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Generalized Universality for Compositions of the Riemann Zeta-Function in Short Intervals

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Abstract: In the paper, the approximation of analytic functions on compact sets of the strip $\{s = \sigma + it \in \mathbb{C} \mid 1/2 < \sigma < 1\}$ by shifts $F(\zeta(s + iu_1(\tau)), \dots, \zeta(s + iu_r(\tau)))$, where $\zeta(s)$ is the Riemann zeta-function, $u_1, \dots, u_r$ are certain differentiable increasing functions, and F is a certain continuous operator in the space of analytic functions, is considered. It is obtained that the set of the above shifts in the interval $[T, T+H]$ with $H = o(T)$, $T \to \infty$, has a positive lower density. Additionally, the positivity of a density with a certain exceptional condition is discussed. Examples of considered operators F are given.

Keywords: Riemann zeta-function; space of analytic functions; joint universality; weak convergence of probability measures

MSC: 11M06; 11M99; 33C15

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1. Introduction

Let $s = \sigma + it$ be the complex variable with $\sigma, t \in \mathbb{R}$ and let $\mathbb{P}$ be the set of all the primes. The celebrated Riemann zeta-function $\zeta(s)$ is defined by

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - p^{-s}\right)^{-1} = \sum_{m=1}^{\infty} m^{-s}$$

for $\sigma > 1$, where both the Euler product and Dirichlet series are absolutely convergent, and hence analytic in that right half-plane. It is continued meromorphically over the whole complex plane with the unique simple pole at $s = 1$ with residue 1 by way of the functional equation and the meromorphic continuation in the critical strip $\{s \in \mathbb{C} \mid 0 < \sigma < 1\}$. It is known that $\zeta(s)$ has infinitely many zeros in the critical strip, called non-trivial zeros, which are essentially connected with the distribution of primes. The famous Riemann hypothesis states that all of the non-trivial zeros lie on the critical line $\sigma = \frac{1}{2}$, which gives the best bound for the error term for the prime number theorem. Since the introduction by Riemann, the function $\zeta(s)$ has been the main impetus for the development of analytic number theory in the area of distribution of primes. The situation has been drastically changed by S. M. Voronin's dicovery [1] of universality of $\zeta(s)$, i.e., attention has been drawn to function-theoretic properties. By symmetry (functional equation), it is enough to consider the right half of the critical strip $\Delta := \{s \in \mathbb{C} \mid \frac{1}{2} < \sigma < 1\}$. Then, the Voronin universality means that the zeta shifts $\zeta(s + i\tau)$ approximate all analytic non-vanishing functions defined in D. The universality of $\zeta(s)$ has been also discovered by A. A. Karatsuba and S. M. Voronin [2], and developed by S. M. Gonek [3], B. Bagchi [4], J. Steuding [5], K. Matsumoto [6], J.-L. Mauclaire [7], the first author [8,9], their students, and others. A wide survey of universality of zeta functions and its applications is given in [10].

To state the modern version of the Voronin universality theorem, we introduce notation, which will be used throughout. Let $\mathcal{K}$ be a family of compact subsets of Δ with connected complements, and let $\mathcal{A}(K)$ (with $K \in \mathcal{K}$) be the class of continuous functions on K that are analytic in the interior of K. Let $\mathcal{A}_0(K)$ denote the subspace of $\mathcal{A}(K)$ consisting of non-vanishing functions. Then, the modern version of the Voronin universality theorem says (see, for example, [5,8]) that for every $K \in \mathcal{K}$, $f(s) \in \mathcal{A}_0(K)$ and $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{mes}\left\{ \tau \in I \mid \sup_{s\in K} |f(s) - \zeta(s+i\tau)| < \varepsilon \right\} > 0, \tag{1}$$

where $I = I_T := [0, T]$ and $\mathrm{mes}\{A\}$ means the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Furthermore, the lower limit may be replaced by a limit for all but at most countably many $\varepsilon > 0$. This proviso is valid in the Theorems 1–2 and 4–6, and it will be omitted.

The impact of (1) was enhanced by B. Bagchi's result [11] that inequality (1) with $f(s) = \zeta(s)$ is equivalent to the Riemann hypothesis that all non-trivial zeros of $\zeta(s)$ $\big($zeros in the strip $\{s \in \mathbb{C} \mid 0 < \sigma < 1\}\big)$ lie on the critical line $\sigma = \frac{1}{2}$.

From (1) arose an enormous number of new problems. The shifts $\zeta(s + i\tau)$ in (1) can be replaced by more general shifts $\zeta(s + iu(\tau))$ with a certain function $u(\tau)$. In [12], the function $u(\tau) = \tau^\alpha(\log \tau)^\beta$, $\alpha, \beta \in \mathbb{R}$, was considered; in [13], a more general differentiable function $u(\tau)$ was used. Using generalized shifts also allows one to investigate a simultaneous approximation of several analytic functions $(f_1(s), \dots, f_r(s))$, say, by r-tuple of zeta-shifts $\big(\zeta(s + iu_1(\tau)), \dots, \zeta(s + iu_r(\tau))\big)$. For example, in [14], the joint approximation by shifts $\big(\zeta(s + ia_1\tau), \dots, \zeta(s + ia_r\tau)\big)$, where $a_1, \dots, a_r$ are real algebraic numbers linearly independent over the field of rational numbers, was obtained. Moreover, inequality (1) means that there are infinitely many shifts $\zeta(s + i\tau)$ approximating a given analytic function $f(s)$ with accuracy ε. Although, such a theorem claims the existence of shifts approximating a given analytic function, it does not give any concrete approximating shift, which is inevitable due to the metrical nature of the assertion. As a rephrase, the effectivity of universality theorems is interpreted as the specification of the interval I_T containing τ with approximating property. The first attempt to solve this problem was made by A. Good [15]; R. Garunkštis applied and developed Good's ideas for the effective approximation of analytic functions in small discs [16]. The mentioned and other effective results connected to the universality of zeta-functions can be found in the survey paper [17]. In this regard, we note that (1) is implied by its short interval version, i.e., with I_T replaced by $[T, 2T]$. This is a standard notion in many aspects of analytic number theory.

From the point of view of effectivity, it is desirable to specify the shortest possible interval containing τ such that $\zeta(s + i\tau)$ approximates a given analytic function. Thus, we arrive at the notion of universality theorems in short intervals $I_{T,H} := [T, T + H]$ with $H = o(T)$ as $T \to \infty$. A joint universality theorem for a short interval with generalized shifts has been obtained in [18].

Denote by $\mathcal{U}_r$ the class of tuples of real differentiable functions $(u_1(\tau), \dots, u_r(\tau))$ satisfying the following hypotheses:

1° $u_1(\tau), \dots, u_r(\tau)$ are increasing functions on $[T_0, \infty]$, $T_0 > 0$, tending to $+\infty$;

2° $u_1(\tau), \dots, u_r(\tau)$ have continuous derivatives such that

$$u_j'(\tau) = \widehat{u}_j(\tau)(1 + o(1)), \quad \tau \to \infty,$$

where the functions $\widehat{u}_j(\tau)$ are monotonic, $j = 1, \dots, r$, and, as $\tau \to \infty$,

$$\widehat{u}_j(\tau) = o(\widehat{u}_{j+1}(\tau)),$$

successively, $j = 1, \dots, r - 1$;

3° the estimates

$$\begin{cases} \frac{\widehat{u}_j(2\tau)}{\widehat{u}_j(\tau)} = O(1) & \text{if} \quad \widehat{u}_j(\tau) \text{ is increasing,} \\ \frac{\widehat{u}_j(\tau)}{\widehat{u}_j(2\tau)} = O(1) & \text{if} \quad \widehat{u}_j(\tau) \text{ is decreasing,} \end{cases}$$

$j = 1, \dots, r$, are valid.

For $(u_1, \dots, u_r) \in \mathcal{U}_r$, let

$$\psi_j(\tau) = (u_j(\tau))^{1/3}(\log u_j(\tau))^{26/15},$$

$$H_{1j}(\tau) = \begin{cases} \frac{\psi_j(\tau)}{\widehat{u}_j(2\tau)} & \text{if } \widehat{u}_j(\tau) \text{ is increasing,} \\ \frac{\psi_j(\tau)}{\widehat{u}_j(\tau)} & \text{if } \widehat{u}_j(\tau) \text{ is decreasing, } j = 1, \dots, r, \end{cases}$$

$$H_{2j}(\tau) = \begin{cases} \frac{u_j(\tau)}{2\widehat{u}_j(2\tau)} & \text{if } \widehat{u}_j(\tau) \text{ is increasing,} \\ \frac{u_j(\tau)}{2\widehat{u}_j(\tau)} & \text{if } \widehat{u}_j(\tau) \text{ is decreasing, } j = 1, \dots, r, \end{cases}$$

and $\widetilde{H}(\tau) = \max_{1 \leqslant j \leqslant r} H_{1j}(\tau)$ and $\widetilde{\widetilde{H}}(\tau) = \min_{1 \leqslant j \leqslant r} H_{2j}(\tau) \leqslant T$. In all subsequent theorems, we assume that H satisfies

$$\widetilde{H}(T) \leqslant H \leqslant \widetilde{\widetilde{H}}(T). \tag{2}$$

Then, in [18], the following theorem has been obtained.

Theorem 1. *Suppose that $(u_1, \dots, u_r) \in \mathcal{U}_r$ and that the length H lies in (2). For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in \mathcal{A}_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{H} \mathrm{mes}\left\{ \tau \in I_{T,H} \mid \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |f_j(s) - \zeta(s + iu_j(\tau))| < \varepsilon \right\} > 0.$$

The universality of the Dirichlet series is a very useful property. Therefore, it is natural to ask if there is a possibility to extend the class of universal functions. One of the ways to achieve this is by using compositions of universal functions.

Denote by $\mathcal{A}(G)$ the space of analytic $G \subset \mathbb{C}$ functions that has the topology of uniform convergence on compacta, and let $\mathcal{A}^r(G)$ be the direct product of r-copies of $\mathcal{A}(G)$. Hence, every element of $\mathcal{A}^r(G)$ is the r-dimensional vector $\boldsymbol{g} = (g_1, \cdots, g_r)$. Moreover, let $\mathcal{S}_G = \{g \in \mathcal{A}(G) \mid g(s) \neq 0\} \cup \{0\}$, where $\{0\}$ is the zero-map, let $\mathcal{S}_G^r$ be the direct product of r-copies of $\mathcal{S}_G$, and let $\mathcal{A}(K)$ be as above. In [18], one theorem on the approximation of analytic functions by shifts $F(\zeta(s + iu_1(\tau)), \dots, \zeta(s + iu_r(\tau)))$ for some classes of operators $F : \mathcal{A}^r(\Delta) \to \mathcal{A}(\Delta)$ was obtained.

Moreover, we will use the vector notation $\boldsymbol{u}(\tau) = (u_1(\tau), \dots, u_r(\tau))$ to mean the r-tuple of admissible shifts, and let

$$F(\zeta(s + i\boldsymbol{u}(\tau))) = F(\zeta(s + iu_1(\tau)), \cdots, \zeta(s + iu_r(\tau))).$$

Theorem 2. *Suppose that $\boldsymbol{u}(\tau) \in \mathcal{U}_r$; the length H lies in (2), and $F : \mathcal{A}^r(\Delta) \to \mathcal{A}(\Delta)$ is a continuous operator subject to the condition that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap \mathcal{S}_\Delta^r$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in \mathcal{A}(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{H} \mathrm{mes}\left\{ \tau \in I_{T,H} \mid \sup_{s \in K} |f(s) - F(\zeta(s + i\boldsymbol{u}(\tau)))| < \varepsilon \right\} > 0. \tag{3}$$

Theorem 2 is theoretical; it is difficult to present examples of the operators F. The aim of the paper is to give other sub-classes of the operators in Theorem 2. We start with a modified Lipschitz class. Let $\alpha_1, \ldots, \alpha_r$ be fixed positive numbers, and let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$.

Definition 1. *The operator $F : \mathcal{A}^r(\Delta) \to \mathcal{A}(\Delta)$ belongs to the class $Lip(\boldsymbol{\alpha})$ if the following hypotheses are satisfied:*

1° For every polynomial $p = p(s)$ and any sets $K_j \in \mathcal{K}$, $j = 1, \ldots, r$, the r-dimensional vector $\boldsymbol{g} = (g_1, \ldots, g_r) \in F^{-1}\{p\} \subset \mathcal{A}^r(\Delta)$ exists such that $g_j(s) \neq 0$ for $s \in K_j$;

2° For every $K \in \mathcal{K}$ and $\boldsymbol{g}_k = (g_{k1}, \ldots, g_{kr}) \in \mathcal{A}^r(\Delta)$, $k = 1, 2$, a constant $c > 0$ and sets $K_1, \ldots, K_r \in \mathcal{K}$ exists such that

$$\sup_{s \in K} |F(\boldsymbol{g}_1) - F(\boldsymbol{g}_2)| \leq c \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| g_{1j}(s) - g_{2j}(s) \right|^{\alpha_j}.$$

Theorem 3. *Suppose that $\boldsymbol{u}(\tau) \in \mathcal{U}_r$; the length H lies in (2), and $F \in Lip(\boldsymbol{\alpha})$. Let $K \in \mathcal{K}$ and $f(s) \in \mathcal{A}(K)$. Then, for every $\varepsilon > 0$ inequality (3) is valid.*

It is very important is to be able to provide concrete examples of the investigated operators F, for $F \in Lip(\boldsymbol{\alpha})$; it is not difficult.

Example 1. *Let $c_1, \ldots, c_r \in \mathbb{C} \setminus \{0\}$, $m_1, \ldots, m_r \in \mathbb{N}$, and $g_j^{(m_j)}$ denote the m_jth derivative of g_j. Define the operator $F : \mathcal{A}^r(\Delta) \to \mathcal{A}(\Delta)$ by*

$$F(\boldsymbol{g}) = \sum_{j=1}^{r} c_j g_j^{(m_j)}, \quad \boldsymbol{g} = (g_1, \ldots, g_r) \in \mathcal{A}^r(\Delta).$$

Now, we take an arbitrary polynomial

$$p(s) = \sum_{k=0}^{l} a_k s^k, \quad a_k \neq 0,$$

and sets $K_1, \ldots, K_r \in \mathcal{K}$. We may take one of the components whose derivative coincides with $p(s)$, e.g., set

$$(g_1, \ldots, g_r) = (1, \ldots, 1, g_r), \tag{4}$$

where

$$g_r(s) = \frac{1}{c_r} \sum_{k=0}^{l} \frac{a_k}{(k+1) \cdots (k+m_r)} s^{k+m_r} + C.$$

where the constant $C \in \mathbb{C}$ is chosen so that $g_r(s) \neq 0$ for $s \in K_r$. Hence, $\boldsymbol{g}$ in (4) satisfies the condition 1° in Definition 1.

To check condition 2°, we apply the Cauchy integral theorem. Let $K \in \mathcal{K}$. Then, there exists an open set U and $\widehat{K} \in \mathcal{K}$ such that $K \subset U \subset \widehat{K}$. We take a simple closed contour C lying in $\widehat{K} \setminus U$ and enclosing K. Then, the Cauchy integral formula shows that, for $\mathbf{g}_k = (g_{k1}, \ldots, g_{kr}) \in \mathcal{A}^r(\Delta)$, $k = 1, 2$,

$$|F(\boldsymbol{g}_1) - F(\boldsymbol{g}_2)| = \left| \sum_{j=1}^{r} \frac{m_j! c_j}{2\pi i} \int_C \frac{g_{1j}(z) - g_{2j}(z)}{(z-s)^{m_j+1}} \mathrm{d}z \right|.$$

This can be bounded by $c \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| g_{1j}(s) - g_{2j}(s) \right|$ for some constant $c > 0$. Thus, the condition 2° holds with $\alpha_j = 1$ and $K_j = \widehat{K}$, $j = 1, \ldots, r$. Hence, $F \in Lip(\boldsymbol{\alpha})$.

For a given $B > 0$, denote the finite part of Δ with imaginary parts being bounded by B, i.e., $\Delta_B = \{s \in \Delta \mid \ |t| < B\}$. Denote by $\mathcal{K}_B$ the class of compact subsets of Δ_B with connected complement, and by $\mathcal{A}(K_B)$ the class of continuous on K_B functions that are analytic in the interior of K_B ($K_B \in \mathcal{K}_B$).

Theorem 4. *Suppose that $\boldsymbol{u}(\tau) \in \mathcal{U}_r$; the length H lies in (2); and $F : \mathcal{A}^r(\Delta_B) \to \mathcal{A}(\Delta_B)$ is a continuous operator so that for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap \mathcal{S}_B^r$ is not empty. Let $K \in \mathcal{K}_B$ and $p(s) \in \mathcal{A}(\Delta_B)$. Then, the same statement as in Theorem 2 holds true.*

Example 2. *Define the operator $F : \mathcal{A}^r(\Delta_B) \to \mathcal{A}(\Delta_B)$ by*

$$F(\boldsymbol{g}) = \sum_{j=1}^{r} g_j.$$

For a given polynomial $p = p(s)$, let

$$(g_1, \ldots, g_r) = (0, \ldots, 0, -C, p + C).$$

If $C \in \mathbb{C}$ is with a large enough $|C|$, then the latter collection lies in $S^r_{\Delta_B}$, and $F(\boldsymbol{g}) = p$. Thus, F satisfies the hypothesis of Theorem 4.

Now, we will approximate the functions from certain subsets of $\mathcal{A}(\Delta)$. Let a_j, $j = 1, \ldots k$ be distinct complex numbers, and

$$\mathcal{A}_k(\Delta) = \{g \in \mathcal{A}(\Delta) \mid g(s) \neq a_j,\, j = 1, \ldots, k\}.$$

Theorem 5. *Suppose that $\boldsymbol{u}(\tau) \in \mathcal{U}_r$; the length H lies in (2), and $F : \mathcal{A}^r(\Delta) \to \mathcal{A}(\Delta)$ is a continuous operator such that $F(\mathcal{S}^r) \supset \mathcal{A}_k(\Delta)$. For $k = 1$, let $K \in \mathcal{K}$, $f(s) \in \mathcal{A}(K)$ and $f(s) \neq a_1$ on K. For $k \geq 2$, let K be an arbitrary compact subset of Δ, and $f(s) \in \mathcal{A}_k(K)$. Then, the same statement as in Theorem 2 holds true.*

Example 3. *Let $k = 2$, $a_1 = 1$, $a_2 = -1$, and*

$$F(\boldsymbol{g}) = \cosh \sum_{j=1}^{r} g_j.$$

For brevity, denote $w = \sum_{j=1}^{r} g_j$, and consider the equation

$$\frac{\mathrm{e}^{w} + \mathrm{e}^{-w}}{2} = f, \quad f \in \mathcal{A}_2(\Delta).$$

Since $\cosh^{-1} f = \log\left(f + \sqrt{f^2 - 1}\right) = w$, taking $g_r = \log\left(f + \sqrt{f^2 - 1}\right)$ with other components $g_j = 0$, $j = 1, \ldots r - 1$, we have that $F(\boldsymbol{g}) = F(0, \ldots 0, w) = f$ with $\boldsymbol{g} \in \mathcal{S}^r$. This shows that $F(\mathcal{S}^r) \supset \mathcal{A}_2(\Delta)$; therefore, the operator F satisfies the condition of Theorem 5.

The last theorem is on the approximation of analytic functions from the set $F(\mathcal{S}^r)$.

Theorem 6. *Suppose that $\boldsymbol{u}(\tau) \in \mathcal{U}_r$; the length H lies in (2), and $F : \mathcal{A}^r(\Delta) \to \mathcal{A}(\Delta)$ is a continuous operator. Let $K \subset \Delta$ be a compact set, and $f(s) \in F(\mathcal{S}^r)$. Then, the same statement as in Theorem 2 holds true.*

Proofs of Theorems 4–6 are of a probabilistic character, while that of Theorem 3 is direct and based on properties of the class $Lip(\boldsymbol{\alpha})$. Moreover, as we mentioned above, in all of these theorems, the lower limit can be replaced with the limit for all but at most countably many $\varepsilon > 0$, and we will prove it.

2. Proof of Theorem 3

For convenience, we start by recalling the Mergelyan theorem on the approximation of analytic functions by polynomials [19].

Lemma 1. *Suppose that $K \subset \mathbb{C}$ is a compact set with a connected complement and that $g(s)$ is a continuous function on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, a polynomial $p(s)$ exists such that*

$$\sup_{s \in K} |p(s) - g(s)| < \varepsilon.$$

Proof of Theorem 3. Suppose that $F \in Lip(\boldsymbol{\alpha})$. In view of the condition 1° in Definition 1, for sets $K_j \in \mathcal{K}$, $j = 1, \dots, r$, and a polynomial $p(s)$, the r-dimensional vector $\boldsymbol{g} \in F^{-1}\{p\}$ exists with $g_j(s) \neq 0$, $s \in K_j$. Let the sets K_j, $j = 1, \dots, r$ correspond to the set K in condition 2° of Definition 1, and

$$A(\tau) := \sup_{1 \le j \le r} \sup_{s \in K_j} \left| g_j(s) - \zeta(s + u_j(\tau)) \right| < \left(\frac{\varepsilon}{2c} \right)^{\frac{1}{\alpha}},$$

where $\alpha = \min_{1 \le j \le r} \alpha_j$. Then, by Theorem 1,

$$\liminf_{T \to \infty} \frac{1}{H} \mathrm{mes}\{\tau \in I_{T,H} : \tau \in A(\tau)\} > 0. \tag{5}$$

The condition in Definition 1 implies that, for $\tau \in A(\tau)$,

$$\begin{aligned} & \sup_{s \in K} \left| p(s) - F(\zeta(s + \boldsymbol{u}(\tau)) \right| = \sup_{s \in K} \left| F(\mathbf{g}) - F(\zeta(s + \boldsymbol{u}(\tau)) \right| \\ \le \quad & c \sup_{1 \le j \le r} \sup_{s \in K_j} \left| g_j(s) - \zeta(s + u_j(\tau)) \right|^{\alpha_j} \le c \frac{\varepsilon}{2c}. \end{aligned} \tag{6}$$

Using Lemma 1, choose the polynomial $p(s)$ such that

$$\sup_{s \in K} |p(s) - f(s)| < \frac{\varepsilon}{2}. \tag{7}$$

Since the expression (6) is bounded by $A(\tau)^\alpha$, we conclude that

$$\sup_{s \in K} \left| f(s) - F(\zeta(s + \boldsymbol{u}(\tau))) \right| < \varepsilon$$

with (7). This and inequality (5) prove the theorem. □

3. Proofs of Theorems 4–6

We will use limit theorems on the weakly convergence of probability measures in the space of analytic functions to prove Theorems 4–6.

Denote by $\mathcal{B}(\mathcal{X})$ the Borel σ-field of the topological space $\mathcal{X}$. Let l denote the unit circle on $\mathbb{C}$, and let $l = l_p$ for all $p \in \mathbb{P}$. Define the set

$$\mathcal{T} = \prod_{p \in \mathbb{P}} l_p.$$

By Tikhonov theorem (see [8] Theorem 5.1.4), the torus $\mathcal{T}$ is a compact topological Abelian group. Let $\mathcal{T}^r$ be the direct product of $\mathcal{T}_j$, where $\mathcal{T}_j = \mathcal{T}$ for all $j = 1, \dots, r$. Then, again, $\mathcal{T}^r$ is a compact topological Abelian group. Thus, on $(\mathcal{T}^r, \mathcal{B}(\mathcal{T}^r))$ we can define the probability Haar measure μ_H. This fact allows us to construct the probability space $(\mathcal{T}^r, \mathcal{B}(\mathcal{T}^r), \mu_H)$. Denote by $\boldsymbol{w} = (w_1, \dots, w_r)$ the elements of $\mathcal{T}^r$, and on the latter probability space, define the $\mathcal{A}^r(\Delta)$-valued random element $\zeta(s, \boldsymbol{w})$ by

$$\zeta(s, \boldsymbol{w}) = (\zeta(s, w_1), \dots, \zeta(s, w_r)),$$

where

$$\zeta(s, w_j) = \prod_{p \in \mathbb{P}} \left(1 - \frac{w_j(p)}{p^s}\right)^{-1}, \quad j = 1, \dots, r.$$

Let P_ζ be the distribution of an element $\zeta(s, \boldsymbol{w})$, i. e.,

$$P_\zeta(A) = \mu_H\{\boldsymbol{w} \in \mathcal{T}^r \mid \zeta(s, \boldsymbol{w}) \in A\}, \quad A \in \mathcal{B}(\mathcal{A}^r(\Delta)).$$

Then, in [18], the following statement (Theorem 4, Lemma 5) has been obtained.

We will use one more notation to formulate this and other statements below. Let P_T and P be probability measures defined on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. The weak convergence of P_T to P as $T \to \infty$; we will denote by $P_T \xrightarrow[T\to\infty]{w} P$.

Lemma 2. *Suppose that $\boldsymbol{u}(\tau) \in \mathcal{U}_r$ and the length H lies in (2). Define*

$$P_{T,H}(A) := \frac{1}{H}\text{mes}\{\tau \in I_{T,H} \mid \zeta(s + i\boldsymbol{u}(\tau)) \in A\}, \quad A \in \mathcal{B}(\mathcal{A}^r(\Delta)),$$

where, as before,

$$\zeta(s + i\boldsymbol{u}(\tau)) = (\zeta(s + iu_1(\tau)), \dots \zeta(s + iu_r(\tau))).$$

Then, $P_{T,H}(A) \xrightarrow[T\to\infty]{w} P_\zeta$. Moreover, the support of the measure P_ζ is the set $\mathcal{S}^r_\Delta$.

In what follows, a property of preservation of weak convergence of probability measures under certain mappings will be useful. Let P be a probability measure on $(\mathcal{X}_1, \mathcal{B}(\mathcal{X}_1))$, and $v : \mathcal{X}_1 \to \mathcal{X}_2$ a $(\mathcal{B}(\mathcal{X}_1), \mathcal{B}(\mathcal{X}_2))$-measurable mapping, i.e., $v^{-1}\mathcal{B}(\mathcal{X}_2) \subset \mathcal{B}(\mathcal{X}_1)$. Then, the measure P induces on $(\mathcal{X}_2, \mathcal{B}(\mathcal{X}_2))$ the unique probability measure Pv^{-1} given by

$$Pv^{-1}(A) = P(v^{-1}A), \quad A \in \mathcal{B}(\mathcal{X}_2).$$

Moreover, every continuous mapping is $(\mathcal{B}(\mathcal{X}_1), \mathcal{B}(\mathcal{X}_2))$-measurable, and the following statement is valid.

Lemma 3. *Suppose that P_n and P, $n \in \mathbb{N}$ are probability measures on $(\mathcal{X}_1, \mathcal{B}(\mathcal{X}_1))$, and $P_n \xrightarrow[n\to\infty]{w} P$. Let $v : \mathcal{X}_1 \to \mathcal{X}_2$ be a continuous mapping. Then, $P_n v^{-1} \xrightarrow[n\to\infty]{w} Pv^{-1}$ as well.*

A proof of the lemma can be found, for example, in [20], Theorem 5.1.

Lemmas 2 and 3 imply the following lemma. For $A \in \mathcal{B}(\mathcal{A}^r_{\Delta_B})$, set

$$P_{T,H,B}(A) := \frac{1}{H}\text{mes}\{\tau \in I_{T,H} \mid \zeta(s + i\boldsymbol{u}(\tau)) \in A\}$$

and

$$P_{\zeta,B}(A) = \mu_H\{\boldsymbol{w} \in \mathcal{T}^r \mid \zeta(s, \boldsymbol{w}) \in A\}.$$

Lemma 4. *Suppose that $\boldsymbol{u}(\tau) \in \mathcal{U}_r$ and the length H lies in (2). Then, $P_{T,H,B} \xrightarrow[T\to\infty]{w} P_{\zeta,B}$, and the support of the measure $P_{\zeta,B}$ is the set $\mathcal{S}^r_B$.*

Proof. Let the mapping $v_B : \mathcal{A}^r(\Delta) \to \mathcal{A}^r(\Delta_B)$ be given by

$$v_B(\boldsymbol{g}) = (g_1(s), \dots, g_r(s)) \,|_{s \in \Delta_B}, \quad \boldsymbol{g} \in \mathcal{A}^r(\Delta).$$

Then, v_B is continuous, and $P_{T,H,B} = P_{T,H} v_B^{-1}$. Therefore, in view of Lemmas 2 and 3, $P_{T,H,B} \xrightarrow[T\to\infty]{w} P_\zeta v_B^{-1}$. Then, $P_\zeta v_B^{-1} = P_{\zeta,B}$.

Let $\boldsymbol{g} \in \mathcal{S}_B^r$ and an open neighbourhood G of $\boldsymbol{g}$ be arbitrary. Since v_B is continuous, the set $v_B^{-1} G$ is open as well and contains an element $\boldsymbol{g}_1 \in \mathcal{S}_\Delta^r$. Therefore, $P_\zeta(v_B^{-1} G) > 0$. Thus, by Lemma 2,

$$P_{\zeta,B}(G) = P_\zeta v_B^{-1}(G) = P_\zeta(v_B^{-1} G) > 0. \tag{8}$$

Moreover, since $v_B(\mathcal{S}_\Delta^r) \subset \mathcal{S}_{\Delta_B}^r$ and $P_\zeta(\mathcal{S}_\Delta^r) = 1$, we have

$$P_{\zeta,B}(\mathcal{S}_{\Delta_B}^r) = P_\zeta v_B^{-1}(\mathcal{S}_{\Delta_B}^r) \geq P_\zeta(v_B^{-1}(\mathcal{S}_\Delta^r)) = P_\zeta(\mathcal{S}_\Delta^r) = 1.$$

This and (8) show that the support of $P_{\zeta,B}$ is the set $\mathcal{S}_{\Delta_B}^r$. □

Lemma 5. *Define*

$$P_{T,H,B,F}(A) := \frac{1}{H} \text{mes}\{\tau \in I_{T,H} \mid (\zeta(s + i\boldsymbol{u}(\tau))) \in A\}, \quad A \in \mathcal{B}(\mathcal{A}(\Delta_B)).$$

Under hypotheses of Theorem 4, $P_{T,H,B,F}(A) \xrightarrow[T\to\infty]{w} P_{\zeta,B} F^{-1}$, and the support of $P_{\zeta,B} F^{-1}$ is the whole space $\mathcal{A}(\Delta_B)$.

Proof. Since the operator F is continuous, $P_{T,H,B,F} \xrightarrow[T\to\infty]{w} P_{\zeta,B} F^{-1}$ follows from the Lemma 4. Let $g \in \mathcal{A}(\Delta_B)$ and its open neighbourhood G be arbitrary. Then, the preimage $F^{-1} G$ is also an open set. Suppose that $F^{-1} G$ contains an element of the set $\mathcal{S}_B^r$. Then, the Lemma 4 implies that

$$P_{\zeta,B} F^{-1}(G) = P_{\zeta,B}(F^{-1} G) > 0.$$

Since $P_{\zeta,B} F^{-1}(\mathcal{A}(\Delta_B)) = 1$, this proves that the support of $P_{\zeta,B} F^{-1}$ is $\mathcal{A}(\Delta_B)$. Thus, it remains to show that the hypothesis $(F^{-1}\{p\}) \bigcap \mathcal{S}_B^r \neq \emptyset$ of the theorem implies that, for an open set $G \subset \mathcal{A}(\Delta_B)$, $(F^{-1} G) \bigcap \mathcal{S}_B^r \neq \emptyset$ as well.

Recall the metric in $\mathcal{A}(\Delta_B)$ describing its topology of uniform convergence on compact sets. There is a sequence of embedded compact subsets $\{K_l\} \subset \Delta$ with connected complements such that $\bigcup\limits_{l=1}^{\infty} K_l = \Delta$, and every compact set $K \subset \Delta$ is contained for some set K_l. Then, for $g_1, g_2 \in \mathcal{A}(\Delta_B)$,

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},$$

gives the desired metric in $\mathcal{A}(\Delta_B)$.

Fix $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ satisfying

$$\sum_{k > k_0}^{\infty} 2^{-k} < \frac{\varepsilon}{2}. \tag{9}$$

Let $g \in \mathcal{A}(\Delta_B)$ and G be its open neighbourhood. Since $K_1 \subset K_2 \subset \cdots \subset K_{k_0}$, by Lemma 1, we can choose a polynomial $p = p(s)$ such that

$$\sup_{s \in K_k} |p(s) - g(s)| < \frac{\varepsilon}{2}, \quad k = 1, \dots, n_0.$$

This, together with (9), shows that $\rho(p,g) < \varepsilon$. Therefore, if $\varepsilon > 0$ is small enough, the polynomial $p(s)$ belongs to G, and its preimage lies in $\mathcal{S}_B^r$ and lies in $F^{-1}G$. Hence, $(F^{-1}G) \bigcap \mathcal{S}_B^r \neq \emptyset$. The lemma is proved. □

The definition of weak convergence of probability measures has equivalents in terms of various sets. We will use these equivalents in terms of some classes of sets, and we will present them in the following lemma (the proof of these given statements can be found, for example, in [20] Theorem 2.1). Denote by $\mathrm{bd}(A)$ the boundary of a set A. We say that A is a continuity set of P if $P(\mathrm{bd}(A)) = 0$.

Lemma 6. *Let P and P_n, $n \in \mathbb{N}$ be probability measures defined on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. Then, the following statements are equivalent:*

1° $P_n \xrightarrow[n\to\infty]{w} P$;

2° $\liminf\limits_{n\to\infty} P_n(G) \geqslant P(G)$, *for every open set* $G \subset \mathcal{X}$;

3° $\lim\limits_{n\to\infty} P_n(A) = P(A)$, *for every continuity set A of P.*

Proof of Theorem 4. By Lemma 1, we can choose a polynomial $p(s)$ such that inequality (7) holds. In view of Lemma 5, the polynomial $p(s)$ is an element of the support of the measure $P_{\zeta,B}F^{-1}$. Hence,

$$P_{\zeta,B}F^{-1}(\mathcal{J}_\varepsilon) > 0. \tag{10}$$

where

$$\mathcal{J}_\varepsilon = \left\{ g \in \mathcal{A}(\Delta_B) | \sup_{s\in K} |p(s) - g(s)| < \frac{\varepsilon}{2} \right\}.$$

Since $\mathcal{J}_\varepsilon$ is an open set, by Lemma 5, and 1° and 2° of Lemma 6, we have

$$\liminf_{T\to\infty} P_{T,H,B,F}(\mathcal{J}_\varepsilon) \geqslant P_{\zeta,B}F^{-1}(\mathcal{J}_\varepsilon).$$

This, the definitions of $P_{T,H,B,F}$ and $\mathcal{J}_\varepsilon$, and (10), (7) prove the inequality (3) in Theorem 4. To replace "lim inf" by "lim" in the inequality (3) of the theorem, we observe that

$$\widehat{\mathcal{J}}_\varepsilon = \left\{ g \in \mathcal{A}(\Delta_B) \mid \sup_{s\in K} |f(s) - g(s)| < \varepsilon \right\}$$

is a continuity set of the measure $P_{\zeta,B}$ for all but at most countably many $\varepsilon > 0$. Actually, the boundary $\mathrm{bd}(\widehat{\mathcal{J}}_\varepsilon)$ lies in the set

$$\left\{ g \in \mathcal{A}(\Delta_B) \mid \sup_{s\in K} |f(s) - g(s)| = \varepsilon \right\}.$$

Therefore, for different positive ε_1 and ε_2, $\mathrm{bd}(\widehat{\mathcal{J}}_{\varepsilon_1})$ and $\mathrm{bd}(\widehat{\mathcal{J}}_{\varepsilon_2})$ do not intersect. Hence, $P_{\zeta,B}(\mathrm{bd}(\widehat{\mathcal{J}}_\varepsilon)) > 0$ for at most countably many $\varepsilon > 0$, that is, the set $\widehat{\mathcal{J}}_\varepsilon$ is a continuity set of the measure $P_{\zeta,B}$ for all but at most countably many $\varepsilon > 0$. Hence, by Lemma 5, and 1° and 3° of Lemma 6,

$$\lim_{T\to\infty} P_{T,H,B,F}(\widehat{\mathcal{J}}_\varepsilon) = P_{\zeta,B}F^{-1}(\widehat{\mathcal{J}}_\varepsilon) \tag{11}$$

for all but at most countably many $\varepsilon > 0$. Moreover, (7) shows that $\mathcal{J}_\varepsilon \subset \widehat{\mathcal{J}}_\varepsilon$. Thus, $P_{\zeta,B}F^{-1}(\widehat{\mathcal{J}}_\varepsilon) > 0$ by (10). This, (11), and the definitions of $P_{T,H,B,F}$ and $\widehat{\mathcal{J}}_\varepsilon$ prove the assertion on density for at most countably many $\varepsilon > 0$. □

For $A \in \mathcal{B}(\mathcal{A}(\Delta))$ and $F : \mathcal{A}^r(\Delta) \to \mathcal{A}(\Delta)$, define

$$P_{T,H,F}(A) = \frac{1}{H}\mathrm{mes}\{\tau \in I \mid F(\zeta(s + i\boldsymbol{u}(\tau))) \in A\}.$$

Lemma 7. *Under hypotheses of Theorem 5,* $P_{T,H,F} \xrightarrow[T\to\infty]{w} P_\zeta F^{-1}$*. Moreover, the closure of the set* $\mathcal{A}_k(\Delta)$ *lies in the support of* $P_\zeta F^{-1}$*.*

Proof. Lemmas 2 and 3, and the continuity of the operator F show that $P_{T,H,F} \xrightarrow[T\to\infty]{w} P_\zeta F^{-1}$.

We take an arbitrary element $g \in F(\mathcal{S}^r_\Delta)$ and any open neighbourhood $\mathcal{J}$ of g. Since F is continuous, $F^{-1}\mathcal{J}$ is an open neighbourhood of some element of the set $\mathcal{S}^r_\Delta$. By Lemma 2, the set $\mathcal{S}^r_\Delta$ is the support of the measure P_ζ. Therefore, $P_\zeta(F^{-1}\mathcal{J}) > 0$. Thus, $P_\zeta F^{-1}(\mathcal{J}) > 0$. Moreover,

$$P_\zeta F^{-1}(F(\mathcal{S}_\Delta)) = P_\zeta(F^{-1}F(\mathcal{S}_\Delta)) = P_\zeta(\mathcal{S}_\Delta) = 1.$$

Furthermore, the support of $P_\zeta F^{-1}$ is a closed set; this shows that the support of $P_\zeta F^{-1}$ is the closure of the set $F(\mathcal{S}_\Delta)$. By the hypothesis of the theorem, $F(\mathcal{S}^r_\Delta) \supset \mathcal{A}_k(\Delta)$. Therefore, the closure of the set $\mathcal{A}_k(\Delta)$ lies in the support of $P_\zeta F^{-1}$. □

Proof of Theorem 5. Let $k = 1$. By Lemma 1, we find a polynomial $p(s)$ such that

$$\sup_{s\in K} |p(s) - f(s)| < \frac{\varepsilon}{4}. \tag{12}$$

By the hypothesis of the theorem, for $s \in K$, $f(s) \neq b_1$. Therefore, $p(s) \neq b_1$ for $s \in K$. Thus, an application of Lemma 1 once more implies that we can choose a polynomial $\widehat{p}(s)$ such that

$$\sup_{s\in K} \left| \mathrm{e}^{\widehat{p}(s)} + b_1 - p(s) \right| < \frac{\varepsilon}{4}. \tag{13}$$

The function $g_{b_1}(s) := \mathrm{e}^{\widehat{p}(s)} + b_1 \in \mathcal{A}_1(\Delta)$. Thus, by Lemma 7, the closure of $\mathcal{A}_1(\Delta)$ lies in the support of $P_\zeta F^{-1}$, and the function $g_{b_1}(s)$ belongs to the support of $P_\zeta F^{-1}$. Hence, taking

$$\mathcal{J}_\varepsilon = \left\{ g \in \mathcal{A}(\Delta) \mid \sup_{s\in K} |g_{b_1}(s) - g(s)| < \frac{\varepsilon}{2} \right\},$$

we have $P_\zeta F^{-1}(\mathcal{J}_\varepsilon) > 0$. This, Lemma 7, and 1° and 2° of Lemma 6 together with (12) and (13) prove the assertion on lower density of the theorem in the case $k = 1$.

Let

$$\widetilde{\mathcal{J}}_\varepsilon = \left\{ g \in \mathcal{A}(\Delta) \mid \sup_{s\in K} |f(s) - g(s)| < \varepsilon \right\}.$$

Then, similarly as in the case of $\widehat{\mathcal{J}}_\varepsilon$ in the proof of Theorem 4, we have that $\widetilde{\mathcal{J}}_\varepsilon$ is a continuity set of the measure $P_\zeta F^{-1}$ for all but at most countably many $\varepsilon > 0$. Moreover, inequalities (12) and (13) show that $\mathcal{J}_\varepsilon \subset \widetilde{\mathcal{J}}_\varepsilon$. Thus, $P_\zeta F^{-1}(\widetilde{\mathcal{J}}_\varepsilon) > 0$. This; Lemma 7; and 1° and 3° of Lemma 6 prove the assertion on density of the theorem in the case $k = 1$.

Now, let $k \geq 2$. Since $f(s) \in \mathcal{A}_k(\Delta)$, we have by Lemma 7 that the set $\widetilde{\mathcal{J}}_\varepsilon$ is an open neighbourhood of an element of the support of the measure $P_\zeta F^{-1}$. Thus, $P_\zeta F^{-1}(\widetilde{\mathcal{J}}_\varepsilon) > 0$. Therefore, by the first part of Lemma 7 and 1°, 2° of Lemma 6,

$$\liminf_{T\to\infty} P_{T,H,F}(\widetilde{\mathcal{J}}_\varepsilon) = P_\zeta F^{-1}(\widetilde{\mathcal{J}}_\varepsilon) > 0,$$

and the definitions of $P_{T,H,F}$ and $\widetilde{\mathcal{J}}_\varepsilon$ prove the assertion in the case of lower density of the theorem.

For the proof of the second assertion of the theorem, it suffices to observe that the set $\widetilde{\mathcal{J}}_\varepsilon$ is a continuity set of the measure $P_\zeta F^{-1}$ for all but at most countably many $\varepsilon > 0$. This; Lemma 7; and 1°, 3° of Lemma 6 give the assertion in the case of density of the theorem. □

Proof of Theorem 6. We use Lemmas 6 and 7 and repeat the arguments of the proof of Theorem 5 in the case $k \geq 2$. □

4. Concluding Remarks

The universality in the approximation of classes of analytic functions is a property of Dirichlet series. However, the theorems of the paper show that the class of universal functions can be extended significantly by using the compositions of universal functions with certain operators F in the space of analytic functions. Moreover, some new aspects of approximation by shifts

$$F\big(\zeta(s+iu_1(\tau)),\dots,\zeta(s+iu_r(\tau))\big) \tag{14}$$

are introduced. First, in approximation the shifts of only one function $\zeta(s)$ are used. Secondly, generalized shifts $\zeta(s+iu_j(\tau))$ with a certain class of tuples $(u_1,\dots,u_r)$ are applied. However, the most important new feature of approximation is investigation of the density of approximating shifts (14) in short intervals $[T, T+H]$, i.e., with H smaller than T.

The universality theorems of the present paper as well as of [18] are of the continuous type because τ in shifts can take arbitrary real values. Additionally, the discrete type of universality in approximation of analytic functions when τ takes values from some discrete sets is known. This type of universality was introduced in [21], and, in the case of $\zeta(s)$, the shifts $\zeta(s+ikh)$, $k\in\mathbb{N}$ were used, with a fixed $h>0$. We have a plan in subsequent papers to obtain universality theorems in short intervals for shifts

$$\zeta(s+i\boldsymbol{u}(k)) = \big(\zeta(s+iu_1(k)),\dots,\zeta(s+iu_r(k))\big), \quad k\in\mathbb{N},$$

for some functions $(u_1,\dots,u_r)$, as well as for compositions $F(\zeta(s+i\boldsymbol{u}(k))$ with certain operators F in the space of analytic functions. We note that in the discrete case, some additional problems connected to the mean square estimates for $\zeta(s)$ arise.

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