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Nonexistence and Existence of Solutions with Prescribed Norms for Nonlocal Elliptic Equations with Combined Nonlinearities

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Abstract: In this paper, we study the nonlocal equation $\left(\int_{\mathbb{R}^N} |u(x)|^2 dx\right)^{\gamma} \Delta u = \lambda u + \mu |u|^{q-2}u + |u|^{p-2}u$, x in $\mathbb{R}^N$ having a prescribed mass $\int_{\mathbb{R}^N} |u(x)|^2 dx = c^2$, where $N \geq 3$, μ, $\gamma \in (0, +\infty)$, $q \in (2, 2^*)$, c is a positive constant, $p, q \in (2, 2^*)$ with $p \neq q$ and $2^* = \frac{2N}{N-2}$. This research is meaningful from a physical point of view. Using variational methods, we present some results on the nonexistence and existence of solutions under different cases p and q which improve upon the previous ones via topological theory.

Keywords: constrained minimization; Pohozaev's identity; nonlocal equation; existence

MSC: 35J20; 35J60

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1. Introduction

In this paper, we consider the following nonlocal elliptic problem:

$$-\left(\int_{\mathbb{R}^N} |u(x)|^2 dx\right)^{\gamma} \Delta u = \lambda u + \mu |u|^{q-2}u + |u|^{p-2}u, \ \ x \text{ in } \mathbb{R}^N, \tag{1}$$

where $N \geq 3$, $\mu, \gamma \in (0, +\infty)$, $q \in (2, 2^*)$, $p \in (2, 2^*)$ and $2^* = \frac{2N}{N-2}$.

In [1,2], Almeida et al. considered the following nonlocal degenerate parabolic equation:

$$\begin{cases} u_t - \left(\int_{\Omega} u^2(z,t) dz\right)^{\gamma} \Delta u = f(x,t), & \text{in } \Omega \times (0,T), \\ u(x,t) = 0, \ \ (x,t) \text{ on } \Omega \times (0,T), \\ u(x,0) = u_0(x), \ \ x \text{ in } \Omega. \end{cases} \tag{2}$$

They proved the existence and uniqueness of weak solutions, and they also presented the convergence and error bounds of the solutions for a linearized Crank–Nicolson–Galerkin finite element method with polynomial approximations of a degree $k \geq 1$. It is easy to see that Equation (1) is related to the stationary analogue of Equation (2). Some other nonlocal degenerate parabolic equations can be found in [3–5]. In recent years, there have been many papers on nonlocal problems (see [6–12] and the references therein). For instance, in [7], Corrêa et al. considered the special case

$$\begin{cases} -\triangle u = u^q f\left(\lambda, \int_{\Omega} u^{\gamma} dx\right), \ \ x \in \Omega, \\ u > 0, \ \ x \in \Omega; u|_{\partial\Omega} = 0. \end{cases} \tag{3}$$

By transforming the above problem into an algebraic nonlinear equation, they gave a complete description of the set of positive solutions. In [6], Alves and Covei investigated the following problems:

$$\begin{cases} -a\left(\int_{\Omega}|u|^{\gamma}dx\right)\triangle u = h_1(x,u)f\left(\int_{\Omega}|u|^{\gamma}dx\right) + h_2(x,u)g\left(\int_{\Omega}|u|^{\gamma}dx\right), \ x\in\Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{4}$$

Using the method of sub-super solutions, they showed the existence of positive solutions. In [10,13], Chipot et al. considered the functional elliptic problems

$$\begin{cases} -A(x,u)\triangle u = \lambda f(u), \ \ x\in\Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{5}$$

Using the Schauder fixed point theorem and a comparison principle, they obtained the existence of at least one positive solution or n distinct solutions. Note that Equation (1) is a special case for Equations (3)–(5), and since this problem lacks a variational structure, it is difficult to discuss it via variational methods directly. However, when $\gamma = 0$, Equation (1) is changed to

$$-\Delta u = \lambda u + \mu|u|^{q-2}u + |u|^{p-2}u, \ \ x \text{ in } \mathbb{R}^N. \tag{6}$$

In [14,15], using a minimizing sequence, Soave obtained some interesting results for the existence of Equation (6) under different assumptions for q, p and c.

Physicists are often interested in normalized solutions, so it is of interest to study solutions to equations having a prescribed L^2 norm (see [16–21] and the references therein). In this paper, our aim is to study the nonexistence and existence of positive solutions with prescribed norms for Equation (1), and we have following main results.

Let Q be the unique ground state solution of

$$-\frac{N(p-2)}{4}\triangle u + \left(1+\frac{p-2}{4}(2-N)\right)u = |u|^{p-2}u, \ \ x\in\mathbb{R}^N \tag{7}$$

with

$$\|Q\|_2 = (\int_{\mathbb{R}^N}|Q(x)|^2)^{\frac{1}{2}}.$$

In addition, for a positive constant $c > 0$, define

$$J_c(u) = \frac{1}{2}c^{2\gamma}\|\nabla u\|_2^2 - \frac{\mu}{q}\|u\|_q^q - \frac{1}{p}\|u\|_p^p, \ \ u\in H^1(\mathbb{R}^N), \tag{8}$$

where $\|\nabla u\|_2 = \left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^{\frac{1}{2}}$, $\|u\|_q = \left(\int_{\mathbb{R}^N}|u|^q dx\right)^{\frac{1}{q}}$ and $\|u\|_p = \left(\int_{\mathbb{R}^N}|u|^p dx\right)^{\frac{1}{p}}$.

Now, we have the following main results in this paper:

Theorem 1. *Assume that $p, q \in (2, 2^*)$, $q < p$. Then, Equation (1) has only a trivial solution in $H^1(\mathbb{R}^N)$ for all $\lambda \geq 0$.*

Theorem 2. *If $p, q \in (2, \frac{2N+4}{N})$, $q < p$, then for any $c > 0$, Equation (1) has at least one couple solution (λ_c, u_c) with $\lambda_c < 0$.*

Theorem 3. *Assume that $q \in (2, \frac{2N+4}{N})$ and $p = \frac{2N+4}{N}$. If $N\gamma < 2$ and $0 < c < \|Q\|_2^{\frac{2}{N\gamma}}$, or if $N\gamma > 2$ and $c > \|Q\|_2^{\frac{2}{N\gamma}}$, then Equation (1) has at least one couple solution (λ_c, u_c) with $\lambda_c < 0$.*

Theorem 4. *Assume that* $q = \frac{2N+4}{N}$ *and* $p \in (\dfrac{2N+4}{N}, 2^*)$. *If* $N\gamma < 2$ *and* $c < \left(\frac{\|Q\|_2^{\frac{4}{N}}}{\mu}\right)^{\frac{N}{2(N\gamma-2)}}$, *or if* $N\gamma > 2$ *and* $c > \left(\frac{\|Q\|_2^{\frac{4}{N}}}{\mu}\right)^{\frac{N}{2(N\gamma-2)}}$, *then Equation* (1) *has a couple solution* (u_c, λ_c) *with* $\lambda_c < 0$.

Theorem 5. *If* $p, q \in (\dfrac{2N+4}{N}, 2^*)$ *with* $p < q$, *then Equation* (1) *has a couple solution* (u_c, λ_c) *for each* $c > 0$ *with* $\lambda_c < 0$.

Theorem 6. *For* $2 < q < \dfrac{2N+4}{N} < p < 2^*$, *if* $N\gamma < 2$ *and*

$$c < \frac{1}{\mu^{\frac{N(p-2)-4}{2(p-q)(2-\gamma N)}}} \|Q\|_2^{\frac{2}{2-\gamma N}} \left(\frac{4-N(q-2)}{N(p-q)}\right)^{\frac{4-N(q-2)}{2(p-q)(2-\gamma N)}} \left(\frac{N(p-2)-4}{N(p-q)}\right)^{\frac{N(p-2)-4}{2(p-q)(2-\gamma N)}} \tag{9}$$

or if $N\gamma > 2$ *and*

$$c > \frac{1}{\mu^{\frac{N(p-2)-4}{2(p-q)(2-\gamma N)}}} \|Q\|_2^{\frac{2}{2-\gamma N}} \left(\frac{4-N(q-2)}{N(p-q)}\right)^{\frac{4-N(q-2)}{2(p-q)(2-\gamma N)}} \left(\frac{N(p-2)-4}{N(p-q)}\right)^{\frac{N(p-2)-4}{2(p-q)(2-\gamma N)}}, \tag{10}$$

then Equation (1) *has at least two couple solutions* (u_c, λ_1) *and* (v_c, λ_2) *with* $J_c(u_c) < 0$ *and* $J_c(v_c) > 0$, *respectively, where* J_c *is defined in Equation* (8).

This paper is organized as follows. In Section 2, we present some preliminary results and prove the Pohozaev identity for Equation (1). In Section 3, we obtain some new lemmas, and using the obtained lemmas, we prove our main theorems. Some ideas in this paper came from [22–27].

2. Preliminaries

In this section, in order to present our results, we list some preliminaries and prove some new lemmas. First, we transform the existence of Equation (1) into that of another problem. For $c > 0$, we consider the following problem:

$$-c^{2\gamma}\Delta u = \lambda u + \mu |u|^{q-2}u + |u|^{p-2}u, \quad x \text{ in } \mathbb{R}^N, \tag{11}$$

where $N \geq 3$.

The solutions to Equation (1) are obtained by looking for the critical points of the following C^1 functional J_c in $H^1(\mathbb{R}^N)$ (or simply H^1) constrained on the L^2 sphere

$$S_r := \left\{u \in H^1(\mathbb{R}^N) : \|u\|_2^2 = r^2, r > 0\right\}.$$

For any fixed $r > 0$, we call $(u_r, \lambda_r) \in H^1(\mathbb{R}^N) \times (0, +\infty)$ a couple solution to Equation (1) if u_r is a critical point of $J_c|_{S_r}$ and λ_r is the associated Lagrange multiplier:

Remark 1. *If* $(u_c, \lambda_c) \in S_c \times (0, +\infty)$ *is a couple solution to Equation* (1), *then* (u_c, λ_c) *is a couple solution to Equation* (1).

Our aim is to obtain the existence of the solutions to Equation (1) from the existence of solutions to Equation (1) on S_c. Now, we list some known lemmas:

Lemma 1 (see [28])**.** *Assume that $p \in (2, 2^*)$ if $N \geq 3$, $p \in (1, +\infty)$. Then, we have*

$$\|u\|_p \leq \left(\frac{p}{2\|Q\|_2^{p-2}} \right)^{\frac{1}{p}} \|\nabla u\|_2^{\frac{N(p-2)}{2p}} \|u\|_2^{1-\frac{N(p-2)}{2p}}$$

with equality only for $u = Q$, where Q is defined in Equation (7).

Lemma 2 (see [29])**.** *Let $1 \leq p < +\infty$, $1 \leq q < +\infty$ with $q \neq \frac{Np}{N-p}$ if $p < N$. Assume that $\{u_n\}$ is bounded in $L^q(\mathbb{R}^N)$, $\{\nabla u_n\}$ is bounded in $L^p(\mathbb{R}^N)$ and*

$$\sup_{y \in \mathbb{R}^N} \int_{y+B_R} |u_n(x)|^q dx \to 0, \quad \textit{for some } R > 0.$$

Then, $u_n \to 0$ in $L^\alpha(\mathbb{R}^N)$ for α between q and $\frac{Np}{N-p}$.

Lemma 3 (see [30])**.** *Suppose $f_n \to f$ a.e. and $\|f_n\|_p \leq C < +\infty$ for all n and for some $0 < p < +\infty$. Then, $f \in L^p(\mathbb{R}^N)$ and*

$$\lim_{n \to +\infty} (\|f_n\|_p^p - \|f_n - f\|_p^p) = \|f\|_p^p.$$

Set $H^1_{rad}(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) | u \text{ is a radially symmetric function}\}$.

Lemma 4 (Compactness Lemma; see [28])**.** *For $2 < p < 2^*$, the embedding $H^1_{rad}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is compact.*

Finally, we prove some new lemmas. Let

$$F_\lambda(u) = \frac{\lambda}{2}u^2 + \mu\frac{1}{q}|u|^q + \frac{1}{p}|u|^p = \int_0^u (\lambda s + \mu|s|^{q-2}s + |s|^{p-2}s) ds.$$

Now, we give the following Pohozaev identity for Equation (1):

Lemma 5. *Assume that $u \in H^1(\mathbb{R}^N)$ is a solution to Equation (1). Then, u satisfies*

$$(N-2)\left(\int_{\mathbb{R}^N} |u(x)|^2 dx \right)^\gamma \int_{\mathbb{R}^N} |\nabla u|^2 dx = 2N \int_{\mathbb{R}^N} F_\lambda(u(x)) dx.$$

Proof. We use the ideas in [31,32]. Set $T := \left(\int_{\mathbb{R}^N} |u(x)|^2 dx \right)^\gamma$. On a ball $B_R = \{x \in \mathbb{R}^N : |x| < R\}$, by multiplying Equation (1) by $x_i \frac{\partial u}{\partial x_i}$ and integrating on B_R, we have

$$-T \int_{B_R} \triangle u x_i \frac{\partial u}{\partial x_i} dx = \int_{B_R} (\lambda u + \mu|u|^{q-2} + |u|^{p-2}u) x_i \frac{\partial u}{\partial x_i} dx. \tag{12}$$

Let $\frac{\partial u}{\partial n}$ be the exterior normal derivative of u at $x \in \partial B_R$, dS be the differential area at $x \in \partial B_R$ and $\overrightarrow{n}_0 = (\cos\alpha_1, \cos\alpha_2, \cdots, \cos\alpha_N)$ be the unit vector of the outward normal direction at $x \in \partial B_R$. Obviously, $\frac{\partial u}{\partial n} = \sum_{i=1}^N \frac{\partial u}{\partial x_i} \cos\alpha_i = \nabla u \cdot \overrightarrow{n}_0$ and $|\overrightarrow{n}_0| = 1$. Since

$$\begin{aligned}\int_{B_R} \triangle u x_i \frac{\partial u}{\partial x_i} dx &= \int_{\partial B_R} x_i \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial n} dS - \int_{B_R} \nabla u \nabla (x_i \frac{\partial u}{\partial x_i}) dx \\ &= \int_{\partial B_R} x_i \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial n} dS - \int_{B_R} \nabla u [\frac{\partial u}{\partial x_i}\nabla x_i + x_i \nabla \frac{\partial u}{\partial x_i}] dx \\ &= \int_{\partial B_R} x_i \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial n} dS - \int_{B_R} (\frac{\partial u}{\partial x_i})^2 dx - \frac{1}{2}\int_{B_R} x_i \frac{\partial \sum_{j=1}^N (\frac{\partial u}{\partial x_j})^2}{\partial x_i} dx \\ &= \int_{\partial B_R} x_i \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial n} dS - \int_{B_R} (\frac{\partial u}{\partial x_i})^2 dx - \frac{1}{2}\int_{\partial B_R} x_i \cos\alpha_i |\nabla u|^2 dS \\ &\quad + \frac{1}{2}\int_{B_R} \sum_{j=1}^N (\frac{\partial u}{\partial x_j})^2 \frac{\partial x_i}{\partial x_i} dx \\ &= \int_{\partial B_R} x_i \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial n} dS - \int_{B_R} (\frac{\partial u}{\partial x_i})^2 dx - \frac{1}{2}\int_{\partial B_R} x_i \cos\alpha_i |\nabla u|^2 dS \\ &\quad + \frac{1}{2}\int_{B_R} \sum_{j=1}^N (\frac{\partial u}{\partial x_j})^2 dx,\end{aligned}$$

by summing from $i = 1$ to N, one has

$$\begin{aligned}\int_{B_R} \triangle u (x\nabla u) dx &= \int_{\partial B_R} \sum_{i=1}^N x_i \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial n} dS - \int_{B_R} |\nabla u|^2 dx - \frac{1}{2}\int_{\partial B_R} \sum_{i=1}^N (x_i \cos\alpha_i)|\nabla u|^2 dS \\ &\quad + \frac{N}{2}\int_{B_R} |\nabla u|^2 dx \\ &= \frac{1}{2}\int_{\partial B_R} \sum_{i=1}^N (x_i \cos\alpha_i)|\nabla u|^2 dS - \frac{2-N}{2}\int_{B_R} |\nabla u|^2 dx \\ &= \frac{1}{2} R \int_{\partial B_R} |\nabla u|^2 dS - \frac{2-N}{2}\int_{B_R} |\nabla u|^2 dx.\end{aligned}$$

Then, we have

$$-T\int_{B_R} \triangle u (x\nabla u) dx = T\Big(\frac{2-N}{2}\int_{B_R} |\nabla u|^2 dx - \frac{1}{2} R \int_{\partial B_R} |\nabla u|^2 dS\Big). \tag{13}$$

Since

$$\begin{aligned}\int_{B_R} (\lambda u + \mu |u|^{q-2} + |u|^{p-2} u) x_i \frac{\partial u}{\partial x_i} dx &= \int_{B_R} \frac{\partial F_\lambda(u)}{\partial x_i} x_i dx \\ &= \int_{\partial B_R} x_i \cos\alpha_i F_\lambda(u) dS - \int_{B_R} F_\lambda(u(x)) dx,\end{aligned}$$

then summing from $i = 1$ to N yields that

$$\begin{aligned}&\int_{B_R} (\lambda u + \mu |u|^{q-2} + |u|^{p-2} u)(x \cdot \nabla u) dx \\ &= \int_{\partial B_R} \sum_{i=1}^N (x_i \cos\alpha_i) F_\lambda(u(x)) dS - N \int_{B_R} F_\lambda(u(x)) dx \\ &= R\int_{\partial B_R} F_\lambda(u(x)) dS - N\int_{B_R} F_\lambda(u(x)) dx.\end{aligned} \tag{14}$$

By combining Equations (12)–(14), we have

$$T\Big(\frac{2-N}{2}\int_{B_R} |\nabla u|^2 dx - \frac{1}{2} R \int_{\partial B_R} |\nabla u|^2 dS\Big) = R\int_{\partial B_R} F_\lambda(u(x)) dx - N\int_{B_R} F_\lambda(u(x)) dx,$$

i.e.,

$$T\frac{2-N}{2}\int_{B_R}|\nabla u|^2dx+N\int_{B_R}F_\lambda(u(x))dx=\frac{1}{2}RT\int_{\partial B_R}|\nabla u|^2dS+R\int_{\partial B_R}F_\lambda(u(x))dS.$$

Since

$$\int_{\mathbb{R}^N}[T|\nabla u|^2+F_\lambda(u(x))]dx=T\int_0^{+\infty}\int_{\partial B_R}R|\nabla u|^2dSdR+\int_0^{+\infty}\int_{\partial B_R}RF_\lambda(u(x))dSdR<+\infty,$$

then there exists a sequence $R_n\to+\infty$ such that

$$TR_n\int_{\partial B_{R_n}}|\nabla u|^2dS+R_n\int_{\partial B_{R_n}}F_\lambda(u(x))dS\to 0 \text{ as } n\to+\infty.$$

Hence, we have

$$T\frac{2-N}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx+N\int_{\mathbb{R}^N}F_\lambda(u(x))dx=0,$$

In other words, we have

$$(N-2)\left(\int_{\mathbb{R}^N}|u(x)|^2dx\right)^\gamma\int_{\mathbb{R}^N}|\nabla u|^2dx=2N\int_{\mathbb{R}^N}F_\lambda(u(x))dx.$$

The proof is completed. □

3. Some Lemmas and the Proofs of Our Theorems

In order to simplify the equations in this section, define the functional $G : H^1(\mathbb{R}^N)\to\mathbb{R}$, where

$$G(u)=c^{2\gamma}\|\nabla u\|_2^2-\mu\frac{N(q-2)}{2q}\|u\|_q^q-\frac{N(p-2)}{2p}\|u\|_p^p.$$

For $r>0$, define

$$I_{r^2}=\inf_{u\in S_r}J_c,\ M_r=\{u\in S_r:G(u)=0\},\ m_r=\inf_{u\in M_r}J_c.$$

Let

$$S_{r,rad}:=\left\{u\in H^1_{rad}(\mathbb{R}^N):\|u\|_2^2=r^2,r>0\right\}$$

and

$$A_c(u):=\frac{1}{2}c^{2\gamma}\|\nabla u\|_2^2,\ \ B_1(u)=\frac{\mu}{q}\|u\|_q^q,\ \ B_2(u)=\frac{1}{p}\|u\|_p^p,\ \ u\in H^1(\mathbb{R}).$$

Then, we have

$$J_c(u):=A_c(u)-B_1(u)-B_2(u).$$

Lemma 6. *For $2<q<p<2^*$, if the functional $J_c|_{S_c}$ has a constraint critical point $u\in S_c$, then*

$$2A_c(u)-\frac{N(q-2)}{2}B_1(u)-\frac{N(p-2)}{2}B_2(u)=0$$

In addition, there exists $\lambda_c<0$ such that $J_c'(u)-\lambda_c u=0$.

Proof. Since $(J_c|_{S_c})'(u)=0$, there exists a $\lambda_c\in\mathbb{R}$ such that $J_c'(u)=\lambda_c u$ in $H^{-1}(\mathbb{R}^N)$, and hence

$$2A_c(u)-qB_1(u)-pB_2(u)=\lambda_c c^2. \tag{15}$$

Moreover, u satisfies Lemma 5 (Pohozaev identity):

$$(N-2)A_c(u) - N(B_1(u)+B_2(u)) = \frac{N\lambda_c}{2}c^2.$$

Then, we have

$$4A_c(u) - (q-2)NB_1(u) - (p-2)NB_2(u) = 0.$$

By putting the above equation into Equation (15), we have

$$\lambda_c = \frac{(q(N-2)-2N)B_1(u) + (p(N-2)-2N)B_2(u)}{2c^2}.$$

Now, $2 < q < p < \frac{2N}{N-2}$ implies $\lambda_c < 0$.
The proof is completed. □

Lemma 7. *Assume that $u \in S_c$:*

(1) *For $\frac{2N+4}{N} = q < p < 2^*$, if $N\gamma < 2$ and $c < \left(\frac{\mu}{\|Q\|_2^{\frac{4}{N}}}\right)^{\frac{N}{2(N\gamma-2)}}$, or if $N\gamma > 2$ and $c > \left(\frac{\mu}{\|Q\|_2^{\frac{4}{N}}}\right)^{\frac{N}{2(N\gamma-2)}}$, then $M_c \neq \emptyset$.*

(2) *If $\frac{2N+4}{N} < q < p < 2^*$, then $M_c \neq \emptyset$.*

(3) *For $2 < q < \frac{2N+4}{N} < p < 2^*$, if $N\gamma < 2$ and*

$$c < \frac{1}{\mu^{\frac{N(p-2)-4}{2(p-q)(2-\gamma N)}}}\|Q\|_2^{\frac{2}{2-\gamma N}}\left(\frac{4-N(q-2)}{N(p-q)}\right)^{\frac{4-N(q-2)}{2(p-q)(2-\gamma N)}}\left(\frac{N(p-2)-4}{N(p-q)}\right)^{\frac{N(p-2)-4}{2(p-q)(2-\gamma N)}}$$

or if $N\gamma > 2$ and

$$c > \frac{1}{\mu^{\frac{N(p-2)-4}{2(p-q)(2-\gamma N)}}}\|Q\|_2^{\frac{2}{2-\gamma N}}\left(\frac{4-N(q-2)}{N(p-q)}\right)^{\frac{4-N(q-2)}{2(p-q)(2-\gamma N)}}\left(\frac{N(p-2)-4}{N(p-q)}\right)^{\frac{N(p-2)-4}{2(p-q)(2-\gamma N)}},$$

then there exist two sets $E_1, E_2 \subset S_c$ and a positive constant $c_0 > 0$ such that

$$E_1 = \{u \in M_c : J_c(u) < 0\} \neq \emptyset \tag{16}$$

and

$$E_2 = \{u \in M_c : J_c(u) \geq c_0 > 0\} \neq \emptyset.$$

Proof. For $u \in S_c$, let $u^t := t^{\frac{N}{2}}u(tx)$. Then, $u^t \in S_c$. We consider the following path $w : [0, +\infty) \to \mathbb{R}$, defined as

$$w(t) = \frac{1}{2}c^{2\gamma}\|\nabla u\|_2^2 t^2 - \frac{\mu}{q}\|u\|_q^q t^{\frac{N(q-2)}{2}} - \frac{1}{p}\|u\|_p^p t^{\frac{N(p-2)}{2}},$$

In other words, $w(t) = J_c(u^t)$, and Lemma 1 implies that

$$w(t) \geq \frac{1}{2}c^{2\gamma}\|\nabla u\|_2^2 t^2 - \mu\frac{1}{2\|Q\|_2^{q-2}}\|\nabla u\|_2^{\frac{N(q-2)}{2}} c^{q-\frac{N(q-2)}{2}} t^{\frac{N(q-2)}{2}} - \frac{1}{2\|Q\|_2^{p-2}}\|\nabla u\|_2^{\frac{N(p-2)}{2}} c^{p-\frac{N(p-2)}{2}} t^{\frac{N(p-2)}{2}}.$$

Set

$$h(t) = \frac{1}{2}c^{2\gamma}t^2 - \mu\frac{1}{2\|Q\|_2^{q-2}}c^{q-\frac{N(q-2)}{2}} t^{\frac{N(q-2)}{2}} - \frac{1}{2\|Q\|_2^{p-2}}c^{p-\frac{N(p-2)}{2}} t^{\frac{N(p-2)}{2}}. \tag{17}$$

Now, we consider three cases for different p and q values:

(1) $\frac{2N+4}{N} = q < p < 2^*$.

If $N\gamma < 2$ and $c < \left(\frac{\mu}{\|Q\|_2^{\frac{4}{N}}}\right)^{\frac{N}{2(N\gamma-2)}}$, or if $N\gamma > 2$ and $c > \left(\frac{\mu}{\|Q\|_2^{\frac{4}{N}}}\right)^{\frac{N}{2(N\gamma-2)}}$, then we have

$$c^{2\gamma} > \mu \frac{1}{\|Q\|_2^{\frac{4}{N}}} c^{\frac{4}{N}},$$

which implies that

$$\frac{1}{2}c^{2\gamma}\|\nabla u\|_2^2 t^2 - \frac{\mu}{q}\|u\|_q^q t^2 \geq \frac{1}{2}\|\nabla u\|_2^2 t^2 (c^{2\gamma} - \mu \frac{1}{\|Q\|_2^{\frac{4}{N}}} c^{\frac{4}{N}}) > 0.$$

Then, for

$$w(t) = \frac{1}{2}c^{2\gamma}\|\nabla u\|_2^2 t^2 - \frac{\mu}{q}\|u\|_q^q t^2 - \frac{1}{p}\|u\|_p^p t^{\frac{N(p-2)}{2}},$$

there exists a unique positive $t_0 := \left(\frac{4p\left(\frac{1}{2}c^{2\gamma}\|\nabla u\|_2^2 - \frac{\mu}{q}\|u\|_q^q\right)}{N(p-2)\|u\|_p^p}\right)^{\frac{1}{\frac{N(p-2)}{2}-2}}$ such that $w'(t_0) = 0$ and $w'(t) > 0$ for all $t \in (0, t_0)$ and $w'(t) < 0$ for all $t \in (t_0, +\infty)$, which implies that $J_c(u^{t_0}) = \max_{t>0} J_c(u^t)$. From $w'(t_0) = 0$ (note that $\frac{2N+4}{N} = q$), we have

$$0 = w'(t_0) = c^{2\gamma}\|\nabla u\|_2^2 t_0 - 2\frac{\mu}{q}\|u\|_q^q t_0 - \frac{1}{p}\frac{N(p-2)}{2}\|u\|_p^p t_0^{\frac{N(p-2)}{2}-1},$$

In other words, we have

$$\begin{aligned} 0 &= c^{2\gamma}\|\nabla u\|_2^2 t_0^2 - 2\frac{\mu}{q}\|u\|_q^q t_0^2 - \frac{1}{p}\frac{N(p-2)}{2}\|u\|_p^p t_0^{\frac{N(p-2)}{2}} \\ &= c^{2\gamma}\int_{\mathbb{R}^N}|\nabla u^{t_0}(x)|^2 dx - 2\frac{\mu}{q}\int_{\mathbb{R}^N}|u^{t_0}(x)|^q dx - \frac{1}{p}\frac{N(p-2)}{2}\int_{\mathbb{R}^N}|u^{t_0}(x)|^p dx. \end{aligned}$$

Since $u^{t_0} \in S_c$ for $u \in S_c$, we have

$$\{u \in S_c | G(u) = 0\} \neq \emptyset.$$

(2) Since $\frac{2N+4}{N} < q < p$, we have $\frac{N(q-2)}{2} - 2 > 0$, $\frac{N(p-2)}{2} - 2 > 0$ and

$$w'(t) = t(c^{2\gamma}\|\nabla u\|_2^2 - \frac{N(q-2)}{2}t^{\frac{N(q-2)}{2}-2}\frac{\mu}{q}\|u\|_q^q - \frac{N(p-2)}{2}t^{\frac{N(p-2)}{2}-2}\frac{1}{p}\|u\|_p^p),$$

then there exists a unique $t_0 > 0$ such that $w'(t_0) = 0$ and $w'(t) > 0$ for all $t \in (0, t_0)$ and $w'(t) < 0$ for all $t \in (t_0, +\infty)$, which implies that $J_c(u^{t_0}) = \max_{t>0} J_c(u^t)$ and

$$t_0^2 c^{2\gamma}\|\nabla u\|_2^2 = \mu t_0^{\frac{N(q-2)}{2}}\frac{N(q-2)}{2q}\|u\|_q^q + t_0^{\frac{N(p-2)}{2}}\frac{N(p-2)}{2p}\|u\|_p^p.$$

Then, we have

$$\{u \in S_c | G(u) = 0\} \neq \emptyset.$$

(3) For $2 < q < \frac{2N+4}{N} < p < 2^*$, since one of $N\gamma < 2$ and

$$c < \frac{1}{\mu^{\frac{N(p-2)-4}{2(p-q)(2-\gamma N)}}}\|Q\|_2^{\frac{2}{2-\gamma N}}(\frac{4-N(q-2)}{N(p-q)})^{\frac{4-N(q-2)}{2(p-q)(2-\gamma N)}}(\frac{N(p-2)-4}{N(p-q)})^{\frac{N(p-2)-4}{2(p-q)(2-\gamma N)}}$$

or $N\gamma > 2$ and

$$c > \frac{1}{\mu^{\frac{N(p-2)-4}{2(p-q)(2-\gamma N)}}}\|Q\|_2^{\frac{2}{2-\gamma N}}\left(\frac{4-N(q-2)}{N(p-q)}\right)^{\frac{4-N(q-2)}{2(p-q)(2-\gamma N)}}\left(\frac{N(p-2)-4}{N(p-q)}\right)^{\frac{N(p-2)-4}{2(p-q)(2-\gamma N)}},$$

is true, we find that

$$\mu c^{\frac{4(p-q)}{N(p-2)-4}} < c^{2\gamma\frac{N(p-q)}{N(p-2)-4}}\|Q\|_2^{\frac{4(p-q)}{N(p-2)-4}}\left(\frac{4-N(q-2)}{N(p-q)}\right)^{\frac{4-N(q-2)}{N(p-2)-4}}\frac{N(p-2)-4}{N(p-q)}. \tag{18}$$

The discussion is divided into three steps:

Step 1. We claim that there exists an interval (a_c, b_c) such that $h(t) < 0$ for all $t \in (0, a_c)$, $h(t') > 0$ for some point $t' \in (a_c, b_c)$ and $h(t) < 0$ for all $t > +\infty$.

Obviously, $h(t) > 0$ if and only if

$$\varphi(t) > \mu\frac{1}{2\|Q\|^{q-2}}c^{q-\frac{N(q-2)}{2}},$$

Here, we have

$$\varphi(t) = t^{2-\frac{N(q-2)}{2}}\frac{1}{2}c^{2\gamma} - t^{\frac{N(p-2)}{2}-\frac{N(q-2)}{2}}\frac{1}{2\|Q\|_2^{p-2}}c^{p-\frac{N(p-2)}{2}}.$$

Since $2 < q < \frac{2N+4}{N} < p < 2^*$, it can be seen that $\varphi(t)$ has a unique critical point $\bar{t}$, which is a global maximum point at a positive level, where

$$\bar{t} = \left(\frac{c^{2\gamma}\|Q\|_2^{p-2}(4-N(q-2))}{N(p-q)c^{p-\frac{N(p-2)}{2}}}\right)^{\frac{2}{N(p-2)-4}},$$

and the maximum of $\varphi(t)$ on $(0, +\infty)$ is

$$\varphi(\bar{t}) = \bar{t}^{2-\frac{N(q-2)}{2}}\frac{1}{2}c^{2\gamma} - \bar{t}^{\frac{N(p-2)}{2}-\frac{N(q-2)}{2}}\frac{1}{2\|Q\|_2^{p-2}}c^{p-\frac{N(p-2)}{2}}.$$

From Equation (18), we have

$$\varphi(\bar{t}) > \mu\frac{1}{2\|Q\|^{q-2}}c^{q-\frac{N(q-2)}{2}},$$

Thus, $h(\bar{t}) > 0$. Since $h(t) < 0$ for a small enough t and $\lim_{\to+\infty} h(t) = -\infty$, there exists an interval (a_c, b_c) such that $h(t) < 0$ for all $t \in (0, a_c) \cup (b_c, +\infty)$ and $h(t') > 0$ for some $t' \in (a_c, b_c)$.

Step 2. We claim that there exists an interval (a_u, b_u) such that $w(t) < 0$ for all $t \in (0, a_u)$, $w(t) > 0$ for some point $t'' \in (a_u, b_u)$ and $h(t) < 0$ for all $t > b_u$.

Lemma 1 implies that

$$\begin{aligned} w(t) &\geq t^2\frac{1}{2}c^{2\gamma}\|\nabla u\|_2^2 - t^{\frac{N(q-2)}{2}}\mu\frac{1}{2\|Q\|_2^{q-2}}\|\nabla u\|_2^{\frac{N(q-2)}{2}}c^{q-\frac{N(q-2)}{2}} \\ &\quad - t^{\frac{N(p-2)}{2}}\frac{1}{2\|Q\|_2^{p-2}}\|\nabla u\|_2^{\frac{N(p-2)}{2}}c^{p-\frac{N(q-2)}{2}} \\ &= \frac{1}{2}c^{2\gamma}(t\|\nabla u\|_2)^2 - t^{\frac{N(q-2)}{2}}\mu\frac{1}{2\|Q\|_2^{q-2}}(\|\nabla u\|_2 t)^{\frac{N(q-2)}{2}}c^{q-\frac{N(q-2)}{2}} \\ &\quad - t^{\frac{N(p-2)}{2}}\frac{1}{2\|Q\|_2^{p-2}}(\|\nabla u\|_2 t)^{\frac{N(p-2)}{2}}c^{p-\frac{N(q-2)}{2}} \\ &= \frac{1}{2}c^{2\gamma}s^2 - \mu\frac{1}{2\|Q\|_2^{q-2}}s^{\frac{N(q-2)}{2}}c^{q-\frac{N(q-2)}{2}} - t^{\frac{N(p-2)}{2}}\frac{1}{2\|Q\|_2^{p-2}}s^{\frac{N(p-2)}{2}}c^{p-\frac{N(q-2)}{2}} \\ &= h(s), \end{aligned}$$

where $s := t\|\nabla u\|_2$ and $h(s)$ is defined in Equation (17).

Since $w(t) < 0$ for small enough values of t, $\lim_{t\to+\infty}\gamma(t) = -\infty$, and there is a $t' \in (\frac{1}{\|\nabla u\|_2}a_c, \frac{1}{\|\nabla u\|_2}b_c)$ such that $w(t') > 0$, then there exists an interval (a_u, b_u) such that $w(t) < 0$ for all $t \in (0, a_u) \cup (b_u, +\infty)$ and $g(t'') > 0$ for some $t'' \in (a_u, b_u)$.

Step 3. We claim that there exist two sets E_1, $E_2 \subset S_c$ and a positive constant $c_0 > 0$ such that

$$E_1 = \{u \in M_c : J_c(u) < 0\} \neq \emptyset$$

and

$$E_2 = \{u \in M_c : J_c(u) \geq c_0 > 0\} \neq \emptyset.$$

Claim 2 implies that there exists an interval (a_u, b_u) such that $w(t) < 0$ for all $t \in (0, a_u)$, $w(t) > 0$ for some point $t'' \in (a_u, b_u)$ and $h(t) < 0$ for all $t > b_u$.

Define $t_* = \min\{s|w(s) = \min_{t\in(0,a_u)} w(t)\}$ and $t^* = \max\{s|w(s) = \max_{t\in(a_u,b_u)} w(t)\}$. Obviously, we have

$$w'(t_*) = 0, \;\; w'(t^*) = 0,$$

In other words, we have

$$(t_*)^2c^{2\gamma}\|\nabla u\|_2^2 = \mu(t_*)^{\frac{N(q-2)}{2}}\frac{N(q-2)}{2q}\|u\|_q^q + (t_*)^{\frac{N(p-2)}{2}}\frac{N(p-2)}{2p}\|u\|_p^p$$

and

$$(t^*)^2c^{2\gamma}\|\nabla u\|_2^2 = \mu(t^*)^{\frac{N(q-2)}{2}}\frac{N(q-2)}{2q}\|u\|_q^q + (t^*)^{\frac{N(p-2)}{2}}\frac{N(p-2)}{2p}\|u\|_p^p.$$

Then, we have

$$c^{2\gamma}\|\nabla u^{t_*}\|_2^2 = \mu\frac{N(q-2)}{2q}\|u^{t_*}\|_q^q + \frac{N(p-2)}{2p}\|u^{t_*}\|_p^p$$

and

$$c^{2\gamma}\|\nabla u^{t^*}\|_2^2 = \mu\frac{N(q-2)}{2q}\|u^{t^*}\|_q^q + \frac{N(p-2)}{2p}\|u^{t^*}\|_p^p.$$

Set $c_0 := \max_{t\in[0,+\infty)} h(t)$. From $w(t) \geq h(t\|\nabla u\|_2)$ for all $t > 0$, we have

$$w(t^*) \geq c_0.$$

The proof is completed. □

Lemma 8. *Assume that $u \in S_r$:*

(1) *If* $\frac{2N+4}{N} = q < p < 2^*$ *and* $0 < r < \left(\frac{\|Q\|_2^{\frac{4}{N}}}{\mu} c^{2\gamma}\right)^{\frac{N}{4}} = \frac{\|Q\|_2}{\mu^{\frac{N}{4}}} c^{\frac{N\gamma}{2}}$, *then* $M_c \neq \emptyset$.

(2) *If* $\frac{2N+4}{N} < q < p < 2^*$, *then* $M_c \neq \emptyset$.

(3) *If* $2 < q < \frac{2N+4}{N} < p < 2^*$ *and*

$$\mu r^{\frac{4(p-q)}{N(p-2)-4}} < c^{2\gamma \frac{N(p-q)}{N(p-2)-4}} \|Q\|_2^{\frac{4(p-q)}{N(p-2)-4}} \left(\frac{4-N(q-2)}{N(p-q)}\right)^{\frac{4-N(q-2)}{N(p-2)-4}} \frac{N(p-2)-4}{N(p-q)}, \tag{19}$$

then there exist two sets $E_{1,r}$, $E_{2,r} \subset S_r$ *and a positive constant* $c_{0,r} > 0$ *such that*

$$E_{1,r} = \{u \in M_r : J_c(u) < 0\} \neq \emptyset \tag{20}$$

and

$$E_{2,r} = \{u \in M_r : J_c(u) \geq c_{0,r} > 0\} \neq \emptyset.$$

The proof is the same as that in Lemma 8, so we omitted it.

For $r > 0$, set

$$I_{r^2} := \inf_{u \in S_r} J_c(u).$$

Lemma 9. *(1)* *If* $2 < q < p < \frac{2N+4}{N}$, *then for each* $r > 0$, I_{r^2} *is well-defined, and* $I_{r^2} < 0$ *for* $r \in (0, +\infty)$, *while the function* $r \to I_{r^2}$ *is continuous on* $(0, +\infty)$;

(2) *If* $2 < q < p = \frac{2N+4}{N}$, *then* I_{r^2} *is well-defined, and* $I_{r^2} < 0$ *for* $r \in (0, c^{\frac{N\gamma}{2}} \|Q\|_2)$, *while the function* $r \to I_{r^2}$ *is continuous on* $(0, c^{\frac{N\gamma}{2}} \|Q\|_2)$;

(3)

$$I_{r^2} < I_{r^2-\alpha^2} + I_{\alpha^2}, \forall 0 < \alpha < r.$$

Proof. For $u \in S_r$, set $u^t(x) := t^{\frac{N}{2}} u(tx)$ for $t > 0$. Then, $u^t \in S_r$ and

$$I_{r^2} \leq J_c(u^t) = \frac{1}{2} c^{2\gamma} t^2 \|\nabla u\|_2^2 - \mu t^{\frac{N(q-2)}{2}} \frac{1}{q} \|u\|_q^q - t^{\frac{N(p-2)}{2}} \frac{1}{p} \|u\|_p^p, t > 0.$$

(1) In the case $2 < q < p < \frac{2N+4}{N}$, for $u \in S_r$, we have

$$\begin{aligned} J_c(u) &= \frac{1}{2} c^{2\gamma} \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - \frac{\mu}{q} \|u\|_q^q - \frac{1}{p} \|u\|_p^p \\ &\geq \frac{1}{2} c^{2\gamma} \|\nabla u\|_2^2 - \mu \frac{r^{q-\frac{N(q-2)}{2}}}{2\|Q\|_2^{q-2}} \|\nabla u\|_2^{\frac{N(q-2)}{2}} - \frac{r^{p-\frac{N(p-2)}{2}}}{2\|Q\|_2^{p-2}} \|\nabla u\|_2^{\frac{N(p-2)}{2}}. \end{aligned} \tag{21}$$

Since q, $p \in (2, \frac{2N+4}{N})$, we have $\frac{N(p-2)-2}{2} < 2$ and $\frac{N(q-2)-2}{2} < 2$, which implies that $J_c(u)$ is bounded from below and $J_c(u^t) < 0$ for small enough t values. Hence, $I_{r^2} < 0$ is well-defined for all $r \in (0, +\infty)$.

The proof of continuity of I_{r^2} is the same as that in Theorem 2.1 in [33], so we omitted it.

(2) Take the case $2 < q < p = \frac{2N+4}{N}$.

For $u \in S_r$, we have

$$\begin{aligned} J_c(u) &\geq \frac{1}{2} c^{2\gamma} \|\nabla u\|_2^2 - \mu \frac{r^{q-\frac{N(q-2)}{2}}}{2\|Q\|_2^{q-2}} \|\nabla u\|_2^{\frac{N(q-2)}{2}} - \frac{r^{\frac{4}{N}}}{2\|Q\|_2^{p-2}} \|\nabla u\|_2^2 \\ &\geq \frac{1}{2} \left(c^{2\gamma} - \frac{r^{\frac{4}{N}}}{\|Q\|_2^{\frac{4}{N}}}\right) \|\nabla u\|_2^2 - \mu \frac{r^{q-\frac{N(q-2)}{2}}}{2\|Q\|_2^{q-2}} \|\nabla u\|_2^{\frac{N(q-2)}{2}}. \end{aligned} \tag{22}$$

Since $q \in (2, \frac{2N+4}{N})$, we have $\dfrac{N(q-2)-2}{2} < 2$, which together with $r < c^{\frac{N\gamma}{2}}\|Q\|_2$ implies that $J_c(u)$ is bounded from below and $J_c(u^t) < 0$ for small enough t values. Hence, $I_{r^2} < 0$ is well-defined for all $r \in (0, c^{\frac{N\gamma}{2}}\|Q\|_2)$.

The proof of continuity of I_{r^2} is the same as that in Theorem 2.1 in [33], so we omitted it.

(3) We can obtain our results from Theorem 2.1 in [16,33], so we omitted the proof.

The proof is completed. □

Set

$$m_r := \inf_{u\in M_r} J_c(u)$$

and

$$m_c := \inf_{u\in M_c} J_c(u).$$

Lemma 10. *Assume that one case of the following conditions hold:*

(1) $\frac{2N+4}{N} = q < p < 2^*$, $N\gamma < 2$ *and* $c < \left(\frac{\mu}{\|Q\|_2^{\frac{4}{N}}}\right)^{\frac{N}{2(N\gamma-2)}}$.

(2) $\frac{2N+4}{N} = q < p < 2^*$, $N\gamma > 2$ *and* $c > \left(\frac{\mu}{\|Q\|_2^{\frac{4}{N}}}\right)^{\frac{N}{2(N\gamma-2)}}$.

(3) $\frac{2N+4}{N} < q < p < 2^*$.

Then, M_c is a C^1 manifold.

Proof. From Lemma 7, $M_c \neq \emptyset$, provided that one condition of our lemma holds.

We show that $G'(u) \neq 0$ for $u \in M_c$.

For any $u \in M_c$, if $p > q \geq \dfrac{2N+4}{2}$, we have

$$\begin{aligned}
(G'(u), u) &= 2c^{2\gamma}\|\nabla u\|_2^2 - \mu\frac{N(q-2)}{2}\|u\|_q^q - \frac{N(p-2)}{2}\|u\|_p^p \\
&= 2[\mu\frac{N(q-2)}{2q}\|u\|_q^q + \frac{N(p-2)}{2p}\|u\|_p^p] \\
&\quad -\mu\frac{N(q-2)}{2}\|u\|_q^q - \frac{N(p-2)}{2}\|u\|_p^p \\
&= \mu\frac{N(q-2)}{2}(\frac{2}{q}-1)\|u\|_q^q + \frac{N(p-2)}{2}(\frac{2}{p}-1)\|u\|_p^p \\
&< 0.
\end{aligned}$$

Hence, M_c is a C^1 manifold. □

Lemma 11. *Assume that one of the following conditions holds:*

(1) $\frac{2N+4}{N} = q < p < 2^*$, $N\gamma < 2$ *and* $c < \left(\frac{\mu}{\|Q\|_2^{\frac{4}{N}}}\right)^{\frac{N}{2(N\gamma-2)}}$.

(2) $\frac{2N+4}{N} = q < p < 2^*$, $N\gamma > 2$ *and* $c > \left(\frac{\mu}{\|Q\|_2^{\frac{4}{N}}}\right)^{\frac{N}{2(N\gamma-2)}}$.

(3) $\frac{2N+4}{N} < q < p < 2^*$.

Then, any critical point of $J_c|_{M_c}$ is also a a critical point of $J_c|_{S_c}$.

Proof. Suppose that u is a critical point of $J_c|_{M_c}$ (i.e., $u \in M_c$ and $(J_c|_{M_c})'(u) = 0$), which implies that there exist λ_1 and λ_2 such that

$$J_c'(u) - \lambda_1 G'(u) - \lambda_2 u = 0 \text{ in } H^{-1}(\mathbb{R}^N). \tag{23}$$

Now, we prove that $\lambda_1 = 0$ and $\lambda_2 \neq 0$.
In fact, if $\lambda_1 \neq 0$, then it follows from Equation (23) that

$$(1-2\lambda_1)c^{2\gamma}\|\nabla u\|_2^2 - \mu(1-\frac{N(q-2)}{2}\lambda_1)\|u\|_q^q - (1-\frac{N(p-2)}{2}\lambda_1)\|u\|_p^p - \lambda_2 c^2 = 0 \quad (24)$$

Additionally, the Pohozaev identity for Equation (23) implies that

$$\frac{N-2}{2}(1-2\lambda_1)c^{2\gamma}\|\nabla u\|_2^2 - \mu\frac{N}{q}(1-\frac{N(q-2)}{2}\lambda_1)\|u\|_q^q - \frac{N}{p}(1-\frac{N(p-2)}{2}\lambda_1)\|u\|_p^p - \frac{N}{2}\lambda_2 c^2 = 0. \quad (25)$$

From Equations (24) and (25) and $G(u) = 0$, we have that

$$\begin{cases} c^{2\gamma}\|\nabla u\|_2^2 - \mu\frac{N(q-2)}{2q}\|u\|_q^q - \frac{N(p-2)}{2p}\|u\|_p^p = 0, \\ (1-2\lambda_1)c^{2\gamma}\|\nabla u\|_2^2 - \mu\frac{2}{q}(1-\frac{N(q-2)}{2}\lambda_1)\|u\|_q^q - \frac{N(p-2)}{2p}(1-\frac{N(p-2)}{2}\lambda_1)\|u\|_p^p = 0, \end{cases}$$

and then

$$\begin{cases} c^{2\gamma}\|\nabla u\|_2^2 - \mu\frac{N(q-2)}{2q}\|u\|_q^q - \frac{N(p-2)}{2p}\|u\|_p^p = 0, \\ 2c^{2\gamma}\|\nabla u\|_2^2 - \mu\frac{N(q-2)}{2q}\frac{N(q-2)}{2}\|u\|_q^q - \frac{N(p-2)}{2p}\frac{N(p-2)}{2}\|u\|_p^p = 0. \end{cases}$$

Hence, we have

$$\mu 2\frac{N(q-2)}{2q}\|u\|_q^q + 2\frac{N(p-2)}{2p}\|u\|_p^p = \mu\frac{N(q-2)}{2q}\frac{N(q-2)}{2}\|u\|_q^q + \frac{N(p-2)}{2p}\frac{N(p-2)}{2}\|u\|_p^p. \quad (26)$$

Since $\frac{2N+4}{N} \leq q < p < 2^*$, we have $2 \leq \frac{N(q-2)}{2}$ and $2 < \frac{N(p-2)}{2}$, which implies that

$$\mu 2\frac{N(q-2)}{2q}\|u\|_q^q + 2\frac{N(p-2)}{2p}\|u\|_p^p < \mu\frac{N(q-2)}{2q}\frac{N(q-2)}{2}\|u\|_q^q + \frac{N(p-2)}{2p}\frac{N(p-2)}{2}\|u\|_p^p.$$

This contradicts Equation (26).
The proof is completed. □

Now, we assume condition (3) of Lemma 7 and construct a set of some paths.
Let

$$H(u,s)(x) = e^{\frac{sN}{2}}u(e^s x), \ \forall (u,s) \in H^1 \times \mathbb{R}$$

and

$$\bar{J}_c(u,s) = \frac{e^{2s}}{2}c^{2\gamma}\|\nabla u\|_2^2 - [\mu\frac{1}{q}e^{\frac{N(q-2)}{2}s}\|u\|_q^q + \frac{1}{p}e^{\frac{N(p-2)}{2}s}\|u\|_p^p].$$

Obviously, we have

$$\bar{J}_c(u,s) = J_c(H(u,s))$$

In addition, for $u \in S_c$, we have

$$\bar{J}_c(u,s) = \gamma_u(e^s) \geq h(e^s).$$

According to the definitions of E_1 and E_2 in Lemma 7, for $u \in S_c$, there exists $s_1 < \bar{s} < s_2$ such that $H(u,s_1) \in E_1$, $H(u,\bar{s}) \in E_2$ and $H(u,s_2) \in S_c$ with $J_c(H(u,s_2)) = \bar{J}_c(u,s_2) < 0$.
Define

$$\Gamma = \{\psi \in C([0,1],S_c) | \psi(0) = H(u,s_1), \psi(1) = H(u,s_2)\}$$

and

$$\bar{\Gamma} = \{\bar{\psi} \in C([0,1],S_c \times \mathbb{R}) | \bar{\psi}(0) = (H(u,s_1),0), \bar{\psi}(1) = (H(u,s_2),0)\}.$$

Set

$$c_1 = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J_c(\psi(t)) > 0.$$

and

$$\bar{c}_1 = \inf_{\gamma\in\overline{\Gamma}} \sup_{t\in[0,1]} \overline{J}_c(\gamma(t)) > 0.$$

Obviously, $c_1 = \bar{c}_1 > 0$.

Our ideas came from [14,18].

Lemma 12 (see [18]). *For $2 < q < \frac{2N+4}{N} < p < 2^*$, let $\{g_n\} \subseteq \overline{\Gamma}$ be such that*

$$\max_{t\in[0,1]} \overline{J}_c(g_n(t)) \le c_1 + \frac{1}{n}.$$

Then, there exists a sequence $\{(u_n, s_n)\} \subseteq S_c \times \mathbb{R}$ such that the following are true:

(1) $\overline{J}_c(u_n, s_n) \in [c_1 - \frac{1}{n}, c_1 + \frac{1}{n}]$*;*

(2) $\min_{t\in[0,1]} \|(u_n, s_n) - g_n(t)\|_{\mathrm{H}^1} \le \frac{1}{\sqrt{n}}$*;*

(3) $\|\overline{J}_c'|_{S_c\times\mathbb{R}}(u_n, s_n)\| \le \frac{2}{\sqrt{n}}$*; in other words, we have*

$$|(\overline{J}_c'(u_n, s_n), z)_{(\mathrm{H}^1\times\mathbb{R})^{-1}\times \mathrm{H}^1\times\mathbb{R}}| \le \frac{2}{\sqrt{n}} \|z\|_{(\mathrm{H}^1\times\mathbb{R})}$$

for all

$$z \in \overline{T}_{(u_n,s_n)} = \{(z_1, z_2) \in \mathrm{H}^1 \times \mathbb{R} : (u_n, z_1)_{L^2(R^N)} = 0\}.$$

Remark 2. *Although the conditions are different from those in Proposition 2.2 [18], we can find the conclusion above via the same proof as in [18], so we omitted the proof.*

Lemma 13. *For $2 < q < \frac{2N+4}{N} < p < 2^*$, there exists a sequence $\{v_n\} \subseteq S_c$ such that the following are true:*

(1) $J_c(v_n) \to c_1 > 0$*;*

(2) $\{v_n\}$ *is bounded in* H^1*;*

(3) $\|J_c'(v_n)\|_{(H^1)^{-1}} \to 0$*;*

(4) $c^{2\gamma}\|\nabla v_n\|_2^2 - \mu\frac{1}{q}\frac{N(q-2)}{2}\|v_n\|_q^q - \frac{1}{p}\frac{N(p-2)}{2}\|v_n\|_p^p \to 0$*, as* $n \to +\infty$*.*

Proof. Now, we use Lemma 12 to show that there exists $\{v_n\} \subseteq S_c$ such that as $n \to +\infty$, the following is true:

$$J_c(v_n) \to c_1 \text{ and } \|J_c'(v_n)\| \to 0, \quad \text{as } n \to +\infty.$$

In Lemma 12, let $g_n(t) = ((g_n)_1(t), 0) \in H^1 \times \mathbb{R}$, $\forall t \in [0,1]$ such that $\overline{J}_c(g_n) \in [c_1 - \frac{1}{n}, c_1 + \frac{1}{n}]$.

Let $\partial_s \overline{J}_c(u_n, s_n) = (\overline{J}_c(s_n, u_n), (0,1))_{(H^1\times\mathbb{R})^{-1}\times(H^1\times\mathbb{R})}$. From point (3) of Lemma 12, we see that

$$\partial_s \overline{J}_c(u_n, s_n) \to 0 \text{ as } n \to +\infty$$

with

$$\partial_s \overline{J}_c(u_n, s_n) = c^{2\gamma}\|\nabla v_n\|_2^2 - \mu\frac{1}{q}\frac{N(q-2)}{2}\|v_n\|_q^q - \frac{1}{p}\frac{N(p-2)}{2}\|v_n\|_p^p,$$

where $v_n = H(u_n, s_n)$. Then, point (4) of our lemma is true.

Since

$$\overline{J}_c(u_n, s_n) = \frac{1}{2}c^{2\gamma}\|\nabla v_n\|^2 - \mu\frac{1}{q}\|v_n\|_q^q - \frac{1}{p}\|v_n\|_p^p$$

is also bounded, there exists a constant $C > 0$ independent of n such that

$$|N\bar{J}_c(u_n, s_n) + \partial_s \bar{J}'_c(u_n, s_n)| \le C,$$

which together with

$$N\bar{J}_c(u_n, s_n) + \partial_s \bar{J}'_c(u_n, s_n) = \frac{N+2}{2}\|\nabla v_n\|^2 - \mu\frac{1}{q}(\frac{N(q-2)}{2} + N)\|v_n\|_q^q - \frac{1}{p}(\frac{N(p-2)}{2} + N)\|v_n\|_p^p$$

implies that

$$-C \le N\bar{J}_c(u_n, s_n) + \partial_s \bar{J}'_c(u_n, s_n) = \frac{N+2}{2}\|\nabla v_n\|^2 - \mu\frac{1}{q}(\frac{N(q-2)}{2} + N)\|v_n\|_q^q - \frac{1}{p}(\frac{N(p-2)}{2} + N)\|v_n\|_p^p. \quad (27)$$

On the other hand, using the boundedness of $\{\bar{J}_c(u_n, s_n)\}$, it follows that

$$\|\nabla v_n\| \le 2C + 2\mu\frac{1}{q}\|v_n\|_q^q + \frac{2}{p}\|v_n\|_p^p. \quad (28)$$

From Equations (27) and (28), we deduce that

$$(N - \frac{2}{p}(N+2))\|v_n\|_p^p \le C + \mu(\frac{2}{q}(N+2) - N)\|v_n\|_q^q. \quad (29)$$

It follows from Equations (28) and (29) that

$$\begin{aligned} \|\nabla v_n\|^2 &\le C' + \left(2\mu\frac{1}{q} + \frac{2}{p}\frac{\mu(\frac{2}{q}(N+2) - N)}{(N - \frac{2}{p}(N+2))}\right)\|v_n\|_q^q \\ &\le C' + \left(2\mu\frac{1}{q} + \frac{2}{p}\frac{\mu(\frac{2}{q}(N+2) - N)}{(N - \frac{2}{p}(N+2))}\right)\left(\frac{q}{2\|Q\|_2^{q-2}}\right)\|\nabla v_n\|_2^{\frac{N(q-2)}{2}}\|v_n\|_2^{1-\frac{N(q-2)}{2}}. \end{aligned}$$

Then, $\left\{\int_{R^N}|\nabla v_n|^2 dx\right\}$ is bounded, and from $\{v_n\} \in S_c$, we have $\{v_n\}$ bounded in H^1. Then, point (2) is true.

Now, point (1) is trivial since $J_c(v_n) = J_c(H(u_n, s_n)) == \bar{J}_c(u_n, s_n)$.

Let $h_n \in T_{v_n}$. Now, we show that $\|J'_c(v_n)|_{S_c}\| \le \frac{4}{\sqrt{n}} \to 0$; in other words, we have

$$(J'_c(v_n), z)_{(H^1)^{-1}\times H^1} \le \frac{4}{\sqrt{n}}\|z\|_{H^1}.$$

for all $z \in T_{v_n} = \{z \in E, (v_n, z)_{L^2(\mathbb{R}^N)} = 0\}$.

Since

$$\begin{aligned} &(J'_c(v_n), h_n)_{(H^1)^{-1}\times H^1} \\ &= c^{2\gamma}\int_{R^N}\nabla v_n(x)\nabla h_n(x)dx - \int_{R^N}[\mu v_n^q(x) + v_n^p(x)]h_n(x)dx \\ &= c^{2\gamma}\int_{R^N}\nabla e^{\frac{s_n(N+2)}{2}}u_n(e^{s_n}x)\nabla h_n(x)dx - \int_{R^N}[\mu(e^{\frac{s_nN}{2}}u_n(e^{s_n}x))^q + (e^{\frac{s_nN}{2}}u_n(e^{s_n}x))^p]h_n(x)dx \\ &= c^{2\gamma}e^{-Ns_n}\int_{R^N}\nabla e^{\frac{s_n(N+2)}{2}}u_n(x)\nabla h_n(e^{-s_n}x)dx \\ &\quad -e^{-Ns_n}\int_{R^N}[\mu(e^{\frac{s_nN}{2}}u_n(x)^q + (e^{\frac{s_nN}{2}}u_n(x))^p]h_n(e^{-s_n}x)dx \\ &= e^{2s_n}c^{2\gamma}\int_{R^N}\nabla u_n(x)e^{-s_n\frac{N+2}{2}}\nabla h_n(e^{-s_n}x)dx \\ &-e^{-\frac{s_nN}{2}}\int_{R^N}[\mu(e^{\frac{s_nN}{2}}u_n(x)^q + (e^{\frac{s_nN}{2}}u_n(x))^p]e^{-\frac{s_nN}{2}}h_n(e^{-s_n}x)dx, \end{aligned}$$

the by setting $(\overline{h}_n(x)) = e^{-\frac{s_n N}{2}} h_n(e^{-s_n} x)$, we have that

$$(J_c(v_n), h_n)_{(H^1)^{-1}\times H^1} = (\overline{J}_c'(u_n, s_n), (\overline{h}_n, 0))_{(H^1\times\mathbb{R})^{-1}\times(H^1\times\mathbb{R})}.$$

Let us show that $(\overline{h}_n, 0) \in \overline{T}(u_n, s_n)$. Indeed, we have

$$\begin{aligned} (\overline{h}_n, 0) \in \overline{T}(u_n, s_n) \quad &\Longleftrightarrow (h_n, u_n)_{L^2(\mathbb{R})} = 0 \\ &\Longleftrightarrow \int_{R^N} u_n e^{-\frac{s_n N}{2}} h(e^{-s_n} x) dx = 0 \\ &\Longleftrightarrow \int_{R^N} e^{\frac{s_n N}{2}} u_n(e^{s_n} x) h_n(x) dx = 0 \\ &\Longleftrightarrow (h_n, v_n) = 0 \\ &\Longleftrightarrow h_n \in T_{v_n}. \end{aligned}$$

Using this observation and point (3) of Lemma 12, we see that

$$|(J_c'(v_n), h_n)_{(H^1)^{-1}\times H^1}| \le \frac{2}{\sqrt{n}} \|(\overline{h}_n, 0)_{H^1}\|.$$

Since

$$\begin{aligned} \|(\overline{h}_n, 0)\|^2 \quad &= \|\overline{h}_n\| \\ &= \int_{R^N} |\overline{h}_n|^2 dx + \int_{R^N} |\nabla \overline{h}_n|^2 dx \\ &= \int_{\mathbb{R}^N} |h_n|^2 dx + e^{-2s_n} \int_{\mathbb{R}^N} |\nabla h_n|^2 dx \\ &\le 2\|h_n\| \end{aligned}$$

if $e^{-2s_n} \le 2$. This is the case for when $n \in N$ is large, since

$$|s_n| = |s_n - 0| \le \max_{t\in[0,1]} |(u_n, s_n) - g_n(t)| \le \frac{1}{\sqrt{n}}.$$

Then, point (2) of our lemma is true.

The proof is completed. □

Lemma 14. *Assume that one case of the following conditions hold:*

(1) $\frac{2N+4}{N} = q < p < 2^*$ *and* $0 < r < \left(\frac{\|Q\|_2^{\frac{4}{N}}}{\mu} c^{2\gamma} \right)^{\frac{N}{4}}$;

(2) $\frac{2N+4}{N} < q < p < 2^*$ *holds.*

Then, $J_c(u)$ *is bounded from below on* M_r. *Additionally,* $m_r > 0$, *and the function* $r \to m_r$ *is strictly decreasing.*

Proof. Now, Lemma 8 guarantees $M_r \ne \emptyset$, provided that one condition of our lemma is true.

For any $u \in M_r$, from $G(u) = 0$, we have

$$c^{2\gamma} \|\nabla u\|_2^2 = \mu \frac{N(q-2)}{2q} \|u\|_q^q + \frac{N(p-2)}{2p} \|u\|_p^p. \tag{30}$$

The proof is divided into three steps:

Step 1. We show that for $\frac{2N+4}{N} = q < p < 2^*$, $J_c(u)$ is bounded from below.

Since

$$\begin{aligned} J_c(u) &= \frac{1}{2}c^{2\gamma}\|\nabla u\|_2^2 - \mu\frac{1}{q}\|u\|_q^q - \left[\frac{2c^{2\gamma}}{N(p-2)}\|\nabla u\|_2^2 - \mu\frac{4}{N(p-2)q}\|u\|_q^q\right] \\ &= \frac{1}{2}c^{2\gamma}\frac{N(p-2)-4}{N(p-2)}\|\nabla u\|_2^2 - \frac{\mu}{q}\frac{N(p-2)-4}{N(p-2)}\int_{\mathbb{R}^{\mathbb{N}}}|u|^q dx \\ &\geq \frac{1}{2}c^{2\gamma}\frac{N(p-2)-4}{N(p-2)}\|\nabla u\|_2^2 - \mu\frac{N(p-2)-4}{N(p-2)}\frac{1}{2\|Q\|_2^{\frac{4}{N}}}\|\nabla u\|_2^2 r^{\frac{4}{N}} \\ &= \left(c^{2\gamma} - \frac{\mu}{\|Q\|_2^{\frac{4}{N}}}r^{\frac{4}{N}}\right)\frac{N(p-2)-4}{2N(p-2)}\|\nabla u\|_2^2, \end{aligned} \tag{31}$$

for $0 < r < \left(\frac{\|Q\|_2^{\frac{4}{N}}}{\mu}c^{2\gamma}\right)^{\frac{N}{4}} = \frac{\|Q\|_2}{\mu^{\frac{N}{4}}}c^{\frac{N\gamma}{2}}$, we know that $J_c(u)$ is coercive on M_r. Moreover, Lemma 1, together with Equation (30), guarantees that

$$c^{2\gamma}\|\nabla u\|_2^2 \leq 2\mu\frac{1}{2\|Q\|_2^{\frac{4}{N}}}\|\nabla u\|_2^2 r^{\frac{4}{N}} + \frac{N(p-2)}{2}\frac{r^{p-\frac{N(p-2)}{2}}}{2\|Q\|_2^{p-2}}\|\nabla u\|_2^{\frac{N(p-2)}{2}},$$

In other words, we have

$$\|\nabla u\|_2 \geq \left(\frac{4(c^{2\gamma} - \mu\frac{1}{\|Q\|_2^{\frac{4}{N}}}r^{\frac{4}{N}})\|Q\|_2^{p-2}}{N(p-2)r^{p-\frac{N(p-2)}{2}}}\right)^{\frac{2}{N(p-2)-4}}. \tag{32}$$

From Equations (31) and (32), we have

$$J_c(u) \geq \left(c^{2\gamma} - \frac{\mu}{\|Q\|_2^{\frac{4}{N}}}r^{\frac{4}{N}}\right)\frac{N(p-2)-4}{2N(p-2)}\left(\frac{4(c^{2\gamma} - \mu\frac{1}{\|Q\|_2^{\frac{4}{N}}}r^{\frac{4}{N}})\|Q\|_2^{p-2}}{N(p-2)r^{p-\frac{N(p-2)}{2}}}\right)^{\frac{2}{N(p-2)-4}}..$$

Hence, we obtain

$$\inf_{u\in M_r} J_c(u) = m_r > 0.$$

Step 2. We show that for $\frac{2N+4}{N} < q < p < 2^*$, $J_c(u)$ is bounded from below.
From Equation (30), we have that

$$\begin{aligned} J_c(u) &= \frac{1}{2}c^{2\gamma}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p - \frac{2}{N(q-2)}[c^{2\gamma}\|\nabla u\|_2^2 - \frac{N(p-2)}{2p}\|u\|_p^p] \\ &= \frac{1}{2}c^{2\gamma}\|\nabla u\|_2^2 - \frac{2c^{2\gamma}}{N(p-2)}\|\nabla u\|_2^2 \\ &= c^{2\gamma}\frac{N(q-2)-4}{2N(q-2)}\|\nabla u\|_2^2 + \frac{1}{p}(\frac{p-2}{q-2} - 1)\|u\|_p^p. \end{aligned} \tag{33}$$

From $\frac{2N+4}{N} < q < p < 2^*$, we find that $J_c(u)$ is coercive on M_r. Moreover, Lemma 1 together with Equation (30) guarantees that

$$\begin{aligned} c^{2\gamma}\|\nabla u\|_2^2 &= \mu\frac{N(q-2)}{2q}\|u\|_q^q + \frac{N(p-2)}{2p}\|u\|_p^p \\ &\leq \mu\frac{N(q-2)}{2}\frac{r^{q-\frac{N(q-2)}{2}}}{2\|Q\|_2^{q-2}}\|\nabla u\|_2^{\frac{N(q-2)}{2}} + \frac{N(p-2)}{2}\frac{r^{p-\frac{N(p-2)}{2}}}{2\|Q\|_2^{p-2}}\|\nabla u\|_2^{\frac{N(p-2)}{2}}, \end{aligned}$$

In other words, we have

$$c^{2\gamma} \leq \mu \frac{N(q-2)}{2} \frac{r^{q-\frac{N(q-2)}{2}}}{2\|Q\|_2^{q-2}} \|\nabla u\|_2^{\frac{N(q-2)}{2}-2} + \frac{N(p-2)}{2} \frac{r^{p-\frac{N(p-2)}{2}}}{2\|Q\|_2^{p-2}} \|\nabla u\|_2^{\frac{N(p-2)}{2}-2}.$$

From $\frac{N(p-2)}{2} - 2 > \frac{N(q-2)}{2} - 2 > 0$, we see that there exists $c_0 > 0$ such that

$$\|\nabla u\|_2^2 \geq c_0. \tag{34}$$

Then, we have

$$J_c(u) \geq c^{2\gamma} \frac{N(p-2)-4}{2N(p-2)} c_0.$$

Hence, we obtain

$$\inf_{u \in M_r} J_c(u) = m_r > 0.$$

Step 3. We show that the function $r \to m_r$ is strictly decreasing.
We only prove the case $\frac{2N+4}{N} = q < p < 2^*$.

For $0 < r_1 < r_2 < \left(\frac{\|Q\|_2^{\frac{4}{N}}}{\mu} c^{2\gamma} \right)^{\frac{N}{4}}$, from point (1) of Lemma 8, there exist $\{u_n\} \subseteq S_{r_1}$ such that $u_n^{t_n} \in M_{r_1}$:

$$m_{r_1} \leq J_c(u_n^{t_n}) = \max_{t>0} J_c(u_n^t) \leq m_{r_1} + \frac{1}{n}.$$

It follows from Equations (30)–(32) that there exist $k_i > 0$, $i = 1, 2$ independent of n such that

$$k_1 \leq \|u_n^{t_n}\|_{L^p} \leq k_2, \;\; k_1 \leq \|\nabla u_n^{t_n}\|_{L^2} \leq k_2.$$

By setting $v_n = (\frac{r_2}{r_1})^{1-\frac{N}{2}} u_n^{t_n}(\frac{r_1}{r_2}x)$ and $\|v_n\|_{L^2} = r_2$, we obtain

$$\|\nabla v_n\|_{L^2} = \|\nabla u_n^{t_n}\|_{L^2}, \;\; \|v_n\|_{L^p}^p = (\frac{r_2}{r_1})^{\frac{(2-N)p}{2}+N} \|u_n^{t_n}\|_{L^p}^p. \tag{35}$$

Moreover, point (1) of Lemma 8 guarantees there exists $t_n' > 0$ such that $v_n^{t_n'} \in M_{r_2}$ and $J_c(v_n^{t_n'}) = \max_{t>0} J_c(v_n^{t_n'})$ with

$$t_n' = \left(\frac{4p\left(\frac{1}{2}c^{2\gamma}\|\nabla v_n\|_2^2 dx - \frac{\mu}{q}\|v_n\|_q^q\right)}{N(p-2)\|v_n\|_p^p} \right)^{\frac{1}{\frac{N(p-2)}{2}-2}}.$$

It follows from $G(u_n^{t_n}) = 0$ and Equation (35) that

$$\begin{aligned} t_n' &= \left(\frac{4p\left(\frac{1}{2}c^{2\gamma}\|\nabla u_n^{t_n}\|_2^2 - \frac{\mu}{q}(\frac{r_2}{r_1})^{\frac{4}{N}}\|u_n^{t_n}|_q^q\right)}{N(p-2)(\frac{r_2}{r_1})^{\frac{(2-N)p}{2}+N}\|u_n^{t_n}\|_p^p} \right)^{\frac{1}{\frac{N(p-2)}{2}-2}} \\ &= \left(\frac{4p\left(\frac{1}{2}c^{2\gamma}\|\nabla u_n^{t_n}\|_2^2 - \frac{\mu}{q}(\frac{r_2}{r_1})^{\frac{4}{N}}\|u_n^{t_n}\|_q^q\right)}{N(p-2)(\frac{r_2}{r_1})^{\frac{(2-N)p}{2}+N}\frac{2p}{N(p-2)}(c^{2\gamma}\|\nabla u_n^{t_n}\|_2^2 - 2\mu\frac{1}{q}\|u_n^{t_n}\|_q^q)} \right)^{\frac{1}{\frac{N(p-2)}{2}-2}} \\ &\leq (\frac{r_1}{r_2})^{(\frac{(2-N)p}{2}+N)\frac{1}{\frac{N(p-2)}{2}-2}}. \end{aligned}$$

Then, $t'_n < 1$, and thus

$$\begin{aligned} m_{r_2} &\leq J_c(v_n^{t'_n}) \\ &= (t'_n)^2 J_c(u_n^{t_n t'_n}) + [(t'_n)^2 - (\frac{r_2}{r_1})^{\frac{(2-N)q}{2}+N}] \frac{\mu}{q} \|u_n^{t_n t'_n}\|_{L^q}^q \\ &\quad + [(t'_n)^2 - (\frac{r_2}{r_1})^{\frac{(2-N)p}{2}+N}] \frac{1}{p} \|u_n^{t_n t'_n}\|_{L^p}^p \\ &< (t'_n)^2 J_c(u_n^{t_n t'_n}) \\ &< m_{r_1} + \frac{1}{n}. \end{aligned}$$

By letting $n \to +\infty$, we have $m_{r_2} < m_{r_1}$. We can prove the case $\frac{2N+4}{N} < q < p < 2^*$ in the same way, and thus we omitted the proof.

The proof is completed. $\square$

Lemma 15. *Assume that one of the following conditions holds:*

(1) $\frac{2N+4}{N} = q < p < 2^*$, $N\gamma < 2$ *and* $c < \left(\frac{\mu}{\|Q\|_2^{\frac{4}{N}}} \right)^{\frac{N}{2(N\gamma-2)}}$;

(2) $\frac{2N+4}{N} = q < p < 2^*$, $N\gamma > 2$ *and* $c > \left(\frac{\mu}{\|Q\|_2^{\frac{4}{N}}} \right)^{\frac{N}{2(N\gamma-2)}}$;

(3) $\frac{2N+4}{N} < q < p < 2^*$.

Let $\{u_n\} \subseteq M_c$ be a minimizing sequence for m_c. Then, there exists $y_n \in \mathbb{R}^N$ such that for any $\varepsilon > 0$, there exists an $R > 0$, satisfying

$$\int_{\mathbb{R}^N - B(y_n,R)} |u_n(x)|^q dx < \varepsilon, \ \int_{\mathbb{R}^N - B(y_n,R)} |u_n(x)|^p dx < \varepsilon.$$

Proof. If conditions (1) and (2) hold, then from Equation (31), we have

$$J_c(u_n) \geq \left(c^{2\gamma} - \frac{\mu}{\|Q\|_2^{\frac{4}{N}}} c^{\frac{4}{N}} \right) \frac{N(p-2)-4}{2N(p-2)} \|\nabla u_n\|_2^2$$

Additionally, if condition (3) holds, then from Equation (33), we have

$$J_c(u_n) \geq c^{2\gamma} \frac{N(q-2)-4}{2N(q-2)} \|\nabla u_n\|_2^2 + \frac{1}{p} (\frac{p-2}{q-2} - 1) \|u_n\|_p^p.$$

Then, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Now, Lemma 1 guarantees that

$$\{\|u_n\|_s\} \text{ is bounded for each } s \in (2, 2^*)$$

Without loss of generality, assume that

$$\|u_n\|_n^p \to a_1 \geq 0, \|u_n\|_q^q \to a_2 \geq 0, \ \ n \to +\infty. \tag{36}$$

Set

$$\rho_n(x) := \mu \frac{N(q-2)-4}{4q} |u_n(x)|^q + \frac{N(p-2)-4}{4p} |u_n(x)|^p > 0, \ \ x \in \mathbb{R}^N.$$

It is easy to see that $\rho_n(x) > 0$ because of $p > q \geq \frac{2N+4}{N}$. From $G(u_n) = 0$, we have

$$\int_{\mathbb{R}^N} \rho_n(x) dx = J_c(u_n) - \frac{1}{2} G(u_n) \to m_r > 0,$$

which implies that

$$\text{one of } a_1, a_2 \text{ is not } 0 \tag{37}$$

and thus

$$\frac{1}{2}c^{2\gamma}\|\nabla u_n\|_2^2 = J_c(u_n) + \mu\frac{1}{q}\|u_n\|_q^q + \frac{1}{p}\|u_n\|_p^p \to a_3 > 0. \tag{38}$$

Now, we use the compactness-concentration principle in [29,34]:

(1) We claim that vanishing does not occur.

Suppose by contradiction that, for all $R > 0$, $\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \rho_n(x)dx = 0$, and then

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n(x)|^q dx = 0.$$

According to Lemma 2, $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $q < s < 2^*$. Since $u_n \in S_c$, Hölder inequalities imply that $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$, which contradicts Equation (37).

(2) We claim that dichotomy does not occur.

Suppose by contradiction that there exists $\alpha \in (0, m_c)$ and $\{y_n\} \subseteq \mathbb{R}^N$ such that for all $\varepsilon_n \to 0$, there exists $\{R_n\}$ with $R_n \to +\infty$, satisfying

$$\limsup_{n\to+\infty}\left(\left|\alpha - \int_{B_{R_n}(y_n)} \rho_n(x)dx\right| + \left|(m_r - \alpha) - \int_{\mathbb{R}^N - B_{2R_n}(y_n)} \rho_n(x)dx\right|\right) < \varepsilon_n. \tag{39}$$

Let $\zeta : R_+ \to [0,1]$ be a cutoff function such that $\zeta(s) = 1$ for $s \le 1$, $\zeta(s) = 0$ for $s \ge 2$ and $|\zeta'(s)| \le 2$ for all $s \in [1,2]$, while

$$v_n(x) := \zeta(\frac{|x-y_n|}{R_n})u_n(x), w_n(x) := (1 - \zeta(\frac{|x-y_n|}{R_n}))u_n(x)$$

and

$$P(u) := J_c(u) - \frac{1}{2}G(u).$$

It is easy to see that for any $r > 0$, $P(u) = J_c(u)$ for all $u \in M_r$. We deduce from Equation (39) that $\liminf_{n\to+\infty} P(v_n) \ge \alpha$ and $\liminf_{n\to+\infty} P(w_n) \ge m_c - \alpha$.

Denote $\Omega_n = B_{2R_n}(y_n) - B_{R_n}(y_n)$. It follows from Equation (36) that

$$\int_{\Omega_n} \rho_n(x)dx = o(1), n \to +\infty.$$

From Equation (38), we have that

$$\int_{\Omega_n} |\nabla v_n|^2 dx = o(1), \int_{\Omega_n} |\nabla w_n(x)|^2 dx = o(1), n \to +\infty. \tag{40}$$

Hence, we have

$$\|\nabla u_n\|_2^2 = \|v_n\|_2^2 + \|\nabla w_n\|_2^2 + o(1), n \to +\infty \tag{41}$$

and

$$\|u_n\|_p^p = \|v_n\|_p^p + \|w_n\|_p^p + o(1), n \to +\infty. \tag{42}$$

We deduce from Equations (40), (41) and (42) that

$$P(u_n) = P(v_n) + P(w_n) + o(1), n \to +\infty.$$

Then, we obtain

$$m_c = \liminf_{n\to+\infty} P(u_n) \geq \liminf_{n\to+\infty} P(u_n) + \liminf_{n\to+\infty} P(w_n) \geq \alpha + m_c - \alpha$$

and

$$\lim_{n\to+\infty} P(v_n) = \alpha, \ \lim_{n\to+\infty} P(w_n) = m_c - \alpha. \tag{43}$$

Moreover, from Equations (40)–(42), we have that

$$0 = G(u_n) \geq G(w_n) + G(v_n) + o(1), n \to +\infty. \tag{44}$$

Then, there are two cases for $\{G(v_n)\}$ and $\{G(w_n)\}$.

Case 1: There exists a negative subsequence of $\{G(v_n)\}$ or $\{G(w_n)\}$. Without loss of generality, we suppose that $G(v_n) \leq 0$. Now, Lemma 8 guarantees that for each n, there exists $t_n > 0$ such that $v_n^{t_n} \in M_{\|v_n\|_2}$, and then $G(v_n^{t_n}) = 0$. Hence, we have

$$\begin{aligned} 0 &\geq t_n^{\frac{N(q-2)-4}{2}} G(v_n) - t_n^{-2} G(v_n^{t_n}) \\ &= c^{2\gamma}(t_n^{\frac{N(q-2)-4}{2}} - 1)\|\nabla v_n\|_2^2 + \frac{N(p-2)}{2p}(t_n^{\frac{N(p-2)-4}{2}} - t_n^{\frac{N(q-2)-4}{2}})\|v_n\|_p^p, \end{aligned}$$

which implies that $t_n \leq 1$. Now, we obtain

$$m_{\|v_n\|_2} \leq J_c(v_n^{t_n}) = P(v_n^{t_n}) \leq P(v_n) \to \alpha < m_c \leq m_{\|v_n\|_2}. \tag{45}$$

This is a contradiction.

Case 2: For each n, $G(v_n) > 0$ and $G(w_n) > 0$.

From Equation (44), we see that $G(v_n) \to 0$ and $G(w_n) \to 0$. Now, Lemma 8 guarantees that for each n, there exists $t_n > 0$ such that $v_n^{t_n} \in M_{\|v_n\|_2}$. If $\limsup t_n \leq 1$, and then $G(v_n^{t_n}) = 0$, then we can obtain the same contradiction as in Equation (45). Suppose now that $\lim t_n = t_0 > 1$. We will show a contradiction for different cases:

(1) If $\frac{2N+4}{N} = q < p < 2^*$, then

$$\begin{aligned} G(v_n) &= G(v_n) - t_n^{-2} G(v_n^{t_n}) \\ &= \frac{N(p-2)}{2p}(t_n^{\frac{N(p-2)}{2}-2} - 1)\|v_n\|_p^p, \end{aligned}$$

which implies

$$\|v_n\|_p \to 0, \quad \text{as } n \to +\infty.$$

We deduce from Equation (43) that there exists $b_1 > 0$ such that

$$\|v_n\|_q \to b_1. \tag{46}$$

On the other hand, we have

$$\begin{aligned} 0 &\leftarrow t_n^{\frac{N(p-2)-4}{2}} G(v_n) - t_n^{-2} G(v_n^{t_n}) \\ &= c^{2\gamma}(t_n^{\frac{N(p-2)-4}{2}} - 1)\|\nabla v_n\|_2^2 + \mu\frac{N(q-2)}{2q}(t_n^{\frac{N(q-2)-4}{2}} - t_n^{\frac{N(p-2)-4}{2}})\|v_n\|_q^q, \end{aligned}$$

which, together with Equation (46), indicates that there exists $b_2 > 0$ such that

$$\|\nabla v_n\|_2 \to b_2. \tag{47}$$

By combining Equaitons (46), (48) and (47) and letting $n \to +\infty$, we have

$$c^{2\gamma}(t_0^{\frac{N(p-2)-4}{2}} - 1)b_2 + \mu\frac{N(q-2)}{2q}(t_0^{\frac{N(q-2)-4}{2}} - t_0^{\frac{N(p-2)-4}{2}})b_1 = 0. \tag{48}$$

Let $s > \frac{N(p-2)}{2} - 2$. Then, we obtain

$$\begin{aligned} 0 \quad &\leftarrow t_n^s G(v_n) - t_n^{-2} G(v_n^{t_n}) \\ &= c^{2\gamma}(t_n^s - 1)\|\nabla v_n\|_2^2 + \mu \frac{N(q-2)}{2q}(t_n^{\frac{N(q-2)-4}{2}} - t_n^s)\|v_n\|_q^q \\ &\quad + \frac{N(p-2)}{2p}(t_n^{\frac{N(p-2)-4}{2}} - t_n^s)\|v_n\|_p^p. \end{aligned}$$

By combining Equations (50), (46) and (47) and letting $n \to +\infty$, we have

$$c^{2\gamma}(t_0^s - 1)b_2 + \mu \frac{N(q-2)}{2q}(t_0^{\frac{N(q-2)-4}{2}} - t^s)b_1 = 0. \tag{49}$$

Since $s > \frac{N(p-2)}{2} - 2$ is arbitrary, Equation (49) contradicts Equation (48).

(2) If $\frac{2N+4}{N} < q < p < 2^*$, then

$$\begin{aligned} G(v_n) \quad &= G(v_n) - t_n^{-2} G(v_n^{t_n}) \\ &= \mu \frac{N(q-2)}{2q}(t_n^{\frac{N(q-2)}{2}-2} - 1)\|v_n\|_q^q + \frac{N(p-2)}{2p}(t_n^{\frac{N(p-2)}{2}-2} - 1)\|v_n\|_p^p, \end{aligned}$$

which implies

$$\|v_n\|_q \to 0 \text{ and } \|v_n\|_p \to 0, \quad \text{as } n \to +\infty. \tag{50}$$

On the other hand, we have

$$\begin{aligned} 0 \quad &\leftarrow t_n^{\frac{N(q-2)-4}{2}} G(v_n) - t_n^{-2} G(v_n^{t_n}) \\ &= c^{2\gamma}(t_n^{\frac{N(q-2)-4}{2}} - 1)\|\nabla v_n\|_2^2 + \frac{N(p-2)}{2p}(t_n^{\frac{N(p-2)-4}{2}} - t_n^{\frac{N(q-2)-4}{2}})\|v_n\|_p^p, \end{aligned}$$

which implies

$$\|\nabla v_n\|_2 \to 0 \text{ and } \|v_n\|_q \to 0, \quad \text{as } n \to +\infty. \tag{51}$$

By combining Equations (50) and (51), we have $P(v_n) \to 0$ as $n \to +\infty$. This contradicts Equation (43).

Therefore, the compactness holds for the sequence ρ_n; that is, there exists $\{y_n\} \in \mathbb{R}^N$ such that for any $\varepsilon > 0$, there exists $R > 0$, satisfying

$$\limsup_{n \to +\infty} \int_{B_R(y_n)} \rho_n(x)dx \geq m_r - \varepsilon.$$

Hence, we obtain

$$\int_{\mathbb{R}^N - B_R(y_n)} |u|^q dx \leq \varepsilon \text{ and } \int_{\mathbb{R}^N - B_R(y_n)} |u|^p dx \leq \varepsilon.$$

The proof is completed. □

Lemma 16. *Assume that $2 < q < \frac{2N+4}{N} < p < 2^*$ and Equation (19) holds. Let $\{u_n\} \subseteq S_c$ be a Palais–Smale sequence for $J_c|_{S_c}$ at a level $v \neq 0$ which satisfies that*

$$c^{2\gamma}\|\nabla u_n\|_2^2 - \mu \frac{N(q-2)}{2q}\|u_n\|_q^q - \frac{N(p-2)}{2p}\|u_n\|_p^p = o(1), \quad \text{as } n \to +\infty. \tag{52}$$

Then, up to a subsequence $u_n \to u$ strongly in H^1, $u \in S_c$ is a real-valued radial solution to Equation (1) for some $\lambda < 0$.

Proof. The proof is divided into four steps:

(1) We show that $\{u_n\}$ is bounded.

Since
$$J_c(u_n) = \frac{1}{2}c^{2\gamma}\|\nabla u_n\|_2^2 - \mu\frac{1}{q}\|u_n\|_q^q - \frac{1}{p}\|u_n\|_p^p, \tag{53}$$
then by putting Equation (52) into Equation (53), we have

$$\begin{aligned} v &\leftarrow J_c(u_n) \\ &= \frac{1}{2}c^{2\gamma}\|\nabla u_n\|_2^2 - \mu\frac{1}{q}\|u_n\|_q^q - \left[\frac{2c^{2\gamma}}{N(p-2)}\|\nabla u_n\|_2^2 - \mu\frac{4}{N(p-2)q}\|u_n\|_q^q\right] \\ &= \frac{1}{2}c^{2\gamma}\frac{N(p-2)-4}{N(p-2)}\|\nabla u_n\|_2^2 - \frac{\mu}{q}\frac{N(p-2)-4}{N(p-2)}\int_{\mathbb{R}^N}|u_n|^q dx \\ &\geq \frac{1}{2}c^{2\gamma}\frac{N(p-2)-4}{N(p-2)}\|\nabla u_n\|_2^2 - \mu\frac{N(p-2)-4}{N(p-2)}\frac{1}{2\|Q\|_2^{q-2}}\|\nabla u_n\|_2^{\frac{N(q-2)}{2}}c^{q-\frac{N(q-2)}{2}}. \end{aligned} \tag{54}$$

Since $2 < q < \frac{2N+4}{N} < p < 2^*$, we have that $\{u_n\}$ is bounded.

(2) We show that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with a relative convergent $\{\lambda_{n_2,2}\}$ such that

$$(J_c'(u_{n_k}), v) - \lambda_{n_k,2}(u_{n_k}, v) = o(1), \ \ \forall v \in H^1(\mathbb{R}^N).$$

Since $N \geq 3$, Lemma 4 guarantees the embedding $H^1_{rad}(\mathbb{R}^N) \hookrightarrow H^s(\mathbb{R}^N)$ is compact for $s \in (2, 2^*)$, and we deduce that there exists $u \in H^1_{rad}$ such that, up to a subsequence, $u_n \to u$ is weakly in H^1, $u_n \to u$ is strongly in $L^s(\mathbb{R}^N)$ for $s \in (2, 2^*)$, and it is a.e. in $\mathbb{R}^N$.

If $u = 0$, then from Equation (52), we have $\|\nabla u_n\|_2 \to 0$, and then $J_c(u_n) \to 0$. This contradicts $J_c(u_n) \to v \neq 0$. Now, since $\{u_n\}$ is a bounded Palais–Smale sequence of $J_c|_{M_c}$, by the Lagrange multipliers rule, there exists $\lambda_{n,2} \in \mathbb{R}$ such that

$$(J_c'(u_n), v) - \lambda_{n,2}(u_n, v) = o(1) \tag{55}$$

for every $v \in H^1$, where $o(1) \to 0$ as $n \to +\infty$.

By letting $v = u_n$ in Equation (55), we have

$$\lambda_{n,2}c^2 = (J_c'(u_n), u_n) + o(1).$$

The boundedness of $\{u\}$ guarantees that $\{\lambda_{n,2}\}$ is bounded. We can assume that $\lambda_{n,2} \to \lambda$ as $n \to +\infty$.

(3) We show that $\lambda < 0$.

By putting Equation (52) into Equation (55), we have

$$\lambda_{n,2}c^2 = \mu(\frac{N(q-2)}{2q} - 1)\|u_n\|_q^q + (\frac{N(p-2)}{2p} - 1)\|u_n\|_p^p + o(1). \tag{56}$$

Since $0 < \frac{N(p-2)}{2p} - 1 < 1$ and $0 < \frac{N(q-2)}{2q} - 1 < 1$, we have $\lambda \leq 0$. If $\lambda = 0$, then we have $u = 0$, which together with Equation (52) implies $J_c(u_n) \to 0$. This is a contradiction.

(4) We show that $u_n \to u$ strongly.

By weak convergence, Equation (55) implies that

$$(J_c'(u), v) - \lambda(u, v) = 0 \tag{57}$$

for every $v \in H^1$. By choosing $v = u_n - u$ in Equations (55) and (57), we obtain

$$(J_c'(u_n) - J_c'(u), u_n - u) - \lambda(u_n - u, u_n - u) = o(1).$$

Using the strong L^p and L^q convergence of u_n, we infer that

$$c^{2\gamma}\|\nabla(u_n-u)\|_2^2-\lambda\|u_n-u\|_2=o(1).$$

which, noting that $\lambda<0$, establishes the strong convergence in H^1.

The proof is completed. □

3.1. The Proof of Theorem 1

Proof of Theorem 1. Assume that u is a solution to Equation (1) for $\lambda\ge 0$. Then, we have

$$(N-2)\left(\|u\|_2^2\right)^\gamma\|\nabla u\|_2^2=N\lambda\|v_n\|_2^2+\mu\frac{2N}{q}\|u\|_q^q+\frac{2N}{p}\|u\|_p^p \tag{58}$$

and

$$\left(\|u\|_2^2\right)^\gamma\|\nabla u\|_2^2=\lambda\|v_n\|_2^2+\mu\|u\|_q^q+\|u\|_p^p. \tag{59}$$

By multiplying Equation (59) by $\frac{2N}{p}$, we have

$$\frac{2N}{p}\left(\|u\|_2^2\right)^\gamma\|\nabla u\|_2^2=\lambda\frac{2N}{p}\|v_n\|_2^2+\mu\frac{2N}{p}\|u\|_q^q+\frac{2N}{p}\|u\|_p^p.$$

Using above equation minus Equation (58), we have that

$$\mu 2N(\frac{1}{q}-\frac{1}{p})\|u\|_q^q+\lambda 2N(\frac{1}{2}-\frac{1}{p})\int_{\mathbb{R}^N}u^2(x)dx=\frac{p(N-2)-2N}{p}\left(\|u\|_2^2\right)^\gamma\|\nabla u\|_2^2. \tag{60}$$

Under our assumptions, Equation (60) implies that if $\lambda\ge 0$, then necessarily $u=0$.

The proof is completed. □

3.2. The Proof of Theorem 2

Proof of Theorem 2. Now, point (1) of Lemma 9 guarantees that $I_{r^2}=\inf_{u\in S_r}J_c(u)<0$ is well-defined for $r\in(0,+\infty)$, which implies that $I_{c^2}=\inf_{u\in S_c}J_c(u)<0$. Let $\{u_n\}\subseteq S_c$ be a minimizing sequence for $I_{c^2}<0$. From Equation (21) and $\{u_n\}\subseteq S_c$, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$.

Set

$$\sigma:=\lim_{n\to+\infty}\sup\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|u_n|^2dx.$$

If $\sigma=0$, then using Lemma 2, $u_n\to 0$ in $L^s(\mathbb{R}^N)$ for any $2<s<2^*$. Hence, $0\le\liminf_{n\to+\infty}A_c(u_n)=\liminf_{n\to+\infty}(A_c(u_n)-B_1(u_n)-B_2(u_n))=\lim_{n\to+\infty}J_c(u_n)=I_{c^2}<0$. This is a contradiction.

Therefore, $\sigma>0$, and then there exists y_n such that

$$\int_{B_1(y_n)}|u_n|^2dx\ge\frac{\sigma}{2}. \tag{61}$$

Denote $\overline{u}_n=u_n(\cdot+y_n)$. Then, $\{\overline{u}_n\}\subseteq S_c$ is also a bounded minimizing sequence for I_{c^2}, which implies that there exists a weakly convergent subsequence of $\{\overline{u}_n\}$ in $H^1(\mathbb{R}^N)$. Without loss of generality, we may assume that for some $\overline{u}\in H^1(\mathbb{R}^N)$, we have

$$\begin{cases}\overline{u}_n\rightharpoonup\overline{u}, & \text{in } H^1(\mathbb{R}^N),\\ \overline{u}_n\to\overline{u}, & \text{in } L^q_{loc}(\mathbb{R}^N), q\in(1,2^*];\\ \overline{u}_n(x)\to\overline{u}(x), & \text{a.e. in } \mathbb{R}^N,\end{cases} \tag{62}$$

which, together with Equation (61), implies that $\|\overline{u}\|_2\neq 0$. Fatou's Lemma guarantees that

$$\sigma_1:=\|\overline{u}\|_2=\left(\int_{\mathbb{R}^N}|\overline{u}|^2\right)^{\frac{1}{2}}\le\liminf_{n\to+\infty}\left(\int_{\mathbb{R}^N}|\overline{u}_n|^2dx\right)^{\frac{1}{2}}=c,\text{ i.e., }\sigma_1\in(0,c].$$

Now, we claim that $\sigma_1 = c$. If $\sigma_1 < c$, then Equation (62) gives

$$\begin{aligned}\|\overline{u}_n\|_2^2 &= \int_{\mathbb{R}^N} |\overline{u}(x) + \overline{u}_n(x) - \overline{u}(x)|^2 dx \\ &= \int_{\mathbb{R}^N} \overline{u}^2(x)dx + \int_{\mathbb{R}^N} (\overline{u}_n(x) - \overline{u}(x))^2 dx + 2\int_{\mathbb{R}^N} \overline{u}(x)(\overline{u}_n(x) - \overline{u}(x))dx \\ &= \|\overline{u}\|_2^2 + \|\overline{u}_n - \overline{u}\|_2^2 + o(1), n \to +\infty.\end{aligned}$$

Hence, we have

$$\|\overline{u}_n - \overline{u}\|_2^2 \to c^2 - \sigma_1^2, n \to +\infty.$$

From the boundedness of $\{u_n\}$ in $H^1(\mathbb{R}^N)$, Lemma 1 guarantees that $\{\|u_n\|_q\}$ and $\{\|u_n\|_p\}$ are bounded. Now, Equation (62) and Lemma 3 guarantee that

$$\lim_{n\to+\infty} (\|\overline{u}_n\|_q^q - \|\overline{u}_n - \overline{u}\|_q^q) = \|\overline{u}\|_q^q. \tag{63}$$

and

$$\lim_{n\to+\infty} (\|\overline{u}_n\|_p^p - \|\overline{u}_n - \overline{u}\|_p^p) = \|\overline{u}\|_p^p. \tag{64}$$

Therefore, we obtain

$$\begin{aligned}\lim_{n\to+\infty} J_c(\overline{u}_n) &= \lim_{n\to+\infty}\left(\frac{1}{2}c^{2\gamma}\|\nabla\overline{u}_n - \overline{u} + \overline{u}|_2^2 - \frac{\mu}{q}\|\overline{u}_n|_q^q - \frac{1}{p}\|\overline{u}_n\|_p^p\right) \\ &= \lim_{n\to+\infty}\left(\frac{1}{2}c^{2\gamma}\|\nabla(\overline{u}_n - \overline{u})\|_2^2 + \|\nabla\overline{u}\|_2^2 - \frac{\mu}{q}\|\overline{u}_n\|_q^q - \frac{1}{p}\|\overline{u}_n\|_p^p\right) \\ &= \lim_{n\to+\infty}\left(\frac{1}{2}c^{2\gamma}\|\nabla(\overline{u}_n - \overline{u})\|_2^2 - \frac{\mu}{q}\|\overline{u}_n - \overline{u}\|_q^q - \frac{1}{p}\|\overline{u}_n - \overline{u}\|_p^p\right) \\ &\quad + \lim_{n\to+\infty}[\frac{1}{2}c^{2\gamma}\|\nabla\overline{u}\|^2 - \frac{\mu}{q}\left(\|\overline{u}_n\|_q^q - \frac{\mu}{q}\|\overline{u}_n - \overline{u}\|_q^q\right) \\ &\quad - \frac{1}{p}\left(\|\overline{u}_n\|_p^p - \|\overline{u}_n - \overline{u}|_p^p\right)] \\ &= J_c(\overline{u}) + \lim_{n\to+\infty} J_c(\overline{u}_n - \overline{u}),\end{aligned}$$

which together with Equations (63) and (64) implies that

$$I_{c^2} = \lim_{n\to+\infty} J_c(\overline{u}_n) \quad = J_c(\overline{u}) + \lim_{n\to+\infty} J_c(\overline{u}_n - \overline{u}) \geq I_{\sigma_1^2} + I_{c^2-\sigma_1^2}.$$

This contradicts point (3) of Lemma 9.

Thus, $\|\overline{u}\|_2 = c$, and so

$$I_{c^2} \leq J_c(\overline{u}) \leq \lim_{n\to+\infty} I(\overline{u}_n) = I_{c^2},$$

In other words, $J_c(\overline{u}) = I_{c^2} = \inf_{u\in S_c} J_c(u)$. Therefore, $\overline{u}$ is a critical point of $J_c|_{S_c}$. Now, Lemma 6 guarantees that there exists $\lambda_c < 0$ such that $(\overline{u}, \lambda_c)$ is a couple solution to Equation (1).

The proof is completed. □

3.3. The Proof of Theorem 3

Proof of Theorem 3. From $N\gamma < 2$ and $0 < c < \|Q\|_2^{\frac{2}{N\gamma}}$ or $N\gamma > 2$ and $c > \|Q\|_2^{\frac{2}{N\gamma}}$, we have

$$c^{2\gamma} - \frac{1}{\|Q\|_2^{\frac{4}{N}}} c^{\frac{4}{N}} > 0,$$

which guarantees that for $u \in S_c$, we have

$$\begin{aligned} J_c(u) &= \frac{1}{2}c^{2\gamma}\int_{\mathbb{R}^N}|\nabla u(x)|^2dx - \frac{\mu}{q}\|u\|_q^q - \frac{1}{p}\|u\|_p^p \\ &\geq \frac{1}{2}c^{2\gamma}\|\nabla u\|_2^2 - \mu\frac{c^{q-\frac{N(q-2)}{2}}}{2\|Q\|_2^{q-2}}\|\nabla u\|_2^{\frac{N(q-2)}{2}} - \frac{c^{\frac{4}{N}}}{2\|Q\|_2^{p-2}}\|\nabla u\|_2^2 \\ &\geq \frac{1}{2}(c^{2\gamma} - \frac{c^{\frac{4}{N}}}{\|Q\|_2^{\frac{4}{N}}})\|\nabla u\|_2^2 - \mu\frac{c^{q-\frac{N(q-2)}{2}}}{2\|Q\|_2^{q-2}}\|\nabla u\|_2^{\frac{N(q-2)}{2}}. \end{aligned} \tag{65}$$

Then, from $q \in (2, \frac{2N+4}{N})$, we have that $J_c(u)$ is bounded from below and $J_c(u^t) < 0$ for small enough t values. Hence, $I_{c^2} < 0$ is well-defined.

Let $\{u_n\} \subseteq S_c$ be a minimizing sequence for $I_{c^2} < 0$. From Equation (65) and $\{u_n\} \subseteq S_c$, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$.

Set

$$\sigma := \lim_{n\to+\infty}\sup\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|u_n|^2dx.$$

If $\sigma = 0$, then using Lemma 2, $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for any $2 < s < 2^*$. Hence, $0 \leq \liminf_{n\to+\infty} A_c(u_n) = \liminf_{n\to+\infty}(A_c(u_n) - B_1(u_n) - B_2(u_n)) = \lim_{n\to+\infty} J_c(u_n) = I_{c^2} < 0$. This is a contradiction.

Therefore, $\sigma > 0$, and then there exists y_n such that

$$\int_{B_1(y_n)}|u_n|^2dx \geq \frac{\sigma}{2}. \tag{66}$$

Denote $\overline{u}_n = u_n(\cdot + y_n)$. Then, $\{\overline{u}_n\} \subseteq S_c$ is also a bounded minimizing sequence for I_{r^2}, which implies that there exists a weakly convergent subsequence of $\{\overline{u}_n\}$ in $H^1(\mathbb{R}^N)$. Without loss of generality, we may assume that for some $\overline{u} \in H^1(\mathbb{R}^N)$, we have

$$\begin{cases} \overline{u}_n \rightharpoonup \overline{u}, & \text{in } H^1(\mathbb{R}^N), \\ \overline{u}_n \to \overline{u}, & \text{in } L^q_{loc}(\mathbb{R}^N), q \in (1, 2^*]; \\ \overline{u}_n(x) \to \overline{u}(x), & \text{a.e. in } \mathbb{R}^N, \end{cases} \tag{67}$$

which together with Equation (66) implies that $\|\overline{u}\|_2 \neq 0$. Fatou's Lemma guarantees that $\sigma_1 := \|\overline{u}\|_2 = \left(\int_{\mathbb{R}^N}|\overline{u}|^2\right)^{\frac{1}{2}} \leq \liminf_{n\to+\infty}\left(\int_{\mathbb{R}^N}|\overline{u}_n|^2dx\right)^{\frac{1}{2}} = c$ (i.e., $\sigma_1 \in (0, c]$).

Now, we claim that $\sigma_1 = c$. If $\sigma_1 < c$, then by using a similar computation as in the proof of Theorem 2, we obtain that

$$I_{c^2} = \lim_{n\to+\infty} J_c(\overline{u}_n) \quad = J_c(\overline{u}) + \lim_{n\to+\infty} J_c(\overline{u}_n - \overline{u}) \geq I_{\sigma_1^2} + I_{c^2-\sigma_1^2}.$$

This contradicts point (3) of Lemma 9.

Thus, $\|\overline{u}\|_2 = c$, and so

$$I_{c^2} \leq J_c(\overline{u}) \leq \lim_{n\to+\infty} I(\overline{u}_n) = I_{c^2},$$

In other words, $J_c(\overline{u}) = I_{c^2} = \inf_{u\in S_c} J_c(u)$. Therefore, $\overline{u}$ is a critical point of $J_c|_{S_c}$. Now, Lemma 6 guarantees that there exists $\lambda_c < 0$ such that $(\overline{u}, \lambda_c)$ is a couple solution to Equation (1).

The proof is completed. □

3.4. The Proof of Theorem 4

Proof of Theorem 4. From $N\gamma < 2$ and $c < \left(\frac{\|Q\|_2^{\frac{4}{N}}}{\mu}\right)^{\frac{N}{2(N\gamma-2)}}$ or $N\gamma > 2$ and $c > \left(\frac{\|Q\|_2^{\frac{4}{N}}}{\mu}\right)^{\frac{N}{2(N\gamma-2)}}$, we have

$$c^{2\gamma} - \frac{\mu}{\|Q\|_2^{\frac{4}{N}}} c^{\frac{4}{N}} > 0,$$

which implies that for $u \in M_c$, we find that

$$\begin{aligned} J_c(u) &= \frac{1}{2}c^{2\gamma}\|\nabla u\|_2^2 - \mu\frac{1}{q}\|u\|_q^q - \left[\frac{2c^{2\gamma}}{N(p-2)}\|\nabla u\|_2^2 - \mu\frac{4}{N(p-2)q}\|u\|_q^q\right] \\ &= \frac{1}{2}c^{2\gamma}\frac{N(p-2)-4}{N(p-2)}\|\nabla u\|_2^2 - \frac{\mu}{q}\frac{N(p-2)-4}{N(p-2)}\int_{\mathbb{R}^N}|u|^q dx \\ &\geq \frac{1}{2}c^{2\gamma}\frac{N(p-2)-4}{N(p-2)}\|\nabla u\|_2^2 - \mu\frac{N(p-2)-4}{N(p-2)}\frac{1}{2\|Q\|_2^{\frac{4}{N}}}\|\nabla u\|_2^2 c^{\frac{4}{N}} \\ &= \left(c^{2\gamma} - \frac{\mu}{\|Q\|_2^{\frac{4}{N}}}c^{\frac{4}{N}}\right)\frac{N(p-2)-4}{2N(p-2)}\|\nabla u\|_2^2, \end{aligned} \tag{68}$$

is coercive on M_c. Moreover, Lemma 1 together with Equation (30) guarantees that

$$c^{2\gamma}\|\nabla u\|_2^2 \leq 2\mu\frac{1}{2\|Q\|_2^{\frac{4}{N}}}\|\nabla u\|_2^2 c^{\frac{4}{N}} + \frac{N(p-2)}{2}\frac{c^{p-\frac{N(p-2)}{2}}}{2\|Q\|_2^{p-2}}\|\nabla u\|_2^{\frac{N(p-2)}{2}},$$

which guarantees that there exists a $c_0 > 0$ such that

$$\|\nabla u\| \geq c_0 > 0$$

for all $u \in M_c$. Hence, we have

$$m_c = \inf_{u \in S_c} J_c(u) > 0.$$

Suppose that $\{u_n\} \subseteq M_c$ is a minimizing sequence for m_c. It follows from Lemma 15 that $\{u_n\} \subseteq H^1(\mathbb{R}^N)$ is bounded, and there exists $y_n \in \mathbb{R}^N$ such that for any $\varepsilon > 0$, there exists $R > 0$, satisfying

$$\int_{\mathbb{R}^N - B(y_n,R)} |u_n(x)|^q dx < \varepsilon, \int_{\mathbb{R}^N - B(y_n,R)} |u_n(x)|^p dx < \varepsilon. \tag{69}$$

Let $\overline{u}_n(x) =: u_n(x+y_n)$ for $x \in \mathbb{R}^N$. Then, $\{\overline{u}_n\} \subseteq M_c$ is a bounded minimizing sequence for mc. According to Equation (69), we see that

$$\int_{\mathbb{R}^N - B(0,R)} |\overline{u}_n(x)|^q dx < \varepsilon, \int_{\mathbb{R}^N - B(0,R)} |\overline{u}_n(x)|^p dx < \varepsilon. \tag{70}$$

Without loss of generality, assume that there exists $\overline{u}$ in $H^1(\mathbb{R}^N)$ such that

$$\begin{cases} \overline{u}_n \rightharpoonup \overline{u}, \text{ in } H^1(\mathbb{R}^N) \\ \overline{u}_n \to \overline{u}, \text{ in } L^s(\mathbb{R}^N), \text{ for all } 2 < s < 2^* \\ \overline{u}_n \to \overline{u}, \text{ a.e. } \mathbb{R}^N. \end{cases} \tag{71}$$

Now, Fatou's Lemma implies that

$$\int_{\mathbb{R}^N - B_R(0)} |\overline{u}(x)|^q \leq \varepsilon, \int_{\mathbb{R}^N - B_R(0)} |\overline{u}(x)|^p \leq \varepsilon. \tag{72}$$

It follows from Equations (70)–(72) that

$$\overline{u}_n \to \overline{u}, \text{ in } L^q(\mathbb{R}^N) \text{ and } \overline{u}_n \to \overline{u}, \text{ in } L^p(\mathbb{R}^N).$$

Since $G(\overline{u}_n) = 0$, we deduce from Equation (32) that there exist two positive constants C_1 and C_2 independent of n such that

$$\int_{\mathbb{R}^N} |\overline{u}|^q dx = \lim_{n\to+\infty} \int_{\mathbb{R}^N} |\overline{u}_n|^q = C_1 \text{ and } \int_{\mathbb{R}^N} |\overline{u}|^p dx = \lim_{n\to+\infty} \int_{\mathbb{R}^N} |\overline{u}_n|^p = C_2,$$

In other words, $\overline{u} \neq 0$. Set $\mu := \|\overline{u}\|_2 \in (0, c]$. From Equaitons (71) and (72) and $\overline{u}_n \in M_c$, we have $G(\overline{u}) \leq \lim_{n\to+\infty} G(\overline{u}_n) = 0$, and thus

$$c^{2\gamma} \int_{\mathbb{R}^N} |\nabla \overline{u}(x)|^2 dx - 2\mu \frac{1}{q} \int_{\mathbb{R}^N} |\overline{u}(x)|^q dx \leq \frac{N(p-2)}{2p} \|u\|_p^p. \tag{73}$$

Lemma 8 infers that that there exists $t_0 = \left(\frac{2p(\frac{1}{2}c^{2\gamma} \int_{\mathbb{R}^N} |\nabla \overline{u}(x)|^2 - \mu \frac{1}{q} \int_{\mathbb{R}^N} |\overline{u}(x)|^q dx)}{\frac{N(p-2)}{2} \int_{\mathbb{R}^N} |\overline{u}(x)|^p dx} \right)^{\frac{1}{\frac{N(p-2)}{2}-2}}$ such that $\overline{u}^{t_0} \in M_\mu$. Now, Equation (73) implies that

$$\begin{aligned}
t_0 &= \left(\frac{2p(\frac{1}{2}c^{2\gamma} \int_{\mathbb{R}^N} |\nabla \overline{u}(x)|^2 dx - \mu \frac{1}{q} \int_{\mathbb{R}^N} |\overline{u}(x)|^q dx)}{\frac{N(p-2)}{2} \int_{\mathbb{R}^N} |\overline{u}(x)|^p dx} \right)^{\frac{1}{\frac{N(p-2)}{2}-2}} \\
&\leq \left(\frac{2p \frac{N(p-2)}{4p} \int_{\mathbb{R}^N} |\overline{u}(x)|^p dx}{\frac{N(p-2)}{2} \int_{\mathbb{R}^N} |\overline{u}(x)|^p dx} \right)^{\frac{1}{\frac{N(p-2)}{2}-2}} \\
&= 1.
\end{aligned}$$

Hence, from Lemma 15, we have

$$\begin{aligned}
m_\mu &\leq J_c(\overline{u}^{t_0}) \\
&= J_c(\overline{u}^{t_0}) - \frac{2}{N(q-2)} G(\overline{u}^{t_0}) \\
&= t_0^2 c^{2\gamma} \frac{N(q-2)-4}{2N(q-2)} \|\nabla \overline{u}\|_2^2 + t_0^{\frac{N(p-2)}{2}} \frac{1}{p} (\frac{p-2}{q-2} - 1) \|\overline{u}\|_p^p \\
&\leq c^{2\gamma} \frac{N(q-2)-4}{2N(q-2)} \|\nabla \overline{u}\|_2^2 + \frac{1}{p} (\frac{p-2}{q-2} - 1) \|\overline{u}\|_p^p \\
&\leq \liminf_{n\to+\infty} \left(c^{2\gamma} \frac{N(q-2)-4}{2N(q-2)} \|\nabla \overline{u}_n\|_2^2 + \frac{1}{p} (\frac{p-2}{q-2} - 1) \|\overline{u}_n\|_p^p \right) \\
&\leq \frac{N(q-2)-4}{N(q-2)} \left(\frac{1}{2} c^{2\gamma} \|\nabla \overline{u}\|_2^2 + \frac{1}{p} \|\overline{u}\|_p^p \right) \\
&= \liminf_{n\to+\infty} (J_c(\overline{u}_n) - \frac{2}{N(q-2)} G(\overline{u}_n)) \\
&= m_c \\
&\leq m_\mu,
\end{aligned}$$

In other words, $m_c = m_\mu$. Lemma 15 implies that $\mu = c$ (i.e., $\overline{u} \in S_c$). Now, $t_0 = 1$, $J_c(\overline{u}) = m_c$ (i.e., $J_c|_{M_c}$ attains its minimum at $\overline{u}$). Hence, $\overline{u}$ is a nontrivial critical point of $J_c|_{M_c}$. It follows from Lemma 6 and Lemma 11 that there exists $\lambda_c < 0$ such that $J_c(\overline{u}) - \lambda_c \overline{u} = 0$ (i.e., Equation (1) has a couple solution $(\overline{u}, \lambda_c)$).

The proof is completed. □

3.5. The Proof of Theorem 5

Proof of Theorem 5. For $c > 0$, point (2) of Lemma 14 implies that $\inf_{u \in M_c} J_c(u) = m_c > 0$. Suppose that $\{u_n\} \subseteq M_c$ is a minimizing sequence for m_c. It follows from Lemma 15 that there exists $\{y_n\} \subseteq \mathbb{R}^N$ such that for any $\varepsilon > 0$, there exists $R > 0$, satisfying

$$\int_{\mathbb{R}^N - B_R(y_n)} |u_n(x)|^q dx < \varepsilon \text{ and } \int_{\mathbb{R}^N - B_R(y_n)} |u_n(x)|^p dx < \varepsilon. \tag{74}$$

Define $\overline{u}_n(x) := u_n(x + y_n)$. Now, $\overline{u}_n \in M_c$ is a bounded minimizing sequence for m_c. From Equation (74), we have

$$\int_{\mathbb{R}^N - B_R(0)} |u_n(x)|^p dx < \varepsilon \text{ and } \int_{\mathbb{R}^N - B_R(0)} |u_n(x)|^p dx < \varepsilon. \tag{75}$$

Without loss of generality, assume that there exists $\{\overline{u}_n\}$ in $H^1(\mathbb{R}^N)$ such that

$$\begin{cases} \overline{u}_n \rightharpoonup \overline{u}, \text{ in } H^1(\mathbb{R}^N) \\ \overline{u}_n \to \overline{u}, \text{ in } L^s(\mathbb{R}^N), \text{for all} 2 < s < 2^* \\ \overline{u}_n \to \overline{u}, \text{ a.e. } \mathbb{R}^N. \end{cases} \tag{76}$$

Now, Fatou's Lemma implies that

$$\int_{\mathbb{R}^N - B_R(0)} |\overline{u}(x)|^q \leq \varepsilon \text{ and } \int_{\mathbb{R}^N - B_R(0)} |\overline{u}(x)|^p \leq \varepsilon. \tag{77}$$

It follows from Equations (75)–(77) that

$$\overline{u}_n \to \overline{u}, \;\; \overline{u}_n \to \overline{u} \text{ in } L^p(\mathbb{R}^N) \cap L^p(\mathbb{R}^N). \tag{78}$$

Since $G(\overline{u}_n) = 0$, we deduce from Equations (78) and (34) that there exist two positive constants C_1 and C_2 independent of n such that

$$\int_{\mathbb{R}^N} |\overline{u}|^q dx = \lim_{n \to +\infty} \int_{\mathbb{R}^N} |\overline{u}_n|^q = C_1 \text{ and } \int_{\mathbb{R}^N} |\overline{u}|^p dx = \lim_{n \to +\infty} \int_{\mathbb{R}^N} |\overline{u}_n|^p = C_2,$$

In other words, $\overline{u} \neq 0$. Then, let $\tau := \|\overline{u}\|_2 \in (0, c]$. From Equations (76) and (78) and $\overline{u}_n \in M_c$, we have $G(\overline{u}) \leq \lim_{n \to +\infty} G(\overline{u}_n) = 0$. In other words, we obtain

$$c^{2\gamma} \int_{\mathbb{R}^N} |\nabla \overline{u}(x)|^2 dx \leq \mu \frac{N(q-2)}{2q} \int_{\mathbb{R}^N} |\overline{u}(x)|^q dx + \frac{N(p-2)}{2p} \int_{\mathbb{R}^N} |\overline{u}(x)|^p dx. \tag{79}$$

Now, point (2) of Lemma 8 infers that that there exists $t_0 > 0$:

$$t_0^2 c^{2\gamma} \int_{\mathbb{R}^N} |\nabla \overline{u}(x)|^2 dx = \mu t_0^{\frac{N(q-2)}{2}} \frac{N(q-2)}{2q} \int_{\mathbb{R}^N} |\overline{u}(x)|^q dx + t_0^{\frac{N(p-2)}{2}} \frac{N(p-2)}{2p} \int_{\mathbb{R}^N} |\overline{u}(x)|^p dx,$$

In other words, there is

$$c^{2\gamma} \int_{\mathbb{R}^N} |\nabla \overline{u}(x)|^2 dx = \mu t_0^{\frac{N(q-2)}{2} - 2} \frac{N(q-2)}{2q} \int_{\mathbb{R}^N} |\overline{u}(x)|^q dx + t_0^{\frac{N(p-2)}{2} - 2} \frac{N(p-2)}{2p} \int_{\mathbb{R}^N} |\overline{u}(x)|^p dx, \tag{80}$$

From $\frac{N(p-2)}{2} - 2 > 0$, $\frac{N(q-2)}{2} - 2 > 0$ and Equations (79) and (80), we have $t_0 \leq 1$, which together with Lemma 15 infers that

$$\begin{aligned}
m_\mu &\leq J_c(\overline{u}^{t_0}) \\
&= J_c(\overline{u}^{t_0}) - \frac{2}{N(q-2)} G(\overline{u}^{t_0}) \\
&= t_0^2 c^{2\gamma} \frac{N(q-2)-4}{2N(q-2)} \int_{\mathbb{R}^N} |\nabla \overline{u}(x)|^2 dx + t_0^{\frac{N(p-2)}{2}} \frac{1}{p} (\frac{p-2}{q-2} - 1) \int_{\mathbb{R}^N} |\overline{u}|^q dx \\
&\leq c^{2\gamma} \frac{N(q-2)-4}{2N(q-2)} \int_{\mathbb{R}^N} |\nabla \overline{u}(x)|^2 dx + \frac{1}{p} (\frac{p-2}{q-2} - 1) \int_{\mathbb{R}^N} |\overline{u}|^q dx \\
&\leq \liminf_{n \to +\infty} \Big(c^{2\gamma} \frac{N(q-2)-4}{2N(q-2)} \int_{\mathbb{R}^N} |\nabla \overline{u}(x)|^2 dx + \frac{1}{p} (\frac{p-2}{q-2} - 1) \int_{\mathbb{R}^N} |\overline{u}|^q dx \Big) \\
&= \liminf_{n \to +\infty} (J_c(\overline{u}_n) - \frac{2}{2N(q-2)} G(\overline{u}_n)) \\
&= m_c \\
&\leq m_\mu,
\end{aligned}$$

In other words, $m_c = m_\mu$. Lemma 15 implies that $\mu = c$ (i.e., $\overline{u} \in S_c$). Now, $t_0 = 1$, $J_c(\overline{u}) = m_c$ (i.e., $J_c|_{M_c}$ attains its minimum at $\overline{u}$). Hence, $\overline{u}$ is a nontrivial critical point of $J_c|_{M_r}$. It follows from Lemma 6 and Lemma 10 that there exists $\lambda_c < 0$ such that $J_c(\overline{u}) - \lambda_c \overline{u} = 0$; that is, Equation (1) has a couple solution $(\overline{u}, \lambda_c)$ for each $c > 0$.

The proof is completed. □

3.6. The Proof of Theorem 6

Proof of Theorem 6. The proof is divided into two steps:

Step 1. We show that Equation (1) has at least one couple solution (u_c, λ_1) with $J_c(u_c) < 0$ under our assumptions.

For $u \in M_c$, since $G(u) = 0$, we have

$$\begin{aligned}
J_c(u) &= \frac{1}{2} c^{2\gamma} \|\nabla u\|_2^2 |\nabla u(x)|^2 dx - \frac{\mu}{q} \|u\|_q^q - \frac{2}{N(p-2)} [c^{2\gamma} \|\nabla u\|_2^2 - \mu \frac{N(q-2)}{2q} \|u\|_q^q] \\
&= c^{2\gamma} \frac{N(p-2)-4}{2N(p-2)} \|\nabla u\|_2^2 |\nabla u(x)|^2 dx - \frac{\mu}{q} (1 - \frac{q-2}{p-2}) \|u\|_q^q \\
&\geq c^{2\gamma} \frac{N(p-2)-4}{2N(p-2)} \|\nabla u\|_2^2 - \mu (1 - \frac{q-2}{p-2}) \|u\|_q^q \frac{1}{2\|Q\|_2^{q-2}} \|\nabla u\|_2^{\frac{N(q-2)}{2}} c^{q - \frac{N(q-2)}{2}}.
\end{aligned} \tag{81}$$

Since $q \in (2, \frac{2N+4}{N})$, we have $\dfrac{N(q-2)-2}{2} < 2$, and thus $J_c(u)$ is bounded from below.

Since $E_1 \subseteq M_c$, we have

$$I_{c^2,rad} \leq \inf_{u \in S_{c,rad}, u \in E_1} J_c(u) < 0.$$

Then, $I_{c^2,rad} < 0$ is well-defined under our assumptions. There exists a sequence $\{u_n\}$ such that $J_c(u_n) \to I_{c^2,rad}$, and the Ekeland variational principle guarantees that $J_c'(u_n) \to 0$. Under Equation (81), $\{u_n\}$ is bounded. Now, Lemma 16 implies that $u_n \to u$ strongly in H^1, and $u_c \in M_c$ is a real-valued radial solution to Equation (1) for some $\lambda_1 < 0$ and $J_c(u_c) < 0$.

(2) We show that Equation (1) has at least one couple solution (v_c, λ_2) with $J_c(v_c) > 0$ under our assumptions.

From Lemma 13, there exists a bounded Palais–Smale sequence $\{v_n\} \subseteq S_c$ satisfying Equation (52). Lemma 16 guarantees that there exists (v_c, λ_2) which satisfies Equation (1) with $v_c \in S_c$, $\lambda_2 < 0$ and $J_c(v_c) > 0$.

Consequently, Equation (1) then has at least two positive couple solutions (u_c, λ_1) and (v_c, λ_2) satisfying $J_c(u_c) < 0$ and $J_c(v_c) > 0$, respectively.

The proof is completed. □

4. Conclusions

In this paper, we studied the nonexistence and existence of solutions with prescribed norms for nonlocal elliptic equations with combined nonlinearities. The results show that the nonexistence and existence of normalized solutions are not only related to the nonlinearities but also related to the prescribed mass.

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