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Linear Algebraic Relations among Cardinalities of Sets of Matroid Functions

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Abstract: We introduce a unifying approach for invariants of finite matroids that count mappings to a finite set. The aim of this paper is to show that if the cardinalities of mappings with fixed values on a restricted set satisfy contraction–deletion rules, then there is a relation among them that can be expressed in terms of linear algebra. In this way, we study regular chain groups, nowhere-zero flows and tensions on graphs, and acyclic and totally cyclic orientations of oriented matroids and graphs.

Keywords: matroid; graph; nowhere-zero flow; tension; orientation

MSC: 05C50; 05B35; 05C21; 05C20; 05C30

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1. Introduction

A Tutte–Grothendieck invariant Φ is a mapping from a class $\mathcal{M}$ of matroids to a commutative ring with the property that there are elements a_1, b_1, a_2, b_2 from the ring such that for a matroid $M \in \mathcal{M}$ on the ground set E, we have

$$\begin{array}{ll} \Phi(M) = 1 & \text{if } E = \emptyset, \\ \Phi(M) = a_1\Phi(M-e) & \text{if } e \text{ is an isthmus of } M, \\ \Phi(M) = b_1\Phi(M-e) & \text{if } e \text{ is a loop of } M, \\ \Phi(M) = a_2\Phi(M/e) + b_2\Phi(M-e) & \text{otherwise.} \end{array}$$

(Notice that $e \in E$ is an isthmus of M if it is contained in each base of M, and $e \in E$ is a loop of M if it belongs to no bases of M.) The best known Tutte–Grothendieck invariant is the Tutte polynomial:

$$t(M; x, y) = \sum_{X \subseteq E} (x-1)^{r(M)-r_M(X)}(y-1)^{|X|-r_M(X)},$$

which maps matroids to the ring of integral polynomials with two variables, x and y. This invariant was introduced by Tutte in [1] for graphs and encodes many properties of graphs and matroids. Applications of the Tutte polynomial in combinatorics, knot theory, statistical physics, and coding theory are surveyed in [2–5].

We consider classes of matroids whose ground sets contain a fixed subset B and study functions from the matroids to finite sets. For each of the matroids, consider the cardinality of a set of functions with fixed values on B. We show that if the cardinalities satisfy contraction–deletion rules, then there exist relations among these numbers that can be expressed in terms of linear algebra. In this way, we study numbers of regular chain groups, nowhere-zero flows and tensions on graphs, and acyclic and totally cyclic orientations of oriented matroids and graphs. These results generalize the approach that we introduced in [6,7].

2. Matroids and B-Classes

Throughout this paper, we consider finite matroids on finite sets. The ground set of a matroid M we denote by $E(M)$.

Let B be a finite set. A class of matroids $\mathcal{M}$ is called a *B-class* if

$$\begin{array}{l} B \subseteq E(M) \text{ for each } M \in \mathcal{M}, \\ M - e, M/e \in \mathcal{M} \text{ for each } M \in \mathcal{M} \text{ and } e \in E(M) \setminus B. \end{array}$$

There exist only finitely many matroids on B. Thus, there exists a finite set $\mathcal{M}_B$ consisting of pairwise nonisomorphic matroids on B belonging to $\mathcal{M}$. (For example, if $\mathcal{M}$ is a class of matroids closed under contraction and deletion, then it is an $\emptyset$-class and $\mathcal{M}_\emptyset = \{\emptyset\}$.)

The collection of mappings from E to a finite set S is denoted by S^E. Assume that M is a matroid from $\mathcal{M}$ with the ground set $E = E(M)$. By an *S-function* on M, we mean any function $f \in S^E$. Then, $f|B$ denotes the restriction of f to B (i.e., $f|B \in S^B$ so that $[f|B](x) = f(x)$ for each $x \in B$). Let $\mathcal{S}$ be a class of S-functions on matroids from $\mathcal{M}$. If $M \in \mathcal{M}$, then $\mathcal{S}_M$ denotes the set of S-functions on M belonging to $\mathcal{S}$. For every $g \in S^B$ and $M \in \mathcal{M}$, let

$$\mathcal{S}_{M,g} = \{f \in \mathcal{S}_M : f|B = g\}.$$

If $M \in \mathcal{M}_B$ and $\mathcal{S}_M = \emptyset$, then M is called $\mathcal{S}$-*trivial*; otherwise, it is called $\mathcal{S}$-*nontrivial*. In this paper, we denote by $\mathcal{M}_{B,\mathcal{S}}$ an ordered n-tuple $(M_1, \dots, M_n)$ of all $\mathcal{S}$-nontrivial elements of $\mathcal{M}_B$. For each $g \in S^B$ and $i = 1, \dots, n$, let $\chi_{i,g} = 1$ ($\chi_{i,g} = 0$) if $g \in \mathcal{S}_{B_i}$, ($g \notin \mathcal{S}_{B_i}$). Let $\boldsymbol{\chi}_{n,g} = (\chi_{1,g}, \dots, \chi_{n,g})$.

Let $\mathbf{e}_{i,n}$ denote the standard basis vector of $\mathbb{R}^n$ and let $\mathbf{0}_n$ denote the zero vector of $\mathbb{R}^n$. Vectors from $\mathbb{R}^n$ are considered row vectors. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then the dot product $\mathbf{x} \cdot \mathbf{y}$ can be expressed as a matrix multiplication $\mathbf{x}\mathbf{y}^T$.

Assume that there exist rational numbers a_1, b_1, a_2, b_2 such that for each $M \in \mathcal{M}$, each $e \in E(M) \setminus B$, and each $g \in S^B$, we have

$$\begin{array}{ll} |\mathcal{S}_{M,g}| = a_1|\mathcal{S}_{M-e,g}| & \text{if } e \text{ is an isthmus of } M, \\ |\mathcal{S}_{M,g}| = b_1|\mathcal{S}_{M-e,g}| & \text{if } e \text{ is a loop of } M, \\ |\mathcal{S}_{M,g}| = a_2|\mathcal{S}_{M/e,g}| + b_2|\mathcal{S}_{M-e,g}| & \text{otherwise.} \end{array} \tag{1}$$

In this case, we say that $\mathcal{S}$ is (a_1, b_1, a_2, b_2)-*regular*. Since the cardinalities of sets are nonnegative integers, a_1, b_1 must be nonnegative, but only one of a_2, b_2 can be negative (but not both).

Theorem 1. *Let $\mathcal{M}$ be a B-class of matroids, with B finite, $\mathcal{S}$ be an (a_1, b_1, a_2, b_2)-regular class of S-functions on matroids from $\mathcal{M}$, with S finite, and $\mathcal{M}_{B,\mathcal{S}} = (M_1, \dots, M_n)$. Then, for each $M \in \mathcal{M}$, there exists a vector $\mathbf{x}_M = (x_1, \dots, x_n)$ such that for every $g \in S^B$, $|\mathcal{S}_{M,g}| = \boldsymbol{\chi}_{n,g} \cdot \mathbf{x}_M$, i.e., $|\mathcal{S}_{M,g}| = \sum_{i=1}^n \chi_{i,g} x_i$. Furthermore, if a_1, b_1, a_2, b_2 are integers, then $\mathbf{x}_M$ can be chosen to be an integral vector.*

Proof. We apply induction on $|E(M)| \geq |B|$. Let $|E(M)| = |B|$ and $g \in S^B$. If M is $\mathcal{S}$-trivial, then $|\mathcal{S}_{M,g}| = 0$, and we can set $\mathbf{x}_M = \mathbf{0}_n$. If M is $\mathcal{S}$-nontrivial, then $M = M_i$, where $i \in \{1, \dots, n\}$, and $|\mathcal{S}_{M,g}| = \chi_{i,g}$. Thus, $\mathbf{x}_M = \mathbf{e}_{i,n}$ satisfies the assumptions.

If $|E(G)| > |B|$, then there exists $e \in E(M) \setminus B$. By the induction hypothesis, there are vectors $\mathbf{x}_{M-e}$ and $\mathbf{x}_{M/e}$ such that for every $g \in S^B$, $|\mathcal{S}_{M-e,g}| = \boldsymbol{\chi}_{n,g} \cdot \mathbf{x}_{M-e}$ and $|\mathcal{S}_{M/e,g}| = \boldsymbol{\chi}_{n,g} \cdot \mathbf{x}_{M/e}$.

If e is an isthmus, then from the first row of (1), $|\mathcal{S}_{M,g}| = a_1|\mathcal{S}_{M-e,g}| = a_1\boldsymbol{\chi}_{n,g} \cdot \mathbf{x}_{M-e}$, where the vector $\mathbf{x}_M = a_1\mathbf{x}_{M-e}$ satisfies the assumptions.

If e is a loop, then from the second row of (1), $|\mathcal{S}_{M,g}| = b_1|\mathcal{S}_{M-e,g}| = b_1\boldsymbol{\chi}_{n,g} \cdot \mathbf{x}_{M-e}$, where the vector $\mathbf{x}_M = b_1\mathbf{x}_{M-e}$ satisfies the assumptions.

If e is neither an isthmus nor a loop, then from (1), $|\mathcal{S}_{M,g}| = a_2|\mathcal{S}_{M/e,g}| + b_2|\mathcal{S}_{M-e,g}| = a_2\boldsymbol{\chi}_{n,g} \cdot \mathbf{x}_{M/e} + b_2\boldsymbol{\chi}_{n,g} \cdot \mathbf{x}_{M-e} = \boldsymbol{\chi}_{n,g} \cdot (a_2\mathbf{x}_{M/e} + b_2\mathbf{x}_{M-e})$, where the vector $\mathbf{x}_M = a_2\mathbf{x}_{M/e} + b_2\mathbf{x}_{M-e}$ satisfies the assumptions.

If a_1, b_1, a_2, b_2 are integers, then all vectors $\mathbf{x}_M$ considered in the proof are integral. This proves the statement. $\square$

Let $Z = (\mathbf{z}_1, \ldots, \mathbf{z}_m)$, $m \le n$, be an ordered basis of the linear hull of $\{\chi_{n,g}; g \in S^B\}$. Denote $\chi_{Z,g} = (t_1, \ldots t_m)$ such that $\chi_{n,g} = \sum_{i-1}^{m} t_i \mathbf{z}_i$. For example, if $n = m$, then we can choose $\mathbf{z}_i = \mathbf{e}_{i,n}$ and then $\chi_{Z,g} = \chi_{n,g}$.

Theorem 2. *Let $\mathcal{M}$ be a B-class of matroids, with B finite, $\mathcal{S}$ be an (a_1, b_1, a_2, b_2)-regular class of S-functions on matroids from $\mathcal{M}$, with S finite, $\mathcal{M}_{B,\mathcal{S}} = (M_1, \ldots, M_n)$, and $Z = (\mathbf{z}_1, \ldots, \mathbf{z}_m)$, $m \le n$, be an ordered basis of the linear hull of $\{\chi_{n,g}; g \in S^B\}$. Then, for each $M \in \mathcal{M}$, there exists a unique vector $\mathbf{y}_M = (y_1, \ldots, y_m)$ such that for every $g \in S^B$, $|\mathcal{S}_{M,g}| = \chi_{Z,g} \cdot \mathbf{y}_M$. Furthermore:*
If $E(M) = B$ and M is trivial, then $\mathbf{y}_M = \mathbf{0}_m$;
If $E(M) = B$ and $M = M_i$, $i \in \{1, \ldots, n\}$, then $\mathbf{y}_M = (y_1, \ldots, y_m)$ such that y_j is the i-th coordinate of $\mathbf{z}_j$, $j \in \{1, \ldots, m\}$;
If $E(M) \neq B$, then $\mathbf{y}_M$ satisfies the following recursive rules:

$$\begin{array}{ll} \mathbf{y}_M = a_1 \mathbf{y}_{M-e} & \text{if } e \text{ is an isthmus of } M, \\ \mathbf{y}_M = b_1 \mathbf{y}_{M-e} & \text{if } e \text{ is a loop of } M, \\ \mathbf{y}_M = a_2 \mathbf{y}_{M/e} + b_2 \mathbf{y}_{M-e} & \text{otherwise.} \end{array} \tag{2}$$

Finally, if a_1, b_1, a_2, b_2 are integers and $\mathbf{z}_1, \ldots, \mathbf{z}_m$ are integral vectors, then $\mathbf{y}_M$ is an integral vector for each M from $\mathcal{M}$.

Proof. We prove the existence of $\mathbf{y}_M$ by induction on $|E(M)| \ge |B|$. Let $|E(M)| = |B|$ and $g \in S^B$. If M is $\mathcal{S}$-trivial, then $|\mathcal{S}_{M,g}| = 0$, and we can set $\mathbf{y}_M = \mathbf{0}_m$. If M is $\mathcal{S}$-nontrivial and $M = M_i$, $i \in \{1, \ldots, n\}$, then from the proof of Theorem 1, $|\mathcal{S}_{M,g}| = \chi_{n,g} \cdot \mathbf{e}_{i,n}$. Let A be an $m \times n$-matrix with the j-th row equal to $\mathbf{z}_j$, $j \in \{1, \ldots, m\}$. Using matrix multiplication, we can express $\chi_{n,g} = \chi_{Z,g} A$ and $|\mathcal{S}_{M,g}| = \chi_{n,g} \mathbf{e}_{i,n}^T$, where $|\mathcal{S}_{M,g}| = (\chi_{Z,g} A)\mathbf{e}_{i,n}^T = \chi_{Z,g}(A\mathbf{e}_{i,n}^T)$. Thus, $A\mathbf{e}_{i,n}^T = \mathbf{y}_M^T$, where $\mathbf{y}_M = (y_1, \ldots, y_m)$ such that y_j is the i-th coordinate of $\mathbf{z}_j$, $j \in \{1, \ldots, m\}$.

If $|E(G)| > |B|$, then there exists $e \in E(M) \setminus B$. By the induction hypothesis, there are integral vectors $\mathbf{y}_{M-e}$ and $\mathbf{y}_{M/e}$ such that for every $g \in S^B$, $|\mathcal{S}_{M-e,g}| = \chi_{n,g} \cdot \mathbf{y}_{M-e}$ and $|\mathcal{S}_{M/e,g}| = \chi_{n,g} \cdot \mathbf{y}_{M/e}$.

If e is an isthmus, then from the first row of (1), $|\mathcal{S}_{M,g}| = a_1 |\mathcal{S}_{M-e,g}| = a_1 \chi_{n,g} \cdot \mathbf{y}_{M-e}$, where $\mathbf{y}_M = a_1 \mathbf{y}_{M-e}$.

If e is a loop, then from the second row of (1), $|\mathcal{S}_{M,g}| = b_1 |\mathcal{S}_{M-e,g}| = b_1 \chi_{n,g} \cdot \mathbf{y}_{M-e}$, where $\mathbf{y}_M = b_1 \mathbf{y}_{M-e}$.

If e is neither an isthmus nor a loop, then from (1), $|\mathcal{S}_{M,g}| = a_2 |\mathcal{S}_{M/e,g}| + b_2 |\mathcal{S}_{M-e,g}| = a_2 \chi_{n,g} \cdot \mathbf{y}_{M/e} + b_2 \chi_{n,g} \cdot \mathbf{y}_{M-e} = \chi_{n,g} \cdot (a_2 \mathbf{y}_{M/e} + b_2 \mathbf{y}_{M-e})$, where the vector $\mathbf{y}_M = a_2 \mathbf{y}_{M/e} + b_2 \mathbf{y}_{M-e}$ satisfies the assumptions. This proves (2).

The uniqueness of $\mathbf{y}_M$ follows from the fact that Z is a basis of the linear hull of $\{\chi_{n,g}; g \in S^B\}$.

Furthermore, if $\mathbf{z}_1, \ldots, \mathbf{z}_m$ are integral vectors, then $\mathbf{y}_M$ is integral for each $M \in \mathcal{M}$ such that $E(M) = B$. If a_1, b_1, a_2, b_2 are also integers, then from (2), $\mathbf{y}_M$ are integral vectors for each M from $\mathcal{M}$. $\square$

We apply Theorem 1 for various S-functions of B-classes of matroids. Analogously, we can apply Theorem 2.

3. Regular Chain Groups

If R is a ring, the elements of R^E are considered vectors indexed by E, and we will use the notation $f + g$, $-f$, and sf for $f, g \in R^E$, and $s \in R$. A *chain* on E (over R, or simply an *R-chain*) is $f \in R^E$, and the *support* of f is $\sigma(f) = \{e \in E; f(e) \neq 0\}$. The *zero chain* has null support. Given $X \subseteq E$ and $f \in R^E$, define $f^{\setminus X} \in R^{E \setminus X}$ such that $[f^{\setminus X}](e) = f(e)$ for each $e \in E \setminus X$.

A matroid M on E of rank $r(M)$ is *regular* if there exists an $r \times n$ ($r = r(M)$, $n = |E|$) totally unimodular matrix D (called a *representative* matrix of M) such that independent sets of M correspond to independent sets of columns of D.

We recall properties of regular matroids presented in [1,8–11]). For any basis B of M, D can be transformed to a form $(I_r|U)$ such that I_r corresponds to B and U is totally unimodular. The dual of M is a regular matroid M^* with a representative matrix $(-U^T|I_{n-r})$ (where I_{n-r} corresponds to $E \setminus B$).

By a *regular chain group* N on E (*associated with* D), we mean a set of chains on E over $\mathbb{Z}$ that are orthogonal to each row of D (i.e., are integral combinations of rows of a representative matrix of M^*). The set of chains orthogonal to every chain of N is a chain group called *orthogonal* to N and denoted by $N^\perp$ (clearly, $N^\perp$ is the set of integral combinations of rows of D). By the *rank* of N, we mean $r(N) = n - r(M) = r^*(M)$. Then, $r(N^\perp) = n - r(N) = r(M)$. We always assume that a regular chain group N is associated with a matrix $D = D(N)$ representing a matroid $M = M(N)$.

For any $X \subseteq E$, let

$$\begin{aligned} N{-}X &= \left\{ f^{\setminus X}; f \in N,\ \sigma(f) \cap X = \emptyset \right\}, \\ N/X &= \left\{ f^{\setminus X}; f \in N \right\}. \end{aligned} \tag{3}$$

We have $M(N{-}X) = M - X$ and $M(N/X) = M/X$. Clearly, $D(N{-}X)$ arises from $D(N)$ after deleting the columns corresponding to X. Furthermore, $(N{-}X)^\perp = N^\perp/X$ and $(N/X)^\perp = N^\perp{-}X$.

A chain f of N is *elementary* if there is no nonzero f' of N such that $\sigma(f') \subset \sigma(f)$. An elementary chain f is called a *primitive* chain of N if the coefficients of f are restricted to the values 0, 1, and -1. (Notice that the set of supports of primitive chains of N is the set of circuits of $M(N)$.) We say that a chain g *conforms* to a chain f if $g(e)$ and $f(e)$ are nonzero and have the same sign for each $e \in E$ such that $g(e) \neq 0$. From [1] (Section 6.1),

$$\begin{gathered} \text{every chain } f \text{ of } N \text{ can be expressed as a sum of} \\ \text{primitive chains in } N \text{ that conform to } f. \end{gathered} \tag{4}$$

Let A be an Abelian group with additive notation. We shall consider A as a (right) $\mathbb{Z}$-module such that the scalar multiplication $a \cdot z$ of $a \in A$ by $z \in \mathbb{Z}$ is equal to 0 if $z = 0$, $\sum_1^z a$ if $z > 0$, and $\sum_1^{-z}(-a)$ if $z < 0$. Similarly, if $a \in A$ and $f \in \mathbb{Z}^E$, then define $a \cdot f \in A^E$ so that $(a \cdot f)(e) = a \cdot f(e)$ for each $e \in E$. If N is a regular chain group on E, define

$$\begin{aligned} A(N) &= \{\textstyle\sum_{i=1}^m a_i \cdot f_i; a_i \in A, f_i \in N, m \geq 1\}, \\ A[N] &= \{f \in A(N); \sigma(f) = E\}. \end{aligned}$$

Notice that $A(N) = N$ if $A = \mathbb{Z}$. From [8] (Proposition 1),

$$\begin{gathered} g \in A^E \text{ is from } A(N) \text{ if and only if for each } f \in N^\perp, \\ \textstyle\sum_{e \in E} g(e) f(e) = 0. \end{gathered} \tag{5}$$

Suppose that $\mathcal{M}$ is a B-class of regular matroids, with B finite. Denote by $\mathcal{R}$ the class of $A{-}\{0\}$-functions on matroids from $\mathcal{M}$ such that $A[N] = \mathcal{R}_{M(N)}$ for each $M(N) \in \mathcal{M}$. In other words, $\mathcal{R}$ is the class of $A[N]$ where $M(N) \in \mathcal{M}$. We claim that $\mathcal{R}$ is $(0, k{-}1, 1, -1)$-regular.

Lemma 1. *For each $M \in \mathcal{M}$, $e \in E(M) \setminus B$, and $g : B \to A - \{0\}$, we have*

$$\begin{array}{lll} |\mathcal{R}_{M,g}| = 0|\mathcal{R}_{M-e,g}| & & \text{if } e \text{ is an isthmus of } M, \\ |\mathcal{R}_{M,g}| = (k-1)|\mathcal{R}_{M-e,g}| & & \text{if } e \text{ is a loop of } M, \\ |\mathcal{R}_{M,g}| = |\mathcal{R}_{M/e,g}| - |\mathcal{R}_{M-e,g}| & & \text{otherwise.} \end{array} \tag{6}$$

Proof. Notice that e is a loop (isthmus) of $M = M(N)$ e if $\chi_e \in N$ ($\chi_e \in N^\perp$). Thus, if e is an isthmus of M, then from (5), each $f \in \mathcal{R}_{M,g}$ satisfies $f(e) = 0$, where $|\mathcal{R}_{M,g}| = 0$.

Given $f \in A^{E\setminus e}$ and $x \in A$, let $f_x \in A^E$ be defined so that $f_x^{\setminus e} = f$ and $f_x(e) = x$.

If e is a loop of M, then from (5), for each $f \in \mathcal{R}_{M-e,g}$ and $x \in A-\{0\}$, $f_x \in \mathcal{R}_{M,g}$. Similarly, if $f \in \mathcal{R}_{M,g}$, then $f^{\setminus e} \in \mathcal{R}_{M-e,g}$. Thus, $|\mathcal{R}_{M,g}| = (k-1)|\mathcal{R}_{M-e,g}|$.

If e is neither an isthmus nor a loop of $M = M(N)$, then there exists $\tilde{f} \in N^\perp$ such that $\tilde{f}(e) \neq 0$ and $\tilde{f} \neq \chi_e$. From (3), for any $f \in A[N/e]$, there exists $a \in A$ such that $f_a \in A(N)$. From (5), f_a must be orthogonal to $\tilde{f}$, where a is unique. Furthermore, if $a = 0$ (resp. $a \neq 0$), then from (3), $f \in A[N-e]$ (resp. $f_a \in A[N]$); i.e., $f \mapsto f_a$ is a bijection from $\mathcal{R}_{M/e,g}$ to the disjoint union of $\mathcal{R}_{M,g}$ and $\mathcal{R}_{M-e,g}$. This implies the last row of (6). □

Corollary 1. *Suppose that $\mathcal{M}$ is a B-class of regular matroids, with B finite, and let $\mathcal{R}$ be the class of $A[N]$ where $M(N) \in \mathcal{M}$. Assume that $\mathcal{M}_{B,\mathcal{R}} = (M_1, \ldots, M_n)$. Then, for each $M \in \mathcal{M}$, there exists an integral vector $\mathbf{x}_M = (x_1, \ldots, x_n)$ such that for every $g : B \to A-\{0\}$, $|\mathcal{R}_{M,g}| = \boldsymbol{\chi}_{n,g} \cdot \mathbf{x}_M$.*

Proof. It follows from (6) and Theorem 1. □

4. Nowhere-Zero Flows and Tensions on Graphs

We deal with finite undirected graphs with multiple edges and loops. If G is a graph, then $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. Every edge e of G determines two opposite arcs arising from it after endowing e with two distinct orientations. All arcs obtained in this way are called *arcs* of G, and the set of them is called the *arc set* of G and denoted by $D(G)$. Clearly, $|D(G)| = 2|E(G)|$. If x is an arc of G, then denote by x^{-1} the second arc arising from the same edge. Clearly, $(x^{-1})^{-1} = x$ and $x \neq x^{-1}$ for every arc x of G. If $X \subseteq D(G)$, then let X^{-1} denote $\{x \in D(G); x^{-1} \in X\}$. For any vertex v of G, denote by $\omega_G^+(v)$ the set of arcs from $D(G)$ directed out of v. If A is an Abelian group, then a *nowhere-zero A-chain* in G is a mapping $\varphi : D(G) \to A - \{0\}$ such that $\varphi(x^{-1}) = -\varphi(x)$ for every $x \in D(G)$.

By an *orientation* of G, we mean any $X \subseteq D(G)$ such that $X \cup X^{-1} = D(G)$ and $X \cap X^{-1} = \emptyset$. In other words, an orientation of G can be considered a directed graph arising from G after endowing each edge with an orientation.

Let A be an Abelian group with additive notation. A nowhere-zero A-chain φ in G is called a *nowhere-zero A-flow* if $\sum_{x\in\omega_G^+(v)} \varphi(x) = 0$ for every vertex v of G. Considering φ as a mapping on an arbitrary but fixed orientation of G, we obtain the usual definition of nowhere-zero A-flows. Such nowhere-zero A-flows on G coincide with $A[N]$, where N is the regular chain group associated with $M(N)$, the cycle matroid of G (edge sets of subforests of G forming independent sets of $M(N)$).

By a B-class of graphs, we mean a class $\mathcal{G}$ such that for each $G \in \mathcal{G}$, $B \subseteq E(G)$, and for each $e \in E(G) \setminus B$, $G-e, G/e \in \mathcal{G}$. Then, the class of cycle matroids of graphs from $\mathcal{G}$ is a B-class of matroids $\mathcal{M}$. Denote by $\mathcal{F}$ the class of nowhere-zero A-flows on a graph from $\mathcal{G}$. Clearly, $\mathcal{F}$ coincides with the class $\mathcal{R}$ associated with $\mathcal{M}$ described in the previous section. Analogously, we write $\mathcal{G}_{B,\mathcal{F}} = (G_1, \ldots, G_n)$ instead of $\mathcal{M}_{B,\mathcal{R}} = (M_1, \ldots, M_n)$, where M_i is the cycle matroid of G_i for $i = 1, \ldots, n$.

Lemma 2. *For each $G \in \mathcal{G}$, $e \in E(G) \setminus B$, and $g : B \to A - \{0\}$, we have*

$$\begin{array}{ll} |\mathcal{F}_{G,g}| = 0|\mathcal{F}_{G-e,g}| & \text{if } e \text{ is an isthmus of } M, \\ |\mathcal{F}_{G,g}| = (k-1)|\mathcal{F}_{G-e,g}| & \text{if } e \text{ is a loop of } M, \\ |\mathcal{F}_{G,g}| = |\mathcal{F}_{G/e,g}| - |\mathcal{F}_{G-e,g}| & \text{otherwise.} \end{array}$$

Proof. Apply Lemma 1 for a class $\mathcal{M}$ of cycle matroids of graphs from $\mathcal{G}$. □

Corollary 2. *Suppose that $\mathcal{G}$ is a B-class of graphs, with B finite, and let $\mathcal{F}$ be the class of nowhere-zero A-flows on graphs from $\mathcal{G}$. Assume that $\mathcal{G}_{B,\mathcal{F}} = (G_1, \dots, G_n)$. Then, for each $G \in \mathcal{G}$, there exists an integral vector $\mathbf{x}_G = (x_1, \dots, x_n)$ such that for every $g : B \to A - \{0\}$, $|\mathcal{F}_{G,g}| = \chi_{n,g} \cdot \mathbf{x}_G$.*

Proof. Apply Corollary 1 for a class $\mathcal{M}$ of cycle matroids of graphs from $\mathcal{G}$. □

We applied the idea of Corollary 2 in [6,7,12,13] and proved that the smallest counterexample to the 5-flow conjecture of Tutte (that every bridgeless graph has a nowhere-zero 5-flow) must be cyclically 6-edge-connected and has a girth of at least 11.

A *circuit* C of G is a connected 2-regular subgraph of G (notice that the loop is a circuit of order 1). By a *directed circuit* of G, we mean an orientation X of C such that $|X \cap \omega_G^+(v)| = 1$ for each vertex v of C.

A nowhere-zero A-chain φ in G is called a *nowhere-zero A-tension* if $\sum_{x \in X} \varphi(x) = 0$ for every directed circuit X of G. Considering φ as a mapping on an arbitrary but fixed orientation of G, we obtain nowhere-zero A-tensions on G that coincide with $A[N]$ such that $M(N)$ is the bond matroid of G (dual of the cycle matroid of G). Denote by $\mathcal{T}$ the class of nowhere-zero A-tensions on graphs from $\mathcal{G}$. Clearly, $\mathcal{T}$ coincides with the class $\mathcal{R}$ associated with the class of bond matroids of graphs from $\mathcal{G}$. Therefore, $\mathcal{T}$ is $(k-1, 0, -1, 1)$-regular.

Lemma 3. *For each $G \in \mathcal{G}$, $e \in E(G) \setminus B$, and $g : B \to A - \{0\}$, we have*

$$\begin{aligned} |\mathcal{T}_{G,g}| &= (k-1)|\mathcal{T}_{G-e,g}| && \text{if } e \text{ is an isthmus of } M, \\ |\mathcal{T}_{G,g}| &= 0|\mathcal{T}_{G-e,g}| && \text{if } e \text{ is a loop of } M, \\ |\mathcal{T}_{G,g}| &= |\mathcal{T}_{G-e,g}| - |\mathcal{T}_{G/e,g}| && \text{otherwise.} \end{aligned}$$

Proof. Apply Lemma 1 for the class of bond matroids of graphs from $\mathcal{G}$. □

Corollary 3. *Suppose that $\mathcal{G}$ is a B-class of graphs, with B finite, and let $\mathcal{T}$ be the class of nowhere-zero A-tensions on graphs from $\mathcal{G}$. Assume that $\mathcal{G}_{B,\mathcal{T}} = (G_1, \dots, G_n)$. Then, for each $G \in \mathcal{G}$, there exists an integral vector $\mathbf{x}_G = (x_1, \dots, x_n)$ such that for every $g : B \to A - \{0\}$, $|\mathcal{T}_{G,g}| = \chi_{n,g} \cdot \mathbf{x}_G$.*

Proof. Apply Corollary 1 for the class of bond matroids of graphs from $\mathcal{G}$. □

5. Orientations in Oriented Matroids

In this section, we use notation and results from [14,15] (see also [9,16,17]). We define a signed set X to be a set $\underline{X}$, called the set *underlying* X, and the mapping $\mathrm{sg}_X(x) : \underline{X} \to \{1, -1\}$, called the *signature* of X. Let X be a signed set. Then, $\underline{X}$ is partitioned into two distinguished subsets: $X^+ = \{x \in \underline{X}; \mathrm{sg}_X(x) = 1\}$ and $X^- = \{x \in \underline{X}; \mathrm{sg}_X(x) = -1\}$. The *opposite* $-X$ of X is defined by $(-X)^+ = X^-$ and $(-X)^- = X^+$. If $\underline{X}$ is a subset of E, then X will be called a *signed subset* of E, and if $\underline{X} = \emptyset$, then we write $X = \emptyset$.

An *oriented matroid* M on E is a couple $(E, \mathcal{O})$, where $\mathcal{O}$ is a collection of signed sets satisfying

$$X \in \mathcal{O} \text{ implies } X \neq \emptyset \text{ and } -X \in \mathcal{O}; \tag{7}$$

$$X_1, X_2 \in \mathcal{O} \text{ and } \underline{X_1} \subseteq \underline{X_2} \text{ imply } X_1 = X_2 \text{ or } X_1 = -X_2; \tag{8}$$

$$\begin{aligned} &\text{for all } X_1, X_2 \in \mathcal{O}, x \in X_1^+ \cap X_2^- \text{ and } y \in X_1^+ \setminus X_2^- \text{ there exists } X_3 \in \mathcal{O} \\ &\text{such that } y \in X_3, X_3^+ \subseteq (X_1^+ \cup X_2^+) \setminus \{x\} \text{ and } X_3^- \subseteq (X_1^- \cup X_2^-) \setminus \{x\}. \end{aligned} \tag{9}$$

Signet sets from $\mathcal{O}$ are called *signed circuits* of M. Let $\underline{\mathcal{O}} = \{\underline{X}; X \in \mathcal{O}\}$. Then, $\underline{\mathcal{O}}$ is a collection of circuits of a matroid $\underline{M}$ on E. The circuits of the dual matroid $\underline{M}^*$ (i.e., the cocircuits of $\underline{M}$) can be oriented in a unique way such that the $\mathcal{O}^*$ of signed cocircuits of M satisfies the *orthogonality property*: for all $X \in \mathcal{O}$ and $Y \in \mathcal{O}^*$ such that $|\underline{X} \cap \underline{Y}| = 2$, both $(X^+ \cap Y^+) \cup (X^- \cap Y^-)$ and $(X^+ \cap Y^-) \cup (X^- \cap Y^+)$ are non-empty. Then, $\mathcal{O}^*$ satisfies (7)–(9) and defines an oriented matroid M^*, the *dual* of M. The orthogonality property holds for all $X \in \mathcal{O}$ and $Y \in \mathcal{O}^*$ such that $\underline{X} \cap \underline{Y} \neq \emptyset$. We have $(M^*)^* = M$. Thus, the class of oriented matroids is a minor and dual closed class of matroids.

A circuit $X \in \mathcal{O}$ is *positive* if $X^- = \emptyset$. We say that $\mathcal{O}$ is *totally cyclic* if each $e \in E$ is contained in a positive circuit $X \in \mathcal{O}$ and that $\mathcal{O}$ is *acyclic* if no $X \in \mathcal{O}$ is positive. From [14] (Theorem 2.2),

$$\mathcal{O} \text{ is acyclic if and only if } \mathcal{O}^* \text{ is totally cyclic.} \tag{10}$$

For any $Z \subseteq E$, denote by $_{\overline{Z}}M$ the oriented matroid obtained from M by reversing signs on Z, i.e., $_{\overline{Z}}M = (E, {}_{\overline{Z}}\mathcal{O})$, where $_{\overline{Z}}\mathcal{O} = \{{}_{\overline{Z}}X; X \in \mathcal{O}\}$, supposing that $_{\overline{Z}}X$ satisfies $({}_{\overline{Z}}X)^+ = (X^+ \setminus Z) \cup (X^- \cap Z)$ and $({}_{\overline{Z}}X)^- = (X^- \setminus Z) \cup (X^+ \cap Z)$. Set $\chi_{Z,E} : E \to \{1, -1\}$ such that $\chi_{Z,E}(x) = -1$ if $x \in Z$ and $\chi_{Z,E}(x) = 1$ if $x \in E \setminus Z$. If X is a directed circuit of $\mathcal{O}$ with the signature sg_X, then the signature of $_{\overline{Z}}X$ ($\in {}_{\overline{Z}}\mathcal{O}$) satisfies $\mathrm{sg}_{{}_{\overline{Z}}X}(x) = \mathrm{sg}_X(x)\chi_{Z,E}(x)$ for each $x \in \underline{X}$. Thus, $\chi_{Z,E}$ uniquely determines $_{\overline{Z}}M$.

Let M be an oriented matroid on E and $e \in E$. From [15] (Lemma 3.1.1),

$$\text{if both } M \text{ and } {}_{\overline{e}}M \text{ are acyclic, then both } M{-}e \text{ and } M/e \text{ are acyclic;} \tag{11}$$

$$\text{if } M \text{ is acyclic and } {}_{\overline{e}}M \text{ is not acyclic, then } M{-}e \text{ is acyclic and } M/e \text{ is not acyclic;} \tag{12}$$

$$\text{if } e \text{ is not a loop of } M \text{ and both } M \text{ and } {}_{\overline{e}}M \text{ are not acyclic, then both } M{-}e \text{ and } M/e \text{ are not acyclic.} \tag{13}$$

Suppose that $\mathcal{M}$ is a B-class of oriented matroids. For any $M \in \mathcal{M}$ and $Y \subseteq B$, denote by $\mathcal{A}_{M,Y}$ the set of subsets Z of $E \setminus B$ such that $_{\overline{Z \cup Y}}M$ is acyclic. Since $_{\overline{Z \cup Y}}M$ is uniquely determined by $\chi_{Z \cup Y, E(M)}$, $\mathcal{A}_{M,Y}$ can be considered a set of $\{1, -1\}$-functions $\chi_{Z \cup Y, E(M)}$ corresponding to acyclic orientations. Denote by $\mathcal{A}$ the union of $\mathcal{A}_{M,Y}$, where M runs through $\mathcal{M}$ and Y runs through the subsets of B. We claim that $\mathcal{A}$ is $(2, 0, 1, 1)$-regular.

Lemma 4. *For any $M \in \mathcal{M}$, $e \in E \backslash B$, and $Y \subseteq B$,*

$$\begin{array}{ll} |\mathcal{A}_{M,Y}| = 2|\mathcal{A}_{M-e,Y}| & \textit{if } e \textit{ is an isthmus of } M, \\ |\mathcal{A}_{M,Y}| = 0|\mathcal{A}_{M-e,Y}| & \textit{if } e \textit{ is a loop of } M, \\ |\mathcal{A}_{M,Y}| = |\mathcal{A}_{M-e,Y}| + |\mathcal{A}_{M-e,Y}| & \textit{otherwise.} \end{array}$$

Proof. The statement is obvious if e is an isthmus or a loop of M. Let $e \in E$ be neither an isthmus nor a loop of M. For a subset Z of $E \setminus B$, set $f(M; Z) = 0$ if $_{\overline{Z \cup Y}}M$ is not acyclic and $f(M; Z) = 1$ if $_{\overline{Z \cup Y}}M$ is acyclic. We have

$$|\mathcal{A}_{M,Y}| = \sum_{Z \subseteq (E \backslash B)} f(M; Z).$$

If $e \in E \setminus B$ is not a loop of M and Z is a subset of $(E \setminus B) \setminus \{e\}$, then from (11)–(13), we have

$$f(M; Z) + f({}_{\overline{e}}M; Z) = f(M - e; Z) + f(M/e; Z).$$

Now, $f({}_{\overline{e}}M; Z) = f(M; Z \cup \{e\})$. Summing up for all subsets Z of $(E \setminus B) \setminus \{e\}$, we obtain $|\mathcal{A}_{M,Y}| = |\mathcal{A}_{M-e,Y}| + |\mathcal{A}_{M-e,Y}|$ as required. □

Considering $\mathcal{A}$ as the class of $\{1, -1\}$-functions on matroids from $\mathcal{M}$ corresponding to acyclic orientations, any $g : B \to \{1, -1\}$ coincides with $Y \subseteq B$ such that $g = \chi_{Y,B}$. Thus, we can write $\mathcal{A}_{M,Y}$ and $\chi_{n,Y}$ instead of $\mathcal{A}_{M,g}$ and $\chi_{n,g}$, respectively. We apply this notation in the following corollary of Theorem 1.

Corollary 4. *Suppose that $\mathcal{M}$ is a B-class of oriented matroids, with B finite, and let $\mathcal{A}$ be the class of acyclic orientations of oriented matroids from $\mathcal{M}$. Assume that $\mathcal{M}_{B,\mathcal{A}} = (M_1, \ldots, M_n)$. Then, for each $M \in \mathcal{M}$, there exists an integral vector $\mathbf{x}_M = (x_1, \ldots, x_n)$ such that for every $Y \subseteq B$, $|\mathcal{A}_{M,Y}| = \chi_{n,Y} \cdot \mathbf{x}_M$.*

Proof. It follows from Lemma 4 and Theorem 1. □

For any $Y \subseteq B$, denote by $\mathcal{C}_{M,Y}$ the set of subsets Z of $E \setminus B$ such that $_{\overline{Z \cup Y}}M$ is totally cyclic. Since $_{\overline{Z \cup Y}}M$ is uniquely determined by $\chi_{Z\cup Y, E(M)}$, $\mathcal{C}_{M,Y}$ can also be considered set of $\{1,-1\}$-functions $\chi_{Z\cup Y,E(M)}$ corresponding to totally cyclic orientations. Denote by $\mathcal{C}$ the union of $\mathcal{C}_{M,Y}$, where M runs through $\mathcal{M}$ and Y runs through the subsets of B. We claim that $\mathcal{C}$ is $(0,2,1,1)$-regular.

Lemma 5. *For any $M \in \mathcal{M}$, $e \in E \setminus B$, and $Y \subseteq B$,*

$$\begin{array}{ll} |\mathcal{C}_{M,Y}| = 0|\mathcal{C}_{M-e,Y}| & \textit{if e is an isthmus of } M, \\ |\mathcal{C}_{M,Y}| = 2|\mathcal{C}_{M-e,Y}| & \textit{if e is a loop of } M, \\ |\mathcal{C}_{M,Y}| = |\mathcal{C}_{M-e,Y}| + |\mathcal{C}_{M-e,Y}| & \textit{otherwise.} \end{array}$$

Proof. It follows from Lemma 4 and (10). □

Similar to the above, $\mathcal{C}$ can be considered the class of $\{1,-1\}$-functions on matroids from $\mathcal{M}$ corresponding to totally cyclic orientations. Any $g : B \to \{1,-1\}$ coincides with $Y \subseteq B$ such that $g = \chi_{Y,B}$, and we can write $\mathcal{C}_{M,Y}$ and $\chi_{n,Y}$ instead of $\mathcal{C}_{M,g}$ and $\chi_{n,g}$, respectively.

Corollary 5. *Suppose that $\mathcal{M}$ is a B-class of oriented matroids, with B finite, and let $\mathcal{C}$ be the class of totally cyclic orientations of oriented matroids from $\mathcal{M}$. Assume that $\mathcal{M}_{B,\mathcal{C}} = (M_1, \ldots, M_n)$. Then, for each $M \in \mathcal{M}$, there exists an integral vector $\mathbf{x}_M = (x_1, \ldots, x_n)$ such that for every $Y \subseteq B$, $|\mathcal{C}_{M,Y}| = \chi_{n,Y} \cdot \mathbf{x}_M$.*

Proof. It follows from Corollaries 4 and (10). □

Let M be a regular matroid on E associated with a totally unimodular matrix D and N be the regular chain group associated with D. The set of circuits of M coincides with the set of supports of primitive chains of N. If fact, each circuit $C \subseteq E$ of M corresponds to exactly one primitive function f_C of N such that $\sigma(f_C) = C$. The set of primitive functions forms a set of oriented circuits of an oriented matroid (see [15]). Thus, we can apply Lemmas 4 and 5 and Corollaries 4 and 5 for any B-class of regular matroids.

6. Orientations of Graphs

Consider a fixed orientation D of a graph G. Each circuit C in G indicates two directed circuits; we denote one of them by Q and the other one by Q^{-1}. The edges of C and Q indicate a signed set X such that $\underline{X} = E(C)$, X^+ consists of the edges having the same orientation in D and Q, and X^- consists of the edges having different orientations in D and Q. Then, Q^{-1} indicates $-X$ in an analogous way. Applying this process for each circuit of G, we generate a set $\mathcal{O}$ such that $(E(G), \mathcal{O})$ is an oriented matroid M on $E(G)$, and the underlying matroid $\underline{M}$ is the cycle matroid of G; i.e., $\underline{\mathcal{O}}$ is the set of circuits of G.

If $Z \subseteq E(G)$, then denote by $_{\overline{Z}}D$ the orientation of G arising from D after changing the orientation of edges from Z. Clearly, $_{\overline{Z}}D$ corresponds to $_{\overline{Z}}M$. Analogously, an orientation D of G is *totally cyclic* if each edge of G is covered by a directed circuit and is *acyclic* if no edge of G is covered by a directed circuit.

Recall that a B-class of graphs is a class $\mathcal{G}$ such that for each $G \in \mathcal{G}$, $B \subseteq E(G)$, and for each $e \in E(G) \setminus B$, $G-e, G/e \in \mathcal{G}$. The class $\mathcal{A}$ (resp. $\mathcal{C}$) of acyclic (resp. totally cyclic) orientations of digraphs from $\mathcal{G}$ is the class of acyclic (resp. totally cyclic) orientations of matroids from the class of cyclic matroids of graphs from $\mathcal{G}$. Similarly, we write $\mathcal{A}_{G,Y}$ (resp. $\mathcal{C}_{G,Y}$) instead of $\mathcal{A}_{M,Y}$ (resp. $\mathcal{C}_{M,Y}$), supposing that M denotes the cyclic matroid of G. Analogously, we write $\mathcal{G}_{B,\mathcal{A}} = (G_1, \ldots, G_n)$ instead of $\mathcal{M}_{B,\mathcal{A}} = (M_1, \ldots, M_n)$, where M_i is the cycle matroid of G_i for $i = 1, \ldots, n$.

Lemma 6. *For each $G \in \mathcal{M}$, $e \in E(G) \setminus B$, and $Y \subseteq B$,*

$$\begin{array}{ll} |\mathcal{A}_{G,Y}| = 2|\mathcal{A}_{G-e,Y}| & \textit{if } e \textit{ is an isthmus of } G, \\ |\mathcal{A}_{G,Y}| = 0|\mathcal{A}_{G-e,Y}| & \textit{if } e \textit{ is a loop of } G, \\ |\mathcal{A}_{G,Y}| = |\mathcal{A}_{G/e,Y}| + |\mathcal{A}_{G-e,Y}| & \textit{otherwise.} \end{array} \tag{14}$$

Proof. It follows from Lemma 4. □

Corollary 6. *Suppose that $\mathcal{B}$ is a B-class of graphs, with B finite, and let $\mathcal{A}$ be the class of acyclic orientations of graphs from $\mathcal{G}$. Assume that $\mathcal{G}_{B,\mathcal{A}} = (G_1, \ldots, G_n)$. Then, for each $G \in \mathcal{G}$, there exists an integral vector $\mathbf{x}_G = (x_1, \ldots, x_n)$ such that for every $Y \subseteq B$, $|\mathcal{A}_{G,Y}| = \chi_{n,Y} \cdot \mathbf{x}_G$.*

Proof. It follows from Corollary 4. □

Lemma 7. *For each $G \in \mathcal{M}$, $e \in E(G) \setminus B$, and $Y \subseteq B$,*

$$\begin{array}{ll} |\mathcal{C}_{G,Y}| = 0|\mathcal{C}_{G-e,Y}| & \textit{if } e \textit{ is an isthmus of } G, \\ |\mathcal{C}_{G,Y}| = 2|\mathcal{C}_{G-e,Y}| & \textit{if } e \textit{ is a loop of } G, \\ |\mathcal{C}_{G,Y}| = |\mathcal{C}_{G/e,Y}| + |\mathcal{C}_{G-e,Y}| & \textit{otherwise.} \end{array} \tag{15}$$

Proof. It follows from Lemma 5. □

Corollary 7. *Suppose that $\mathcal{B}$ is a B-class of graphs, with B finite, and let $\mathcal{C}$ be the class of totally cyclic orientations of graphs from $\mathcal{G}$. Assume that $\mathcal{G}_{B,\mathcal{C}} = (G_1, \ldots, G_n)$. Then, for each $G \in \mathcal{G}$, there exists an integral vector $\mathbf{x}_G = (x_1, \ldots, x_n)$ such that for every $Y \subseteq B$, $|\mathcal{C}_{G,Y}| = \chi_{n,Y} \cdot \mathbf{x}_G$.*

Proof. It follows from Corollary 5. □

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