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Positive Periodic Solution for Neutral-Type Integral Differential Equation Arising in Epidemic Model

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Abstract: This paper is devoted to investigating a class of neutral-type integral differential equations arising in an epidemic model. By using Mawhin's continuation theorem and the properties of neutral-type operators, we obtain the existence conditions for positive periodic solutions of the considered neutral-type integral differential equation. Compared with previous results, the existence conditions in this paper are less restricted, thus extending the results of the existing literature. Finally, two examples are given to show the effectiveness and merits of the main results of this paper. Our results can be used to obtain the existence of a positive periodic solution to the corresponding non-neutral-type integral differential equation.

Keywords: positive periodic solution; existence; neutral-type; time-varying delay

MSC: 45D05; 45G10; 47H30

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1. Introduction

In this paper, we consider the following two classes of neutral-type integral differential equations arising in an epidemic model:

$$u(t) = au(t-\sigma) + b\int_{t-\sigma}^{t} f(s, u(s), u'(s))ds \tag{1}$$

and

$$u(t) = au(t-\tau(t)) + b\int_{t-\tau(t)}^{t} f(s, u(s), u'(s))ds. \tag{2}$$

For Equation (1), $u(t)$ represents the population of infectious individuals at time t, $a > 0$ is the effective contraction rate, $b \in \mathbb{R}$ represents the impact rate of the external environment, $f(t, u(t), u'(t)) \in C(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}, (0, \infty))$ is the instantaneous rate of infection, and $f(t, u(t), u'(t))dt$ is the fraction of individuals infected within the period $[t, t + dt]$. The constant delay σ can be interpreted as the duration of an infection. The number of all infected individuals is the total number of infections between $t - \sigma$ and t. The meanings of u, a, b and f in Equation (2) are similar to the corresponding ones in Equation (1). The time-varying delay $\tau(t) \in C^1(\mathbb{R}, \mathbb{R})$ is a $\omega-$periodic function that represents the duration of infectivity, and the number of all infected individuals is the total number of infections between $t - \tau(t)$ and t. Time delay is an inherent feature of the equation and becomes one of the main sources for causing existence and stability. Particularly, when the delay is a constant, Equation (1) is equivalent to Equation (7), and we can use Lemma 1 to study Equation (7). In addition, when the delay is time-varying, Equation (2) is equivalent to Equation (30), and we can use Lemma 2 to study Equation (30). Therefore, the research methods for different types of time delays are completely different.

Models similar to Equations (1) and (2) have been extensively studied. In 1990, Fink and Gatica [1] firstly studied the following equation:

$$u(t) = \int_{t-\sigma}^{t} f(s, u(s))ds \tag{3}$$

where the delay σ is a constant. The existence results of positive almost periodic solutions to (3) have been obtained. When the delay σ in (3) is a time-varying $\sigma(t)$, related research can be found in [2–5]. Specially, for the existence of a positive pseudo almost periodic solution, see [2,6]; for the existence of a positive almost periodic solution, see [3,5]; for the existence of a positive almost automorphic solution, see [4,7,8]. For σ in (3) as a state-dependent delay $\sigma(x(t))$, Torrejón [9] dealt with the positive almost periodic solution of (3). In [10], the authors studied the synchronization problem for an epidemic system with a Neumann boundary value under delayed impulse. Stability analysis of multi-point boundary conditions for a fractional differential equation with a non-instantaneous integral impulse was considered in [11]. Zhao and Zhu [12] investigated stabilization of stochastic highly nonlinear delay systems with a neutral term. Wang and Yao [13] studied a class impulsive stochastic food chain system with time-varying delays and obtained practical exponential stability conditions. For more results about functional differential and integral equations, see, e.g., [14–18].

This article focuses on neutral-type nonlinear integral equations arising in an epidemic model. In [19], the authors considered the existence of positive almost automorphic solutions to the neutral-type integral differential equation as follows:

$$u(t) = au(t-\tau) + (1-a)\int_{t-\tau}^{t} f(s, u(s))ds,$$

where $0 \le a < 1$, $\tau > 0$ is a constant. Furthermore, in [20], they studied the following neutral-type integral differential equation with time-varying delay:

$$u(t) = au(t-\tau(t)) + (1-a)\int_{t-\tau(t)}^{t} f(s, u(s), u'(s))ds,$$

where $0 \le a < 1$, $\tau(t)$ is a time-varying delay. We note that the research method in the above papers is based on the fixed point theorem. In this article, we use Mawhin's continuity theorem to study the existence of positive periodic solutions for Equations (1) and (2). The existence conditions obtained in this article are easy to verify, thus promoting the study of Equations (1) and (2).

The main contributions are summarized in the following two aspects:

(1) We extend the scope of the parameter a from $0 < a < 1$ to $|a| \neq 1$ with $a > 0$ and obtain sufficient conditions for the existence of a positive periodic solution to Equations (1) and (2).

(2) We innovatively use Mawhin's continuation theorem to study the existence of positive periodic solutions for Equations (1) and (2).

The following sections are organized as follows: Section 2 gives some preliminaries. We obtain the existence of positive periodic solutions for Equations (1) and (2) in Sections 3 and 4, respectively. Section 5 discusses two examples that show the feasibility of our results. Finally, Section 6 concludes the paper.

2. Preliminaries

Lemma 1 ([21,22])**.** *Let:*

$$D : P_\omega \to P_\omega, \ \ [Du](t) = u(t) - au(t-\tau), \ \ \forall t \in \mathbb{R},$$

where P_ω is a $\omega-$periodic continuous function space, and a and $\tau > 0$ are constants. If $|a| \neq 1$, then the operator D has a continuous inverse D^{-1} on P_ω satisfying:

(1) $[D^{-1}u](t) = \begin{cases} \sum_{n\geq 0} a^n u(t-n\tau), & \text{for } |a|<1,\ \forall u \in P_\omega, \\ \sum_{n\geq 0} a^{-n-1} u(t+n\tau), & |a|>1,\ \forall u \in P_\omega, \end{cases}$

(2) $|[D^{-1}u](t)| \leq \frac{1}{|1-|a||}|u(t)|, \quad \forall u \in P_\omega,$

(3) $\int_0^\omega |[D^{-1}u](t)|dt \leq \frac{1}{|1-|a||}\int_0^\omega |u(t)|dt, \quad \forall u \in P_\omega.$

Lemma 2 ([23])**.** *Let:*

$$D : P_\omega \to P_\omega,\ [Du](t) = u(t) - \delta(t)u(t-\gamma(t)),\ \forall t \in \mathbb{R},$$

where P_ω is an $\omega-$periodic continuous function space, and $\delta(t)$ and $\gamma(t)$ are $\omega-$periodic continuous functions. If $|\delta(t)| \neq 1$, then operator D has a continuous inverse D^{-1} on P_ω satisfying:

(1) $[D^{-1}u](t) = \begin{cases} u(t) + \sum_{j=1}^{\infty} \prod_{i=1}^{j} \delta(A_i) u(t - \prod_{i=1}^{j} \gamma(A_i)), & \text{for } |\delta(t)|<1,\ \forall u \in P_\omega, \\ -\frac{u(t+\gamma(t))}{\delta(t+\gamma(t))} - \sum_{j=1}^{\infty} \frac{u(t+\gamma(t)+\sum_{i=1}^{j}\gamma(A_i'))}{\delta(t+\gamma(t))\prod_{i=1}^{j}\delta(A_i')}, & \text{for } |\delta(t)|>1,\ \forall u \in P_\omega, \end{cases}$

(2) $||D^{-1}u|| \leq \begin{cases} \frac{1}{1-\delta_0}||u(t)||, \text{ for } \delta_0 < 1,\ \forall u \in P_\omega, \\ \frac{1}{\delta_1 - 1}||u(t)||, \text{ for } \delta_1 > 1,\ \forall u \in P_\omega, \end{cases}$

(3) $\int_0^\omega |[D^{-1}u](t)|dt \leq \begin{cases} \frac{1}{1-\delta_0}\int_0^\omega |u(t)|dt, \text{ for } \delta_0 < 1,\ \forall u \in P_\omega, \\ \frac{1}{\delta_1-1}\int_0^\omega |u(t)|dt, \text{ for } \delta_1 > 1,\ \forall u \in P_\omega, \end{cases}$

where:

$$\delta_0 = \max_{t\in[0,\omega]} |\delta(t)|,\ \delta_1 = \min_{t\in[0,\omega]} |\delta(t)|,\ D_1 = t,\ D_{j+1} = t - \sum_{i=1}^{j} \gamma(D_i),\ j = 1, 2, \cdots.$$

Now, we give the famous Mawhin's continuation theorem.

Lemma 3 ([24])**.** *Let A and B be two Banach spaces. Let $F : Dom(F) \subset A \to B$, be a Fredholm operator with index zero, where $Dom(F)$ is the domain of F. Furthermore, $\Theta \subset A$ is an open bounded set and $G : \bar{\Theta} \to B$ is L-compact on $\bar{\Theta}$. If the following conditions hold:*

(1) $Fu \neq \lambda Gu, \forall u \in \partial\Omega \cap D(F), \forall \mu \in (0,1),$

(2) $Gu \notin ImF, \forall u \in \partial\Theta \cap KerF,$

(3) $deg\{RG, \Theta \cap KerF, 0\} \neq 0,$

then equation $Fu = Gu$ has a solution on $\bar{\Theta} \cap Dom(F)$.

Remark 1. *In Lemm 1, when a is a constant in the $D-$operator (neutral-type operator) and the delay τ is a constant, the authors obtained the properties of the $D-$operator. In Lemm 2, when a is a continuous function $a(t)$ in the $D-$operator (neutral-type operator) and the delay τ is a continuous function $\tau(t)$, the authors obtained the properties of the $D-$operator. Obviously, Lemm 2 extends the results of Lemma 1 and has wider applications. Lemma 3 is the famous Mawhin's continuation theorem that has been widely used to study the periodic solution problem of functional differential equations.*

In the present paper, we need the following assumptions:

$(\mathcal{A}_1)$ $|a| \neq 1$ with $a > 0$.

$(\mathcal{A}_2)$ There exist positive constants $k_1,\ k_2,\ k_3$ and k_4 such that:

$$|f(t,u_1,v_1) - f(s,u_2,v_2)| \leq k_1|u_1-u_2| + k_2|v_1-v_2| \text{ for all } t,s,v_1,v_2 \in \mathbb{R},\ u_1,u_2 \in \mathbb{R}^+$$

and

$$f(t,u,v) \leq k_3 u - k_4 \text{ for all } t,v \in \mathbb{R},\ u \in \mathbb{R}^+.$$

($\mathcal{A}_3$) There exist positive constants c and M such that:

$$b\Big(f(t,c,0) - f(t-\sigma,c,0)\Big) \not\equiv 0 \text{ for all } t \in \mathbb{R},\ c > M.$$

($\mathcal{A}_4$) There exist positive constants c and M such that:

$$b\Big(f(t,c,0) - (1-\tau'(t))f(t-\tau(t),c,0)\Big) \not\equiv 0 \text{ for all } t \in \mathbb{R},\ c > M.$$

For obtaining the existence of positive periodic solutions to Equation (1), we need the assumptions $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$; for obtaining the existence of positive periodic solutions to Equation (2), we need the assumptions $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_4$.

3. Positive Periodic Solution for Equation (1)

Theorem 1. *Assume that ($\mathcal{A}_1$)–($\mathcal{A}_3$) hold. Then Equation (1) has at least one positive ω-periodic solution if:*

$$|1-|a||(1-k_2|b|) > k_2|b(a-1)|, \tag{4}$$

$$|1-|a||(1-k_1\omega|b|) > k_1\omega|ba| + |b|\sigma k_3, \tag{5}$$

$$\frac{(|1-|a||k_1|b|) + k_1|b(a-1)|}{|1-|a||(1-k_2|b|) - k_2|b(a-1)|}\,\frac{(|1-|a||k_2\omega|b|) + k_2\omega|ba|}{|1-|a||(1-k_1\omega|b|) - |b|\sigma k_3 - k_1\omega|ba|} < 1. \tag{6}$$

Proof. Taking the derivative on both sides of Equation (1) yields:

$$\Big(u(t) - au(t-\sigma)\Big)' = bf(t,u(t),u'(t)) - bf(t-\sigma,u(t-\sigma),u'(t-\sigma)). \tag{7}$$

Since Equation (1) is equivalent to Equation (7), we only need to consider the existence of positive periodic solutions for Equation (7). Let $(Du)(t) = u(t) - au(t-\sigma)$ in (7); then:

$$(Du)'(t) = bf(t,u(t),u'(t)) - bf(t-\sigma,u(t-\sigma),u'(t-\sigma)). \tag{8}$$

Let:

$$F : D(F) \subset P_\omega \to P_\omega,\ (Fu)(t) = (Du)'(t) \tag{9}$$

and

$$G : P_\omega \to P_\omega,\ (Gu)(t) = bf(t,u(t),u'(t)) - bf(t-\sigma,u(t-\sigma),u'(t-\sigma)). \tag{10}$$

Then Equation (8) can be represented by:

$$(Fu)(t) = (Gu)(t),$$

where F and G are defined by (9) and (10), respectively. Set:

$$\Theta_1 = \{u | u \in Dom(F),\ Fu = \mu Gu,\ \mu \in (0,1)\}.$$

For each $u \in \Theta_1$, we have:

$$(Du)'(t) = b\mu f(t,u(t),u'(t)) - b\mu f(t-\sigma,u(t-\sigma),u'(t-\sigma)). \tag{11}$$

From ($\mathcal{A}_2$), Lemm 1 and (11), we have:

$$
\begin{aligned}
|(Du)'(t)| &\leq k_1|b||u(t)-u(t-\sigma)| + k_2|b||u'(t)-u'(t-\sigma)| \\
&\leq k_1|b||(Du)(t)| + k_1|b(a-1)||u(t-\sigma)| + k_2|b||(Du)'(t)| + k_2|b(a-1)||u'(t-\sigma)| \\
&\leq k_1|b||(Du)(t)| + \frac{k_1|b(a-1)|}{|1-|a||}|(Du)(t-\sigma)| + k_2|b||(Du)'(t)| + \frac{k_2|b(a-1)|}{|1-|a||}|(Du)'(t-\sigma)|.
\end{aligned} \tag{12}
$$

In view of (12) and (4), we get:

$$
||(Du)'|| \leq \frac{|1-|a||k_1|b| + k_1|b(a-1)|}{|1-|a|||(1-k_2|b|) - k_2|b(a-1)|}||Du||. \tag{13}
$$

We note that Equation (7) is equivalent to the following equation:

$$
(Du)(t) = b\mu \int_{t-\sigma}^{t} f(s,u(s),u'(s))ds. \tag{14}
$$

Let $t = 0$ in (14); then:

$$
(Du)(0) = b\mu \int_{-\sigma}^{0} f(s,u(s),u'(s))ds
$$

and

$$
\begin{aligned}
|(Du)(0)| &\leq |b|\sigma k_3|u| + |b|\sigma k_4 \\
&\leq \frac{|b|\sigma k_3}{|1-|a||}|Du| + |b|\sigma k_4.
\end{aligned} \tag{15}
$$

Integrate both sides of (7) on $[0,t]$; then:

$$
(Du)(t) = (Du)(0) + \int_0^t b\mu\Big(f(s,u(s),u'(s)) - f(s-\sigma,u(s-\sigma),u'(s-\sigma))\Big)ds. \tag{16}
$$

In view of (15), (16), ($\mathcal{A}_2$) and Lemm 1, we get:

$$
||Du|| \leq \frac{|b|\sigma k_3}{|1-|a||}||Du|| + |b|\sigma k_4 + k_1\omega|b|||Du|| + \frac{k_1\omega|ba|}{|1-|a||}||Du|| + k_2\omega|b|||(Du)'|| + \frac{k_2\omega|ba|}{|1-|a||}||(Du)'||. \tag{17}
$$

By (5) and (17), we have:

$$
||Du|| \leq \frac{|b|\sigma k_4|1-|a||}{|1-|a||(1-k_1\omega|b|) - |b|\sigma k_3 - k_1\omega|ba|} + \frac{(|1-|a||k_2\omega|b|) + k_2\omega|ba|}{|1-|a||(1-k_1\omega|b|) - |b|\sigma k_3 - k_1\omega|ba|}||(Du)'||. \tag{18}
$$

In view of (13) and (18), we get:

$$
||(Du)'|| \leq \lambda_1 + \lambda_2||(Du)'||, \tag{19}
$$

where:

$$
\lambda_1 = \frac{(|1-|a||k_1|b|) + k_1|b(a-1)|}{|1-|a||(1-k_2|b|) - k_2|b(a-1)|} \frac{|b|\sigma k_4|1-|a||}{|1-|a||(1-k_1\omega|b|) - |b|\sigma k_3 - k_1\omega|ba|},
$$

$$
\lambda_2 = \frac{(|1-|a||k_1|b|) + k_1|b(a-1)|}{|1-|a||(1-k_2|b|) - k_2|b(a-1)|} \frac{(|1-|a||k_2\omega|b|) + k_2\omega|ba|}{|1-|a||(1-k_1\omega|b|) - |b|\sigma k_3 - k_1\omega|ba|}.
$$

Using (19) and (6), we get:

$$
||(Du)'|| \leq \frac{\lambda_1}{1-\lambda_2}. \tag{20}
$$

Obviously, we have:

$$(Du)(t) = (Du)(0) + \int_0^t (Du)'(s)ds$$

and

$$||Du|| \leq |(Du)(0)| + \omega||(Du)'||. \tag{21}$$

It follows by (15), (20) and (21) that:

$$||Du|| \leq \frac{|b|\sigma k_3}{|1-|a||}||Du|| + |b|\sigma k_4 + \frac{\lambda_1 \omega}{1-\lambda_2}. \tag{22}$$

Using (5), (22) and Lemma 1, we have:

$$||Du|| \leq \frac{|b|\sigma k_4|1-|a||}{|1-|a||-|b|\sigma k_3} + \frac{\lambda_1\omega|1-|a||}{(1-\lambda_2)(|1-|a||-|b|\sigma k_3)}$$

and

$$||u|| \leq \frac{|b|\sigma k_4}{|1-|a||-|b|\sigma k_3} + \frac{\lambda_1\omega}{(1-\lambda_2)(|1-|a||-|b|\sigma k_3)}. \tag{23}$$

From $(\mathcal{A}_2)$, we have:

$$||u|| \geq \frac{k_3}{k_4}. \tag{24}$$

Due to (23) and (24), Θ_1 is a bounded set. In view of (9), we have $KerF = \mathbb{R}$ and $ImF = \{u : u \in P_\omega, \ \int_0^\omega u(s)ds = 0\}$. Thus, F is a Fredholm operator with index zero. Define the operators by:

$$S : A \to KerF, \ Su = u(0)$$

and

$$R : B \to ImF, \ Rv = \frac{1}{\omega}\int_0^\omega v(s)ds.$$

Let:

$$F_P : Dom(F) \cap KerS \to ImF.$$

Then F_P has a continuous inverse F_P^{-1} defined by:

$$(F_P^{-1}v)(t) = D^{-1}\left(\int_0^\omega \Gamma(t,s)v(s)ds\right) \text{ for } v \in ImL, \tag{25}$$

where:

$$\Gamma(t,s) = \begin{cases} \frac{s-\omega}{\omega} & \text{for } \ 0 \leq t < s \leq \omega \\ \frac{s}{\omega} & \text{for } \ 0 \leq s \leq t \leq \omega. \end{cases}$$

Set $\Theta_2 = \{u | u \in KerF, \ Gu \in ImF\}$. For each $u \in \Theta_2$, we have $u = c$ and

$$f(t,c,0) - f(t-\sigma,c,0) = 0.$$

Using assumption $(\mathcal{A}_3)$, we see that Θ_2 is also bounded. Therefore, conditions (1) and (2) in Lemma 3 hold. Set $\Theta \supset \Theta_1 \cup \Theta_2$. From (10) and (25), it is easy to see that G is $L-$compact on $\bar{\Theta}$. Define Φ on $P_\omega \times [0,1]$ by:

$$\Phi(u,\lambda) = \lambda u + \frac{(1-\lambda)b}{\omega}\int_0^\omega \Big(f(s,u(s),u'(s)) - f(s-\sigma,u(s-\sigma),u'(s-\sigma))\Big)ds$$

By assumptions $(\mathcal{A}_3)$ for $u \in \partial\Theta \cap KerF$ and $\lambda \in [0,1]$, we have $\Phi(u,\lambda) \neq 0$. Hence,

$$\begin{aligned}\deg\{RG, \Theta \cap KerF, 0\} &= \deg\{\Phi(\cdot,0), \Theta \cap KerF, 0\} \\ &= \deg\{\Phi(\cdot,1), \Theta \cap KerF, 0\} \\ &= \deg\{I, \Theta \cap KerF, 0\} \\ &\neq 0\end{aligned}$$

and condition (3) of Lemma 3 holds. Using Lemma 3, we obtain that Equation (7) has at least one ω-periodic solution $u(t)$, i.e., Equation (1) has at least one ω-periodic solution $u(t)$. □

4. Positive Periodic Solution for Equation (2)

Theorem 2. *Assume that $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and $(\mathcal{A}_4)$ hold. Then Equation (2) has at least one positive ω-periodic solution if:*

$$|a||1-\tau'(t)|_0 + 2k_2|b| < 1, \tag{26}$$

$$|a| + |b||\tau|_0 k_3 + \lambda_3\omega < 1, \tag{27}$$

where:

$$|1-\tau'(t)|_0 = \max_{t\in\mathbb{R}} |1-\tau'(t)|, \quad |\tau|_0 = \max_{t\in\mathbb{R}} |\tau(t)|,$$

$$\lambda_3 = \frac{1}{1-|a||1-\tau'(t)|_0 - 2k_2|b|}\left(\frac{|b||\tau|_0 k_3}{1-a} + 2k_1|b|\right)|b||\tau|_0 k_3 \text{ for } a < 1 \tag{28}$$

or

$$\lambda_3 = \frac{1}{1-|a||1-\tau'(t)|_0 - 2k_2|b|}\left(\frac{|b||\tau|_0 k_3}{a-1} + 2k_1|b|\right)|b||\tau|_0 k_3 \text{ for } a > 1. \tag{29}$$

Proof. Taking the derivative on both sides of Equation (2) yields:

$$\Big(u(t) - au(t-\tau(t))\Big)' = bf(t,u(t),u'(t)) - b(1-\tau'(t))f(t-\tau(t),u(t-\tau(t)),u'(t-\tau(t))). \tag{30}$$

Let $(\mathcal{D}u)(t) = u(t) - au(t-\tau(t))$ in (30); then:

$$(\mathcal{D}u)'(t) = bf(t,u(t),u'(t)) - b(1-\tau'(t))f(t-\tau(t),u(t-\tau(t)),u'(t-\tau(t))).$$

Let:

$$\mathcal{F} : \mathcal{D}(\mathcal{F}) \subset P_\omega \to P_\omega,\ (\mathcal{F}u)(t) = (\mathcal{D}u)'(t) \tag{31}$$

and

$$\mathcal{G} : P_\omega \to P_\omega,\ (\mathcal{G}u)(t) = bf(t,u(t),u'(t)) - b(1-\tau'(t))f(t-\tau(t),u(t-\tau(t)),u'(t-\tau(t))). \tag{32}$$

Set:

$$\Omega_1 = \{u | u \in Dom(\mathcal{F}),\ \mathcal{F}u = \mu\mathcal{G}u,\ \mu \in (0,1)\},$$

where $\mathcal{F}$ and $\mathcal{G}$ are defined by (31) and (32), respectively. For each $u \in \Omega_1$, we have:

$$(\mathcal{D}u)'(t) = b\mu f(t,u(t),u'(t)) - b(1-\tau'(t))\mu f(t-\tau(t),u(t-\tau(t)),u'(t-\tau(t))). \tag{33}$$

If $a < 1$, from $(\mathcal{A}_2)$, Lemm 2 and (33), we have:

$$\begin{aligned}
|u'(t)| &\leq |a||1-\tau'(t)|_0||u'|| + |b||f(t,u(t),u'(t)) - f(t-\tau(t),u(t-\tau(t)),u'(t-\tau(t)))| \\
&+ |b||\tau|_0 k_3||u|| + |b||\tau|_0 k_4 \\
&\leq |a||1-\tau'(t)|_0||u'|| + k_1|b||u(t) - u(t-\tau(t))| + k_2|b||u'(t) - u'(t-\tau(t))| \\
&+ |b||\tau|_0 k_3||u|| + |b||\tau|_0 k_4 \\
&\leq |a||1-\tau'(t)|_0||u'|| + k_1|b||(\mathcal{D}u)(t)| + k_1|b(a-1)||u(t-\tau(t))| + 2k_2|b|||u'|| \\
&+ |b||\tau|_0 k_3||u|| + |b||\tau|_0 k_4 \\
&\leq |a||1-\tau'(t)|_0||u'|| + 2k_1|b|||\mathcal{D}u|| + 2k_2|b|||u'|| \\
&+ \frac{|b||\tau|_0 k_3}{1-a}||\mathcal{D}u|| + |b||\tau|_0 k_4.
\end{aligned} \tag{34}$$

Using (34) and (27), we have:

$$||u'|| \leq \frac{1}{1-|a||1-\tau'(t)|_0 - 2k_2|b|}\left(\frac{|b||\tau|_0 k_3}{1-a} + 2k_1|b|\right)||\mathcal{D}u|| + \frac{|b||\tau|_0 k_4}{1-|a||1-\tau'(t)|_0 - 2k_2|b|}. \tag{35}$$

We note that Equation (33) is equivalent to the following equation:

$$(\mathcal{D}u)(t) = b\mu \int_{t-\tau(t)}^{t} f(s,u(s),u'(s))ds. \tag{36}$$

In view of $(\mathcal{A}_2)$ and (36), we have:

$$||\mathcal{D}u|| \leq |b||\tau|_0 k_3||u|| + |b||\tau|_0 k_4. \tag{37}$$

From (35) and (37), we have:

$$||u'|| \leq \lambda_3||u|| + \lambda_4, \tag{38}$$

where λ_3 is defined by (28),

$$\begin{aligned}
\lambda_4 = &\frac{1}{1-|a||1-\tau'(t)|_0 - 2k_2|b|}\left(\frac{|b||\tau|_0 k_3}{1-a} + 2k_1|b|\right)|b||\tau|_0 k_4 \\
&+ \frac{|b||\tau|_0 k_4}{1-|a||1-\tau'(t)|_0 - 2k_2|b|}.
\end{aligned}$$

Set $t = 0$ in (36); by $(\mathcal{A}_2)$, then:

$$u(0) = au(-\tau(0)) + b\mu \int_{-\tau(0)}^{0} f(s,u(s),u'(s))ds$$

and

$$|u(0)| \leq \big(|a| + |b||\tau|_0 k_3\big)||u|| + |b||\tau|_0 k_4. \tag{39}$$

We note that:

$$u(t) = u(0) + \int_0^t u'(s)ds. \tag{40}$$

From (27), (39) and (40), we have:

$$\begin{aligned}
|u(t)| &\leq |u(0)| + \omega||u'|| \\
&\leq \big(|a| + |b||\tau|_0 k_3\big)||u|| + |b||\tau|_0 k_4 + \omega||u'||
\end{aligned}$$

and

$$||u|| \leq \frac{\omega}{1-\big(|a| + |b||\tau|_0 k_3\big)}||u'|| + \frac{|b||\tau|_0 k_4}{1-\big(|a| + |b||\tau|_0 k_3\big)}. \tag{41}$$

Using (38), (41) and (27), we get:

$$\begin{aligned} ||u'|| &\leq \lambda_3||u|| + \lambda_4 \\ &\leq \frac{\lambda_3\omega}{1-\big(|a|+|b||\tau|_0k_3\big)}||u'|| + \frac{\lambda_3|b||\tau|_0k_4}{1-\big(|a|+|b||\tau|_0k_3\big)} + \lambda_4 \end{aligned}$$

and

$$||u'|| \leq \frac{\lambda_3|b||\tau|_0k_4}{1-\big(|a|+|b||\tau|_0k_3\big)-\lambda_3\omega} + \frac{1-\big(|a|+|b||\tau|_0k_3\big)\lambda_4}{1-\big(|a|+|b||\tau|_0k_3\big)-\lambda_3\omega} := N_1. \tag{42}$$

From (41) and (42), we have:

$$||u|| \leq \frac{\omega}{1-\big(|a|+|b||\tau|_0k_3\big)}N_1 + \frac{|b||\tau|_0k_4}{1-\big(|a|+|b||\tau|_0k_3\big)} := N_2. \tag{43}$$

If $a > 1$, let:

$$\begin{aligned} \lambda_4 = &\frac{1}{1-|a||1-\tau'(t)|_0-2k_2|b|}\left(\frac{|b||\tau|_0k_3}{a-1}+2k_1|b|\right)|b||\tau|_0k_4 \\ &+\frac{|b||\tau|_0k_4}{1-|a||1-\tau'(t)|_0-2k_2|b|}. \end{aligned} \tag{44}$$

Similar to the above proof, we get:

$$||u'|| \leq \lambda_3||u|| + \lambda_4,$$

where λ_3 and λ_4 are defined by (29) and (44), respectively. Furthermore, similar to the proof of (42) and (43), there exists $N_3 > 0$ such that:

$$||u|| \leq N_3. \tag{45}$$

From $(\mathcal{A}_2)$, we have:

$$||u|| \geq \frac{k_3}{k_4}. \tag{46}$$

Due to (45) and (46), Ω_1 is a bounded set. Thus, condition (1) in Lemma 3 holds. Similar to the proof of Theorem 1, it is easy to see that $\mathcal{F}$ is a Fredholm operator with index zero and $\mathcal{G}$ is L-compact on $\bar{\Omega}$.

Set $\Omega_2 = \{u|u \in Ker\mathcal{F},\ Gu \in Im\mathcal{F}\}$. For each $u \in \Omega_2$, we have $u = c$, where $c > M$ is a constant, and

$$f(t,c,0) - (1-\tau'(t))f(t-\tau(t),c,0) = 0.$$

Using assumption $(\mathcal{A}_4)$, we see that Ω_2 is also bounded. Therefore, condition (2) in Lemma 3 holds. Set $\Omega \supset \Omega_1 \cup \Omega_2$. Similar to the proof of Theorem 1, it is easy to see that $\mathcal{F}$ is a Fredholm operator with index zero and $\mathcal{G}$ is $L-$compact on $\bar{\Omega}$. Define Ψ on $P_\omega \times [0,1]$ by:

$$\Psi(u,\lambda) = \lambda u + \frac{(1-\lambda)b}{\omega}\int_0^\omega \Big(f(s,u(s),u'(s)) - (1-\tau'(s))f(s-\sigma,u(s-\sigma),u'(s-\sigma))\Big)ds$$

By assumptions $(\mathcal{A}_4)$ for $u \in \partial\Omega \cap Ker\mathcal{F}$ and $\lambda \in [0,1]$, we have $\Psi(u,\lambda) \neq 0$. Hence,

$$\begin{aligned} \deg\{R\mathcal{G},\Omega\cap Ker\mathcal{F},0\} &= \deg\{\Psi(\cdot,0),\Omega\cap Ker\mathcal{F},0\} \\ &= \deg\{\Psi(\cdot,1),\Omega\cap Ker\mathcal{F},0\} \\ &= \deg\{I,\Omega\cap Ker\mathcal{F},0\} \\ &\neq 0 \end{aligned}$$

and condition (3) of Lemma 3 holds. Using Lemma 3, we obtain that Equation (30) has at least one ω-periodic solution $u(t)$, i.e., Equation (2) has at least one ω-periodic solution $u(t)$. $\square$

Remark 2. *In [19], the authors showed a fixed point theorem for a mixed monotone operator. When $0 < a < 1$ in Equation (1), they used this fixed point theorem to obtain the existence of positive almost automorphic solutions for Equation (1). In [20], when $0 < a < 1$, the authors used Perov's fixed point theorem to obtain the existence and the uniqueness of a positive periodic solution for Equation (2). In the preset paper, we obtain the existence of a positive periodic solution for Equations (1) and (2) under $|a| \neq 1$ with $a > 0$ that generalize the results in [19,20].*

5. Examples

Example 1. *Consider the following equation:*

$$u(t) = \frac{1}{2}u(t-0.1) + 0.01 \times \int_{t-0.1}^{t} \Big(u(s) + \cos u'(s) + \sin s - 3\Big)ds, \tag{47}$$

where:

$$a = \frac{1}{2},\ b = 0.01,\ \sigma = 0.1,\ \omega = 2\pi,$$

$$f(t,u,v) = u + \cos v + \sin t - 3.$$

Obviously,

$$|f(t,u_1,v_1) - f(s,u_2,v_2)| \leq |u_1 - u_2| + |v_1 - v_2| \text{ for all } t,s,v_1,v_2 \in \mathbb{R},\ u_1,u_2 \in \mathbb{R}^+$$

and

$$f(t,u,v) \leq u - 1 \text{ for all } t,v \in \mathbb{R},\ u > 1$$

where $k_1 = k_2 = k_3 = k_4 = 1$,

$$b\Big(f(t,c,0) - f(t-\sigma,c,0)\Big) = 0.01\sin t - 0.01\sin(t-0.1) \not\equiv 0.$$

Hence, assumptions $\mathcal{A}_1$-$\mathcal{A}_3$ hold. Furthermore,

$$|1-|a||(1-k_2|b|) - k_2|b(a-1)| = 0.49 > 0,$$

$$|1-|a||(1-k_1\omega|b|) - k_1\omega|ba| + |b|\sigma k_3 = 0.1536 > 0,$$

$$\frac{(|1-|a||k_1|b|) + k_1|b(a-1)|}{|1-|a||(1-k_2|b|) - k_2|b(a-1)|}\,\frac{(|1-|a||k_2\omega|b|) + k_2\omega|ba|}{|1-|a||(1-k_1\omega|b|) - |b|\sigma k_3 - k_1\omega|ba|} \approx 0.003 < 1.$$

Thus, conditions (4)–(6) hold. Therefore, all conditions of Theorem 1 hold and Equation (47) has a positive 2π-periodic solution.

Example 2. *Consider the following equation:*

$$u(t) = \frac{1}{2}u(t-0.1\sin t) + 0.01 \times \int_{t-0.1\sin t}^{t} \Big(u(s) + \cos u'(s) + \sin s - 3\Big)ds, \tag{48}$$

where:

$$a = \frac{1}{2},\ b = 0.01,\ \tau(t) = 0.1\sin t$$

$$f(t,u,v) = u + \cos v + \sin t - 3.$$

Obviously,

$$|f(t,u_1,v_1) - f(s,u_2,v_2)| \leq |u_1 - u_2| + |v_1 - v_2| \text{ for all } t,s,v_1,v_2 \in \mathbb{R},\ u_1,u_2 \in \mathbb{R}^+$$

and

$$f(t,u,v) \leq u - 1 \text{ for all } t,v \in \mathbb{R},\ u > 1$$

where $k_1 = k_2 = k_3 = k_4 = 1$,

$$b\Big(f(t,c,0) - (1-\tau'(t))f(t-\tau(t),c,0)\Big) = 0.01\sin t - 0.01\sin(t-\cos t) \not\equiv 0.$$

Hence, assumptions $\mathcal{A}_1$, $\mathcal{A}_2$ *and* $\mathcal{A}_4$ *hold. Furthermore, we get:*

$$|a||1-\tau'(t)|_0 + 2k_2|b| = 0.57 < 1$$

and

$$|a| + |b||\tau|_0 k_3 + \lambda_3\omega \approx 0.0026 < 1.$$

Thus, conditions (26) and (27) hold. Therefore, all conditions of Theorem 2 hold and Equation (48) has a positive 2π*-periodic solution. Figure 1 shows that for Equation (47) there exists a positive* 2π*-periodic solutions when the delay is a constant. The parameters* a, b *and the function* f *in Equation (47) are different from the corresponding ones in [25]. Therefore, our results are more general than those in [25] and have a wider range of applications. Furthermore, when the delay is time-varying, Figure 1 also shows that for Equation (48) there exists a positive* 2π*-periodic solution that greatly improves the existing results; see [4,26,27].*

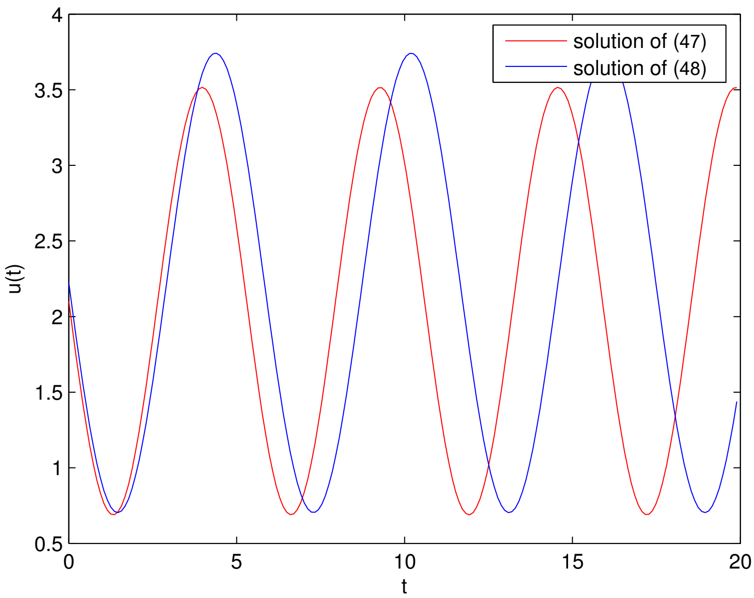

Figure 1. Positive periodic solutions of Equations (47) and (48).

Remark 3. *In [27], Cooke and Kaplan studied Equation (47) for the case of* $a = 0,\ b = 1$ *and* $f(t,u(t),u'(t)) = f(t,u(t))$*. They proved that if the delay* σ *is large enough, there exists a positive periodic solution with period equal to the period of* f*. The considered equation in [1] is a special case of Equation (47); furthermore, the existence of a positive periodic solution for Equation (47) does not require a sufficiently large delay. In [20], Bellour and Dads studied Equation (48) for the case of* $a = 0,\ b = 1$ *and* $f(t,u(t),u'(t)) = f(t,u(t))$*. They obtained the existence and the uniqueness of a positive periodic solution by using Perov's fixed point theorem in generalized metric spaces. Obviously, the considered equation in [2] is a special case of Equation (48). Figure 1 shows that there exists a positive periodic solution for Equations (47) and (48) for the case of* $a \neq 0,\ b \neq 1$.

6. Conclusions and Discussions

In this paper, we obtain some sufficient conditions that guarantee the existence of positive periodic solutions for Equations (1) and (2). It should be pointed out that our results do not depend on monotonicity of the function $f(t,\cdot,\cdot)$. The research methods of Equations (1) and (2) are based on the fixed point theorem and the theory for Hilbert's projective metric; see [19,20,26]. In general, Mawhin's continuity theorem can be used to conveniently study the existence of periodic solutions for delay equations; see, e.g., [28–32]. However, few scholars use this theorem to study the existence of positive periodic solutions. Actually, the study of positive periodic solutions of differential equations can be traced back to the 18th century; see, e.g., [33,34]. In this article, we developed Mawhin's continuity theorem to study the existence of positive periodic solutions. In future work, we will consider using Mawhin's continuity theorem to investigate the existence of an almost periodic solution, a pseudo almost periodic solution and an almost automorphic solution for Equations (1) and (2). The theoretical findings are verified by two examples that show their correctness, effectiveness and feasibility.

The domain of Equations (1) and (2) offers potential for further studies. For example, the methods from this paper can be used for studying Equations (1) and (2) with random perturbations, impulses, different time scales, etc. We can also further study dynamical behavior for Equations (1) and (2), such as progressive stability, exponential stability, synchronization, etc.

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