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A Note on Nearly Sasakian Manifolds

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Abstract: A class of nearly Sasakian manifolds is considered in this paper. We discuss the geometric effects of some symmetries on such manifolds and show, under a certain condition, that the class of Ricci semi-symmetric nearly Sasakian manifolds is a subclass of Einstein manifolds. We prove that a Codazzi-type Ricci nearly Sasakian space form is either a Sasakian manifold with a constant ϕ-holomorphic sectional curvature $\mathcal{H} = 1$ or a 5-dimensional proper nearly Sasakian manifold with a constant ϕ-holomorphic sectional curvature $\mathcal{H} > 1$. We also prove that the spectrum of the operator H^2 generated by the nearly Sasakian space form is a set of a simple eigenvalue of 0 and an eigenvalue of multiplicity 4, and we induce that the underlying space form carries a Sasaki–Einstein structure. We show that there exist integrable distributions with totally geodesic leaves on the same manifolds, and we prove that there are no proper nearly Sasakian space forms with constant sectional curvature.

Keywords: nearly Sasakian space forms; locally symmetric manifold; k-nullity distribution; semi-symmetric manifold; Ricci-symmetric manifold

MSC: 53C15; 53C25

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1. Introduction

Blair, Yano, and Showers introduced in [1] the concept of nearly Sasakian structures as an odd-dimensional counterpart of nearly Kähler structures. They proved that a normal nearly Sasakian structure is Sasakian, and, hence, is contact in particular. Also, in the same paper, it was shown that a hypersurface of a nearly Kähler manifold is nearly Sasakian if and only if it is quasi-umbilical with respect to the (almost) contact form. This result was supported by an example stating that S^5 properly imbedded in S^6 inherits a nearly Sasakian structure, which is not a Sasakian structure. That is why nearly Sasakian manifolds may also be considered as an odd-dimensional analogue of nearly Kähler manifolds. However, it is very difficult to find relationships between the two structures, such as for the duo Sasakian and Kähler structures (see [2] for details).

Nearly Sasakian structures can also be seen as the vanishing of the symmetric part of Sasakian structures. Several authors have studied these structures in [2–5] and the references therein. For instance, Olszak in [4,5] gave a good number of properties for nearly Sasakian structures. He proved that if nearly Sasakian manifolds are not Sasakian, they are of dimension 5 and of a constant curvature. Olszak also proved some equivalent conditions for non-Sasakian nearly Sasakian manifolds to be of dimension 5 and showed that such manifolds are Einstein manifolds.

In Ref. [2], among other results, the authors proved that there are two types of integrable distributions with totally geodesic leaves in a nearly Sasakian manifold, which are Sasakian and 5-dimensional nearly Sasakian manifolds. Note that a $(2n+1)$-dimensional nearly Sasakian with $n \geq 3$ is a Sasakian manifold ([3], Theorem 4.9).

In this paper, we consider the same nearly Sasakian structures by paying attention to certain foliations and curvature properties. We prove that some of these foliations are naturally generated by the symmetry properties on curvature and Ricci tensors.

The study of locally symmetric Riemannian manifolds has a long history, and several authors have worked in this direction. In Ref. [6] and the references therein, a series of results is presented regarding locally symmetric contact manifolds derived under some restrictions. In a direct way, Boeckx and Cho, in [7], proved that a locally symmetric contact manifold is either a Sasakian manifold with a constant sectional curvature 1 or is locally isometric to a unit tangent sphere bundle of a Euclidean space endowed with its standard contact metric structure.

A smooth manifold M is locally symmetric if its Riemannian curvature tensor R is parallel, i.e., $\nabla R = 0$, where ∇ is the Levi–Civita connection on M extended to act on tensors as a derivation. This class of manifolds contains manifolds of a constant curvature. The integrability condition of $\nabla R = 0$ is $R \cdot R = 0$, where again R is extended to act on tensors as a derivation. Manifolds that satisfy the latter condition are called semi-symmetric (see [8,9], for more details). A smooth manifold is said to be Ricci semi-symmetric, if $R \cdot \mathrm{Ric} = 0$. The set of all manifolds that are Ricci semi-symmetric contains the set of manifolds that are semi-symmetric. This means that semi-symmetric conditions imply Ricci semi-symmetric conditions, but the converse is not true, in general.

The present paper studies the two foliations stated by Olszak in papers [4,5]. He proved that, if a proper nearly Sasakian manifold is locally symmetric, then it is of a constant curvature and of dimension 5. These foliations were also investigated by Cappelletti-Montano et al. in [2,3].

The organization of the paper is as follows. Section 2 deals with a definition and properties of a nearly Sasakian manifold and some identity formulas of the underlying tensors, which are supported by two examples. In Section 3, we discuss the two foliations as stated in [2,4]. We establish the geometric effects of semi-symmetry and Ricci semi-symmetry on nearly Sasakian manifolds. Under a certain condition, we show that the class of Ricci-symmetric nearly Sasakian manifolds is a subclass of Einstein manifolds. We prove that these foliations exist canonically in a locally symmetric nearly Sasakian manifold of a constant sectional curvature and k space. Some examples are also established. In Section 4, we derive some algebraic formulas of the curvature tensor for nearly Sasakian manifolds (Proposition 3). We prove that a Codazzi-type Ricci nearly Sasakian space form is either Sasakian with a constant ϕ-holomorphic sectional curvature $\mathcal{H} = 1$ or a 5-dimensional proper nearly Sasakian manifold with a constant ϕ-holomorphic sectional curvature $\mathcal{H} > 1$. In the same settings, we also prove that the spectrum of the operator H^2 has a simple eigenvalue of 0 and an eigenvalue of multiplicity 4, which therefore induces that such a Codazzi-type Ricci nearly Sasakian space form carries a Sasaki–Einstein structure. We show that there exist integrable distributions with totally geodesic leaves (Theorems 9 and 10). Contrary to ([4], Theorem 6.1), we prove that there are no proper nearly Sasakian space forms with a constant sectional curvature (Theorem 12).

2. Preliminaries

Let M be a $(2n+1)$-dimensional manifold equipped with an almost contact structure (ϕ, ξ, η), that is, ϕ is a tensor field of type $(1,1)$, ξ is a vector field, and η is a 1-form satisfying [6]

$$\phi^2 = -\mathbb{I} + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{1}$$

This implies that $\phi\xi = 0$, $\eta \circ \phi = 0$, and $\mathrm{rank}(\phi) = 2n$. In this case, (ϕ, ξ, η, g) is called an almost contact metric structure on M if (ϕ, ξ, η) is an almost contact structure of M and g is a Riemannian metric of M such that [6]

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for any vector field X, Y of M. It is easy to see the $(1,1)$-tensor field ϕ is skew-symmetric, and so $\eta(X) = g(\xi, X)$.

If, moreover,

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X, \tag{3}$$

where ∇ is the Levi–Civita connection for the Riemannian metric g, then M is called a nearly Sasakian manifold. From (3), one has

$$\nabla_X \xi = -\phi X - HX, \tag{4}$$

where

$$HX = \phi\big(\nabla_\xi \phi\big)X. \tag{5}$$

This operator is skew-symmetric and also anti-commutes with ϕ. The tensor field H is of type $(1,1)$ and satisfies $H\xi = 0$, $\eta \circ H = 0$, and

$$\nabla_\xi H = -\nabla_\xi \phi = \phi H = -\frac{1}{3}\mathcal{L}_\xi \phi, \tag{6}$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ. If H vanishes, then a nearly Sasakian manifold is Sasakian (see [10] and the references therein).

It is easy to see that

$$H^2 X = \big(\nabla_\xi \phi\big)^2 X. \tag{7}$$

The divergence of ξ is given by

$$\operatorname{div} \xi = 0. \tag{8}$$

Example 1. *Let M be a 5-dimensional smooth manifold defined as $M = \{(x_1, x_2, \cdots, x_5) \in \mathbb{R}^5 : x_2 \neq 0,\ x_5 \neq 0\}$ with standard coordinates $(x_1, x_2, \cdots, x_5)$. The vector fields*

$$X_1 = 2\left(x_2 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_1}\right), \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = \xi = -\frac{\partial}{\partial x_3},$$
$$X_4 = 2\left(x_5 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}\right), \quad X_5 = \frac{\partial}{\partial x_5},$$

are linearly independent at each point of M. Denote g to be the Riemannian metric of M, defined as $g(X_i, X_j) = \delta_{ij}$, for any $i, j = 1, 2, \cdots, 4$, where δ_{ij} is the Kronecker symbol, and $g(\xi, \xi) = 1$. Locally, the metric g takes the form

$$g = (\frac{1}{4} - x_2^2)dx_1^2 + dx_2^2 + dx_3^2 + (\frac{1}{4} - x_5^2)dx_4^2 + dx_5^2.$$

We define the 1-form η and $(1,1)$-tensor field ϕ, respectively, by, $\eta = -dx_3$ and $\phi X_1 = X_2$, $\phi X_2 = -X_1$, $\phi X_3 = 0$, $\phi X_4 = X_5$, and $\phi X_5 = -X_4$. The relations (1) and (2) are satisfied for $\mathbb{R}^5$ by the linearity of ϕ and g. Thus, the structure (ϕ, ξ, η, g) defines an almost contact metric structure for $\mathbb{R}^5$. Let ∇ be the Levi–Civita connection compatible with the metric g. Then, the non-vanishing Lie brackets are $[X_1, X_2] = [X_4, X_5] = 2\xi$. These lead to the following non-vanishing components of the covariant derivative

$$\nabla_{X_1} X_2 = \xi, \ \ \nabla_{X_1} \xi = -X_2, \ \ \nabla_{X_2} X_1 = -\xi, \ \ \nabla_{X_2} \xi = X_1,$$
$$\nabla_\xi X_1 = -X_2, \ \ \nabla_\xi X_2 = X_1, \ \ \nabla_\xi X_4 = -X_5, \ \ \nabla_\xi X_5 = X_4,$$
$$\nabla_{X_4} \xi = -X_5, \ \ \nabla_{X_4} X_5 = \xi, \ \ \nabla_{X_5} \xi = X_4, \ \ \nabla_{X_5} X_4 = -\xi.$$

Using these covariant derivatives, it is easy to see that relation (3) is satisfied, and, therefore, (ϕ, ξ, η, g) is a nearly Sasakian structure.

Throughout this note, manifolds are assumed to be of class C^∞ and connected, and all tensor fields are of class C^∞. We will denote the $\mathcal{F}(M)$ module of smooth sections of a vector bundle E with $\Gamma(E)$.

A vector field V on M is said to be an affine Killing vector field if it satisfies (see [11], p. 51)

$$\mathcal{L}_V \nabla = 0. \tag{9}$$

Relation (9) reduces to

$$R(V, X)Y + \nabla_X \nabla_Y V - \nabla_{\nabla_X Y} V = 0, \tag{10}$$

where R is the Riemannian curvature tensor R of M defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad \forall\, X,\ Y,\ Z \in \Gamma(TM). \tag{11}$$

Relation (9) is the integrability condition for the Killing vector field V (see [11], for more details). If M is nearly Sasakian, then by using (4), it is easy to see that ξ is a Killing vector. Hence, the vector field ξ is an affine Killing vector field. The converse is not true, in general. In [11], it was proven that the converse holds when the underlying manifold is compact and without a boundary.

Let (M, ϕ, ξ, η, g) be a $(2n+1)$-dimensional nearly Sasakian manifold. Through (10), we obtain

$$R(X, \xi)Y = \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi. \tag{12}$$

Therefore, we have [10]:

$$\begin{aligned} R(\xi, X)Y &= (\nabla_X \phi)Y + (\nabla_X H)Y \\ &= g(X - H^2X, Y)\xi - \eta(Y)(X - H^2X) \\ &= \{g(X, Y) - g(H^2X, Y)\}\xi - \eta(Y)X + \eta(Y)H^2X, \end{aligned} \tag{13}$$

$$\left(\nabla_X H^2\right)Y = \eta(Y)(\phi + H)H^2X + g((\phi + H)H^2X, Y)\xi, \tag{14}$$

$$g((\nabla_X \phi)Y, HZ) = -\eta(Y)g(H^2X, \phi Z) + \eta(X)g(H^2Y, \phi Z) + \eta(Y)g(HX, Z) \tag{15}$$

for any $X, Y, Z \in \Gamma(TM)$.

As proven in [10] and using the relations (13)–(15), we have

$$(\nabla_X \phi)Y = -\eta(X)\phi HY - \eta(Y)(X - \phi HX) + g(X - \phi HX, Y)\xi, \tag{16}$$

$$(\nabla_X H)Y = \eta(X)\phi HY + \eta(Y)(H^2X - \phi HX) - g(H^2X - \phi HX, Y)\xi, \tag{17}$$

$$\begin{aligned} (\nabla_X \phi H)Y &= \eta(Y)(\phi H^2X + HX) - \eta(X)(\phi H^2Y + HY) \\ &\quad - g(HX + \phi H^2X, Y)\xi. \end{aligned} \tag{18}$$

Now, for any vector fields X and Y of M,

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y + \eta(X)H^2Y - \eta(Y)H^2X. \tag{19}$$

Then,

$$\mathrm{Ric}(X, \xi) = (2n - \operatorname{trace} H^2)\eta(X), \quad \forall\, X \in \Gamma(TM). \tag{20}$$

By (13), we have, for any $X, Y \in \Gamma(TM)$,

$$R(X, \xi)Y = -g(X, Y)\xi + \eta(Y)X - \eta(Y)H^2X + g(H^2Y, X)\xi. \tag{21}$$

3. Foliations of a Nearly Sasakian Manifold

In Refs. [2,4], for instance, the authors showed that there are two foliations in any nearly Sasakian manifold with leaves that are Sasakian or 5-dimensional nearly Sasakian non-Sasakian manifolds. This fact is led by the square of a skew-symmetric operator H, i.e., H^2. The latter plays an important role, as well as its spectrum.

Let (M, ϕ, ξ, η, g) be a $(2n+1)$-dimensional nearly Sasakian manifold. Olszak, in [4], showed that, if M satisfies the condition

$$H^2 = \alpha\{\mathbb{I} - \eta \otimes \xi\} \tag{22}$$

for a real number α, then $\dim M = 5$. The converse is true if the real number α is non-zero (see [5], Theorem 4.1 for more details).

We say that M is a *proper nearly Sasakian manifold* if it is a nearly Sasakian non-Sasakian manifold.

Let $D := \ker \eta$ denote the contact distribution, and let $D^\perp$ denote the one spanned structure vector field ξ. Then, the tangent space TM is decomposed as

$$TM = D \oplus D^\perp, \tag{23}$$

where $\oplus$ is the orthogonal direct sum. Through (23), any $X \in \Gamma(TM)$ can be rewritten as

$$X = QX + Q^\perp X, \tag{24}$$

where Q and $Q^\perp$ are the projection morphisms of TM onto D and $D^\perp$, respectively. Then, for any vector field $X \in \Gamma(TM)$, $Q^\perp X = \eta(X)\xi$, and $X = QX + \eta(X)\xi$.

If (22) is satisfied, then, for any non-zero vector field $X \in \Gamma(D)$,

$$-g(HX, HX) = \alpha g(X, X), \text{ i.e., } \alpha = -\frac{g(HX, HX)}{g(X, X)}. \tag{25}$$

This means that there is $\lambda \in \mathbb{R}$ such that $\alpha = -\lambda^2 \leq 0$, and, therefore, (22) becomes

$$H^2 = -\lambda^2\{\mathbb{I} - \eta \otimes \xi\}. \tag{26}$$

As examples for both Sasakian and proper nearly Sasakian manifolds, we have the following.

Example 2. *Let us recall the* 5*-dimensional manifold M considered in Example 1. Then, the components of the tensor field H of the type* $(1,1)$ *are given*

$$\begin{aligned}
H\xi &= \phi(\nabla_\xi \phi)\xi = 0, \\
HX_1 &= \phi\nabla_\xi X_2 - \phi^2 \nabla_\xi X_1 = \phi X_1 + \phi^2 X_2 = X_2 - X_2 = 0, \\
HX_2 &= -\phi\nabla_\xi X_1 - \phi^2 \nabla_\xi X_2 = -X_1 + X_1 = 0, \\
HX_4 &= \phi\nabla_\xi X_5 - \phi^2 \nabla_\xi X_4 = X_5 - X_5 = 0 \\
HX_5 &= -\phi\nabla_\xi X_4 - \phi^2 \nabla_\xi X_5 = -X_4 + X_4 = 0.
\end{aligned}$$

This means that H vanishes everywhere. Therefore, in this case, the structure in (3) reduces to $(\nabla_X \phi)Y = g(X, Y)\xi - \eta(X)Y$, $\forall X,\ Y \in \Gamma(TM)$, which shows that M is a Sasakian manifold.

In [1], the authors showed how to induce a nearly Sasakian structure for S^5. In order to do so, they looked at S^5 as a hypersurface in S^6 equipped with its nearly Kähler structure.

Example 3. *We recall an example of* 5*-dimensional nearly Sasakian manifolds as detailed in [1,2,6]. Let S^6 be the unit sphere in $\mathbb{R}^7$ with its cross product $\times$ induced by Cayley algebra. Let $\mathcal{N} = \sum_{i=1}^7 x_i \frac{\partial}{\partial x_i}$ denote the unit outer normal. We define an almost complex structure J for S^6 as $JX = \mathcal{N} \times X$, which implies,*

$$J^2 = \mathcal{N} \times (\mathcal{N} \times X) = -X, \ \ \forall\, X \in \Gamma(TS^6).$$

It is easy to see that J is almost complex structure and is also nearly Kähler (but non-Kähler) when associated with the induced Riemannian metric. As detailed in [2], now we consider S^5 as a totally umbilical hypersurface of S^6 defined by $x_7 = \frac{\sqrt{2}}{2}$, with unit normal at each point x, which is given by $\omega = x - \sqrt{2}\frac{\partial}{\partial x_7} = \sum_{i=1}^{6} x_i \frac{\partial}{\partial x_i} - \frac{\sqrt{2}}{2}\frac{\partial}{\partial x_7}$ and the shape operator is $A = -\mathbb{I}$. Let (ϕ, ξ, η, g) be the almost induced contact metric structure with

$$\xi = -J\omega = \sqrt{2}\left(x_1 \frac{\partial}{\partial x_6} - x_2 \frac{\partial}{\partial x_5} - x_3 \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_3} + x_5 \frac{\partial}{\partial x_2} - x_6 \frac{\partial}{\partial x_1}\right),$$

and η is given by the restriction of $\sqrt{2}(x_1 dx_6 - x_6 dx_1 + x_5 dx_2 - x_2 dx_5 + x_4 dx_3 - x_3 dx_4)$ to S^5. This is a nearly Sasakian non-Sasakian structure with a constant sectional curvature of 2. The latter means that

$$R(X,Y)\xi = 2\{\eta(Y)X - \eta(X)Y\}, \quad \forall\, X,\ Y \in \Gamma(TS^5),$$

which implies that $-\phi^2 X - H^2 X = 2\{X - \eta(X)\xi\}$; that is, $H^2 X = -\{X - \eta(X)\xi\}$ with $\lambda^2 = 1$.

Next, we present some classes of nearly Sasakian manifolds in which condition (26) is satisfied.

Suppose M is a semi-symmetric nearly Sasakian manifold. Then, the curvature tensor R of M satisfies, for any vector fields X and Y of M, $R(X,Y) \cdot R = 0$, where $R(X,Y)$ operates on R as a derivation of the tensor algebra at each point (see [8,9] for more details). Now, let X and Y be vector fields in D such that $g(X,Y) = 0$. Then, using (19) and (21), we have,

$$\begin{aligned}(R(X,\xi)\cdot R)(X,Y)Y &= R(X,\xi)R(X,Y)Y - R(X,Y)R(X,\xi)Y - R(R(X,\xi)X,Y)Y \\ &\quad - R(X,R(X,\xi)Y)Y \\ &= -g(X,R(X,Y)Y)\xi + \eta(R(X,Y)Y)X - \eta(R(X,Y)Y)H^2X \\ &\quad + g(H^2X,R(X,Y)Y)\xi + \{g(X,X) - g(H^2X,X)\}\{g(Y,Y) - g(H^2Y,Y)\}\xi \\ &\quad - g(X,H^2Y)g(X,H^2Y)\xi.\end{aligned} \tag{27}$$

Hence,

$$\begin{aligned}&-g(X,R(X,Y)Y)\xi + \eta(R(X,Y)Y)X - \eta(R(X,Y)Y)H^2X \\ &+ g(H^2X,R(X,Y)Y)\xi + \{g(X,X) - g(H^2X,X)\}\{g(Y,Y) - g(H^2Y,Y)\}\xi \\ &- g(X,H^2Y)g(X,H^2Y)\xi = 0.\end{aligned} \tag{28}$$

Thus, considering the ξ-component of (28), we obtain

$$\begin{aligned}g(R(X,Y)Y,X) &= g(H^2X,R(X,Y)Y) + g(X,X)g(Y,Y) + g(X,X)g(HY,HY) \\ &\quad + g(Y,Y)g(HX,HX) + g(HX,HX)g(HY,HY) \\ &\quad - g(HX,HY)g(HX,HY).\end{aligned} \tag{29}$$

If condition (26) is satisfied, then, from relation (29), one obtains,

$$(1+\lambda^2)g(R(X,Y)Y,X) = (1 + 2\lambda^2 + \lambda^4)g(X,X)g(Y,Y). \tag{30}$$

That is,

$$g(R(X,Y)Y,X) = (1+\lambda^2)g(X,X)g(Y,Y). \tag{31}$$

Therefore, we have

Theorem 1. *Let (M, ϕ, ξ, η, g) be a nearly Sasakian manifold satisfying the Nomizu's condition, i.e., $R(X,Y) \cdot R = 0$ for any vector fields X and Y of M. If*

$$H^2 = -\lambda^2\{\mathbb{I} - \eta \otimes \xi\}$$

for some real number λ, then M is of a constant curvature $1 + \lambda^2$. Moreover, M is either a Sasakian manifold or a 5-dimensional proper nearly Sasakian manifold.

Let κ be a real constant. Denote $N(\kappa)$ as the κ-nullity distribution of M. Then, $N(\kappa)$ is seen as the function $p \longmapsto N_p(\kappa)$ with $p \in M$, where $N_p(\kappa)$ is the κ-nullity space at p given by (see [12,13] for more details and reference therein)

$$N_p(\kappa) = \{Z \in T_pM : R(X,Y)Z = \kappa(g(Y,Z)X - g(X,Z)Y), \ \forall X, Y \in T_pM\},$$

where T_pM is the tangent space at p. If the vector field ξ on the nearly Sasakian manifold M belongs to $N(\kappa)$, then M is called κ *space*.

Therefore, we have this result.

Theorem 2. *Let (M, ϕ, ξ, η, g) be a nearly Sasakian manifold. Then, M satisfies the condition (26) if and only if M is a $(1 + \lambda^2)$ space.*

Proof. If condition (26) is satisfied, then, for any vector vector fields X and Y of M,

$$\begin{aligned} R(X,Y)\xi &= \eta(Y)X - \eta(X)Y - \lambda^2\eta(X)\{Y - \eta(Y)\xi\} + \lambda^2\eta(Y)\{X - \eta(X)\xi\} \\ &= (1 + \lambda^2)\{\eta(Y)X - \eta(X)Y\}. \end{aligned}$$

The converse is straightforward and this completes the proof. □

If a nearly Sasakian manifold M is Ricci semi-symmetric, then

$$\begin{aligned} (R(X,Y) \cdot \mathrm{Ric})(Z,W) &= -\mathrm{Ric}(R(X,Y)Z, W) - \mathrm{Ric}(Z, R(X,Y)W) \\ &= 0, \quad \forall\, X,\ \ Y,\ \ Z,\ \ W \in \Gamma(TM). \end{aligned} \tag{32}$$

Using (19) and (20), one has

$$\begin{aligned} (R(X,Y) \cdot \mathrm{Ric})(\xi, Z) &= -\mathrm{Ric}(R(X,Y)\xi, Z) - \mathrm{Ric}(\xi, R(X,Y)Z) \\ &= -\eta(Y)\mathrm{Ric}(X,Z) + \eta(X)\mathrm{Ric}(Y,Z) - \eta(X)\mathrm{Ric}(H^2Y, Z) \\ &+ \eta(Y)\mathrm{Ric}(H^2X, Z) - (2n - \mathrm{trace}\, H^2)\eta(R(X,Y)Z). \end{aligned} \tag{33}$$

Now, through relation (20), we obtain

$$\begin{aligned} (R(\xi, X) \cdot \mathrm{Ric})(Y, \xi) &= -\mathrm{Ric}(R(\xi, X)Y, \xi) - \mathrm{Ric}(Y, R(\xi, X)\xi) \\ &= -(2n - \mathrm{trace}\, H^2)g(X,Y) + (2n - \mathrm{trace}\, H^2)g(H^2X, Y) \\ &+ \mathrm{Ric}(X,Y) - \mathrm{Ric}(H^2X, Y). \end{aligned} \tag{34}$$

If condition (26) is satisfied for M, then (34) becomes

$$(R(\xi, X) \cdot \mathrm{Ric})(Y, \xi) = -2n(1 + \lambda^2)^2 g(X,Y) + (1 + \lambda^2)\mathrm{Ric}(X,Y). \tag{35}$$

Therefore, we obtain this result.

Theorem 3. *A Ricci semi-symmetric nearly Sasakian manifold satisfying (26) is Einstein.*

Proof. If M is a Ricci semi-symmetric nearly Sasakian manifold satisfying (26), then, using (35), the Ricci tensor is given by $\mathrm{Ric}(X,Y) = 2n(1+\lambda^2)g(X,Y)$ for any vector fields X and Y of M, and the proof is completed. □

In Ref. [4], Olszak proved that if a nearly Sasakian non-Sasakian manifold is locally symmetric, then it is of a constant curvature and of dimension 5. If we assume that the nearly Sasakian manifold M is of a constant sectional curvature κ, then the curvature tensor R of M satisfies the equation in [14,15]:

$$R(X,Y)Z = \kappa\{g(Y,Z)X - g(X,Z)Y\}, \ \ \forall X,\ Y \in \Gamma(TM). \tag{36}$$

Then, by putting $Z = \xi$ into (36) and using (19), we obtain

$$\eta(Y)\{(\kappa-1)X + H^2X\} = \eta(X)\{(\kappa-1)Y + H^2Y\}. \tag{37}$$

This implies that

$$H^2X = -(\kappa-1)\{X - \eta(X)\xi\}. \tag{38}$$

Therefore, we obtain:

Theorem 4. *Let $(M,\phi,\xi,\eta,\ g)$ be a nearly Sasakian manifold. If M is of a constant sectional curvature κ, then M is either Sasakian or satisfies condition (26), with $\kappa = 1+\lambda^2$, $\lambda \neq 0$, and a $(1+\lambda^2)$ space.*

A nearly Sasakian manifold M is locally symmetric if

$$(\nabla_W R)(X,Y)Z = 0,, \ \ \forall X,\ Y,\ Z,\ W \in \Gamma(TM).$$

We know that the covariant derivative of R, namely, ∇R, is defined as

$$\begin{aligned}(\nabla_Z R)(X,Y,W) = \nabla_Z R(X,Y)W - R(\nabla_Z X,Y)W - R(X,\nabla_Z Y)W \\ - R(X,Y)\nabla_Z W.\end{aligned} \tag{39}$$

By putting $W = \xi$ into (39), one has

$$\begin{aligned}(\nabla_Z R)(X,Y,\xi) = \{g(\phi Z,X) + g(HZ,X)\}Y - \{g(\phi Z,Y) + g(HZ,Y)\}X - \{g(\phi Z,X) \\ + g(HZ,X)\}H^2Y + \{g(\phi Z,Y) + g(HZ,Y)\}H^2X + \eta(X)(\nabla_Z H^2)Y \\ - \eta(Y)(\nabla_Z H^2)X + R(X,Y)\phi Z + R(X,Y)HZ.\end{aligned} \tag{40}$$

By using (14), the term $\eta(X)(\nabla_Z H^2)Y - \eta(Y)(\nabla_Z H^2)X$ becomes

$$\begin{aligned}\eta(X)(\nabla_Z H^2)Y - \eta(Y)(\nabla_Z H^2)X = \eta(X)g(\phi H^2Z,Y)\xi + \eta(X)g(H^3Z,Y)\xi \\ - \eta(Y)g(\phi H^2Z,X)\xi - \eta(Y)g(H^3Z,X)\xi.\end{aligned} \tag{41}$$

Therefore,

$$\begin{aligned}(\nabla_Z R)(X,Y,\xi) = \{g(\phi Z,X) + g(HZ,X)\}Y - \{g(\phi Z,Y) + g(HZ,Y)\}X \\ - \{g(\phi Z,X) + g(HZ,X)\}H^2Y + \{g(\phi Z,Y) + g(HZ,Y)\}H^2X \\ + \eta(X)g(\phi H^2Z,Y)\xi + \eta(X)g(H^3Z,Y)\xi - \eta(Y)g(\phi H^2Z,X) \\ - \eta(Y)g(H^3Z,X)\xi + R(X,Y)\phi Z + R(X,Y)HZ.\end{aligned} \tag{42}$$

If a nearly Sasakian manifold M is locally symmetric, then (42) leads to

$$
\begin{aligned}
0 = &\{g(\phi Z, X) + g(HZ, X)\}g(Y, W) - \{g(\phi Z, Y) + g(HZ, Y)\}g(X, W) \\
&- \{g(\phi Z, X) + g(HZ, X)\}g(H^2Y, W) + \{g(\phi Z, Y) + g(HZ, Y)\}g(H^2X, W) \\
&+ \eta(X)g(\phi H^2Z, Y)\eta(W) + \eta(X)g(H^3Z, Y)\eta(W) - \eta(Y)g(\phi H^2Z, X)\eta(W) \\
&- \eta(Y)g(H^3Z, X)\eta(W) + g(R(X, Y)\phi Z, W) + g(R(X, Y)HZ, W)
\end{aligned} \tag{43}
$$

for any vector field X, Y, Z, and W of M. As a result,

$$
\begin{aligned}
g(R(X, Y)\phi Z, W) + g(R(X, Y)HZ, W) &= -g(R(X, Y)W, \phi Z) - g(R(X, Y)W, HZ) \\
&= -g(R(X, Y)W, \phi Z + HZ).
\end{aligned} \tag{44}
$$

Relation (43) becomes

$$
\begin{aligned}
0 = &\, g(Y, W)g(\phi Z + HZ, X) - g(X, W)g(\phi Z + HZ, Y) \\
&- g(H^2Y, W)g(\phi Z + HZ, X) + g(H^2X, W)g(\phi Z + HZ, Y) \\
&+ \eta(X)\eta(W)g(\phi Z + HZ, H^2Y) - \eta(Y)\eta(W)g(\phi Z + HZ, H^2X) \\
&- g(R(X, Y)W, \phi Z + HZ).
\end{aligned} \tag{45}
$$

Thus,

$$
\begin{aligned}
R(X, Y)W = &\, g(Y, W)X - g(X, W)Y - g(H^2Y, W)X + g(H^2X, W)Y \\
&+ \eta(X)\eta(W)H^2Y - \eta(Y)\eta(W)H^2X \\
&= \{g(Y, W) - g(H^2Y, W)\}X - \{g(X, W) - g(H^2X, W)\}Y \\
&+ \eta(W)\{\eta(X)H^2Y - \eta(Y)H^2X\}.
\end{aligned} \tag{46}
$$

Therefore, we have the following.

Theorem 5. *Let (M, ϕ, ξ, η, g) be a nearly Sasakian manifold. If M is locally symmetric, then the curvature tensor R of M is given by, for any vector fields X, Y, and Z of M,*

$$
\begin{aligned}
R(X, Y)Z = &\, g(Y - H^2Y, Z)X - g(X - H^2X, Z)Y \\
&+ \eta(Z)\{\eta(X)H^2Y - \eta(Y)H^2X\}.
\end{aligned} \tag{47}
$$

Moreover, the Ricci tensor Ric *and scalar curvature* Scal *are given, respectively, by*

$$
\mathrm{Ric}(X, Y) = 2ng(X - H^2X, Y) - \eta(X)\eta(Y)\mathrm{trace}\, H^2, \tag{48}
$$

$$
\textit{and} \quad \mathrm{Scal} = (2n + 1)\left\{2n - \mathrm{trace}\, H^2\right\}. \tag{49}
$$

Proof. Let $\{E_i\}_{1 \le i \le 2n+1}$ be an orthonormal frame with respect to g. Then, the scalar curvature is given by

$$
\mathrm{Scal} = \sum_{i=1}^{2n+1} \mathrm{Ric}(E_i, E_i) = (2n + 1)\left\{2n - \mathrm{trace}\, H^2\right\},
$$

which completes the proof. □

Note that the geometric information of relations (47)–(49) depends on the information of the operator H^2. Let M be a locally symmetric nearly Sasakian manifold. Then, the curvature tensor R of M satisfies Equation (47). In addition, if M is of a constant curvature κ, then, by comparing both (36) and (47), one has,

$$
\begin{aligned}
(\kappa-1)\{g(Y,Z)X-g(X,Z)Y\} = & -g(H^2Y,Z)X+g(H^2X,Z)Y \\
& +\eta(Z)\{\eta(X)H^2Y-\eta(Y)H^2X\}.
\end{aligned}
$$

By letting $Y=Z=\xi$, this equation reduces to $H^2X=-(\kappa-1)\{X-\eta(X)\xi\}$. This means that M is either Sasakian (when $\kappa=1$) or non-Sasakian (when $\kappa\neq 1$), thus satisfying $H^2X=-\lambda^2\{X-\eta(X)\xi\}$, with $\kappa=1+\lambda^2$ and $\lambda\neq 0$. The converse is straightforward; that is, if $H^2X=-(\kappa-1)\{X-\eta(X)\xi\}$, then, using (47), the curvature tensor R satisfies

$$
R(X,Y)Z=\kappa\{g(Y,Z)X-g(X,Z)Y\};
$$

that is, M is of a constant curvature κ. Thus, according to [5], Theorem 4.1 we have the following.

Theorem 6. *Let $(M,\phi,\xi,\eta,\ g)$ be a locally symmetric nearly Sasakian manifold. Then, M is of a constant curvature κ if and only if M is either Sasakian or is a 5-dimensional proper nearly Sasakian manifold.*

As a consequence to this theorem, we remark the following.

Corollary 1. *There exist no locally symmetric nearly Sasakian manifolds of constant sectional curvature such that, for some real number λ,*

$$
H^2\neq-\lambda^2\{\mathbb{I}-\eta\otimes\xi\}.
$$

4. Curvature Tensor Properties

First of all, we shall prove the following propositions.

Proposition 1. *Let (M,ϕ,ξ,η,g) be a nearly Sasakian manifold and R be the Riemannian curvature tensor of M. Then,*

$$
\begin{aligned}
& R(X,Y)\phi Z-\phi R(X,Y)Z=2\{g(\phi X,Y)+g(HX,Y)\}\phi HZ-\eta(Z)\{\eta(X)(\phi H^2Y+HY) \\
& -\eta(Y)(\phi H^2X+HX)\}-g(Y-\phi HY,Z)\{\phi X+HX\}+g(X-\phi HX,Z)\{\phi Y+HY\} \\
& -\{g(\phi Y,Z)+g(HY,Z)\}\{X-\phi HX\}+\{g(\phi X,Z)+g(HX,Z)\}\{Y-\phi HY\} \\
& +\{\eta(X)g(HY+\phi H^2Y,Z)-\eta(Y)g(HX+\phi H^2X,Z)\}\xi,
\end{aligned}
\tag{50}
$$

for any vector fields X, Y and Z on M.

Proof. The proof follows from straightforward calculations. □

From (2), one obtains the following

$$
\begin{aligned}
& g(R(X,Y)\phi Z,\phi W)-g(R(X,Y)Z,W)=-\eta(W)g(R(Z,\xi)X,Y)+2g(\phi X,Y)g(HZ,W) \\
& +2g(HX,Y)g(HZ,W)-\eta(X)\eta(Z)g(H^2Y,W)-\eta(X)\eta(Z)g(HY,\phi W) \\
& +\eta(Y)\eta(Z)g(H^2X,W)+\eta(Y)\eta(Z)g(HX,\phi W)-g(\phi Y,Z)g(X,\phi W) \\
& +g(\phi Y,Z)g(HX,W)-g(HY,Z)g(X,\phi W)+g(HY,Z)g(HX,W)+g(\phi X,Z)g(Y,\phi W) \\
& -g(\phi X,Z)g(HY,W)+g(HX,Z)g(Y,\phi W)-g(HX,Z)g(HY,W)-g(Y,Z)g(X,W) \\
& +\eta(X)\eta(W)g(Y,Z)-g(Y,Z)g(HX,\phi W)+g(\phi HY,Z)g(X,W)-\eta(X)\eta(W)g(\phi HY,Z) \\
& +g(\phi HY,Z)g(HX,\phi W)+g(X,Z)g(Y,W)-\eta(Y)\eta(W)g(X,Z)+g(X,Z)g(HY,\phi W) \\
& -g(\phi HX,Z)g(Y,W)+\eta(Y)\eta(W)g(\phi HX,Z)-g(\phi HX,Z)g(HY,\phi W).
\end{aligned}
\tag{51}
$$

By using the equality, $g(R(X,Y)\phi Z,\phi W)=g(R(\phi Z,\phi W)X,Y)$, the relation (51) reduces to

$$\begin{aligned}
&g(R(\phi Z,\phi W)X,Y)-g(R(Z,W)X,Y)=-\eta(W)g(R(Z,\xi)X,Y)+2g(\phi X,Y)g(HZ,W)\\
&+2g(HX,Y)g(HZ,W)-\eta(X)\eta(Z)g(H^2Y,W)-\eta(X)\eta(Z)g(HY,\phi W)\\
&+\eta(Y)\eta(Z)g(H^2X,W)+\eta(Y)\eta(Z)g(HX,\phi W)-g(\phi Y,Z)g(X,\phi W)\\
&+g(\phi Y,Z)g(HX,W)-g(HY,Z)g(X,\phi W)+g(HY,Z)g(HX,W)+g(\phi X,Z)g(Y,\phi W)\\
&-g(\phi X,Z)g(HY,W)+g(HX,Z)g(Y,\phi W)-g(HX,Z)g(HY,W)-g(Y,Z)g(X,W)\\
&+\eta(X)\eta(W)g(Y,Z)-g(Y,Z)g(HX,\phi W)+g(\phi HY,Z)g(X,W)-\eta(X)\eta(W)g(\phi HY,Z)\\
&+g(\phi HY,Z)g(HX,\phi W)+g(X,Z)g(Y,W)-\eta(Y)\eta(W)g(X,Z)+g(X,Z)g(HY,\phi W)\\
&-g(\phi HX,Z)g(Y,W)+\eta(Y)\eta(W)g(\phi HX,Z)-g(\phi HX,Z)g(HY,\phi W).
\end{aligned} \tag{52}$$

Therefore, we have the following.

Proposition 2. *Let (M,ϕ,ξ,η,g) be a nearly Sasakian manifold and R be the Riemannian curvature tensor of M. Then,*

$$\begin{aligned}
&R(\phi X,\phi Y)Z-R(X,Y)Z=-\eta(Y)g(H^2X,Z)\xi+\eta(Y)\eta(Z)H^2X+2g(HX,Y)\phi Z\\
&+2g(HX,Y)HZ-\eta(Z)\eta(X)H^2Y-\eta(Z)\eta(X)\phi HY+\eta(X)g(H^2Z,Y)\xi\\
&+\eta(X)g(HZ,\phi Y)\xi+g(Z,\phi Y)\phi X-g(HZ,Y)\phi X+g(Z,\phi Y)HX-g(HZ,Y)HX\\
&+g(\phi Z,X)\phi Y+g(\phi Z,Y)HY+g(HZ,X)\phi Y+g(HZ,X)HY-g(Z,Y)X\\
&-g(HZ,\phi Y)X-g(Z,Y)\phi HX+\eta(Z)\eta(Y)\phi HX-g(HZ,\phi Y)\phi HX+g(Z,X)Y\\
&+g(Z,X)\phi HY-g(\phi HZ,X)Y+\eta(Y)g(\phi HZ,X)\xi-g(\phi HZ,X)\phi HY,
\end{aligned} \tag{53}$$

for any vector fields X, Y, and Z of M.

Next, we deal with the ϕ-holomorphic sectional curvature on a nearly Sasakian manifold. A plane section σ in T_pM of a nearly Sasakian manifold M is called a ϕ section if there exists a vector field X for M that is orthogonal to ξ such that the basis $\{X,\phi X\}$ spans σ. The sectional curvature $K(X,\phi X)$ of a ϕ section is called the *ϕ-sectional curvature*, and it is denoted by $\mathcal{H}$. If M has a pointwise constant ϕ-holomorphic sectional curvature $\mathcal{H}=\mathcal{H}(p)$, $p\in M$, then, for any vector fields X and $Y\in D=\ker\eta$, we have

$$g(R(X,\phi X)X,\phi X)=-\mathcal{H}g(X,X)^2. \tag{54}$$

By taking the g-dot with ϕW of (2) and for any X, Y, and Z of D, we have

$$\begin{aligned}
g(R(X,Y)\phi Z,\phi W)=\;&g(R(X,Y)Z,W)+2\{g(\phi X,Y)+g(HX,Y)\}g(HZ,W)\\
&-\{g(\phi Y,Z)+g(HY,Z)\}g(X-\phi HX,\phi W)+\{g(\phi X,Z)\\
&+g(HX,Z)\}g(Y-\phi HY,\phi W)-g(Y-\phi HY,Z)g(\phi X+HX,\phi W)\\
&+g(X-\phi HX,Z)g(\phi Y+HY,\phi W).
\end{aligned} \tag{55}$$

By putting the vector fields $Y=\phi Y$, $Z=\phi X$, and $W=Y$ into (55), one obtains

$$\begin{aligned}
g(R(X,\phi Y)X,\phi Y)=\;&g(R(X,\phi Y)Y,\phi X)+g(X,Y)^2-g(HX,\phi Y)^2+g(X,\phi Y)^2\\
&-g(HX,Y)^2-g(X,X)g(Y,Y).
\end{aligned} \tag{56}$$

Likewise, for any $X,Y\in\Gamma(D)$, we have,

$$g(R(X,\phi X)Y,\phi X)=g(R(X,\phi X)X,\phi Y). \tag{57}$$

By substituting $X+Y$ in (54), and by using (57), the left-hand side of relation (54) becomes

$$
\begin{aligned}
&g(R(X+Y,\phi X+\phi Y)(X+Y),\phi X+\phi Y) = g(R(X,\phi X)X,\phi X) + g(R(Y,\phi Y)Y,\phi Y) \\
&+ g(R(Y,\phi X)X,\phi X) + g(R(X,\phi Y)X,\phi X) + g(R(Y,\phi Y)X,\phi X) + g(R(X,\phi X)Y,\phi X) \\
&+ g(R(Y,\phi X)Y,\phi X) + g(R(X,\phi Y)Y,\phi X) + g(R(Y,\phi Y)Y,\phi X) + g(R(X,\phi X)X,\phi Y) \\
&+ g(R(Y,\phi X)X,\phi Y) + g(R(X,\phi Y)X,\phi Y) + g(R(Y,\phi Y)X,\phi Y) + g(R(X,\phi X)Y,\phi Y) \\
&+ g(R(Y,\phi X)Y,\phi Y) + g(R(X,\phi Y)Y,\phi Y).
\end{aligned} \tag{58}
$$

By using (56) and (57), the Bianchi Identity, i.e., $g(R(Y,\phi Y)X,\phi X) = g(R(X,\phi Y)Y,\phi X) + g(R(\phi X,\phi Y)X,Y)$ and $g(R(X,\phi X)Y,\phi Y) = g(R(X,\phi Y)Y,\phi X) + g(R(\phi X,\phi Y)X,Y)$, one has

$$
\begin{aligned}
g(R(X+Y,\phi X+\phi Y)(X+Y),\phi X+\phi Y)& \\
&= 4g(R(X,\phi Y)Y,\phi X) + 4g(R(Y,\phi Y)Y,\phi X) + 4g(R(X,\phi X)X,\phi Y) \\
&+ 2g(R(\phi X,\phi Y)X,Y) + g(R(Y,\phi X)Y,\phi X) + g(R(X,\phi Y)X,\phi Y) \\
&+ g(R(X,\phi X)X,\phi X) + g(R(Y,\phi Y)Y,\phi Y).
\end{aligned} \tag{59}
$$

Now, by using (54), the relation (59) becomes, for any X and Y of D,

$$
\begin{aligned}
g(R(X+Y,\phi X+\phi Y)(X+Y),\phi X+\phi Y)& \\
&= 4g(R(X,\phi Y)Y,\phi X) + 4g(R(Y,\phi Y)Y,\phi X) + 4g(R(X,\phi X)X,\phi Y) \\
&+ 2g(R(\phi X,\phi Y)X,Y) + g(R(Y,\phi X)Y,\phi X) + g(R(X,\phi Y)X,\phi Y) \\
&- \mathcal{H}g(X,X)^2 - \mathcal{H}g(Y,Y)^2.
\end{aligned} \tag{60}
$$

By substituting $X+Y$ in (54), the right-hand side of relation (54) yields

$$
\begin{aligned}
-\mathcal{H}g(X,X)^2 = -\mathcal{H}\{&g(X,X)^2 + 4g(X,X)g(X,Y) + 2g(X,X)g(Y,Y) \\
&+ 4g(X,Y)^2 + 4g(X,Y)g(Y,Y) + g(Y,Y)^2\}.
\end{aligned} \tag{61}
$$

By calculating equality of (60) and (61), we obtain

$$
\begin{aligned}
&\frac{1}{2}\{4g(R(X,\phi Y)Y,\phi X) + 4g(R(Y,\phi Y)Y,\phi X) + 4g(R(X,\phi X)X,\phi Y) \\
&+ 2g(R(\phi X,\phi Y)X,Y) + g(R(Y,\phi X)Y,\phi X) + g(R(X,\phi Y)X,\phi Y)\} \\
&= -\mathcal{H}\{2g(X,Y)^2 + 2g(X,X)g(X,Y) + 2g(X,Y)g(Y,Y) \\
&+ g(X,X)g(Y,Y)\}.
\end{aligned} \tag{62}
$$

By putting $X = \phi Y$, $Y = X$, and $Z = Y$ into (2), we have

$$
\begin{aligned}
-g(R(Y,\phi X)Y,\phi X) - g(R(\phi Y,X)Y,\phi X) &= g(HY,\phi X)^2 - g(Y,\phi X)^2 \\
&+ g(HY,X)^2 + g(Y,Y)g(X,X) - g(X,Y)^2.
\end{aligned} \tag{63}
$$

Therefore, we have

$$
\begin{aligned}
g(R(Y,\phi X)Y,\phi X) &= g(R(X,\phi Y)Y,\phi X) + g(X,Y)^2 - g(HY,\phi X)^2 \\
&+ g(Y,\phi X)^2 - g(HY,X)^2 - g(X,X)g(Y,Y).
\end{aligned} \tag{64}
$$

By adding (56) and (64), one obtains

$$
\begin{aligned}
g(R(X,\phi Y)X,\phi Y) + g(R(Y,\phi X)Y,\phi X) &= 2g(R(X,\phi Y)Y,\phi X) \\
&+ 2g(X,Y)^2 - 2g(HX,\phi Y)^2 + 2g(X,\phi Y)^2 - 2g(HX,Y)^2 \\
&- 2g(X,X)g(Y,Y).
\end{aligned} \tag{65}
$$

By putting (65) into (62), we have

$$\begin{aligned}3g(R(X,\phi Y)Y,\phi X)&+2g(R(Y,\phi Y)Y,\phi X)+2g(R(X,\phi X)X,\phi Y)\\&+g(R(\phi X,\phi Y)X,Y)+g(X,Y)^2-g(HX,\phi Y)^2+g(X,\phi Y)^2\\&-g(HX,Y)^2-g(X,X)g(Y,Y)\\&=-\mathcal{H}\{2g(X,Y)^2+2g(X,X)g(X,Y)+2g(X,Y)g(Y,Y)\\&+g(X,X)g(Y,Y)\}.\end{aligned}\tag{66}$$

Since

$$\begin{aligned}g(R(\phi X,\phi Y)X,Y)=g(R(X,Y)X,Y)&+g(HX,Y)^2-g(X,\phi Y)^2\\&-g(X,Y)^2+g(HX,\phi Y)^2+g(X,X)g(Y,Y),\end{aligned}\tag{67}$$

the relation (66) becomes

$$\begin{aligned}3g(R(X,\phi Y)Y,\phi X)+2g(R(Y,\phi Y)Y,\phi X)&+2g(R(X,\phi X)X,\phi Y)+g(R(X,Y)X,Y)\\&=-\mathcal{H}\{2g(X,Y)^2+2g(X,X)g(X,Y)+2g(X,Y)g(Y,Y)\\&+g(X,X)g(Y,Y)\}.\end{aligned}\tag{68}$$

By replacing Y with $-Y$ in (68), one obtains

$$\begin{aligned}3g(R(X,\phi Y)Y,\phi X)&-2g(R(Y,\phi Y)Y,\phi X)-2g(R(X,\phi X)X,\phi Y)\\&+g(R(X,Y)X,Y)=-\mathcal{H}\{2g(X,Y)^2-2g(X,X)g(X,Y)\\&-2g(X,Y)g(Y,Y)+g(X,X)g(Y,Y)\}.\end{aligned}\tag{69}$$

By summing the relations (68) and (69), we have

$$3g(R(X,\phi Y)Y,\phi X)+g(R(X,Y)X,Y)=-\mathcal{H}\{2g(X,Y)^2+g(X,X)g(Y,Y)\}.\tag{70}$$

By replacing Y with ϕY in (70) and using curvature identities, we obtain,

$$3g(R(\phi Y,\phi X)Y,X)+g(R(X,\phi Y)X,\phi Y)=-\mathcal{H}\{2g(X,\phi Y)^2+g(X,X)g(Y,Y)\}.\tag{71}$$

Through the relations (56) and (67), the left-hand side of relation (71) becomes

$$\begin{aligned}3g(R(\phi Y,\phi X)Y,X)+g(R(X,\phi Y)X,\phi Y)&=3g(R(X,Y)X,Y)+g(R(X,\phi Y)Y,\phi X)\\&+2g(HX,Y)^2-2g(X,\phi Y)^2-2g(X,Y)^2+2g(HX,\phi Y)^2\\&+2g(X,X)g(Y,Y).\end{aligned}\tag{72}$$

By putting the pieces (71) and (72) together, we have

$$\begin{aligned}g(R(X,\phi Y)Y,\phi X)=-3g(R(X,Y)X,Y)&-\mathcal{H}\{2g(X,\phi Y)^2+g(X,X)g(Y,Y)\}\\&-2g(HX,Y)^2+2g(X,\phi Y)^2+2g(X,Y)^2-2g(HX,\phi Y)^2\\&-2g(X,X)g(Y,Y).\end{aligned}\tag{73}$$

Substituting (73) into (70) leads to

$$\begin{aligned}-8g(R(X,Y)X,Y)&-3\mathcal{H}\{2g(X,\phi Y)^2+g(X,X)g(Y,Y)\}-6g(HX,Y)^2\\&+6g(X,\phi Y)^2+6g(X,Y)^2-6g(HX,\phi Y)^2\\&-6g(X,X)g(Y,Y)=-\mathcal{H}\{2g(X,Y)^2+g(X,X)g(Y,Y)\}.\end{aligned}\tag{74}$$

Therefore, we have

$$4g(R(X,Y)X,Y) = (\mathcal{H}+3)\{g(X,Y)^2 - g(X,X)g(Y,Y)\} \\ - 3(\mathcal{H}-1)g(X,\phi Y)^2 - 3g(HX,Y)^2 - 3g(HX,\phi Y)^2. \tag{75}$$

By replacing X and Y with $X+Z$ and $Y+W$, respectively, in both sides of (75), one has,

$$\begin{aligned} 8g(R(X,Y)Z,W) + 8g(R(Z,Y)X,W) = & -4(\mathcal{H}+3)g(X,Z)g(Y,W) \\ & + 2(\mathcal{H}+3)\{g(X,Y)g(Z,W) + g(X,W)g(Z,Y)\} \\ & - 6(\mathcal{H}-1)\{g(X,\phi Y)g(Z,\phi W) + g(X,\phi W)g(Z,\phi Y)\} \\ & - 6\{g(HX,Y)g(HZ,W) + g(HX,W)g(HZ,Y) \\ & + g(HX,\phi Y)g(HZ,\phi W) + g(HX,\phi W)g(HZ,\phi Y)\}. \end{aligned} \tag{76}$$

In addition, by replacing Y with Z and Z with Y in (76) and then multiplying both sides by -1, we have

$$\begin{aligned} & - 8g(R(X,Z)Y,W) - 8g(R(Y,Z)X,W) = 4(\mathcal{H}+3)g(X,Y)g(Z,W) \\ & - 2(\mathcal{H}+3)\{g(X,Z)g(Y,W) + g(X,W)g(Z,Y)\} + 6(\mathcal{H}-1)\{g(X,\phi Z)g(Y,\phi W) \\ & + g(X,\phi W)g(Y,\phi Z)\} + 6\{g(HX,Z)g(HY,W) + g(HX,W)g(HY,Z) \\ & + g(HX,\phi Z)g(HY,\phi W) + g(HX,\phi W)g(HY,\phi Z)\}. \end{aligned} \tag{77}$$

By adding (76) and (77), we have

$$\begin{aligned} & 8g(R(X,Y)Z,W) + 16g(R(Z,Y)X,W) - 8g(R(X,Z)Y,W) \\ & = 6(\mathcal{H}+3)\{g(X,Y)g(Z,W) - g(X,Z)g(Y,W)\} - 6(\mathcal{H}-1)\{g(X,\phi Y)g(Z,\phi W) \\ & - g(X,\phi Z)g(Y,\phi W) + 2g(X,\phi W)g(Z,\phi Y)\} - 6\{g(HX,Y)g(HZ,W) \\ & - g(HX,Z)g(HY,W) + 2g(HX,W)g(HZ,Y) + g(HX,\phi Y)g(HZ,\phi W) \\ & - g(HX,\phi Z)g(HY,\phi W) + 2g(HX,\phi W)g(HZ,\phi Y)\}. \end{aligned} \tag{78}$$

By using the Bianchi identity, that is, $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ and $g(R(Z,Y)X,W) = g(R(X,W)Z,Y)$, the relation ((78) becomes

$$\begin{aligned} & 24g(R(X,W)Z,Y) = 6(\mathcal{H}+3)\{g(X,Y)g(Z,W) - g(X,Z)g(Y,W)\} \\ & - 6(\mathcal{H}-1)\{g(X,\phi Y)g(Z,\phi W) - g(X,\phi Z)g(Y,\phi W) + 2g(X,\phi W)g(Z,\phi Y)\} \\ & - 6\{g(HX,Y)g(HZ,W) - g(HX,Z)g(HY,W) + 2g(HX,W)g(HZ,Y) \\ & + g(HX,\phi Y)g(HZ,\phi W) - g(HX,\phi Z)g(HY,\phi W) + 2g(HX,\phi W)g(HZ,\phi Y)\}. \end{aligned} \tag{79}$$

By exchanging W and Y in (79), we obtain

$$\begin{aligned} & 24g(R(X,Y)Z,W) = 6(\mathcal{H}+3)\{g(X,W)g(Z,Y) - g(X,Z)g(Y,W)\} \\ & - 6(\mathcal{H}-1)\{g(X,\phi W)g(Z,\phi Y) - g(X,\phi Z)g(W,\phi Y) + 2g(X,\phi Y)g(Z,\phi W)\} \\ & - 6\{g(HX,W)g(HZ,Y) - g(HX,Z)g(HW,Y) + 2g(HX,Y)g(HZ,W) \\ & + g(HX,\phi W)g(HZ,\phi Y) - g(HX,\phi Z)g(HW,\phi Y) + 2g(HX,\phi Y)g(HZ,\phi W)\} \end{aligned} \tag{80}$$

for any X, Y, Z, and $W \in \Gamma(D)$. Now, by considering a vector field X of M as $X = QX + \eta(X)\xi$, where Q is the projection onto D, one has, for any X, Y , Z, and $W \in \Gamma(TM)$,

$$\begin{aligned} & g(R(QX,QY)QZ,QW) = g(R(X,Y)Z,W) - \eta(X)\eta(W)\{g(Y,Z) - g(H^2Z,Y)\} \\ & + \eta(X)\eta(Z)\{g(W,Y) - g(H^2W,Y)\} - \eta(Y)\eta(Z)\{g(W,X) - g(H^2W,X)\} \\ & + \eta(Y)\eta(W)\{g(Z,X) - g(H^2Z,X)\}. \end{aligned} \tag{81}$$

From (80), and by using (81), we have the following

$$
\begin{aligned}
&24g(R(X,Y)Z,W) = 6(\mathcal{H}+3)\{g(X,W)g(Z,Y) - \eta(Z)\eta(Y)g(X,W)\\
&- \eta(X)\eta(W)g(Z,Y) - g(X,Z)g(Y,W) + \eta(Y)\eta(W)g(X,Z)\\
&+ \eta(X)\eta(Z)g(Y,W)\} - 6(\mathcal{H}-1)\{g(X,\phi W)g(Z,\phi Y) - g(X,\phi Z)g(W,\phi Y)\\
&+ 2g(X,\phi Y)g(Z,\phi W)\} - 6\{g(HX,W)g(HZ,Y) - g(HX,Z)g(HW,Y)\\
&+ 2g(HX,Y)g(HZ,W) + g(HX,\phi W)g(HZ,\phi Y) - g(HX,\phi Z)g(HW,\phi Y)\\
&+ 2g(HX,\phi Y)g(HZ,\phi W)\} + 24\eta(X)\eta(W)\{g(Y,Z) - g(H^2Z,Y)\}\\
&- 24\eta(X)\eta(Z)\{g(W,Y) - g(H^2W,Y)\} + 24\eta(Y)\eta(Z)\{g(W,X) - g(H^2W,X)\}\\
&- 24\eta(Y)\eta(W)\{g(Z,X) - g(H^2Z,X)\}.
\end{aligned}
\tag{82}
$$

Therefore, one has the following.

Proposition 3. *Let (M,ϕ,ξ,η,g) be a nearly Sasakian manifold. Then, the necessary and sufficient condition for M to have a pointwise constant ϕ-holomorphic sectional curvature $\mathcal{H}$ is*

$$
\begin{aligned}
R(X,Y)Z = &\frac{\mathcal{H}+3}{4}\{g(Z,Y)X - g(X,Z)Y\} + \frac{\mathcal{H}-1}{4}\{\eta(X)\eta(Z)Y\\
&- \eta(Z)\eta(Y)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Z,Y)\xi + g(Z,\phi Y)\phi X\\
&+ g(X,\phi Z)\phi Y + 2g(X,\phi Y)\phi Z\} - \frac{1}{4}\{g(HZ,Y)HX + g(HX,Z)HY\\
&+ 2g(HX,Y)HZ - g(HZ,\phi Y)\phi HX - g(HX,\phi Z)\phi HY\\
&- 2g(HX,\phi Y)\phi HZ\} + \eta(Z)\{\eta(X)H^2Y - \eta(Y)H^2X\}\\
&+ \{\eta(Y)g(H^2Z,X) - \eta(X)g(H^2Z,Y)\}\xi
\end{aligned}
\tag{83}
$$

for all vector fields X, Y, and Z of M.

From relation (83), the Ricci tensor Ric associated with the Riemannian metric g yields

$$
\begin{aligned}
\mathrm{Ric}(X,Y) = &\frac{n(\mathcal{H}+3)+\mathcal{H}-1}{2}g(X,Y) - \frac{(n+1)(\mathcal{H}-1)}{2}\eta(X)\eta(Y)\\
&- \frac{5}{2}g(X,H^2Y) - \eta(X)\eta(Y)\mathrm{trace}\,H^2.
\end{aligned}
\tag{84}
$$

Moreover, we have the identity for the Ricci curvature:

$$
\mathrm{Ric}(\phi X,\phi Y) = \mathrm{Ric}(X,Y) - (2n - \mathrm{trace}\,H^2)\eta(X)\eta(Y).
\tag{85}
$$

Let τ be the scalar curvature of g. Then, τ is given by

$$
\tau = \frac{1}{2}\{n(2n+1)(\mathcal{H}+3) + n(\mathcal{H}-1)\} - \frac{7}{2}\mathrm{trace}\,H^2.
\tag{86}
$$

Lemma 1. *In a nearly Sasakian manifold, the eigenvalues of the operator H^2 are constant.*

Proof. The proof follows from a direct calculation using (14). □

For any vector field X, Y, Z, and W, one has

$$
\begin{aligned}
(\nabla_X \mathrm{Ric})(Y,Z) = & \frac{n+1}{2}X(\mathcal{H})g(Y,Z) - \left\{\frac{(n+1)(\mathcal{H}-1)}{2} + \operatorname{trace} H^2\right\}\{\eta(Z)(\nabla_X\eta)Y \\
& + \eta(Y)(\nabla_X\eta)Z\} + \frac{5}{2}\{g((\nabla_X H)Y, HZ) + g(HY, (\nabla_X H)Z)\} \\
= & \frac{n+1}{2}X(\mathcal{H})g(Y,Z) + \left\{\frac{(n+1)(\mathcal{H}-1)}{2} + \operatorname{trace} H^2\right\}\{\eta(Z)g(\phi X, Y) \\
& + \eta(Z)g(HX,Y) + \eta(Y)g(\phi X, Z) + \eta(Y)g(HX,Z)\} \\
& + \frac{5}{2}\Big\{\eta(Y)g(H^2X, HZ) - \eta(Y)g(\phi HX, HZ) + \eta(Z)g(H^2X, HY) \\
& - \eta(Z)g(\phi HX, HY)\}.
\end{aligned}
\tag{87}
$$

Let $\{E_i\}_{1\le i\le 2n+1}$ be an arbitrary local orthonormal frame field on M. Then,

$$
\nabla_X \tau = 2\sum_{i=1}^{2n+1}(\nabla_{E_i} Ric)(X, E_i) = (n+1)X(\mathcal{H}). \tag{88}
$$

On the other hand, by using (86) and Lemma 1, one obtains

$$
\nabla_X \tau = \frac{1}{2}\{n(2n+1)X(\mathcal{H}) + nX(\mathcal{H})\} - \frac{1}{2}X(\operatorname{trace} H^2) = n(n+1)X(\mathcal{H}). \tag{89}
$$

From the relations (88) and (89), we have

$$
n(n+1)X(\mathcal{H}) = (n+1)X(\mathcal{H}),\ \forall\, X \in \Gamma(TM).
$$

This leads to $(n-1)X(\mathcal{H}) = 0$. If $n > 1$, and the nearly Sasakian manifold M is connected, then H is constant for M. Therefore, according to Ogiue [16], we obtain the following theorem.

Theorem 7. *Let M be a $(2n+1)$-dimensional nearly Sasakian manifold ($n > 1$). If the ϕ-holomorphic sectional curvature at any point of M is independent of the choice of the ϕ-holomorphic section, then it is constant for M, and the curvature tensor R is given by*

$$
\begin{aligned}
R(X,Y)Z = & \frac{\mathcal{H}+3}{4}\{g(Z,Y)X - g(X,Z)Y\} + \frac{\mathcal{H}-1}{4}\{\eta(X)\eta(Z)Y - \eta(Z)\eta(Y)X \\
& + \eta(Y)g(X,Z)\xi - \eta(X)g(Z,Y)\xi + g(Z,\phi Y)\phi X + g(X,\phi Z)\phi Y + 2g(X,\phi Y)\phi Z\} \\
& - \frac{1}{4}\{g(HZ,Y)HX + g(HX,Z)HY + 2g(HX,Y)HZ - g(HZ,\phi Y)\phi HX \\
& - g(HX,\phi Z)\phi HY - 2g(HX,\phi Y)\phi HZ\} + \eta(Z)\{\eta(X)H^2Y - \eta(Y)H^2X\} \\
& + \{\eta(Y)g(H^2Z,X) - \eta(X)g(H^2Z,Y)\}\xi
\end{aligned}
\tag{90}
$$

for any vector fields X, Y, and Z of M.

Note that a complete and simply connected nearly Sasakian manifold with a constant ϕ-holomorphic sectional curvature $\mathcal{H}$ is said to be a *nearly Sasakian space form*. Thus, we obtain the following result.

Theorem 8. *Let M be a $(2n+1)$-dimensional complete and simply connected nearly Sasakian manifold ($n > 1$). Then, M is a nearly Sasakian space form if and only if the curvature tensor R is given by (90).*

Next, we introduce another class of nearly Sasakian manifolds with a Codazzi-type Ricci tensor in which the condition (26) is naturally derived.

With regard to a *Codazzi-type Ricci tensor*, we mean a Ricci tensor Ric satisfying the Codazzi equation; that is,

$$(\nabla_X \mathrm{Ric})(Y,Z) = (\nabla_Y \mathrm{Ric})(X,Z), \ \ \forall X,\ Y,\ Z \in \Gamma(TM). \tag{91}$$

A manifold with such a tensor is called a *Codazzi-type Ricci manifold*.

Now, from (87) and (91), one has

$$\begin{aligned}
&\left\{\frac{(n+1)(\mathcal{H}-1)}{2} + \mathrm{trace}\, H^2\right\}\{2\eta(Z)g(\phi X, Y + 2\eta(Z)g(HX,Y) \\
&+\eta(Y)g(\phi X,Z) - \eta(X)g(\phi Y,Z) + \eta(Y)g(HX,Z) - \eta(X)g(HY,Z)\} \\
&+\frac{5}{2}\Big\{\eta(Y)g(H^2X,HZ) - \eta(X)g(H^2Y,HZ) + \eta(X)g(\phi HY,HZ) \\
&-\eta(Y)g(\phi HX,HZ) + 2\eta(Z)g(H^2X,HY) - 2\eta(Z)g(\phi HX,HY)\Big\} = 0.
\end{aligned} \tag{92}$$

Letting $X = \xi$ in this equation yields

$$\begin{aligned}
&\left\{\frac{(n+1)(\mathcal{H}-1)}{2} + \mathrm{trace}\, H^2\right\}\{-g(\phi Y,Z) - g(HY,Z)\} \\
&+\frac{5}{2}\Big\{-g(H^2Y,HZ) + g(\phi HY,HZ)\Big\} = 0.
\end{aligned} \tag{93}$$

If $Z = \phi Y$, then this equation becomes

$$\frac{2}{5}\left\{\frac{(n+1)(\mathcal{H}-1)}{2} + \mathrm{trace}\, H^2\right\}\{g(Y,Y) - \eta(Y)\eta(Y)\} + g(HY,HY) = 0. \tag{94}$$

We set

$$\mu = \frac{2}{5}\left\{\frac{(n+1)(\mathcal{H}-1)}{2} + \mathrm{trace}\, H^2\right\}. \tag{95}$$

Then, relation (94) leads to

$$H^2 = \mu\{\mathbb{I} - \eta \otimes \xi\}. \tag{96}$$

In accordance with Lemma 1 and Theorem 8, the function μ defined in (95) on a nearly Sasakian space form is a constant. This is achieved by taking into account the reasoning that led to (26), $\mu = -\lambda^2$. This means that μ is non-positive. According to Theorem 4.1 in [5], we have the following result.

Theorem 9. *A Codazzi-type Ricci nearly Sasakian space form is either a Sasakian manifold with a constant ϕ-holomorphic sectional curvature $\mathcal{H} = 1$ or is a 5-dimensional proper nearly Sasakian manifold with a constant ϕ-holomorphic sectional curvature $\mathcal{H} > 1$.*

Proof. The last assertion follows from (95), (96), and the sign of μ. □

Note that $H\xi = 0$, i.e., 0 is an eigenvalue of H^2. Also, since the operator H is skew-symmetric, the non-vanishing eigenvalues of H^2 are negative, as proven by (26). Thus, the spectrum of H^2 is of the type

$$\mathrm{Spec}(H^2) = \{0, -\lambda^2, \cdots, -\lambda^2\}, \ \ \lambda \neq 0.$$

Let $\mathbb{R}\xi$, $D(0)$, and $D(-\lambda^2)$ denote the distribution of dimension 1 generated by ξ and the distributions of the eigenvectors corresponding to the eigenvalues 0 and $-\lambda^2$, respectively.

If X is an eigenvector of H^2 with a corresponding eigenvalue of $-\lambda^2$, then, from (4), we have

$$H^2\nabla_X\xi = -(\nabla_X H^2)\xi = -(\phi + H)H^2X = -\lambda^2\nabla_X\xi. \tag{97}$$

This means that $\nabla_X\xi$ is an eigenvector corresponding to the eigenvalue $-\lambda^2$. Given that the relation (14) leads to $\nabla_\xi H^2 = 0$, we have

$$H^2\nabla_\xi X = \nabla_\xi H^2 X = -\xi(\lambda^2)X - \lambda^2\nabla_\xi X = -\lambda^2\nabla_\xi X. \tag{98}$$

Thus, $\nabla_\xi X$ is also an eigenvector corresponding to the eigenvalue $-\lambda^2$. If vector fields X and Y are both eigenvectors with the eigenvalue $-\lambda^2$ and are orthogonal to ξ, then, from (14), one obtains

$$H^2(\nabla_X Y) = \nabla_X H^2 Y - (\nabla_X H^2)Y = -\lambda^2\nabla_X Y + \lambda^2 g(\phi X + HX, Y)\xi. \tag{99}$$

If $\lambda = 0$, the $\nabla_X Y$ belongs to $D(0)$. If $\lambda \neq 0$, one obtains

$$H^2(\phi^2\nabla_X Y) = \phi^2(H^2\nabla_X Y) = -\lambda^2\phi^2(\nabla_X Y), \tag{100}$$

and, thus,

$$\nabla_X Y = -\phi^2\nabla_X Y + \eta(\nabla_X Y)\xi \in \mathbb{R}\xi \oplus D(-\lambda^2).$$

Note that, if X is an eigenvector of H^2 with an eigenvalue $-\lambda^2$, then the vector fields X, ϕX, HX, and $H\phi X$ are mutually orthogonal, and they are also eigenvectors of H^2 with the corresponding eigenvalue $-\lambda^2$. By using Theorem 9, 0 becomes a simple eigenvalue, and the multiplicity of the eigenvalue $-\lambda^2$ is 4. Therefore, we obtain the following result.

Theorem 10. *Let M be a Codazzi-type Ricci nearly Sasakian space form. Then, the spectrum of H^2 has the form*

$$\mathrm{Spec}(H^2) = \{0, -\lambda^2,\ -\lambda^2,\ -\lambda^2,\ -\lambda^2\},\ \ \lambda \neq 0,$$

where 0 is a simple eigenvalue, and $-\lambda^2$ is an eigenvalue of multiplicity 4. Moreover, the distributions $D(0)$ and $\mathbb{R}\xi \oplus D(-\lambda^2)$ are integrable with totally geodesic leaves.

Cappelletti-Montano and Dileo proved in ([Theorem 4.3] in [2]) that there is a one-to-one correspondence between a nearly Sasakian space form and $SU(2)$ structures. The latter induces a Sasaki–Einstein structure (see [2] for more details). Therefore, we have the following result.

Theorem 11. *A Codazzi-type Ricci nearly Sasakian space form carries a Sasaki–Einstein structure.*

A similar conclusion from Theorem 11 can also be induced from some of the results found in Section 3. In [Theorem 6.1] in [4], Olszak proved, under the condition (22), that a proper nearly Sasakian space form is a 5-dimensional manifold of a constant sectional curvature. Next, we prove otherwise using the projectively flat notion. First of all, we note that the class of Codazzi-type Ricci manifolds is a subclass of projectively flat manifolds (see [Proposition 5] in [17] for more details). The concept of projectively flat is defined via a tensor called the projective curvature tensor. This plays a role as an important tensor in differential geometry. A manifold M is said to be locally projectively flat if there is a one-to-one correspondence between each coordinate system of M and a subspace of a Euclidean space $\mathbb{E}$ such that any geodesic of M corresponds to a straight line in $\mathbb{E}$. As known in ([17], p. 411), the Levi–Civita connection of a non-degenerate metric g is locally projectively flat if and only if g has a constant sectional curvature.

For $n \geq 1$, a nearly Sasakian manifold M is locally projectively flat if and only if the projective curvature tensor $\mathcal{P}$ vanishes, where $\mathcal{P}$ is given by (see [17])

$$\mathcal{P}(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\{Ric(Y,Z)X - Ric(X,Z)Y\} \tag{101}$$

for any vector fields X, Y, and Z of M.

Theorem 12. *A proper nearly Sasakian space form is not of constant sectional curvature.*

Proof. Let M be a proper nearly Sasakian space form. If we assume that M is of a constant sectional curvature, then it is locally projectively flat; that is, the projective curvature tensor $\mathcal{P}$ in (101) vanishes. A direct calculation of (101) leads to

$$\begin{aligned}
&2nR(X,Y)Z - \{Ric(Y,Z)X - Ric(X,Z)Y\} = -\frac{\mathcal{H}-1}{2}\{g(Z,Y)X - g(X,Z)Y\} \\
&+ \left\{\text{trace } H^2 + \frac{\mathcal{H}-1}{2}\right\}\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\} + \frac{n(\mathcal{H}-1)}{2}\{\eta(Y)g(X,Z)\xi \\
&- \eta(X)g(Z,Y)\xi + g(Z,\phi Y)\phi X + g(X,\phi Z)\phi Y + 2g(X,\phi Y)\phi Z\} \\
&- \frac{n}{2}\{g(HZ,Y)HX + g(HX,Z)HY + 2g(HX,Y)HZ - g(HZ,\phi Y)\phi HX \\
&- g(HX,\phi Z)\phi HY - 2g(HX,\phi Y)\phi HZ\} + \eta(Z)\{\eta(X)H^2Y - \eta(Y)H^2X \\
&+ 2n\{\eta(Y)g(H^2Z,X) - \eta(X)g(H^2Z,Y)\}\xi - \frac{5}{2}g(HY,HZ)X + \frac{5}{2}g(HX,HZ)Y.
\end{aligned} \tag{102}$$

Now, by putting $Y = Z \in \Gamma(D)$ into (102) and considering $X \in \Gamma(D)$ such that $g(X,Y) = 0$, we have

$$2ng(\mathcal{P}(X,Y)Y,Y) = \frac{5}{2}g(HX,HY)g(Y,Y). \tag{103}$$

Since M is locally projectively flat, then (103) vanishes; that is,

$$0 = g(HX,HY)g(Y,Y), \quad \forall\, X \in \Gamma(D).$$

This implies that $H^2Y = 0$, as $g(Y,Y) \neq 0$, $\forall\, Y \in \Gamma(D)$. Since $H\xi = 0$, thus $H^2 = 0$ for M, which is a contradiction, as M is non-Sasakian. This completes the proof. □

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