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New Family of Multi-Step Iterative Methods Based on Homotopy Perturbation Technique for Solving Nonlinear Equations

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Abstract: This research aims to propose a new family of one-parameter multi-step iterative methods that combine the homotopy perturbation method with a quadrature formula for solving nonlinear equations. The proposed methods are based on a higher-order convergence scheme that allows for faster and more efficient convergence compared to existing methods. It aims also to demonstrate that the efficiency index of the proposed iterative methods can reach up to $\sqrt[3]{4} \approx 1.587$ and $\sqrt[4]{8} \approx 1.681$, respectively, indicating a high degree of accuracy and efficiency in solving nonlinear equations. To evaluate the effectiveness of the suggested methods, several numerical examples including their performance are provided and compared with existing methods.

Keywords: homotopy perturbation; second derivative-free; iterative methods; nonlinear equations

MSC: 65H05; 65B99



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1. Introduction

Nonlinear equations are ubiquitous in various scientific and engineering fields and are of great importance in applied mathematics. Finding their solutions analytically is often infeasible or impossible, and therefore numerical iterative methods are commonly employed to obtain approximations. Among the classical iterative formulas, Newton's method and its variants are widely used, but many other techniques have been developed and studied, such as the Adomian decomposition method, the variational iteration method, and the differential transform method, to name a few (see [1–5] and the references therein).

One of the techniques that has gained attention for obtaining iterative methods for nonlinear equations is the homotopy analysis method (HAM) introduced by Liao in 1992 [6]. Another successful iterative method for nonlinear equations is the homotopy perturbation method (HPM), which is a special case of HAM introduced by He in 1999 [7] (see [8–10] for more details). Since then, many developments and extensions of HPM have been proposed by many authors, such as Sehati et al. [11] and others, resulting in a variety of novel iterative methods with higher orders of convergence. For instance, Waheed et al. [12] used the HPM to develop certain iterative methods with high accuracy and efficiency.

Recent advances in artificial intelligence and machine learning techniques have also been applied to develop iterative methods for nonlinear equations, such as using deep learning neural networks, for predicting iterative solutions [13,14].

An essential aspect of iterative methods is their convergence rate, which determines how quickly the method converges to the true solution. Numerous studies have focused on the convergence analysis of iterative methods for nonlinear equations, resulting in various convergence criteria that can be used to improve the efficiency of iterative methods and optimize their convergence rate [11,15–17].

Iterative methods for solving nonlinear equations are a vital research area with many practical applications. The recent developments in this field, including the use of the HPM and artificial intelligence techniques, have expanded the available toolkit for approximating the solutions of nonlinear equations and have led to improved insights into their behavior. As research in this area continues, further advances can be expected in the accuracy and efficiency of iterative methods, enabling researchers to tackle increasingly complex nonlinear problems.

Motivated by the objective of enhancing the precision and efficiency of iterative methods employed in solving nonlinear equations, this paper introduces a novel multi-step iterative method that combines the Haar wavelets quadrature formula and the homotopy perturbation technique. The organization of the paper is as follows: Section 2 presents the construction of the new iterative methods, comprising a novel set of algorithms designed specifically for solving nonlinear equations. The algorithms employed as proposed methods are demonstrated, accompanied by an explanation of their underlying rationale. Section 3 provides a convergence analysis of the proposed methods, accompanied by the declaration of efficiency indices. The performance of the algorithms is further assessed through an analysis of convergence rates. Additionally, efficiency indices, which serve as indicators of the computational cost and accuracy of the methods, are described. In Section 4, a comparative analysis is conducted between the performance of the proposed algorithms and several existing methods found in the literature. Various examples are employed to evaluate the performance of the algorithms, and numerical results are presented to demonstrate their effectiveness in solving the problem at hand. Furthermore, graphical representations are utilized to provide a more comprehensive description of the convergence, accuracy, and reliability of the proposed algorithms. Lastly, Section 5 delves into a discussion of the obtained results, drawing conclusions from the analysis conducted. The strengths and weaknesses of the proposed methods are highlighted, and suggestions for future research directions are provided.

2. The New Iterative Methods Construction

Consider the following nonlinear equation:

$$F(x) = 0, \tag{1}$$

and suppose α be a simple root of Equation (1), and x_0 is an initial approximation near to α . Consider the quadrature formula which is defined by

$$F(x) = F(x_0) + \int_{x_0}^x F'(\mu) d\mu. \tag{2}$$

Approximating the above integral in Equation (2) by using Haar wavelets quadrature formula [18], one could obtain that

$$F(x) \approx F(x_0) + \frac{(x - x_0)}{2r} \sum_{\ell=1}^{2r} F' \left(x_0 + \frac{(\ell - 0.5)(x - x_0)}{2r} \right). \tag{3}$$

Assuming that $F(x)$ is close enough to $F(x_0)$ in Equation (3) leads to

$$\frac{(x - x_0)}{2r} \sum_{\ell=1}^{2r} F' \left(x_0 + \frac{(\ell - 0.5)(x - x_0)}{2r} \right) \approx 0. \tag{4}$$

Based on HPM, the construction of the homotopy $\mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ can be made, which will satisfy

$$H(x, p) = pF(x) + (1 - p) \left[\frac{(x - x_0)}{2r} \sum_{\ell=1}^{2r} F' \left(x_0 + \frac{(\ell - 0.5)(x - x_0)}{2r} \right) \right] = 0, \tag{5}$$

where $p \in [0, 1]$ is an artificial parameter. As a trivial case, one could see that

$$H(x, 0) = \frac{(x - x_0)}{2r} \sum_{\ell=1}^{2r} F' \left(x_0 + \frac{(\ell - 0.5)(x - x_0)}{2r} \right) = 0, \tag{6}$$

and

$$H(x, 1) = F(x) = 0. \tag{7}$$

The solution of Equation (5) is obtained as an infinite series that is given as

$$x = x_0 + px_1 + p^2x_2 + p^3x_3 + \dots = \sum_{k=0}^{\infty} p^k x_k. \tag{8}$$

Consequently, the approximate solution of Equation (1) is given by

$$\tilde{x} = \lim_{p \rightarrow 1} x = \sum_{k=0}^{\infty} x_k, \tag{9}$$

provided the series (8) converges. Series (9) is convergent for most cases, and also the rate of convergence depends on $F(x)$ [7]. Equation (5) can be expressed as follows by using Taylor’s series [19] for $F(x)$ and $F' \left(x_0 + \frac{(\ell - 0.5)(x - x_0)}{2r} \right)$ about x_0 :

$$p \left[F(x_0) + (x - x_0)F'(x_0) + \frac{1}{2}(x - x_0)^2F''(x_0) + \dots \right] + (1 - p) \left[\frac{(x - x_0)}{2r} \sum_{\ell=1}^{2r} \left\{ F'(x_0) + \frac{(\ell - 0.5)(x - x_0)}{2r} F''(x_0) + \dots \right\} \right] = 0.$$

Now, taking $r = 1$ to obtain

$$p \left[F(x_0) + (x - x_0)F'(x_0) + \frac{1}{2}(x - x_0)^2F''(x_0) \right] + (1 - p) \left[(x - x_0)F'(x_0) + \frac{1}{2}(x - x_0)^2F''(x_0) \right] = 0. \tag{10}$$

Equation (8) is substituted into Equation (10), resulting in

$$p \left[F(x_0) + \left(\sum_{k=1}^{\infty} p^k x_k \right) F'(x_0) + \frac{1}{2} \left(\sum_{k=1}^{\infty} p^k x_k \right)^2 F''(x_0) \right] + (1 - p) \left[\left(\sum_{k=1}^{\infty} p^k x_k \right) F'(x_0) + \frac{1}{2} \left(\sum_{k=1}^{\infty} p^k x_k \right)^2 F''(x_0) \right] = 0. \tag{11}$$

Matching powers of the above coefficients of p , one can obtain the following system of equations

$$\begin{aligned}
 p^0 &: 0, \\
 p^1 &: F(x_0) + x_1F'(x_0) = 0, \\
 p^2 &: x_2F'(x_0) + \frac{1}{2}x_1^2F''(x_0) = 0, \\
 p^3 &: x_3F'(x_0) + x_1x_2F''(x_0) = 0, \\
 p^4 &: x_4F'(x_0) + \left(x_1x_3 + \frac{1}{2}x_2^2\right)F''(x_0) = 0, \\
 p^5 &: x_5F'(x_0) + (x_1x_4 + x_2x_3)F''(x_0) = 0, \\
 p^6 &: x_6F'(x_0) + \left(x_1x_5 + x_2x_4 + \frac{1}{2}x_3^2\right)F''(x_0) = 0,
 \end{aligned}$$

and so on. After solving the above system of equations, the following is obtained:

$$\begin{aligned}
 x_1 &= -\frac{F(x_0)}{F'(x_0)}, \quad x_2 = -\frac{F^2(x_0)F''(x_0)}{2(F'(x_0))^3}, \\
 x_3 &= -\frac{F^3(x_0)(F''(x_0))^2}{2(F'(x_0))^5}, \quad x_4 = -\frac{5F^4(x_0)(F''(x_0))^3}{8(F'(x_0))^7}.
 \end{aligned} \tag{12}$$

Substituting the values of x_1, x_2, x_3 and x_4 in (9) obtains

$$\tilde{x} = x_0 - \frac{F(x_0)}{F'(x_0)} - \frac{F^2(x_0)F''(x_0)}{2(F'(x_0))^3} - \frac{F^3(x_0)(F''(x_0))^2}{2(F'(x_0))^5} - \frac{5F^4(x_0)(F''(x_0))^3}{8(F'(x_0))^7}. \tag{13}$$

The above formula is a generalization of Equation (8) in [20]. From the aforementioned formulation, a one-step iterative method is presented for solving nonlinear equations, which is given by the following Algorithm 1.

Algorithm 1: A suggested one-step iterative method.

For a suitable initial approximation x_0 , calculate the next solution x_{i+1} using the below iterative method

$$x_{i+1} = x_i - \frac{F(x_i)}{F'(x_i)} - \frac{F^2(x_i)F''(x_i)}{2(F'(x_i))^3} - \frac{F^3(x_i)(F''(x_i))^2}{2(F'(x_i))^5} - \frac{5F^4(x_i)(F''(x_i))^3}{8(F'(x_i))^7}.$$

As the aim of this study is to develop effective multi-step iterative methods, we rely on Algorithm 1 and the following iterative method proposed by [17]:

$$y_i = x_i - \frac{F(x_i)}{\lambda_i F(x_i) + F'(x_i)},$$

where $\lambda_i \in \mathbb{R}, 0 < |\lambda_i| < +\infty$, and λ_i is selected such that the corresponding function value $\lambda_i F(x_i)$ and $F'(x_i)$ have the same signs.

In addition, to improve the efficiency of the new iterative methods, the first and second derivatives $F'(y_i)$ and $F''(y_i)$ are approximated, respectively, by using a Bernoulli interpolating polynomial of degree 2 and as follows:

$$F'(y_i) = 2F[x_i, y_i] - F'(x_i) := K_1(x_i, y_i), \tag{14}$$

$$F''(y_i) = \frac{2}{x_i - y_i} (F[x_i, y_i] - K_1(x_i, y_i)) := K_2(x_i, y_i). \tag{15}$$

Furthermore, using the Bernoulli interpolating polynomial of degree 3, the derivative $F'(z_i)$ is approximated:

$$F'(z_i) = F[y_i, z_i] + \frac{y_i - z_i}{y_i - x_i} (F[y_i, x_i] - F'(x_i)) + 2(F[x_i, z_i] - F[y_i, x_i]) := K_3(x_i, y_i, z_i). \tag{16}$$

Therefore, the new two and three step iterative methods, which will be referred to as (HM2) and (HM3), respectively, are given by the following Algorithms 2 and 3:

Algorithm 2: A suggested two-step iterative method (HM2).

For a suitable initial approximation x_0 , calculate the next solution x_{i+1} using the following iterative method:

$$y_i = x_i - \frac{F(x_i)}{\lambda_i F(x_i) + F'(x_i)}, \quad i = 0, 1, 2, \dots,$$

$$x_{i+1} = y_i - \frac{F(y_i)}{K_1(x_i, y_i)} - \frac{F^2(y_i)K_2(x_i, y_i)}{2(K_1(x_i, y_i))^3} - \frac{F^3(y_i)(K_2(x_i, y_i))^2}{2(K_1(x_i, y_i))^5} - \frac{5F^4(y_i)(K_2(x_i, y_i))^3}{8(K_1(x_i, y_i))^7}.$$

Algorithm 3: A suggested three-step iterative method (HM3).

For a suitable initial approximation x_0 , calculate the next solution x_{i+1} using the following iterative method:

$$y_i = x_i - \frac{F(x_i)}{\lambda_i F(x_i) + F'(x_i)}, \quad i = 0, 1, 2, \dots,$$

$$z_i = y_i - \frac{F(y_i)}{K_1(x_i, y_i)} - \frac{F^2(y_i)K_2(x_i, y_i)}{2(K_1(x_i, y_i))^3} - \frac{F^3(y_i)(K_2(x_i, y_i))^2}{2(K_1(x_i, y_i))^5} - \frac{5F^4(y_i)(K_2(x_i, y_i))^3}{8(K_1(x_i, y_i))^7},$$

$$x_{i+1} = z_i - \frac{F(z_i)}{\lambda_i F(z_i) + K_3(x_i, y_i, z_i)}.$$

3. Convergence Investigation

In this section, a convergence analysis of Algorithms 2 and 3 is provided and their efficiency indices are stated. Specifically, it is meant to show that both algorithms converge to the exact solution of the nonlinear equation under certain conditions on the initial guess and the nonlinear function. Moreover, the convergence rate of the proposed iterative methods is analyzed by computing their efficiency indices, which reflect the number of iterations required to achieve a given level of accuracy.

Theorem 1. *Suppose that $F : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function with a simple root $\alpha \in D$. If x_0 is sufficiently near to α , then Algorithm 2 exhibits at least fourth-order convergence in its iteration towards the exact root.*

Proof. Let α be a simple root of $F(x) = 0$. (Note that, $F(\alpha) = 0$ and $F'(\alpha) \neq 0$). By applying the Taylor series expansion around α , one can expand $F(x_i)$ and $F'(x_i)$ to obtain the following:

$$F(x_i) = F'(\alpha) \left[e_i + c_2 e_i^2 + c_3 e_i^3 + c_4 e_i^4 + O(e_i^5) \right], \tag{17}$$

where $c_n = \frac{F^{(n)}(\alpha)}{n!F'(\alpha)}$, $n = 2, 3, 4, \dots$ and $e_i = x_i - \alpha$. From Equation (17), it can be shown that

$$F'(x_i) = F'(\alpha) \left[1 + 2c_2e_i + 3c_3e_i^2 + 4c_4e_i^3 + 5c_4e_i^4 + O(e_i^5) \right], \tag{18}$$

and hence,

$$\frac{F(x_i)}{\lambda_i F(x_i) + F'(x_i)} = e_i + (-c_2 - \lambda_i)e_i^2 + (\lambda_i^2 + 3\lambda_i c_2 + 2c_2^2 - \lambda_i c_2 - 2c_3)e_i^3 + O(e_i^4). \tag{19}$$

Moreover,

$$y_i = \alpha + (c_2 + \lambda_i)e_i^2 + (-\lambda_i^2 - 3\lambda_i c_2 - 2c_2^2 + \lambda_i c_2 + 2c_3)e_i^3 + O(e_i^4), \tag{20}$$

$$K_1(x_i, y_i) = F'(\alpha) \left[1 + (2\lambda_i c_2 + 2c_2^2 - c_3)e_i^2 + (2c_2(\lambda_i c_2) - 4c_3^3 - 6\lambda_i c_2^2 + (-2\lambda_i^2 + 6c_3)c_2 + 2\lambda_i c_3 - 2c_4)e_i^3 + O(e_i^4) \right], \tag{21}$$

$$K_2(x_i, y_i) = F'(\alpha) \left[2c_2 + 4c_3e_i + ((2c_2 + 2\lambda_i)c_3 + 6c_4)e_i^2 + (2(\lambda_i c_2)c_3 + 4c_3^2 - 2(\lambda_i + 2c_2)(c_2 + \lambda_i)c_3 + 4\lambda_i c_4 + 4c_2 c_4 + 8c_5)e_i^3 + O(e_i^4) \right]. \tag{22}$$

From Equation (20), it can be seen that

$$F(y_i) = F'(\alpha) \left[(c_2 + \lambda_i)e_i^2 + (-\lambda_i^2 - 3\lambda_i c_2 - 2c_2^2 + \lambda_i c_2 + 2c_3)e_i^3 + O(e_i^4) \right]. \tag{23}$$

In addition,

$$\frac{F(y_i)}{K_1(x_i, y_i)} = (c_2 + \lambda_i)e_i^2 + (-\lambda_i^2 - 3\lambda_i c_2 - 2c_2^2 + \lambda_i c_2 + 2c_3)e_i^3 + O(e_i^4), \tag{24}$$

and

$$\frac{F^2(y_i)K_2(x_i, y_i)}{2(K_1(x_i, y_i))^3} = c_2(c_2 + \lambda_i)^2 e_i^4 - 2(c_2 + \lambda_i) \left(-c_2(\lambda_i c_2) + 2c_2^3 + 3\lambda_i c_2^2 + (\lambda_i^2 - 3c_3)c_2 - \lambda_i c_3 \right) e_i^5 + O(e_i^6), \tag{25}$$

$$\frac{F^3(y_i)(K_2(x_i, y_i))^2}{2(K_1(x_i, y_i))^5} = 2(c_2 + \lambda_i)^3 c_2^2 e_i^6 - 6(c_2 + \lambda_i)^2 \left(-c_2(\lambda_i c_2) + 2c_2^3 + 3\lambda_i c_2^2 + (\lambda_i^2 - \frac{10}{3}c_3)c_2 - \frac{4}{3}\lambda_i c_3 \right) c_2 e_i^7 + O(e_i^8), \tag{26}$$

$$\frac{5F^4(y_i)(K_2(x_i, y_i))^3}{8(K_1(x_i, y_i))^7} = 5(c_2 + \lambda_i)^4 c_2^3 e_i^8 + \left(-20\lambda_i^2 c_2 - 60\lambda_i c_2^2 - 40c_2^3 + 20c_2(\lambda_i c_2) + 30\lambda_i c_3 + 70c_2 c_3 \right) (c_2 + \lambda_i)^3 c_2^2 e_i^9 + O(e_i^{10}). \tag{27}$$

Substituting Equations (24)–(27) in Algorithm 2 obtains

$$x_{i+1} = \alpha - c_3(c_2 + \lambda_i)e_i^4 + O(e_i^5), \tag{28}$$

implying that

$$e_{i+1} = -c_3(c_2 + \lambda_i)e_i^4 + O(e_i^5). \tag{29}$$

As a result, it can be seen that Algorithm 2 has fourth-order convergence. \square

Theorem 2. Suppose that $F : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function with a simple root $\alpha \in D$. If x_0 is sufficiently near to α , then Algorithm 3 exhibits at least eighth-order convergence in its iteration towards the exact root.

Proof. By applying a similar argument as in Theorem 1, one could have the following:

$$z_i = \alpha - c_3(c_2 + \lambda_i)e_i^4 + \left(-\lambda_i c_2 c_3 - 2c_3^2 + (\lambda_i + 2c_2)(c_2 + \lambda_i)c_3 - 2c_4(c_2 + \lambda_i)\right)e_i^5 + O(e_i^6). \tag{30}$$

Expanding $F(z_i)$ about α and using Equation (30) to have

$$F(z_i) = F'(\alpha) \left[-c_3(c_2 + \lambda_i)e_i^4 + \left(-\lambda_i c_2 c_3 - 2c_3^2 + (\lambda_i + 2c_2)(c_2 + \lambda_i)c_3 - 2c_4(c_2 + \lambda_i)\right)e_i^5 + O(e_i^6) \right] \tag{31}$$

and

$$K_3(x_i, y_i, z_i) = F'(\alpha) \left[1 + \left(-2c_2^2 c_3 + (c_4 - 2\lambda_i c_3)c_2 + \lambda_i c_4\right)e_i^4 + O(e_i^5) \right]. \tag{32}$$

From Equations (31) and (32), it can be seen that

$$\frac{F(z_i)}{\lambda_i F(z_i) + K_3(x_i, y_i, z_i)} = -c_3(c_2 + \lambda_i)e_i^4 + \left(-\lambda_i c_2 c_3 - 2c_3^2 + (\lambda_i + 2c_2)(c_2 + \lambda_i)c_3 - 2c_4(c_2 + \lambda_i)\right)e_i^5 + O(e_i^6). \tag{33}$$

Then

$$x_{i+1} = \alpha + c_3 \left(2\lambda_i c_2^2 c_3 - c_2^2 c_4 + \lambda_i^2 c_2 c_3 - \lambda_i^2 c_4 - 2\lambda_i c_2 c_4 + \lambda_i((c_2 + \lambda_i)c_3)c_2 + \lambda_i^2(c_2 + \lambda_i)c_3 + c_2^3 c_3 \right) e_i^8 + O(e_i^9). \tag{34}$$

From Equation (34), one could obtain

$$e_{i+1} = \alpha + c_3 \left(2\lambda_i c_2^2 c_3 - c_2^2 c_4 + \lambda_i^2 c_2 c_3 - \lambda_i^2 c_4 - 2\lambda_i c_2 c_4 + \lambda_i((c_2 + \lambda_i)c_3)c_2 + \lambda_i^2(c_2 + \lambda_i)c_3 + c_2^3 c_3 \right) e_i^8 + O(e_i^9). \tag{35}$$

As a result, it can be seen that Algorithm 3 has eighth-order convergence. \square

In fact, it is essential in numerical analysis to know the behavior of an approximate method. Therefore, the order of convergence of the new iterative methods can be explained by the following definition and remark (more details can be found in [11,21,22]).

Definition 1. Let $F : D \subset \mathbb{R} \rightarrow \mathbb{R}$ is a scalar function on an open interval D with a simple root α of the nonlinear equation. An iterative method is said to have an integer order of convergence s if it produces the sequence $\{x_i\}$ of real numbers such that

$$\lim_{x \rightarrow \infty} \frac{x_{i+1} - \alpha}{(x_i - \alpha)^s} = A \neq 0,$$

for some $A \neq 0$ and $s \geq 1$, then s is said to be the order of convergence of the sequence, and A is known as the asymptotic error constant, or equivalently

$$x_{i+1} - \alpha = A(x_i - \alpha)^s + O\left((x_i - \alpha)^{s+1}\right).$$

Remark 1. Let $e_i = x_i - \alpha$ be the error in the i^{th} iteration. The equation

$$e_{i+1} = ce_i^s + O\left(e_i^{s+1}\right)$$

is called the error equation for the method, s being the order of convergence.

Remark 2. The definition of efficiency index in [16] is analyzed, which is given as $EI = \sqrt[s]{s}$, where s is the order of the iterative method and m is the number of function evaluations in each iteration. Using this definition, the efficiency indices of our suggested Algorithms 2 and 3 are 1.587 and 1.681, respectively.

4. Numerical Applications

The purpose of this section is to compare the performance of the proposed Algorithms 2 and 3 with the Noor method (NR2) [2], the Chun method (CM) [23], and the Sehati et al. method (AL2) [11]. The comparison is aimed at demonstrating the applicability and efficacy of the newly proposed methods.

To evaluate the methods, the Maple16 program is utilized and floating-point arithmetic with 1000 digits (Digits := 1000) is employed. The computations were performed on a desktop machine that runs on Windows 10 Pro (64-bit) operating system. The computer is equipped with an Intel(R) Core(TM) i3-4170 CPU that runs at a speed of 3.70 GHz, and it has a total installed memory of 10.00 GB RAM. The system configuration provided ample resources to ensure a smooth and efficient execution of the Maple16 algorithms used in this study. The stopping criteria $|x_{i+1} - x_i| < \epsilon$ as well as $|F(x_{i+1})| < \epsilon$ are used, where $\epsilon = 10^{-15}$, and (HM2) and (HM3) have been applied for the value of $|\lambda_i| = 0.5$. Table 1 presents various test functions $F_i(x), i = 1, \dots, 5$, and the approximated root α of each equation $F_i(x) = 0$. The table also provides information about the initial guess x_0 , the number of iterations (NI), the value of $|x_{i+1} - x_i|$, the absolute value of the function $|F(x_{i+1})|$, and the computational order of convergence (COC), which is described in [16] and given as the following:

$$\rho = \frac{\log|(x_{i+1} - \alpha)/(x_i - \alpha)|}{\log|(x_i - \alpha)/(x_{i-1} - \alpha)|}.$$

All the results and the performance comparison are included in Table 2.

Table 1. Nonlinear equations and their simple root α .

$F(x)$	α
$F_1(x) = x - e^{-x} - 2 = 0$	2.12002823898764122948468710
$F_2(x) = xe^{x^2} - (\sin x)^2 + 3 \cos x + 5 = 0$	-1.2076478271309189270094168
$F_3(x) = x^2 - (1 - x)^5 = 0$	0.3459548158482420179582044
$F_4(x) = \ln(x^2 + x + 2) - x + 11 = 0$	16.6951567675073761976927688
$F_5(x) = (1 + \cos x)(e^x - 2) = 0$	0.6931471805599453094172322

Table 2. Comparison between the proposed methods and relevant methods.

$F(x)$	x_0	Methods	NI	$ x_{i+1} - x_i $	$ F(x_{i+1}) $	COC
F_1	2.0	NR2	3	$6.8094693 \times 10^{-16}$	$6.3164342 \times 10^{-48}$	3.0104059
		CM	3	$8.93125110 \times 10^{-31}$	$5.4817818 \times 10^{-124}$	4.01084410
		AL2	3	$2.2199937 \times 10^{-78}$	$2.0350825 \times 10^{-548}$	7.0116691
		HM2	3	$9.0762879 \times 10^{-26}$	$6.0604694 \times 10^{-103}$	4.0094105
		HM3	3	$3.9771853 \times 10^{-98}$	$3.1045139 \times 10^{-784}$	8.0082867
F_2	−1.0	NR2	5	$1.8095423 \times 10^{-39}$	$2.4092523 \times 10^{-115}$	3.0000063
		CM	5	$1.5057542 \times 10^{-44}$	$1.7691495 \times 10^{-173}$	3.9998224
		AL2	3	$1.65614610 \times 10^{-22}$	$6.6848624 \times 10^{-150}$	6.7759406
		HM2	4	$4.0189653 \times 10^{-42}$	$1.0630597 \times 10^{-164}$	3.9996505
		HM3	3	$1.9180050 \times 10^{-49}$	$1.8554791 \times 10^{-389}$	7.9462676
F_3	0.2	NR2	4	$8.2499542 \times 10^{-20}$	$2.4019841 \times 10^{-57}$	2.9998339
		CM	4	$5.4788184 \times 10^{-36}$	$1.0139405 \times 10^{-140}$	4.0007370
		AL2	4	$2.6904210 \times 10^{-100}$	$3.71201210 \times 10^{-695}$	6.9990345
		HM2	4	$3.7572186 \times 10^{-44}$	$5.2755324 \times 10^{-174}$	3.9993329
		HM3	3	$1.1771498 \times 10^{-55}$	$2.3415694 \times 10^{-440}$	8.0457563
F_4	16.0	NR2	3	$2.0752478 \times 10^{-17}$	$1.1309175 \times 10^{-54}$	3.0094031
		CM	3	$7.0787489 \times 10^{-36}$	$5.8881265 \times 10^{-148}$	4.0092711
		AL2	diverge	diverge	diverge	diverge
		HM2	3	$3.8856993 \times 10^{-23}$	$1.4531784 \times 10^{-94}$	4.0468837
		HM3	3	$1.9058709 \times 10^{-84}$	$4.3697445 \times 10^{-679}$	8.0447728
F_5	0.5	NR2	4	$1.8521494 \times 10^{-29}$	$5.2002262 \times 10^{-87}$	2.9999240
		CM	3	$4.8721383 \times 10^{-19}$	$2.6686961 \times 10^{-75}$	4.1825251
		AL2	3	$2.9291152 \times 10^{-44}$	$9.7204429 \times 10^{-306}$	6.9492016
		HM2	3	$5.1650339 \times 10^{-16}$	$3.7212218 \times 10^{-62}$	4.0243631
		HM3	3	$9.3624134 \times 10^{-65}$	$1.5920864 \times 10^{-514}$	8.0130984

Moreover, Figures 1–5 are presented to provide a more comprehensive description of the convergence, accuracy, and reliability of the proposed Algorithms 2 and 3. These graphs display the logarithm of residuals at each iteration for the nonlinear equations listed in Table 1. Additionally, Figure 6 is included, which compares the efficiency of the suggested two- and three-step methods with the alternative methods that have been considered.

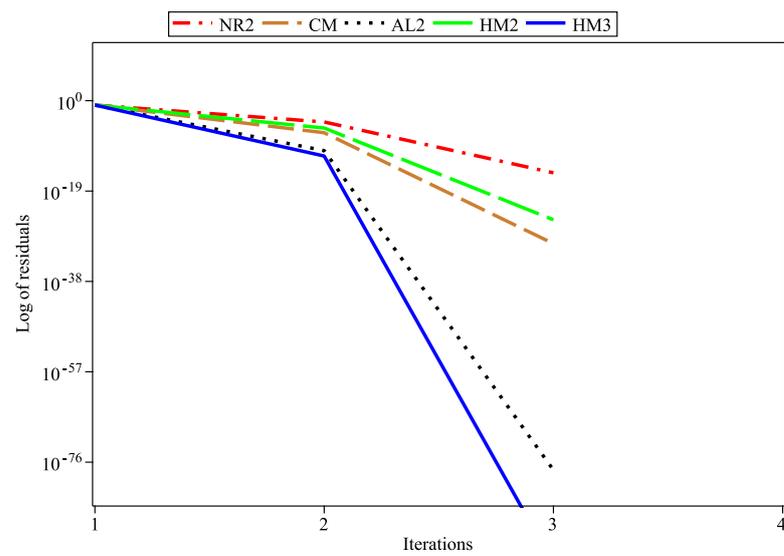


Figure 1. Log of residuals for $F_1(x) = 0$.

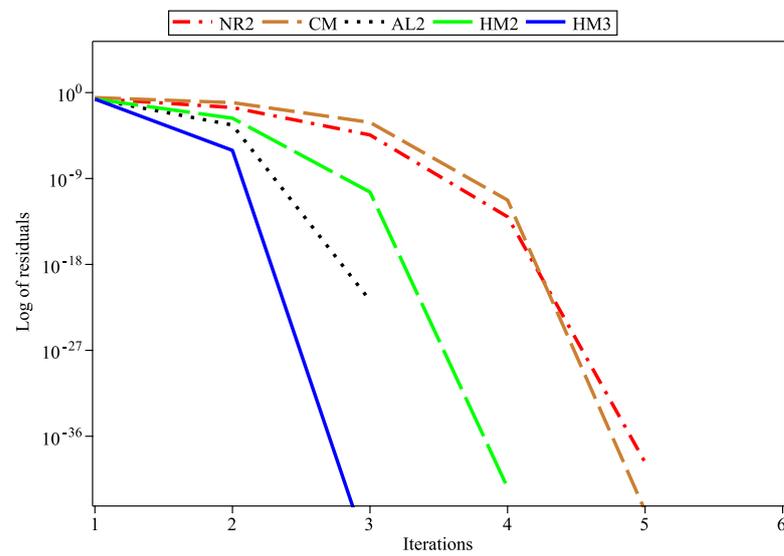


Figure 2. Log of residuals for $F_2(x) = 0$.

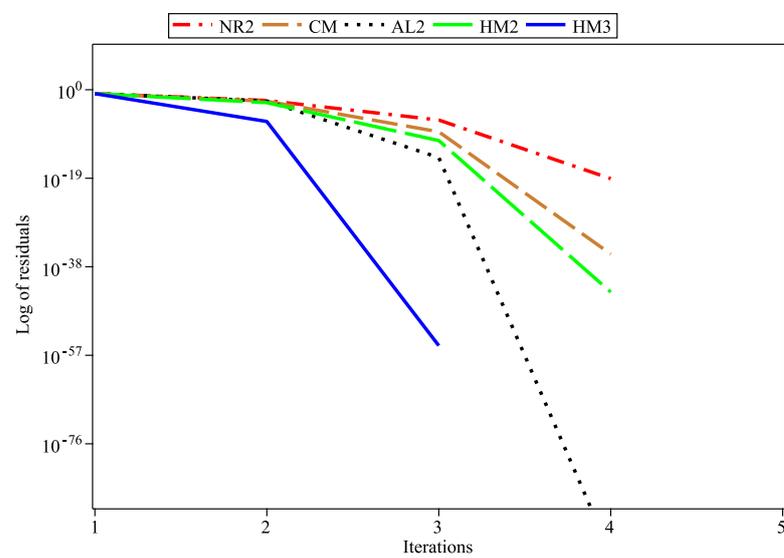


Figure 3. Log of residuals for $F_3(x) = 0$.

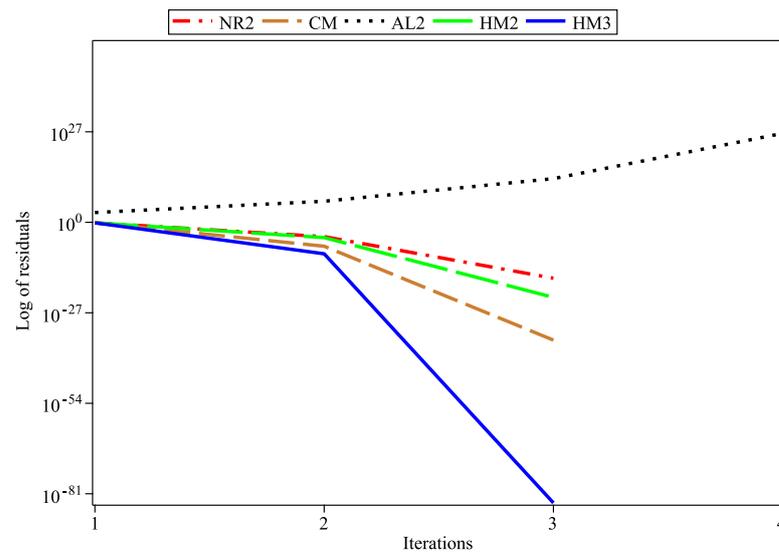


Figure 4. Log of residuals for $F_4(x) = 0$.

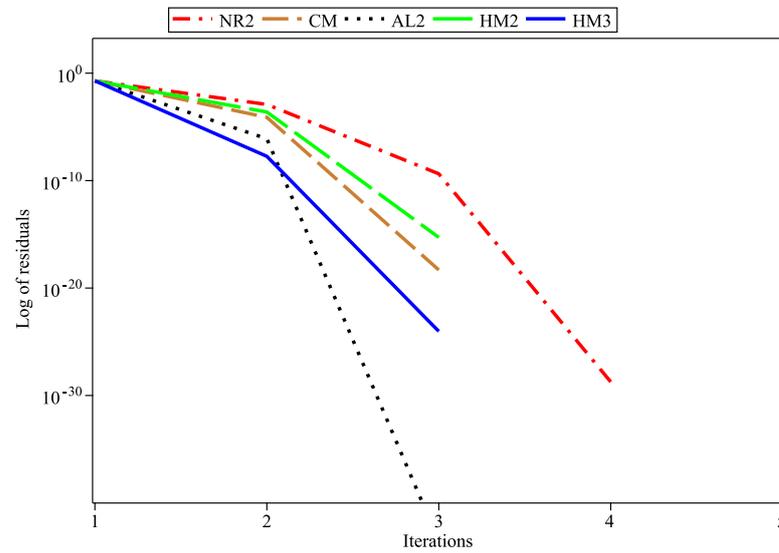


Figure 5. Log of residuals for $F_5(x) = 0$.

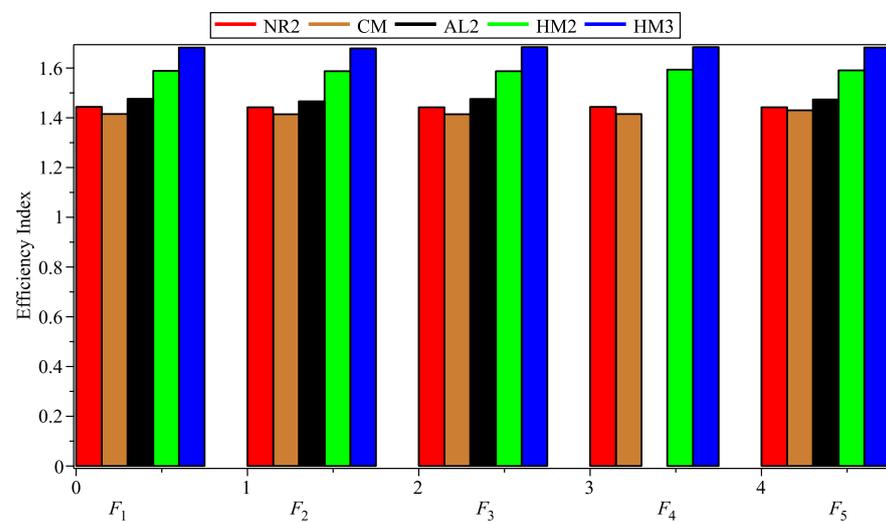


Figure 6. Comparison of efficiency indices among different methods.

5. Discussion and Conclusions

The numerical findings in Table 2 and Figures 1–5 demonstrate that none of the five nonlinear equations above exhibits the failure or divergence of our multi-step iterative methods. The new three-step iterative method proves to be more effective and accurate, and provides solutions faster than the other methods that it is compared against, such as Noor's method, Chun's method, and the Sehati et al. method. Moreover, the suggested iterative methods have a higher computational order of convergence (COC) than any other methods, making them more efficient. Our findings can be viewed as an alternative to comparable methods for solving nonlinear equations as they approximate higher derivatives, enabling us to obtain solutions with fewer computational steps and a higher order of convergence.

This paper introduces a new class of iterative methods free from second derivatives to solve nonlinear equations based on the homotopy perturbation method, Haar wavelets quadrature rule, and some numerical techniques. The suggested iterative methods have a fourth-order convergence for the two-step type with two function evaluations and one first derivative evaluation, while for the three-step type, it has an eighth-order convergence with three function evaluations and one first derivative evaluation. The convergence order using Theorems 1 and 2, efficiency index, computational order of convergence, and convergence criterion of the new multi-step iterative methods are evaluated. The proposed Algorithms 2 and 3 outperform the previously mentioned methods in terms of efficiency index. The eighth-order iterative method (HM3) is the key finding of this study since it is both fast and efficient, as evident through both theoretical evaluation and numerical testing.

The suggested methods are numerically and graphically compared to several currently used methods, and the proposed methods outperform several other previously known methods in terms of convergence rate and computational efficiency. The obtained results show that the proposed methods are promising tools for solving nonlinear equations, as they outperform existing methods. Overall, our work provides a novel approach to solving nonlinear equations that can significantly improve the accuracy and efficiency of numerical methods, indicating its potential for use in a range of scientific and engineering applications.

There are several potential directions for future work. For instance, hybrid algorithms that combine the proposed methods with other existing techniques could be developed to further improve their performance. Additionally, more efficient and accurate versions of the methods could be created to enhance their effectiveness. Furthermore, the methods can be adapted or modified to tackle more complex and diverse problems, including systems of nonlinear equations.

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References

1. Noor, K.I.; Noor, M.A. Predictor-corrector Hally method for nonlinear equations. *Appl. Math. Comput.* **2007**, *188*, 1587–1591.
2. Noor, M.A.; Noor, K.I.; Mohyud-Din, S.T.; Shabbir, A. An iterative method with cubic convergence for nonlinear equations. *Appl. Math. Comput.* **2006**, *183*, 1249–1255 [[CrossRef](#)]

3. Noor, M.A. Some iterative methods for solving nonlinear equations using homotopy perturbation method. *Int. J. Comput. Math.* **2010**, *1*, 141149. [[CrossRef](#)]
4. Noor, M.A.; Khan, W.A. New iterative methods for solving nonlinear equation by using homotopy perturbation method. *Appl. Math. Comput.* **2012**, *219*, 3565–3574. [[CrossRef](#)]
5. Abdul-Hassan, N.Y.; Ali, A.H.; Park, C. A new fifth-order iterative method free from second derivative for solving nonlinear equations. *J. Appl. Math. Comput.* **2022**, *68*, 2877–2886. [[CrossRef](#)]
6. Liao, S.J. The Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problems. Ph.D. Thesis, Shanghai Jiao Tong University, Shanghai, China, 1992.
7. He, J.H. Homotopy perturbation technique. *Comput. Methods Appl. Mech. Eng.* **1999**, *178*, 257–262. [[CrossRef](#)]
8. Liao, S.J. Comparison between the homotopy analysis method and homotopy perturbation method. *Appl. Math. Comput.* **2005**, *169*, 1186–1194. [[CrossRef](#)]
9. Az-Zo'bi, E.A.; Al-Khaled, K.; Darweesh, A. Numeric-analytic solutions for nonlinear oscillators via the modified multi-stage decomposition method. *Mathematics* **2019**, *7*, 550. [[CrossRef](#)]
10. Alim, M.A.; Kawser, M.A. Illustration of the homotopy perturbation method to the modified nonlinear single degree of freedom system. *Chaos Solitons Fractals* **2023**, *171*, 113481. [[CrossRef](#)]
11. Sehati, M.M.; Karbassi, S.M.; Heydari, M.; Loghmani, G.B. several new iterative methods for solving nonlinear algebraic equations incorporating homotopy perturbation method (HPM). *Int. J. Phys. Sci.* **2012**, *12*, 5017–5025. [[CrossRef](#)]
12. Waheed, A.; Din, S.T.M.; Zeb, M.; Usman, M. Some Higher Order Algorithms for Solving Fixed Point Problems. *Commun. Math. Appl.* **2018**, *1*, 41–52.
13. Freno, B.A.; Carlberg, T.K. Machine-learning error models for approximate solutions to parameterized systems of nonlinear equations. *Comput. Methods Appl. Mech. Eng.* **2019**, *348*, 250–296. [[CrossRef](#)]
14. Gong, W.; Liao, Z.; Mi, X.; Wang, L.; Guo, Y. Nonlinear equations solving with intelligent optimization algorithms: A survey. *Complex Syst. Model. Simul.* **2021**, *1*, 15–32. [[CrossRef](#)]
15. Abbasbandy, S. Improving Newton-Raphson method for nonlinear equations by modified Adomian decomposition method. *Appl. Math. Comput.* **2023**, *145*, 887–893. [[CrossRef](#)]
16. Saeed, H.J.; Abdul-Hassan, N.Y. An Efficient Three-Step Iterative Methods Based on Bernstein Quadrature Formula for Solving Nonlinear Equations. *Basrah J. Sci.* **2021**, *3*, 355–383. [[CrossRef](#)]
17. Wu, X.Y. A New Continuation Newton-Like Method and its Deformation. *Appl. Math. Comput.* **2000**, *112*, 75–78. [[CrossRef](#)]
18. Aziz, I.; Siraj-ul-Islam; Khan, W. Quadrature Rules for Numerical Integration Based on Haar wavelets and Hybrid Functions. *Comput. Math. Appl.* **2011**, *61*, 2770–2781. [[CrossRef](#)]
19. Ali, A.H.; Páles, Z. Taylor-type Expansions in Terms of Exponential Polynomials. *Math. Inequalities Appl.* **2022**, *25*, 1123–1141. [[CrossRef](#)]
20. Abbasbandy, S. Modified homotopy perturbation method for nonlinear equations and comparison with Adomian decomposition method. *Appl. Math. Comput.* **2006**, *172*, 431–438. [[CrossRef](#)]
21. Ljajko, E.; Tosic, M.; Kevkic, T.; Stojanovic, V. Application of the Homotopy Perturbations Method in Approximation Probability Distributions of Non-linear Time Series. *Univ. Politeh. Buchar. Sci.-Bull.-Ser.-Appl. Math. Phys.* **2021**, *83*, 177–186.
22. Khan, K.; Syed, H.Z. Semi Analytic Solution of Hodgkin-Huxley Model by Homotopy Perturbation Method. *Punjab Univ. J. Math.* **2021**, *53*, 825–842. [[CrossRef](#)]
23. Chun, C. A new iterative method for solving nonlinear equations. *Appl. Math. Comput.* **2006**, *178*, 415–422. [[CrossRef](#)]

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