mathematics

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On Fractional-Order Discrete-Time Reaction Diffusion Systems

Othman Abdullah Almatroud [1], Amel Hioual [2,*], Adel Ouannas [3] and Giuseppe Grassi [4]

[1] Department of Mathematics, Faculty of Science, University of Ha'il, Ha'il 81451, Saudi Arabia; o.almatroud@uoh.edu.sa
[2] Laboratory of Dynamical Systems and Control, University of Oum EL-Bouaghi, Oum El Bouaghi 04000, Algeria
[3] Department of Mathematics and Computer Science, University of Oum EL-Bouaghi, Oum El Bouaghi 04000, Algeria; ouannas.adel@univ-oeb.dz
[4] Dipartimento Ingegneria Innovazione, Universita del Salento, 73100 Lecce, Italy; giuseppe.grassi@unisalento.it
* Correspondence: amel.hioual@univ-oeb.dz

Abstract: Reaction–diffusion systems have a broad variety of applications, particularly in biology, and it is well known that fractional calculus has been successfully used with this type of system. However, analyzing these systems using discrete fractional calculus is novel and requires significant research in a diversity of disciplines. Thus, in this paper, we investigate the discrete-time fractional-order Lengyel–Epstein system as a model of the chlorite iodide malonic acid (CIMA) chemical reaction. With the help of the second order difference operator, we describe the fractional discrete model. Furthermore, using the linearization approach, we established acceptable requirements for the local asymptotic stability of the system's unique equilibrium. Moreover, we employ a Lyapunov functional to show that when the iodide feeding rate is moderate, the constant equilibrium solution is globally asymptotically stable. Finally, numerical models are presented to validate the theoretical conclusions and demonstrate the impact of discretization and fractional-order on system dynamics. The continuous version of the fractional-order Lengyel–Epstein reaction–diffusion system is compared to the discrete-time system under consideration.

Keywords: Lengyel–Epstein reaction–diffusion system; second order difference operator; fractional-order Caputo $\hbar$-difference operator; Lyapunov function; local stability; global stability

MSC: 39A12; 39A30; 39A60; 39B82

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Citation: Almatroud, O.A.; Hioual, A.; Ouannas, A.; Grassi, G. On Fractional-Order Discrete-Time Reaction Diffusion Systems. *Mathematics* **2023**, *11*, 2447. https://doi.org/10.3390/math11112447

Academic Editor: Yong Xie

Received: 7 April 2023
Revised: 15 May 2023
Accepted: 22 May 2023
Published: 25 May 2023

1. Introduction

A mathematical model is a collection of equations that represents the mathematical representation of hypotheses (or assumptions). Mathematical modeling is widely used in ecological, epidemiological, and biological fields [1–6]. To gain a better understanding of patterns, modeling is widely utilized. There are several studies that use mathematical models as a tool for analysis; for example, ref. [7] addresses the modeling of biological systems using an enhanced fractional Gompertz equation. The dynamical complexity in a time-delayed tumor-immune model was investigated in [8]. A fractional Keller–Segel model was introduced in [9]. In [10], an anoise-assisted tumor-immune system was introduced, and stochastic sensitivity and chaos were investigated. The deterministic and stochastic dynamics of Michaelis–Menten kinetics-based tumor-immune interactions were explored in [11]. In [12], a delayed model of hospital-acquired infection with multidrug-resistant bacterium Acinatobactor baumannii was explored.

Due to their ability to imitate a variety of real-world phenomena and the complexity of their solutions, reaction–diffusion systems have gained significant theoretical interest and are of tremendous value in many scientific and engineering domains [13–16]. Meanwhile, the fractional partial differential equation is commonly employed in practical applications.

Several publications in the topic have been published lately [17,18]. Fractional diffusion equations are employed in the modeling of anomalous diffusion in porous media with rich nano–micro size features, which is an efficient and frequent application.

Many nonlinear processes in nature, such as population models, brain networks, and gene information, have discrete properties. Given this, discrete models may be employed to effectively identify parameters from experimental data. Fractional partial difference equations provide a distinct time-discretization model, especially for anomalous diffusion. A time-discretization difference approach was only recently presented as a discrete fractional modeling. Ref. [19], for example, developed a fractional time-discretization diffusion model in the Caputo-like delta interpretation and also addressed diffusion concentration for different fractional difference orders. On the other hand, ref. [20] suggested a variable-order fractional diffusion equation on discrete timeframes and generated a variable-order function using a chaotic map.

Since 1991, numerous scientists have expressed interest in the investigated Lengyel–Epstein reaction–diffusion system proposed in [21,22] as a model of the chlorite-iodide malonic-acid (CIMA) chemical reaction–diffusion model. The CIMA reaction is of particular relevance since it was one of the first tests to demonstrate Alan Turing's theoretical claims concerning the chemical foundation for morphogenesis and, more broadly, pattern generation in 1952 [23,24]. However, due to the complexities of biochemical reactions, chemical reaction practices are frequently affected by or depend on the historical context of chemical reactions. The Lengyel–Epstein chemical reactions model has been solved by numerous researchers [25–28]. A number of researchers stated that fractional-order methods were more appropriate than classical ones [29–32].

Given that the discreteness and structure of the underlying spatial area have a major effect on dynamical behavior, discrete fractional reaction–diffusion systems are more closely related to biological systems than continuous systems. To show this effect on such systems, this research will focus on the discrete-time fractional form of the Lengyel–Epstein CIMA reaction–diffusion model. To begin, we use the same technique as [19] in order to create a discrete fractional representation of the model. Following the model presentation, we supply adequate conditions for local stability using the same strategy as in [33], which is to linearize the reaction–diffusion system around the steady-state. The Lyapunov function is also used to explore global asymptotic stability. Furthermore, numerical simulations are conducted using a numerical formula and the fact that the conditions considered in the discrete-time fractional reaction–diffusion model are boundary conditions; these conditions are the same as those used in [19] when presenting the discrete version of fractional reaction–diffusion equations. One can clearly see that the main conclusions of the work are similar to those obtained in the study of the continuous version of the model, but the calculations and selection of the Lyapunov functions and the stability criteria differ from one model to another, making the study very interesting and opening the door to many problems that we hope to investigate in the future.

The following is how this document is structured: Section 2 provides some terminology and theory relevant to fractional discrete-time systems. In Section 3, a unique discrete-time fractional-order Lengyel–Epstein reaction–diffusion system is introduced. Section 4 specifies requirements for the proposed system's local asymptotic stability, whereas, Section 5 discusses the global asymptotic stability for the proposed system. Finally, Section 6 uses numerical examples to demonstrate the analytical conditions.

2. Preliminaries

This section begins by introducing the subject's required nomenclature and stability theory.

Definition 1 ([34])**.** *Assume $\varkappa : \mathbb{N} \to \mathbb{R}$. The forward difference operator Δ is then defined.*

$$\Delta\varkappa(\ell) = \varkappa(\ell+1) - \varkappa(\ell); \quad \ell \in \mathbb{N}. \tag{1}$$

In addition, the operators Δ^n, $n = 1, 2, 3, ...$, are recursively identified by $\Delta^n \varkappa(\ell) = \Delta(\Delta^{n-1}\varkappa(\ell))$, $\ell \in \mathbb{N}$.

In particular, the second-order difference operator of function $\varkappa(t)$ is given by

$$\Delta^2 \varkappa(\ell) = \varkappa(\ell+2) - 2\varkappa(\ell+1) + \varkappa(\ell). \tag{2}$$

Lemma 1 ([34])**.** *Here, we give some properties of the difference operator Δ*

- $\Delta c = 0$ *where c is a constant.*
- $\Delta(\varkappa(\ell) + \kappa(\ell)) = \Delta\varkappa(\ell) + \Delta\kappa(\ell)$.
- $\Delta(\varkappa(\ell)\kappa(\ell)) = \varkappa(\ell)\Delta\kappa(\ell) + \kappa(\ell+1)\Delta\varkappa(\ell)$.

Theorem 1 ([34])**.** *Given two functions $\varkappa; \kappa : \mathbb{R} \to \mathbb{R}$ and $a; b \in \mathbb{N}$; $\quad a < b$; we have the summation by parts formulas:*

$$\sum_{\jmath=a}^{b-1} \varkappa(\jmath)\Delta\kappa(\jmath) = \varkappa(\jmath)\kappa(\jmath)|_a^b - \sum_{\jmath=a}^{b-1} \kappa(\jmath+1)\Delta\varkappa(\jmath), \tag{3}$$

$$\sum_{\jmath=a}^{b-1} \varkappa(\jmath+1)\Delta\kappa(\jmath) = \varkappa(\jmath)\kappa(\jmath)|_a^b - \sum_{\jmath=a}^{b-1} \kappa(\jmath)\Delta\varkappa(\jmath). \tag{4}$$

Definition 2 ([35])**.** *Let $\varkappa \in (h\mathbb{N})_a \to \mathbb{R}$. For given $\alpha > 0$, the α-th order h-sum is given by*

$${}_\hbar\Delta_a^{-\alpha}\varkappa(t) = \frac{\hbar}{\Gamma(\alpha)} \sum_{\frac{t}{\hbar}-\alpha}^{s=\frac{a}{\hbar}} (t - \sigma(s\hbar))^{(\alpha-1)} \varkappa(s\hbar), \quad \sigma(s\hbar) = (s+1)\hbar, \quad t \in (h\mathbb{N})_{a+\alpha\hbar}, \tag{5}$$

with $a \in \mathbb{R}$ being the initial value and the $\hbar$-falling factorial function being described by

$$t_\hbar^{(\alpha)} = \hbar^\alpha \frac{\Gamma(\frac{t}{\hbar}+1)}{\Gamma(\frac{t}{\hbar}+1-\alpha)},$$

while

$$(\hbar\mathbb{N})_{a+\alpha\hbar} = \{a + (1-\alpha)\hbar, a + (2-\alpha)\hbar, ...\}.$$

Definition 3 ([36,37])**.** *For a function $\varkappa(t)$ defined on $(h\mathbb{N})_a$ and for a certain $\alpha > 0$ so that $\alpha \in \mathbb{N}$, the Caputo $\hbar$-difference operator is expressed by*

$${}_\hbar^C\Delta_a^\alpha \varkappa(t) =_\hbar \Delta_a^{-(n-\alpha)} \Delta_\hbar^n \varkappa(t), \tag{6}$$

where $\quad \Delta_\hbar^n \varkappa(t) = \dfrac{\varkappa(t+\hbar) - \varkappa(t)}{\hbar}$.

Lemma 2 ([35])**.** *Here are some important properties employed in this work*

- *Discrete Leibniz integral law*

$${}_\hbar\Delta_{a+(1-\alpha)\hbar}^{-\alpha}\, {}_\hbar^C\Delta^\alpha \varkappa(t) = \varkappa(t) - \varkappa(a), \quad 0 < \alpha \leq 1, \quad t \in (\hbar\mathbb{N})_{a+\hbar}. \tag{7}$$

- *Caputo fractional difference of a constant x*

$${}_\hbar^C\Delta^\alpha x = 0, \quad 0 < \alpha \leq 1. \tag{8}$$

Lemma 3 ([35])**.** *The following inequality holds*

$${}_\hbar^C\Delta_a^\alpha \varkappa^2(t) \leq 2\varkappa(t+\alpha\hbar)\, {}_\hbar^C\Delta_a^\alpha \varkappa(t), \quad t \in (\hbar\mathbb{N})_{a+\alpha\hbar}, \tag{9}$$

where $0 < \alpha \leq 1$.

Let us consider the nonlinear fractional-order difference system

$$ {}^{C}_{\hbar}\Delta^{\alpha}_{a}\varkappa(t) = \psi(t+\hbar\alpha, \varkappa(t+\hbar\alpha)), \quad t \in (h\mathbb{N})_{a+\alpha\hbar}, \tag{10} $$

Theorem 2 ([38])**.** *Let $\varkappa^*$ be an equilibrium point of (24). If all the eigenvalues of $\psi'(\varkappa^*)$ are located in $S^{\alpha}_{\hbar}$, then $\varkappa^*$ is asymptotically stable. where*

$$ S^{\alpha}_{\hbar} = \left\{ w \in \mathbb{C} : |\mathrm{Arg}(w)| > \frac{\alpha\pi}{2} \quad or \quad |w| > \frac{2^{\alpha}}{\hbar^{\alpha}} \cos^{\alpha}\left(\frac{\mathrm{Arg}(w)}{\alpha}\right) \right\}. \tag{11} $$

Theorem 3 ([35])**.** *Let $\varkappa = 0$ be the system's equilibrium point (24). The equilibrium point is asymptotically stable if there exists a positive definite and declining scalar function ${}^{C}_{\hbar}\Delta^{\alpha}_{a}V(t, \varkappa(t)) \leq 0$.*

3. The Fractional Discrete Lengyl–Epstein Reaction–Diffusion System

In this part, the models under examination are approximated using two well-known methods. To the best of our knowledge, these discrete models would be the first in the literature.

As known, the Lengyel–Epstein reaction–diffusion system was presented as a model of the chlorite-iodide-malonic-acid chemical reaction (CIMA) that may be characterized by three chemical reaction schemes, which are as follows:

$$ \begin{cases} MA + I_2 \rightarrow IMA + I^- + H^+, & (1) \\ CIO_2 + I^- \rightarrow \frac{1}{2}I_2 + CIO_2^-, & (2) \\ CIO_2^- + 4I^- + 4H^+ \rightarrow CI^- + 2I_2 + 2H_2O. & (3) \end{cases} $$

- (1) describes the iodization of malonic acid (MA).
- (2) describes the oxidation of iodide ions by free chlorine dioxide radicals.
- (3) describes an interaction between chlorite and iodide ions created in the (1) and (2) processes to produce iodine.

Using the empirical rate laws for these processes and disregarding constant factors, the model for this reaction was simplified to the standard Lengyel–Epstein model with two independent variables u and v relating to the iodide concentration (I^-) and the chlorite concentration (ClO_2). This model represents the system that resulted from two differential equations. In [28], the following model was considered

$$ \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + a - u - \dfrac{uv}{1+u^2}, \quad x \in \Omega, t > 0, \\ \dfrac{\partial v}{\partial t} = \sigma\left(c\Delta v + b\left(u - \dfrac{uv}{1+u^2}\right)\right), \quad x \in \Omega, t > 0, \\ \partial_u = \partial_v = 0 \quad , \quad x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) > 0, \quad x \in \Omega, \end{cases} \tag{12} $$

where Ω is a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. u indicates the chemical concentration of the activator iodide and v represents the inhibitor chlorite at time $t < 0$ and a point $x \in \Omega$. The parameters a and b are rotated to the feed concentration, c is the ratio of the diffusion coefficient, and $\sigma > 0$ is a rescaling parameter depending on the concentration of the starch.

Since the time fractional systems have been widely investigated by researchers, a fractional-time Lengyel–Epstein reaction–diffusion system was presented as follows

$$ \begin{cases} {}^{C}_{0}D^{\delta}_{t}u - d_1\Delta u = a - u - \dfrac{4uv}{1+u^2}, \\ {}^{C}_{0}D^{\delta}_{t}v - d_2\Delta v = \sigma b\left(u - \dfrac{uv}{1+u^2}\right), \end{cases} \tag{13} $$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $0 < \delta \leq 1$ is the fractional-order and ${}_0^C D_t^\delta$ describes the Caputo fractional derivative; d_1, d_2 and σ are strictly positive constants with the same initial conditions and Neumann boundary conditions.

Based on the model (13) and with the discretization used in [19], assuming that $x \in [0, L]$, we have $x_{i+1} = x_i + k, \quad i = 0, ..., m$ and using the central difference formula concerning x, $\frac{\partial^2 u(x,t)}{\partial x^2}$ and $\frac{\partial^2 v(x,t)}{\partial x^2}$ can be approximately expended as

$$\begin{cases} \frac{\partial^2 u(x,t)}{\partial x^2} \approx \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{k^2}, \\ \frac{\partial^2 v(x,t)}{\partial x^2} \approx \frac{v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)}{k^2}. \end{cases}$$

Using the definition of the second order difference operator of u_i and v_i, we obtain

$$\begin{cases} \frac{\partial^2 u(x,t)}{\partial x^2} \approx \frac{\Delta^2 u_{i-1}(t)}{k^2}, \\ \frac{\partial^2 v(x,t)}{\partial x^2} \approx \frac{\Delta^2 v_{i-1}(t)}{k^2}. \end{cases}$$

Therefore, we consider the following discrete-time reaction–diffusion fractional Lengyel–Epstein system

$$\begin{cases} {}_\hbar^C\Delta_{t_0}^\alpha u_i(t) = \frac{d_1}{k^2}\Delta^2 u_{i-1}(t+\hbar\alpha) + a - u_i(t+\hbar\alpha) - \frac{4u_i(t+\hbar\alpha)v_i(t+\hbar\alpha)}{1+(u_i(t+\hbar\alpha))^2}, \\ {}_\hbar^C\Delta_{t_0}^\alpha v_i(t) = \frac{d_2}{k^2}\Delta^2 v_{i-1}(t+\hbar\alpha) + \sigma b\left(u_i(t+\hbar\alpha) - \frac{u_i(t+\hbar\alpha)v_i(t+\hbar\alpha)}{1+(u_i(t+\hbar\alpha))^2}\right), \end{cases} \tag{14}$$

where ${}_\hbar^C\Delta_{t_0}^\alpha$ is the Caputo-like difference, $0 < \alpha \leq 1$, $t \in (\hbar\mathbb{N})_{t_0}$, with periodic boundary conditions

$$\begin{cases} u_0(t) = u_m(t), \quad u_1(t) = u_{m+1}(t), \\ v_0(t) = v_m(t), \quad v_1(t) = v_{m+1}(t), \end{cases} \tag{15}$$

and initial conditions

$$u_i(t_0) = \phi_1(x_i) \geq 0, \quad v_i(t_0) = \phi_2(x_i) \geq 0.$$

4. Local Stability

In order to investigate the asymptotic stability of the considered discrete-time fractional Lengyel–Epstein system, we consider the unique equilibrium point which is the solution of the following system

$$\begin{cases} \frac{d_1}{k^2}\Delta^2 u^* + a - u^* - \frac{4u^* v^*}{1+(u^*)^2} = 0, \\ \frac{d_2}{k^2}\Delta^2 v^* + \sigma b\left(u^* - \frac{u^* v^*}{1+(u^*)^2}\right) = 0. \end{cases} \tag{16}$$

Using Lemma 1, the unique equilibrium point of system (14) is given by

$$(u^*, v^*) = \left(\frac{a}{5}, 1 + \left(\frac{a}{5}\right)^2\right). \tag{17}$$

4.1. Local Stability of the Free Diffusions System

In this part, we develop suitable requirements for the local asymptotic stability of the following system:

$$\begin{cases} {}^{C}_{\hbar}\Delta^{\alpha}_{t_0} u(t) = a - u(t+\hbar\alpha) - \dfrac{4u(t+\hbar\alpha)v(t+\hbar\alpha)}{1+(u(t+\hbar\alpha))^2}, \\ {}^{C}_{\hbar}\Delta^{\alpha}_{t_0} v(t) = \sigma b\left(u(t+\hbar\alpha) - \dfrac{u(t+\hbar\alpha)v(t+\hbar\alpha)}{1+(u(t+\hbar\alpha))^2}\right). \end{cases} \tag{18}$$

The characteristic equation for the eigenvalues is obtained using linear stability analysis around this stable state.

$$\mathrm{J} = \begin{pmatrix} \dfrac{\partial \psi}{\partial u} & \dfrac{\partial f}{\partial v} \\ \dfrac{\partial \Psi}{\partial u} & \dfrac{\partial g}{\partial v} \end{pmatrix} = \begin{pmatrix} \dfrac{3\left(\frac{a}{5}\right)^2 - 5}{1+\left(\frac{a}{5}\right)^2} & -\dfrac{4\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2} \\ \dfrac{2b\sigma\left(\frac{a}{5}\right)^2}{1+\left(\frac{a}{5}\right)^2} & -\dfrac{b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2} \end{pmatrix}, \tag{19}$$

where

$$\psi(u,v) = a - u(t+\hbar\alpha) - \frac{4u(t+\hbar\alpha)v(t+\hbar\alpha)}{1+(u(t+\hbar\alpha))^2}, \tag{20}$$

and

$$\Psi(u,v) = \sigma b\left(u(t+\hbar\alpha) - \frac{u(t+\hbar\alpha)v(t+\hbar\alpha)}{1+(u(t+\hbar\alpha))^2}\right). \tag{21}$$

We might observe from the Jacobian matrix that

$$\mathrm{tr}(\mathrm{J}) = \frac{3\left(\frac{a}{5}\right)^2 - 5 - b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2}, \qquad \det(\mathrm{J}) = \frac{5ab\sigma}{1+\left(\frac{a}{5}\right)^2}. \tag{22}$$

The Jacobian matrix has the following characteristic equation:

$$\Lambda^2 - \mathrm{tr}(\mathrm{J})\Lambda + \det(\mathrm{J}) = 0. \tag{23}$$

Its discriminant is

$$\Delta_\Lambda = \mathrm{tr}^2(\mathrm{J}) - 4\det(\mathrm{J}) = \left(\frac{3\left(\frac{a}{5}\right)^2 - 5 - b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2}\right)^2 - \frac{20ab\sigma}{1+\left(\frac{a}{5}\right)^2}. \tag{24}$$

As a result, we may deduce the following:

Theorem 4. *System (18) is locally asymptotically stable at the positive steady-state (u^*, v^*) if the following conditions hold*

- *If* $\left(\dfrac{3\left(\frac{a}{5}\right)^2 - 5 - b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2}\right)^2 - \dfrac{20ab\sigma}{1+\left(\frac{a}{5}\right)^2} \geq 0$ *and* $\dfrac{3\left(\frac{a}{5}\right)^2 - 5 - b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2} < 0.$
- *If* $\left(\dfrac{3\left(\frac{a}{5}\right)^2 - 5 - b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2}\right)^2 - \dfrac{20ab\sigma}{1+\left(\frac{a}{5}\right)^2} < 0$ *and* $\dfrac{3\left(\frac{a}{5}\right)^2 - 5 - b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2} \leq 0.$

Proof. Based on (24), we investigate each case independently.

- If $\Delta_\Lambda > 0$, we can see that $\det(\mathrm{J}) > 0$. As a result, the eigenvalues' negativity is dependent on the sign of $\mathrm{tr}(\mathrm{J})$, and the eigenvalues Λ_1 and Λ_2 are real and may be represented as

$$\Lambda_1 = \frac{\mathrm{tr}(\mathrm{J}) - \sqrt{\Delta_\Lambda}}{2}, \qquad \Lambda_2 = \frac{\mathrm{tr}(\mathrm{J}) + \sqrt{\Delta_\Lambda}}{2}. \tag{25}$$

– If $\mathrm{tr}(\mathrm{J}) < 0$, then, we have

$$\Lambda_1 = \frac{\mathrm{tr}(\mathrm{J}) - \sqrt{\Delta_\Lambda}}{2} < 0, \tag{26}$$

As a result, $\mathrm{Arg}(\Lambda_1) = \pi$. Since both eigenvalues are real, it is obvious that $\mathrm{Arg}(\Lambda_1) = \mathrm{Arg}(\Lambda_2) = \pi$. As a consequence, based on Theorem 2, the equilibrium $(u^*,^v *)$ is asymptotically stable.

– If $\mathrm{tr}(\mathrm{J}) < 0$, then, we have

$$\Lambda_2 = \frac{\mathrm{tr}(\mathrm{J}) + \sqrt{\Delta_\Lambda}}{2} > 0. \tag{27}$$

Therefore, $\mathrm{Arg}(\Lambda_2) = 0$. Based on Theorem 2, system (18) is unstable.

- If $\Delta_\Lambda < 0$, then,

$$\Lambda_1 = \frac{\mathrm{tr}(\mathrm{J}) - i\sqrt{-\Delta_\Lambda}}{2}, \qquad \Lambda_2 = \frac{\mathrm{tr}(\mathrm{J}) + i\sqrt{-\Delta_\Lambda}}{2}. \tag{28}$$

We may discuss the solutions based on the sign of $\mathrm{tr}(\mathrm{J})$.

– If $\mathrm{tr}(\mathrm{J}) < 0$ or $\mathrm{tr}(\mathrm{J}) > 0$, then, following the same case investigated previously, system (18) is asymptotically stable.

– If $\mathrm{tr}(\mathrm{J}) = 0$ then

$$\mathrm{Arg}\left(\frac{-i\sqrt{-\Delta_\Lambda}}{2}\right) = \mathrm{Arg}\left(\frac{i\sqrt{-\Delta_\Lambda}}{2}\right) = \frac{\pi}{2},$$

and system (18) is asymptotically stable.

- If $\Delta_\Lambda = 0$, as $\det(\mathrm{J}) > 0$, $\mathrm{tr}(\mathrm{J})$ cannot be equal to zero. The sign of the eigenvalues is the same as the sign of $\mathrm{tr}(\mathrm{J})$. As a result, (u^*, v^*) is asymptotically stable for all $\alpha \in (0, 1]$ if $\mathrm{tr}(\mathrm{J}) < 0$ and unstable if $\mathrm{tr}(\mathrm{J}) > 0$.

The proof is completed. □

4.2. Local Stability of the Diffusion System

We shall now show that in the presence of diffusion, the steady-state $(u*, v*)$ can be stable under certain parameter circumstances. We will adopt the same approach as in [33], first considering the eigenvalues of the following equation:

$$\Delta^2 \varkappa_{i-1}(t + \hbar\alpha) + \Lambda_i \varkappa_i(t + \hbar\alpha) = 0, \tag{29}$$

with the periodic boundary conditions:

$$\varkappa_0(t) = \varkappa_m(t), \quad \varkappa_1(t) = \varkappa_{m+1}(t). \tag{30}$$

We obtain

$$\begin{cases} {}^{C}_{\hbar}\Delta^{\alpha}_{t_0} u_i(t) = \dfrac{d_1}{k^2}\Lambda_i u_i(t+\hbar\alpha) + a - u_i(t+\hbar\alpha) - \dfrac{4u_i(t+\hbar\alpha)v_i(t+\hbar\alpha)}{1+(u_i(t+\hbar\alpha))^2}, \\ {}^{C}_{\hbar}\Delta^{\alpha}_{t_0} v_i(t) = \dfrac{d_2}{k^2}\Lambda_i v_i(t+\hbar\alpha) + \sigma b\left(u_i(t+\hbar\alpha) - \dfrac{u_i(t+\hbar\alpha)v_i(t+\hbar\alpha)}{1+(u_i(t+\hbar\alpha))^2}\right). \end{cases} \tag{31}$$

We derive the following by linearizing the reaction–diffusion system (31) about the steady-state $(u*, v*)$

$$\mathrm{J}_i = \begin{pmatrix} -\frac{d_1}{k^2}\Lambda_i + \frac{3\left(\frac{a}{5}\right)^2 - 5}{1+\left(\frac{a}{5}\right)^2} & \frac{-4\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2} \\ \sigma b\frac{2\left(\frac{a}{5}\right)^2}{1+\left(\frac{a}{5}\right)^2} & -\frac{d_2}{k^2}\Lambda_i - \sigma b\frac{\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2} \end{pmatrix}. \tag{32}$$

The following result is conducted,

Theorem 5. *We suppose that*

$$\left(\frac{3\left(\frac{a}{5}\right)^2-5-b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2}\right)^2-\frac{20ab\sigma}{1+\left(\frac{a}{5}\right)^2}>0 \quad \textit{and} \quad \frac{3\left(\frac{a}{5}\right)^2-5-b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2}<0,$$

system (14) is asymptotically stable at the steady-state (u^*,v^*) if the following holds:

- *If $d_1<d_2$ and* $\frac{d_1}{k^2}\Lambda_1-\frac{3\left(\frac{a}{5}\right)^2-5}{1+\left(\frac{a}{5}\right)^2}\geq 0.$
- *If $d_1>d_2$ and* $\frac{d_1}{k^2}\Lambda_1-\frac{3\left(\frac{a}{5}\right)^2-5}{1+\left(\frac{a}{5}\right)^2}\geq 0$*, in addition the eigenvalues*

$$\mu_j(\Lambda_i)=\frac{\mathrm{tr}(\mathrm{J_i})\,\underline{+-}\sqrt{\mathrm{tr}(J_i)^2-4\mathrm{det}(\mathrm{J_i})}}{2}, \quad j=1,2,$$

satisfy $\quad \mathrm{Arg}(\mu_j(\Lambda_i))>\frac{\alpha\pi}{2}.$

Proof. To explore the system's local asymptotic stability, we shall linearize it. If the eigenvalues of the linearized system fulfill the conditions of Theorem 2, using fundamental linear operator theory and maintaining the system's fractional structure in mind, we might state that $(u*,v*)$ is asymptotically stable. Assume that (Φ,Ψ) is an eigenfunction of (29) with the eigenvalue Λ. Then, let $u_i=\sum_{j=1}^{n}\kappa_{ij}\Phi_{ij}$ and $v_i=\sum_{j=1}^{n}\delta_{ij}\Psi_{ij}$. We have

$$\begin{cases}\sum_{j=1}^{n}\kappa_{ij}\,{}_{\hbar}^{C}\Delta_{t_0}^{\alpha}\Phi_{ij}=\frac{d_1}{k^2}\xi_i\sum_{j=1}^{n}\kappa_{ij}\Phi_{ij}+a-\sum_{j=1}^{n}\kappa_{ij}\Phi_{ij}-\frac{4\sum_{j=1}^{n}\kappa_{ij}\Phi_{ij}\sum_{j=1}^{n}\delta_{ij}\Psi_{ij}}{1+(\sum_{j=1}^{n}\kappa_{ij}\Phi_{ij})^2},\\ \sum_{j=1}^{n}\delta_{ij}\,{}_{\hbar}^{C}\Delta_{t_0}^{\alpha}\Psi_{ij}=\frac{d_2}{k^2}\xi_i\sum_{j=1}^{n}\delta_{ij}\Psi_{ij}+\sigma b\left(\sum_{j=1}^{n}\kappa_{ij}\Phi_{ij}-\frac{\sum_{j=1}^{n}\kappa_{ij}\Phi_{ij}\sum_{j=1}^{n}\delta_{ij}\Psi_{ij}}{1+(\sum_{j=1}^{n}\kappa_{ij}\Phi_{ij})^2}\right).\end{cases} \tag{33}$$

We obtain

$$\begin{pmatrix}\frac{d_1}{k^2}\Lambda_i-\mu(\Lambda_i)+\frac{3\left(\frac{a}{5}\right)^2-5}{1+\left(\frac{a}{5}\right)^2} & \frac{-4\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2}\\ \sigma b\frac{2\left(\frac{a}{5}\right)^2}{1+\left(\frac{a}{5}\right)^2} & \frac{d_2}{k^2}\xi_i-\mu(\Lambda_i)-\sigma b\frac{\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2}\end{pmatrix}=\mathrm{J_i}-\mu(\Lambda_i)I, \tag{34}$$

which has the eigenvalue equation

$$\mu^2(\Lambda_i)-\mathrm{tr}(\mathrm{J_i})\mu(\Lambda_i)+\mathrm{det}(\mathrm{J_i})=0, \tag{35}$$

where

$$\mathrm{tr}(\mathrm{J_i})=-\left(\frac{d_1}{k^2}+\frac{d_2}{k^2}\right)\Lambda_i+\frac{3\left(\frac{a}{5}\right)^2-5-b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2} \tag{36}$$

and

$$\mathrm{det}(\mathrm{J_i})=\left(\frac{d_1}{k^2}\Lambda_i-\frac{3\left(\frac{a}{5}\right)^2-5}{1+\left(\frac{a}{5}\right)^2}\right)\frac{d_1}{k^2}\Lambda_i+\frac{b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2}\left(\frac{d_1}{k^2}\Lambda_i+5\right). \tag{37}$$

Its discriminant is

$$
\begin{aligned}
\Delta_i &= \mathrm{tr}^2(\mathrm{J_i}) - 4\mathrm{dert}(\mathrm{J_i}) \\
&= \left(\left(\frac{d_1}{k^2}+\frac{d_2}{k^2}\right)\Lambda_i + \frac{3\left(\frac{a}{5}\right)^2-5-b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2}\right)^2 - \left(\left(\frac{d_1}{k^2}\xi_i - \frac{3\left(\frac{a}{5}\right)^2-5}{1+\left(\frac{a}{5}\right)^2}\right)\frac{d_2}{k^2}\Lambda_i + \frac{b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2}\right), \\
&= \left(\frac{d_1}{k^2}-\frac{d_2}{k^2}\right)\Lambda_i^2 + +2\left(\frac{d_1}{k^2}-\frac{d_2}{k^2}\right)\left(\sigma b\frac{\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2} - \frac{3\left(\frac{a}{5}\right)^2-5}{1+\left(\frac{a}{5}\right)^2}\right)\Lambda_i + \left(\frac{3\left(\frac{a}{5}\right)^2-5-b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2}\right)^2 - 4\frac{b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2}, \\
&= \left(\frac{d_1}{k^2}-\frac{d_2}{k^2}\right)\Lambda_i^2 + 2\left(\frac{d_1}{k^2}-\frac{d_2}{k^2}\right)\left(\sigma b\frac{\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2} - \frac{3\left(\frac{a}{5}\right)^2-5}{1+\left(\frac{a}{5}\right)^2}\right)\Lambda_i + \Delta_\Lambda.
\end{aligned}
$$

The sign of Δ_i is important to the stability of $(u*, v*)$. The discriminant of Δ_i in relation to Λ_i is

$$
\begin{aligned}
\Delta_{\Lambda_i} &= 4\left(\frac{d_1}{k^2}-\frac{d_2}{k^2}\right)^2\left(\sigma b\frac{\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2} - \frac{3\left(\frac{a}{5}\right)^2-5}{1+\left(\frac{a}{5}\right)^2}\right)^2 - 4\left(\frac{3\left(\frac{a}{5}\right)^2-5-b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2}\right)^2 + 16\frac{b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2}, \\
&= 32\left(\frac{d_1}{k^2}-\frac{d_2}{k^2}\right)^2\frac{b\sigma\left(\frac{a}{5}\right)^3}{\left(1+\left(\frac{a}{5}\right)^2\right)^2}.
\end{aligned}
$$

Clearly $\Delta_{\Lambda_i} > 0$. Since $d_1 \neq d_2$, we distinguish two cases:

- If $d_1 < d_2$, then, $\left(\frac{3\left(\frac{a}{5}\right)^2-5-b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2}\right)^2 - 4\frac{b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2} > 0$. The two solutions of the equation $\Delta_{\Lambda_i} = 0$ are both negative. Thus, $\Delta_{\Lambda_i} > 0$ and the roots of (35) are

$$
\begin{cases}
\mu_1(\Lambda_i) = \dfrac{\mathrm{tr}(\mathrm{J_i}) + \sqrt{\mathrm{tr}(\mathrm{J_i})^2 - 4\det(\mathrm{J_i})}}{2}, \\
\mu_2(\Lambda_i) = \dfrac{\mathrm{tr}(\mathrm{J_i}) - \sqrt{\mathrm{tr}(\mathrm{J_i})^2 - 4\det(\mathrm{J_i})}}{2}.
\end{cases} \tag{38}
$$

  Note that the solutions are real and $\mu(\Lambda_i)_1 < 0$. In addition, if $\Lambda_1\frac{d_1}{k^2} \geq \frac{3\left(\frac{a}{5}\right)^2-5}{1+\left(\frac{a}{5}\right)^2}$, then $\mu(\Lambda_i)_2 < 0$. This leads to

$$
|\mathrm{Arg}(\mu_1(\Lambda_i))| = |\mathrm{Arg}(\mu_2(\Lambda_i)_2)| = \pi, \tag{39}
$$

  which ensures $(u*, v*)$ is asymptotically stable.
- If $d_1 > d_2$, we have $\left(\frac{3\left(\frac{a}{5}\right)^2-5-b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2}\right)^2 - 4\frac{b\sigma\frac{a}{5}}{1+\left(\frac{a}{5}\right)^2} > 0$. This returns us to the previous scenario, again, for $\frac{d_1}{k^2}\Lambda_1 - \frac{3\left(\frac{a}{5}\right)^2-5}{1+\left(\frac{a}{5}\right)^2} \geq 0$, $\det(\mathrm{J_i}) > 0$. Hence, λ_1 and λ_2 are negative and must meet the conditions of Theorem 2.

□

5. Global Stability

In this part, we demonstrate the global asymptotic stability of the constant steady-state solution.

Theorem 6. *If*

$$
0 < a \leq 27, \tag{40}
$$

then, system (14) is globally asymptotically stable.

Proof. The function stated in Theorem 3 is applied to prove the global asymptotic stability of the unique equilibrium point $(u*, v*)$. For this, we evaluate the following function.

$$L(t) = \sum_{i=1}^{m} \frac{\sigma b}{3} U_i^3(t) + \sigma b u^* U_i^2(t) + 2V_i^2(t).$$

We consider the change of variable $U_i = u_- u^*$ and $V_i = v_i - v^*$. Using Lemmas 1 and 2, system (14) can be expressed as follows

$$\begin{cases} {}_{\hbar}^{C}\Delta_{t_0}^{\alpha} U_i(t) = \dfrac{d_1}{k^2}\Delta^2 U_{i-1}(t+\hbar\alpha) + a - (U_i(t+\hbar\alpha)+u^*) - \dfrac{4(U_i(t+\hbar\alpha)+u^*)(V_i(t+\hbar\alpha)+v^*)}{1+(U_i(t+\hbar\alpha)+u^*)^2}, \\ {}_{\hbar}^{C}\Delta_{t_0}^{\alpha} V_i(t) = \dfrac{d_2}{k^2}\Delta^2 V_{i-1}(t+\hbar\alpha) + \sigma b\Big((U_i(t+\hbar\alpha)+u^*) - \dfrac{(U_i(t+\hbar\alpha)+u^*)(V_i(t+\hbar\alpha)+v^*)}{1+(U_i(t+\hbar\alpha)+u^*)^2}\Big). \end{cases} \tag{41}$$

Taking the Caputo $\hbar$-difference operator and using Lemma 3, we have

$$\begin{aligned}
{}_{\hbar}^{C}\Delta_{t_0}^{\alpha} L(t) &= \sum_{i=1}^{m} \frac{\sigma b}{3}\, {}_{\hbar}^{C}\Delta_{t_0}^{\alpha} U_i^3(t) + \sigma b u^*\, {}_{\hbar}^{C}\Delta_{t_0}^{\alpha} U_i^2(t) + 2\, {}_{\hbar}^{C}\Delta_{t_0}^{\alpha} V_i^2(t), \\
&\leq \sum_{i=1}^{m} \sigma b U_i^2(t+\hbar\alpha)\, {}_{\hbar}^{C}\Delta_{t_0}^{\alpha} U_i(t) + 2\sigma b u^* U_i(t+\hbar\alpha)\, {}_{\hbar}^{C}\Delta_{t_0}^{\alpha} U_i(t) + 4V_i(t+\hbar\alpha)\, {}_{\hbar}^{C}\Delta_{t_0}^{\alpha} V_i(t), \\
&= \sum_{i=1}^{m} \sigma b(U_i(t+\hbar\alpha)+2u^*)U_i(t+\hbar\alpha)\, {}_{\hbar}^{C}\Delta_{t_0}^{\alpha} U_i(t) + 4V_i(t+\hbar\alpha)\, {}_{\hbar}^{C}\Delta_{t_0}^{\alpha} V_i(t), \\
&= \sum_{i=1}^{m} \sigma b(U_i(t+\hbar\alpha)+2u^*)U_i(t+\hbar\alpha)\Big(\frac{d_1}{k^2}\Delta^2 U_{i-1}(t+\hbar\alpha) + a - (U_i(t+\hbar\alpha)+u^*) \\
&\quad - \frac{4(U_i(t+\hbar\alpha)+u^*)(V_i(t+\hbar\alpha)+v^*)}{1+(U_i(t+\hbar\alpha)+u^*)^2}\Big) + 4V_i(t+\hbar\alpha)\Big(\frac{d_2}{k^2}\Delta^2 V_{i-1}(t+\hbar\alpha) \\
&\quad + \sigma b\Big((U_i(t+\hbar\alpha)+u^*) - \frac{(U_i(t+\hbar\alpha)+u^*)(V_i(t+\hbar\alpha)+v^*)}{1+(U_i(t+\hbar\alpha)+u^*)^2}\Big)\Big), \\
&= \sum_{i=1}^{m} \frac{\sigma b d_1}{k^2}(U_i(t+\hbar\alpha)+2u^*)U_i(t+\hbar\alpha)\Delta^2 U_{i-1}(t+\hbar\alpha) + \frac{4d_2}{k^2}\sum_{i=1}^{m} V_i(t+\hbar\alpha)\Delta^2 V_{i-1}(t+\hbar\alpha) \\
&\quad + \sum_{i=1}^{m} \sigma b(U_i(t+\hbar\alpha)+2u^*)U_i(t+\hbar\alpha)\Big(a - (U_i(t+\hbar\alpha)+u^*) - \frac{4(U_i(t+\hbar\alpha)+u^*)(V_i(t+\hbar\alpha)+v^*)}{1+(U_i(t+\hbar\alpha)+u^*)^2}\Big), \\
&\quad + 4\sigma b V_i(t+\hbar\alpha)\Big((U_i(t+\hbar\alpha)+u^*) - \frac{(U_i(t+\hbar\alpha)+u^*)(V_i(t+\hbar\alpha)+v^*)}{1+(U_i(t+\hbar\alpha)+u^*)^2}\Big)\Big), \\
&= J_1(t) + J_2(t).
\end{aligned}$$

where

$$\begin{aligned}
J_1(t) &= \sum_{i=1}^{m} \frac{\sigma b d_1}{k^2}(U_i(t+\hbar\alpha)+2u^*)U_i(t+\hbar\alpha)\Delta^2 U_{i-1}(t+\hbar\alpha) + \frac{4d_2}{k^2}\sum_{i=1}^{m} V_i(t+\hbar\alpha)\Delta^2 V_{i-1}(t+\hbar\alpha), \\
J_2(t) &= \sum_{i=1}^{m} \sigma b(U_i(t+\hbar\alpha)+2u^*)U_i(t+\hbar\alpha)\Big(a - (U_i(t+\hbar\alpha)+u^*) - \frac{4(U_i(t+\hbar\alpha)+u^*)(V_i(t+\hbar\alpha)+v^*)}{1+(U_i(t+\hbar\alpha)+u^*)^2}\Big) \\
&\qquad + 4\sigma b V_i(t+\hbar\alpha)\Big((U_i(t+\hbar\alpha)+u^*) - \frac{(U_i(t+\hbar\alpha)+u^*)(V_i(t+\hbar\alpha)+v^*)}{1+(U_i(t+\hbar\alpha)+u^*)^2}\Big)\Big).
\end{aligned}$$

Using summation by parts and taking into account the periodic boundary conditions as well as Theorem 1, we conclude, for $J_1(t)$,

$$
\begin{aligned}
J_1(t) &= \frac{\sigma b d_1}{k^2}\sum_{i=1}^{m}(U_i(t+\hbar\alpha)+2u^*)U_i(t+\hbar\alpha)\Delta^2 U_{i-1}(t+\hbar\alpha)+\frac{4d_2}{k^2}\sum_{i=1}^{m}V_i(t+\hbar\alpha)\Delta^2 V_{i-1}(t+\hbar\alpha),\\
&= \frac{\sigma b d_1}{k^2}\sum_{i=1}^{m}(U_i(t+\hbar\alpha)+2u^*)U_i(t+\hbar\alpha)\Delta(\Delta U_{i-1}(t+\hbar\alpha))+\frac{4d_2}{k^2}\sum_{i=1}^{m}V_i(t+\hbar\alpha)\Delta(\Delta V_{i-1}(t+\hbar\alpha)),\\
&= \frac{\sigma b d_1}{k^2}(U_{i-1}(t+\hbar\alpha)+2u^*)U_{i-1}(t+\hbar\alpha)\Delta U_{i-1}(t+\hbar\alpha)|_1^{m+1}\\
&\quad -\frac{\sigma b d_1}{k^2}\sum_{i=1}^{m}\Delta((U_{i-1}(t+\hbar\alpha)+2u^*)U_{i-1}(t+\hbar\alpha))\Delta U_{i-1}(t+\hbar\alpha)\\
&\quad +\frac{4d_2}{k^2}V_{i-1}(t+\hbar\alpha)\Delta V_{i-1}(t+\hbar\alpha)|_1^{m+1}-\frac{4d_2}{k^2}\sum_{i=1}^{m}(\Delta V_{i-1}(t+\hbar\alpha))^2,\\
&= -\frac{2\sigma b d_1}{k^2}u^*\sum_{i=1}^{m}(\Delta U_{i-1}(t+\hbar\alpha))^2-\frac{\sigma b d_1}{k^2}\sum_{i=1}^{m}\Delta U_{i-1}^2(t+\hbar\alpha)\Delta U_{i-1}(t+\hbar\alpha)-\frac{4d_2}{k^2}\sum_{i=1}^{m}(\Delta V_{i-1}(t+\hbar\alpha))^2,\\
&= -\frac{2\sigma b d_1}{k^2}u^*\sum_{i=1}^{m}(\Delta U_{i-1}(t+\hbar\alpha))^2-\frac{4d_2}{k^2}\sum_{i=1}^{m}(\Delta V_{i-1}(t+\hbar\alpha))^2\\
&\quad -\frac{4d_2}{k^2}\sum_{i=1}^{m}(U_i(t+\hbar\alpha)\Delta U_{i-1}(t+\hbar\alpha)+\Delta U_{i-1}(t+\hbar\alpha)U_{i-1}(t+\hbar\alpha))\Delta U_{i-1}(t+\hbar\alpha),\\
&= -\left(\frac{2\sigma b d_1}{k^2}u^*\sum_{i=1}^{m}(1+U_i(t+\hbar\alpha)+U_{i-1}(t+\hbar\alpha))(\Delta U_{i-1}(t+\hbar\alpha))^2+\frac{4d_2}{k^2}\sum_{i=1}^{m}(\Delta V_{i-1}(t+\hbar\alpha))^2\right)<0.
\end{aligned}
$$

Now, evaluating $J_2(t)$

$$
\begin{aligned}
J_2(t) &= \sum_{i=1}^{m}4\sigma b\frac{(U_i(t+\hbar\alpha)+u^*)}{1+(U_i(t+\hbar\alpha)+u^*)^2}[(U_i(t+\hbar\alpha)+2u^*)U_i(t+\hbar\alpha)\times\\
&\quad \left((a-(U_i(t+\hbar\alpha)+u^*))\frac{1+(U_i(t+\hbar\alpha)+u^*)^2}{U_i(t+\hbar\alpha)+u^*}-(V_i(t+\hbar\alpha)+v^*)\right)\\
&\quad +V_i(t+\hbar\alpha)\Big(1+(U_i(t+\hbar\alpha)+u^*)^2-(V_i(t+\hbar\alpha)+v^*)\Big)],\\
&= \sum_{i=1}^{m}4\sigma b\frac{(U_i(t+\hbar\alpha)+u^*)}{1+(U_i(t+\hbar\alpha)+u^*)^2}[(U_i(t+\hbar\alpha)+2u^*)U_i(t+\hbar\alpha)\times\\
&\quad \left((a-(U_i(t+\hbar\alpha)+u^*))\frac{1+(U_i(t+\hbar\alpha)+u^*)^2}{U_i(t+\hbar\alpha)+u^*}-\left(a-u^*\frac{1+(u^*)^2}{u^*}\right)\right)+V_i(t+\hbar\alpha)(v^*-(V_i(t+\hbar\alpha)+v^*))],\\
&= \sum_{i=1}^{m}4\sigma b\frac{(U_i(t+\hbar\alpha)+u^*)}{1+(U_i(t+\hbar\alpha)+u^*)^2}[(U_i(t+\hbar\alpha)+2u^*)U_i(t+\hbar\alpha)(g(U_i(t+\hbar\alpha)+u^*)-g(u^*))-V_i^2(t+\hbar\alpha)],
\end{aligned}
$$

where

$$
g(U_i(t+\hbar\alpha))=(a-U_i(t+\hbar\alpha))\frac{1+U_i^2(t+\hbar\alpha)}{U_i(t+\hbar\alpha)}.
$$

As mentioned in [28], g is a strictly decreasing function if $0<a\leq 27$, which means that $g(U_i(t+\hbar\alpha)+u^*)-g(u^*)<0$. Therefore, $J_2(t)\leq 0$

Hence,

$$
{}_{\hbar}^{C}\Delta_{t_0}^{\alpha}L(t)\leq 0,
$$

and ${}_{\hbar}^{C}\Delta_{t_0}^{\alpha}L(t)=0$ if and only if $(U_i,V_i)=(0,0)$. As a result, Theorem 3 indicates that under the condition (40), the constant steady-state $(u*,v*)$ is globally asymptotically stable. □

Remark 1. *We observe that the results reported in Theorem 6 are independent of the parameters b, σ, and α. Remember that a is the iodide's feeding rate. When the feeding rate is exceedingly low, the chemical reaction will stabilize at the unique constant equilibrium. When the feeding rate a rises, the system transforms into an activator–inhibitor system.*

6. Numerical Simulations

In this section, we display some illustrative simulations of the theoretical characteristics of the stability of the discrete-time fractional Lengyel–Epstein reaction–diffusion system to show the effect of α on the dynamics of the fractional Lengyel-–Epstein system (14). By adjusting the system's characteristics and order, we may observe its behavior. We use the following numerical solution, and the system (14) appears as such:

$$\begin{cases} u_i(n\hbar) = \phi_1(x_i) + \dfrac{\hbar^\alpha}{\Gamma(\alpha)} \sum_{j=1}^{n} \dfrac{\Gamma(n-j+\alpha)}{\Gamma(n-j+1)} [d_1 \dfrac{u_{i+1}((j-1)\hbar) - 2u_i((j-1)\hbar) + u_{i-1}((j-1)\hbar)}{k^2} \\ \qquad\qquad +a - u_i((j-1)\hbar) - \dfrac{4u_i((j-1)\hbar)v_i((j-1)\hbar)}{1+(u_i((j-1)\hbar))^2}], \\ v_i(n\hbar) = \phi_2(x_i) + \dfrac{\hbar^\alpha}{\Gamma(\alpha)} \sum_{j=1}^{n} \dfrac{\Gamma(n-j+\alpha)}{\Gamma(n-j+1)} [d_2 \dfrac{v_{i+1}((j-1)\hbar) - 2v_i((j-1)\hbar) + v_{i-1}((j-1)\hbar)}{k^2} \\ \qquad\qquad +\sigma b \left(u_i((j-1)\hbar) - \dfrac{u_i((j-1)\hbar)v_i((j-1)\hbar)}{1+(u_i((j-1)\hbar))^2} \right)], \\ 1 \le i \le m, \\ n > 0. \end{cases} \tag{42}$$

Example 1. *To demonstrate our point, consider the following parameter values: $(a, b, \sigma, d_1, d_2) = (15, 1, 7, 1, 10), N = 150, \hbar = 0.07, t \in [0, 10]$, $x \in [0, 20]$ and the boundary conditions $(u_0(t), v_0(t)) = (5, 10), (u_1(t), v_1(t)) = (5, 10)$. There is thus just one unique positive equilibrium $(u^*, v^*) = (3, 10)$. The conditions in Theorem 6 are clearly verified. Figures 1 and 2 depict the dynamic behavior of the system (14) with the appropriate initial conditions.*

$$\begin{cases} \phi_1(x_i) = 5 + 0.3 \sin \dfrac{x_i}{2}, \\ \phi_2(x_i) = 5 + 0.6 \sin \dfrac{x_i}{2}. \end{cases} \tag{43}$$

The simulations shown in Figures 1 and 2 show that the positive equilibrium is asymptotically stable.

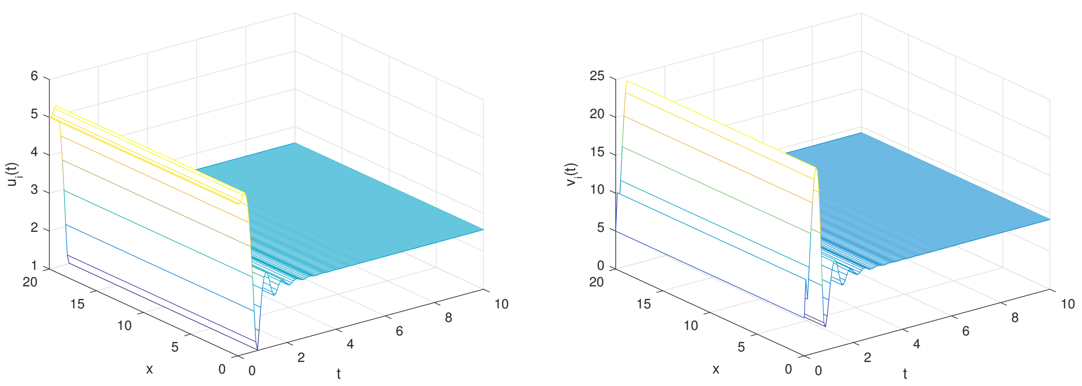

Figure 1. One dimensional concentration $u_i(t)$ and $v_i(t)$ as a solution of (14) with $(a, b, \sigma, d_1, d_2) = (15, 4, 7, 1, 10)$, $N = 100$, initial conditions (43), zero periodic boundary conditions, and fractional-order $\alpha = 0.025$.

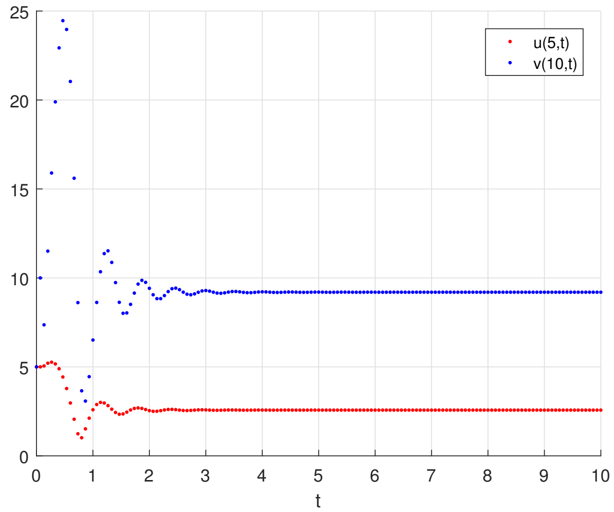

Figure 2. Dynamic behaviors of $u_i(t)$ and $v_i(t)$ for $x = 5$ and $x = 10$, $N = 100$, $(a, b, \sigma, d_1, d_2) = (15, 4, 7, 1, 10)$, initial conditions (43), zero periodic boundary conditions, and fractional-order $\alpha = 0.025$.

Example 2. *Consider the following parameter values of model (14): $N = 150$, $(a, b, \sigma, d_1, d_2) = (20, 9, 7, 7, 8)$ $\hbar = 0.2$, $t \in [0, 30]$, $x \in [0, 25]$, $\alpha = 0.26$ and the boundary conditions $(u_0(t), v_0(t)) = (5, 8)$, $(u_1(t), v_1(t)) = (5, 8)$. Figures 3 and 4 show the dynamics of u_i and v_i for the initial condition.*

$$\begin{cases} \phi_1(x_i) = 6 + \cos \dfrac{x_i}{2}, \\ \phi_2(x_i) = 5 + \cos \dfrac{x_i}{3}. \end{cases} \tag{44}$$

We see that all of our model's solutions converge at some point to the equilibrium point $(u^*, v^*) = (4, 17)$. The unique equilibrium is thus asymptotically stable. This numerical conclusion is consistent with our earlier theoretical results. Figures 3 and 4 display the results mentioned earlier.

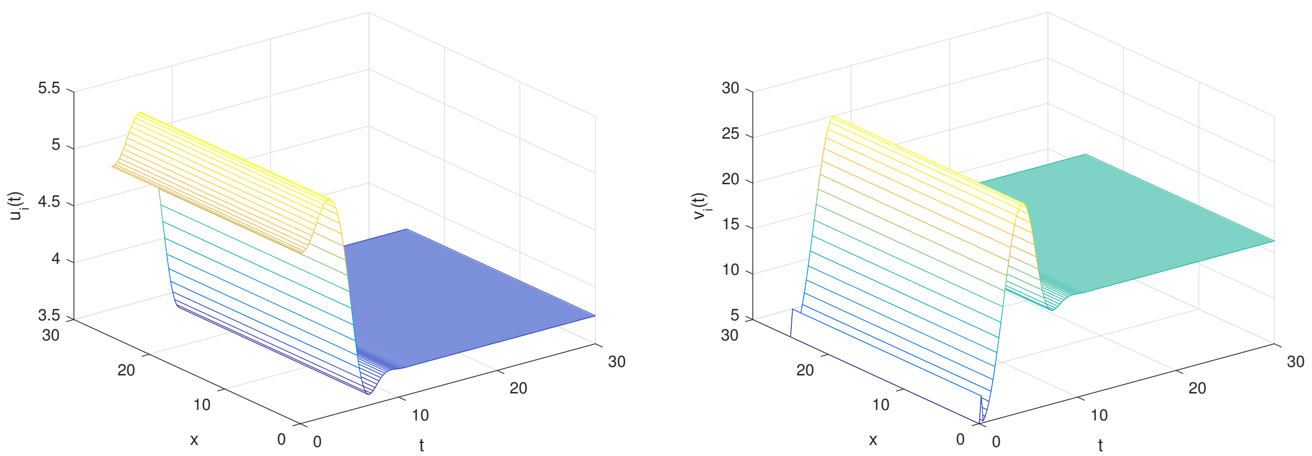

Figure 3. One dimensional concentration $u_i(t)$ and $v_i(t)$ as a solution of (14) with $(a, b, \sigma, d_1, d_2) = (20, 9, 7, 7, 8)$, $N = 150$, initial conditions (44), zero periodic boundary conditions, and fractional-order $\alpha = 0.26$.

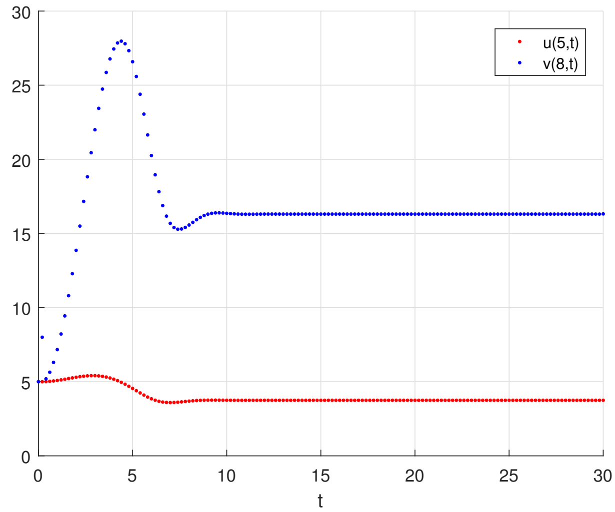

Figure 4. Dynamic behaviors of $u_i(t)$ and $v_i(t)$ for $x = 5$ and $x = 8$, $N = 150$, $(a, b, \sigma, d_1, d_2) = (20, 9, 7, 7, 8)$, initial conditions (44), zero periodic boundary conditions, and fractional-order $\alpha = 0.26$.

Two sets of parameters from both examples were implemented wirh the help of the Matlab environment, and the fractional discrete reaction–diffusion system (14) was numerically solved using the numerial solution (42). Table 1 shows the two sets of parameters and initial conditions.

Table 1. Simulation parameters for the discrete fractional-order Lengyel–Epstein model of the CIMA reaction represented in (14).

a	b	σ	d_1	d_2	α	$\phi_1(x_i)$	$\phi_2(x_i)$	(u^*, v^*)	Case
15	4	7	1	10	0.025	$5 + 0.3 \sin \frac{x_i}{2}$	$5 + 0.6 \sin \frac{x_i}{2}$	$(3, 10)$	stable
20	9	7	7	8	0.26	$6 + \cos \frac{x_i}{2}$	$\phi_2(x_i) = 5 + \cos \frac{x_i}{3}$	$(4, 17)$	stable

Remark 2. *We noticed that when the fractional-order was below one, a periodic solution in the standard case became asymptotically stable. This is a significant finding that deserves more examination and analysis since it gives a new viewpoint on the control and uses of the Lengyel–Epstein system. We have also shown that the existence of diffusion changes the system's stability criteria, which is quite similar to the continuous case. Furthermore, we discovered that the diffusion-driven stability varies with the fractional-order. Future investigations will closer discuss these findings.*

7. Conclusions

In this paper, we investigated the local and global asymptotic stability of a novel discrete-time fractional-order version of the Lengyel–Epstein system that models the chlorite-iodide-malonic acid (CIMA) chemical reaction. Using the specific forward difference operator and an L1 finite difference scheme, we introduce a fractional discrete version of the well-known Lengyel–Epstein reaction–diffusion system. Then, we provided sufficient constraints for the unique equilibrium's local asymptotic stability. Furthermore, the steady-state solution's global asymptotic stability was proven using the direct Lyapunov technique. Finally, the numerical simulations demonstrate all of the theoretical research' results. Indeed, the graphs illustrate that the recommended model's dynamics are compatible with the performance of continuous version of the system.

The results show that the linearization technique and the Lyapunov functional may be used to address the problem of stability in discrete fractional-order reaction–diffusion systems. Furthermore, the study conclusions established in this work may be directly applied to many different types of discrete fractional spatiotemporal systems with reaction–diffusion terms.

Author Contributions: Formal analysis, OA.A. and A.O.; Investigation, A.H.; Software, A.H.; Supervision, A.O.; Validation, G.G.; Visualization, O.A.A.; Writing—original draft, G.G. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Bailey, J.E. Mathematical modeling and analysis in biochemical engineering: Past accomplishments and future opportunities. *Biotechnol. Prog.* **1998**, *14*, 8–20. [CrossRef]
2. Zaheri, S.; Hassanipour, F. A comprehensive approach to the mathematical modeling of mass transport in biological systems: Fundamental concepts and models. *Int. J. Heat Mass Transf.* **2020**, *158*, 119777. [CrossRef]
3. Azizi, T. *Mathematical Modeling with Applications in Biological Systems, Physiology, and Neuroscience*; Kansas State University: Manhattan, KS, USA, 2021.
4. Siettos, C.I.; Russo, L. Mathematical modeling of infectious disease dynamics. *Virulence* **2013**, *4*, 295–306. [CrossRef] [PubMed]
5. Brauer, F.; Castillo-Chavez, C.; Castillo-Chavez, C. *Mathematical Models in Population Biology and Epidemiology*; Springer: New York, NY, USA, 2012; Volume 2, No. 40.
6. Yang, C.; Wang, J. A mathematical model for the novel coronavirus epidemic in Wuhan, China. *Math. Biosci. Eng. MBE* **2020**, *17*, 2708. [CrossRef] [PubMed]
7. Frunzo, L.; Garra, R.; Giusti, A.; Luongo, V. Modeling biological systems with an improved fractional Gompertz law. *Commun. Nonlinear Sci. Numer. Simul.* **2019**, *74*, 260–267. [CrossRef]
8. Das, P.; Upadhyay, R.K.; Das, P.; Ghosh, D. Exploring dynamical complexity in a time-delayed tumor-immune model. *Chaos Interdiscip. J. Nonlinear Sci.* **2020**, *30*, 123118. [CrossRef]
9. Escudero, C. The fractional keller–segel model. *Nonlinearity* **2006**, *19*, 2909. [CrossRef]
10. Das, P.; Mukherjee, S.; Das, P.; Banerjee, S. Characterizing chaos and multifractality in noise-assisted tumor-immune interplay. *Nonlinear Dyn.* **2020**, *101*, 675–685. [CrossRef]
11. Das, P.; Das, P.; Mukherjee, S. Stochastic dynamics of Michaelis–Menten kinetics based tumor-immune interactions. *Phys. Stat. Mech. Its Appl.* **2020**, *541*, 123603. [CrossRef]
12. Das, S.; Das, P.; Das, P. Dynamics and control of multidrug-resistant bacterial infection in hospital with multiple delays. *Commun. Nonlinear Sci. Numer. Simul.* **2020**, *89*, 105279. [CrossRef]
13. Henry, I.; Wearne, S.L. Fractional reaction–diffusion. *Physics A* **2000**, *276*, 448–455. [CrossRef]
14. Henry, B.I.; Wearne, S.L. Existence of Turing instabilities in a two-species fractional reaction–diffusion system. *SIAM J. Appl. Math.* **2002**, *62*, 870–887. [CrossRef]
15. Gafiychuk, V.V.; Datsko, B.Y.; Izmajlova, Y.Y. Analysis of the dissipative structures in reaction–diffusion systems with fractional derivatives. *Math. Methods Phys. Mech. Fields* **2006**, *49*, 109–116.
16. Seki, K.; Wojcik, M.; Tachiya, M. Fractional reaction–diffusion equation, *J. Chem. Phys.* **2003**, *119*, 2165. [CrossRef]
17. West, B.J. *Fractional Calculus View of Complexity: Tomorrow's Science*; CRC Press: Boca Raton, FL, USA, 2016.
18. Baleanu, D.; Mustafa, O.G. *Asymptotic Integration and Stability: For Ordinary, Functional and Discrete Differential Equations of Fractional Order*; World Scientific: Singapore, 2015; Volume 4.
19. Wu, G.C.; Baleanu, D.; Zeng, S.D.; Deng, Z.G. Discrete fractional diffusion equation. *Nonlinear Dyn.* **2015**, *80*, 281–286. [CrossRef]
20. Wu, G.C.; Baleanu, D.; Xie, H.P.; Zeng, S.D. Discrete fractional diffusion equation of chaotic order. *Int. J. Bifurc. Chaos* **2016**, *26*, 1650013. [CrossRef]
21. Lengyel, I.; Epstein, I.R. A chemical approach to designing Turing patterns in reaction-diffusion systems. *Proc. Natl. Acad. Sci. USA* **1992**, *89*, 3977–3979. [CrossRef]
22. Lengyel, I.; Epstein, I.R. Modeling of Turing structures in the chlorite–iodide–malonic acid–starch reaction system. *Science* **1991**, *251*, 650–652. [CrossRef]
23. Kepper, P.D.; Epstein, I.R.; Kustin, K.; Orban, M. Systematic design of chemical oscillators. Part 8. Batch oscillations and spatial wave patterns in chlorite oscillating systems. *J. Phys. Chem.* **1982**, *86*, 170–171. [CrossRef]
24. Turing, A.M. The chemical basis of morphogenesis. *Bull. Math. Biol.* **1990**, *52*, 153–197. [CrossRef]
25. Lisena, B. On the global dynamics of the Lengyel–Epstein system. *Appl. Math. Comput.* **2014**, *249*, 67–75. [CrossRef]
26. Zheng, J. Time optimal controls of the Lengyel–Epstein model with internal control. *Appl. Math. Optim.* **2014**, *70*, 345–371. [CrossRef]
27. Zheng, J. Optimal control problem for Lengyel–Epstein model with obstacles and state constraints. *Nonlinear Anal. Model. Control* **2016**, *21*, 18–39. [CrossRef]
28. Yi, F.; Wei, J.; Shi, J. Global asymptotical behavior of the Lengyel–Epstein reaction–diffusion system. *Appl. Math. Lett.* **2009**, *22*, 52–55. [CrossRef]

29. Zafar, Z.U.A. RETRACTED ARTICLE: Fractional order Lengyel–Epstein chemical reaction model. *Comput. Appl. Math.* **2019**, *38*, 1–16. [CrossRef]
30. Ouannas, A.; Wang, X.; Pham, V.T.; Grassi, G.; Huynh, V. Synchronization results for a class of fractional-order spatiotemporal partial differential systems based on fractional Lyapunov approach. *Bound. Value Probl.* **2019**, *2019*, 74. [CrossRef]
31. Mesdoui, F.; Shawagfeh, N.; Ouannas, A. Global synchronization of fractional-order and integer-order N component reaction diffusion systems: Application to biochemical models. *Math. Methods Appl. Sci.* **2021**, *44*, 1003–1012. [CrossRef]
32. Daşbaşı, B.; Daşbaşı, T. Mathematical analysis of Lengyel-Epstein chemical reaction model by fractional-order differential equation's system with multi-orders. *Int. J. Sci. Eng. Investig.* **2017**, *6*, 78–83.
33. Xu, L.; Zhao, L.J.; Chang, Z.X.; Feng, J.T.; Zhang, G. Turing instability and pattern formation in a semi-discrete Brusselator model. *Mod. Phys. Lett.* **2013**, *27*, 1350006. [CrossRef]
34. Kelley, W.G.; Peterson, A.C. *Difference Equations: An Introduction with Applications*; Academic Press: Cambridge, MA, USA, 2001.
35. Baleanu, D.; Wu, G.C.; Bai, Y.R.; Chen, F.L. Stability analysis of Caputo–like discrete fractional systems. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *48*, 520–530. [CrossRef]
36. Abdeljawad, T. On Riemann and Caputo fractional differences. *Comput. Math. Appl.* **2011**, *62*, 1602–1611. [CrossRef]
37. Abdeljawad, T. Different type kernel h-fractional differences and their fractional h-sums. *Chaos Solitons Fractals* **2018**, *116*, 146–156. [CrossRef]
38. Čermák, J.; Nechvátal, L. On a problem of linearized stability for fractional difference equations. *Nonlinear Dyn.* **2021**, *104*, 1253–1267. [CrossRef]