* \circ \delta^v_{-F_+} \circ L\Big) &= \{(x, \alpha) : L(x) \in -F_+,\ \alpha \in \mathbb{R}^+\}\\ &= L^{-1}(-F_+) \times \mathbb{R}^+, \end{aligned}$$

and as $L^{-1}(-F_+)$ is closed, we deduce that $Epi\Big(g^* \circ \delta^v_{-F_+} \circ L\Big)$ is closed, which yields that $\delta^v_{-F_+} \circ L$ is star G_+-lower semicontinuous. Since H is G_+-convex, star G_+-lower semicontinuous and $a \in domH \cap L^{-1}(-intF_+)$, it follows that $H + \delta^v_{-F_+} \circ L$ is G_+-convex, star

G_+-lower semicontinuous, and proper. We claim that $\delta^v_{-F_+}$ is continuous on $-intF_+$. Indeed, for any neighborhood V of 0_G, we have $\delta^v_{-F_+}(-intF_+) = \{0_G\} \subset V$. As $a \in L^{-1}(-intF_+)$, then $\delta^v_{-F_+}$ is continuous at $L(a)$. Let us note that all assumptions of Proposition 1 are satisfied; therefore, we obtain

$$\partial^s_\eta K(\widetilde{x}) \subset \partial^s_{\eta+\epsilon}\Big(H + \delta^v_{-F_+} \circ L\Big)(\widetilde{x}),\ \forall \eta \in G_+. \tag{11}$$

Let us observe that all hypotheses of Theorem 5 are satisfied and therefore (11) becomes equivalent to

$$\partial^s_\eta K(\widetilde{x}) \subset \bigcup_{\substack{\epsilon_1+\epsilon_2=\eta+\epsilon \\ \epsilon_1,\ \epsilon_2 \in G_+}} \Big\{\partial^s_{\epsilon_1}(H + T \circ L)(\widetilde{x}), T \in \partial^s_{\epsilon_2}\delta^v_{-F_+}(L(\widetilde{x}))\Big\},\ \forall \eta \in G_+,$$

i.e.,

$$\partial^s_\eta K(\widetilde{x}) \subset \bigcup_{\substack{\epsilon_1+\epsilon_2=\eta+\epsilon \\ \epsilon_1,\ \epsilon_2 \in G_+}} \{\partial^s_{\epsilon_1}(H + T \circ L)(\widetilde{x}), T \in N^v_{\epsilon_2}(-F_+, L(\widetilde{x}))\},\ \forall \eta \in G_+.$$

Therefore, by virtue of Lemma 2 we obtain, for any $\eta \in G_+$ and for any $A \in \partial^s_\eta K(\widetilde{x})$, there exist $\epsilon_1,\ \epsilon_2 \in G_+$, and $T \in L_+(F, G)$ satisfying $\epsilon_1 + \epsilon_2 = \eta + \epsilon$, $A \in \partial^s_{\epsilon_1}(H + T \circ L)(\widetilde{x})$, and $-\epsilon_2 \leq_{G_+} T(L(\widetilde{x}))$. This completes the proof. □

5. Application to a Multiobjective Fractional Programming Problem

This section focuses on a general multiobjective fractional programming problem,

$$(Q) \quad \begin{cases} \min\Big(\frac{h_1(x)}{k_1(x)}, \dots, \frac{h_n(x)}{k_n(x)}\Big) \\ \\ L(x) \in -F_+ \end{cases},$$

where the functions $h_i,\ k_i : E \longrightarrow \mathbb{R}$ are convex such that $h_i(x) \geq 0,\ k_i(x) > 0$, for any $x \in E$ $(i = 1, \dots, n)$, and $L : E \longrightarrow F \cup \{+\infty_F\}$ is a proper F_+-convex mapping. The following notation will be required:

$$\begin{aligned} &\epsilon := (\epsilon_1, \dots, \epsilon_n) \in \mathbb{R}^n_+, \\ &\nu_i := \tfrac{h_i(\widetilde{x})}{k_i(\widetilde{x})} - \epsilon_i, \\ &\overline{\epsilon} := (\epsilon_1 k_1(\widetilde{x}), \dots, \epsilon_n k_n(\widetilde{x})). \end{aligned}$$

The finite-dimensional space $G := \mathbb{R}^n$ is equipped with its natural order induced by the positive cone

$$G_+ := \mathbb{R}^n_+ = \{(d_1, \dots, d_n) \in \mathbb{R}^n\ :\ d_i \geq 0,\ \forall i = 1, \dots, n\},$$

i.e.,

$$(c_1, \dots, c_n) \leq_{\mathbb{R}^n_+} (d_1, \dots, d_n) \Longleftrightarrow c_i \leq d_i,\ \forall i = 1, \dots, n.$$

The following definition is equivalent to the one of an ϵ-minimizer.

Definition 2. *Let $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \mathbb{R}^n_+$. We say that a point $\widetilde{x} \in L^{-1}(-F_+)$ is an ϵ-minimizer of the problem (Q) if*

$$\frac{h_i(\widetilde{x})}{k_i(\widetilde{x})} - \epsilon_i \leq \frac{h_i(x)}{k_i(x)},\ \forall x \in L^{-1}(-F_+),\ \forall i = 1, \dots, n.$$

By using a parametric approach, we can equivalently convert the multiobjective fractional programming problem (Q) into a DC vector nonfractional programming problem defined in the following way:

$$(Q_{\widetilde{x}}) \quad \begin{cases} \min(H(x) - K_{\widetilde{x}}(x)) \\ \\ L(x) \in -F_+ \end{cases},$$

where $H : E \longrightarrow \mathbb{R}^n$ and $K_{\widetilde{x}} : E \longrightarrow \mathbb{R}^n$ are two mappings defined for every $x \in E$ by

$$H(x) := (h_1(x), \ldots, h_n(x)),\ K_{\widetilde{x}}(x) := (\nu_1 k_1(x), \ldots, \nu_n k_n(x)).$$

In order to relate the fractional programming problem (Q) to the DC vector optimization problem $(Q_{\widetilde{x}})$, we formulate the following lemma.

Lemma 3. *A point $\widetilde{x} \in L^{-1}(-F_+)$ is an ϵ-minimizer of (Q) if and only if $\widetilde{x}$ is an $\bar{\epsilon}$-minimizer of the problem $(Q_{\widetilde{x}})$.*

Proof. Assume that $\widetilde{x}$ is an ϵ-minimizer of (Q). From Definition 2, we have for each $i = 1, \ldots, n$

$$\frac{h_i(\widetilde{x})}{k_i(\widetilde{x})} - \epsilon_i \leq \frac{h_i(x)}{k_i(x)}, \quad \forall x \in L^{-1}(-F_+). \tag{12}$$

Since $k_i(x) > 0$, we deduce from (12) that $0 \leq h_i(x) - \nu_i k_i(x)$, for any $x \in L^{-1}(-F_+)$ and $i = 1, \ldots, n$. As $h_i(\widetilde{x}) - \nu_i k_i(\widetilde{x}) - \epsilon_i k_i(\widetilde{x}) = 0$, we write

$$0 = h_i(\widetilde{x}) - \nu_i k_i(\widetilde{x}) - \epsilon_i k_i(\widetilde{x}) \leq h_i(x) - \nu_i k_i(x),\ \forall x \in L^{-1}(-F_+),\ \forall i = 1, \ldots, n,$$

i.e.,

$$H(\widetilde{x}) - K_{\widetilde{x}}(\widetilde{x}) - \bar{\epsilon} \leq_{\mathbb{R}^n_+} H(x) - K_{\widetilde{x}}(x),\ \forall x \in L^{-1}(-F_+),$$

which yields that $\widetilde{x}$ is an $\bar{\epsilon}$-minimizer for the problem $(Q_{\widetilde{x}})$.

By using similar arguments as above, we show easily that if $\widetilde{x}$ is an $\bar{\epsilon}$-minimizer for the problem $(Q_{\widetilde{x}})$, then $\widetilde{x}$ is an ϵ-minimizer for the problem (Q).
This completes the proof. □

The problem $(Q_{\widetilde{x}})$ is reduced to the following unconstrained minimization problem:

$$\begin{cases} \min(H(x) + \delta^v_{-F_+} \circ L(x) - K_{\widetilde{x}}(x)) \\ \\ x \in E \end{cases}.$$

Proposition 2. *Let h_i, $k_i : E \longrightarrow \mathbb{R}$ be $2n$ convex and lower semicontinuous functions such that $h_i(x) \geq 0$ and $k_i(x) > 0$, for each $i = 1, \ldots, n$ and for any $x \in E$. Let $L : E \longrightarrow F \cup \{+\infty_F\}$ be a proper F_+-convex mapping. We assume that $L^{-1}(-F_+)$ is closed nonempty and there exists some $x_0 \in E$ such that $(n-1)$ functions k_i are continuous at x_0. Let $\epsilon = (\epsilon_1, ..., \epsilon_n) \in \mathbb{R}^n_+$, $\widetilde{x} \in L^{-1}(-F_+)$, and $\nu_i := \frac{h_i(\widetilde{x})}{k_i(\widetilde{x})} - \epsilon_i \geq 0$ $(i = 1, \ldots, n)$. Then, $\widetilde{x}$ is an ϵ-minimizer of the problem (Q) if and only if for any $\eta_i \geq 0$ and for any $e^*_i \in \partial_{\eta_i}(\nu_i k_i)(\widetilde{x})$, there exist $\epsilon^i_1, \epsilon^i_2 \geq 0$ and $T_i \in L_+(F, \mathbb{R})$ satisfying $\epsilon^i_1 + \epsilon^i_2 = \epsilon_i + \eta_i$, $e^*_i \in \partial_{\epsilon^i_1}(h_i + T_i \circ L)(\widetilde{x})$ and $-\epsilon^i_2 \leq T_i(L(\widetilde{x}))$.*

Proof. Let $\widetilde{x} \in L^{-1}(-F_+)$, $\bar{\epsilon} := (\epsilon_1 k_1(\widetilde{x}), ..., \epsilon_n k_n(\widetilde{x}))$, and $\eta = (\eta_1, ..., \eta_n) \in \mathbb{R}^n_+$. By Lemma 3, we have that $\widetilde{x}$ is an ϵ-minimizer of (Q) if and only if $\widetilde{x}$ is an $\bar{\epsilon}$-minimizer of the problem $(Q_{\widetilde{x}})$ i.e.,

$$0 \in \partial^s_{\epsilon}(H + \delta^v_{-F_+} \circ L - K_{\widetilde{x}})(\widetilde{x}).$$

Let us note that in this situation $G = \mathbb{R}^n$ and $G_+ = \mathbb{R}^n_+$, which is a closed convex cone, and $intG_+ \neq \emptyset$; hence, the $\mathbb{R}^n_+$-convexity of the mappings H and $K_{\widetilde{x}}$ follows easily from the convexity of the functions h_i and k_i for $i = 1, \ldots, n$. For any $g^* = (\alpha_1, \ldots, \alpha_n) \in G^*_+ = \mathbb{R}^n_+$, we have $g^* \circ H = \sum_{i=1}^n \alpha_i h_i$ and, since h_i is lower semicontinuous, we deduce that $g^* \circ H$ is lower semicontinuous, which yields that H is star $\mathbb{R}^n_+$-lower semicontinuous. Similarly, we show that $K_{\widetilde{x}}$ is also star $\mathbb{R}^n_+$-lower semicontinuous.

Let $\gamma \geq 0$, by virtue of [17], the γ-weak subdifferential regularity of $K_{\widetilde{x}} = (\nu_1 k_1, \ldots, \nu_n k_n)$ becomes exactly a famous chain rule of convex analysis, i.e., for any $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_+ \setminus \{0_{\mathbb{R}^n}\}$, we have

$$\partial_\gamma\Big(\sum_{i=1}^n \alpha_i \nu_i k_i\Big) = \bigcup_{\substack{\epsilon_i \geq 0 \\ \sum_{i=1}^n \alpha_i \epsilon_i = \gamma}} \sum_{i=1}^n \alpha_i \partial_{\epsilon_i}(\nu_i k_i)$$

and this formula holds under the popular Moreau–Rockafellar qualification condition, i.e., the functions k_i, $(i = 1, \ldots, n)$ are convex and there exits some $x_0 \in E$ such that $(n-1)$ functions k_i are continuous at x_0. For our purpose, this qualification condition is satisfied. Let us emphasize that all the assumptions of Theorem 6 are fulfilled; therefore, $\widetilde{x}$ is an ϵ-minimizer of the problem (Q) if and only if for any $A \in \partial^s_\eta K_{\widetilde{x}}(\widetilde{x})$, there exist $\epsilon_1, \epsilon_2 \in \mathbb{R}^n_+$ and $T \in L_+(F, \mathbb{R}^n)$ satisfying $\epsilon_1 + \epsilon_2 = \eta + \epsilon$, $A \in \partial^s_{\epsilon_1}(H + T \circ L)(\widetilde{x})$ and $-\epsilon_2 \leq_{\mathbb{R}^n_+} T(L(\widetilde{x}))$.

The strong η-subdifferential $\partial^s_\eta K_{\widetilde{x}}(\widetilde{x})$ reduces to

$$\partial^s_\eta K_{\widetilde{x}}(\widetilde{x}) = \partial_{\eta_1}(\nu_1 k_1)(\widetilde{x}) \times \ldots \times \partial_{\eta_n}(\nu_n k_n)(\widetilde{x}).$$

The condition $T \in L_+(E, \mathbb{R}^n)$ can be written as $T = (T_1, \ldots, T_n)$ where $T_i \in L_+(E, \mathbb{R})$. The composed mapping $T \circ L : E \longrightarrow \mathbb{R}^n \cup \{+\infty_{\mathbb{R}^n}\}$ is defined by

$$(T \circ L)(x) := \begin{cases} T(L(x)) = (T_1(L(x)), \ldots, T_n(L(x))), & \text{if } x \in domL \\ \\ +\infty_{\mathbb{R}^n}, & \text{otherwise} \end{cases}.$$

Now, we can write $A = (e_1^*, \ldots, e_n^*)$ with $e_i^* \in E^*$, and hence we obtain

$$A \in \partial^s_\eta K_{\widetilde{x}}(\widetilde{x}) \Longleftrightarrow e_i^* \in \partial_{\eta_i}(\nu_i k_i)(\widetilde{x}),\ \forall i = 1, \ldots, n.$$

The condition $A \in \partial^s_{\epsilon_1}(H + T \circ L)(\widetilde{x})$ may be rewritten as $e_i^* \in \partial_{\epsilon_1^i}(h_i + T_i \circ L)(\widetilde{x})$ for any $i = 1, \ldots, n$. Obviously, the condition $-\epsilon_2 \leq_{\mathbb{R}^n_+} T(L(\widetilde{x}))$ is equivalent to $-\epsilon_2^i \leq T_i(L(\widetilde{x}))$, for any $i = 1, ..., n$.

The proof is complete. □

6. Conclusions and Discussion

Our investigation in this article aimed to extend within the setting of vector convex mappings a formula [16] dealing with the approximate subdifferential of the difference of two real convex functions. This is obtained by a scalarization process by using this scalar formula, the regular subdifferentiability concept, and the difference star operation. Therefore, the established result allows us to obtain the existence of approximate strong solutions to a constrained vector DC programming problem and a constrained multiobjective fractional problem.

Let us note that a similar result of Proposition 1 was developed by Hiriart-Urruty [3] for an unconstrained scalar DC optimization problem in terms of Fenchel approximate subdifferentials characterizing a global (exact or approximate) solution. Additionally, in [5], a similar condition is established characterizing a weakly efficient solution for the difference of two vector mappings in finite or infinite-dimensional preordered space.

In a forthcoming work, we will try to study a Pareto version (weak and proper) of the above formula and also, we will attempt to find efficient algorithms for solving numerically this class of problems.

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