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A New Projection Method for a System of Fractional Cauchy Integro-Differential Equations via Vieta–Lucas Polynomials

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Abstract: This work presents a projection method based on Vieta–Lucas polynomials and an effective approach to solve a Cauchy-type fractional integro-differential equation system. The suggested established model overcomes two linear equation systems. We prove the existence of the problem's approximate solution and conduct an error analysis in a weighted space. The theoretical results are numerically supported.

Keywords: fractional integro-differential equations; Cauchy kernel; projection approach; Vieta–Lucas polynomials

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1. Introduction

In recent years, numerous issues in mathematics, engineering, physics, and allied fields have been formulated in integral equations, notably, singular integral equations. Singular integral equations with a Cauchy kernel are an important class of such equations.

Fractional-order differential equations have been utilized extensively in quantum mechanics, astrobiology, medical science, chemical engineering, robust control, engineering, physiology, and hydrodynamics to describe various events.

The purpose of [1] is to investigate the dynamical evolution of symmetric oscillator with a fractional Caputo operator. The dynamical properties of the considered model such as equilibria and its stability are also examined. The existence, results, and uniqueness of proposed model solutions are considered using methods from fixed point theory. The authors of [2] decoupled a Lotka–Volterra model to explore the critical typical form coefficients of bifurcations for one-parameter and two-parameter bifurcations using a newly disclosed nonstandard finite difference method. In [3], a neural network strategy for solving the spatiotemporal fractional advection–diffusion equation with a nonlinear source term was described. Utilizing shifted Legendre orthogonal polynomials with variable coefficients, the network is created. The loss function of a neural network can be determined theoretically based on the features of unstable fractional derivatives. Multiple and generic discrete-time planar bifurcations were examined in [4]; using bifurcation theory and numerical continuation approaches, the Hindmarsh–Rose oscillator was analyzed for the study of three types of one-parameter bifurcation and five types of two-parameter bifurcation. Complex dynamics of the Kopel model with non-symmetric response among oligopolies were described in [5]. The studies indicated that a fixed point in a non-symmetric model may undergo fold, transcriptional, pitchfork, and Neimark–Sacker bifurcation under specific parameters.

The most valuable features for simulating functional problems and mathematical modeling are projection methods. These methods are efficient techniques for numerically solving integral and integro-differential equations. Over the past two decades, practical approaches for solving compact operator equations utilizing the Galerkin and Kulkarni approaches have been established. These two strategies inspired [6] to solve the following bounded equation. In [7], the authors recently proved how to solve fuzzy integro-differential equations with weak singularities using airfoil polynomials. The authors of [8] presented a novel projection approach based on Legendre polynomials for evaluating integro-differential equations of the Cauchy type. In addition, they investigated, in [9], a projection approximation for solving integro-differential problems of the Cauchy kind using first-order airfoil polynomials.

The orthogonality property of various significant polynomials, such as Vieta–Lucas polynomials, is used to approach the solution of different functional equations. These polynomials are crucial to solving functional equations.

The recurrence interaction of Vieta–Pell and Vieta–Pell–Lucas polynomials was introduced in [10]. The authors defined the associated sequences and obtained the Binet form and generating functions of Vieta–Pell and Vieta–Pell–Lucas polynomials. They also described some differentiation rules as well as finite summation formulas.

The authors of [11] developed a collocation approach based on a novel family of orthogonal functions for numerically treating a class of second-order singular multi-pantograph delay differential equations. The Vieta–Lucas functions were examined as differential bases, and the maximum norm errors were calculated.

The variable-order fractional form of the coupled nonlinear Ginzburg–Landau equations was formulated using the non-singular variable-order fractional derivative in the Heydari–Hosseininia concept in [12]. A numerical scheme based on shifted Vieta–Lucas polynomials was used to solve this system.

The authors of [13] proposed a fractional model of non-Newtonian Casson and Williamson boundary layer flow in fluid flow that takes the heat flux and slip velocity into account. The governing nonlinear system of PDEs is transformed into a nonlinear set of coupled ODEs, which is then solved using Vieta–Lucas polynomials, which are used to implement the spectral collocation method.

In [14], a new approximation technique combined with Vieta–Lucas orthogonal polynomials is used to solve the advection–dispersion equation, a fractional-order mathematical physics model.

In the last two decades, numerous results for solving compact operator equations using the Galerkin and Kulkarni approaches have been established. These two approaches are used to approach the solution of the following bounded equation, as cited in [6]:

$$x - Ax = y,$$

where A is a bounded linear operator, y is a know function, and x is a unknown function. The author examined an approximation of the following equation:

$$x_n^G - A_n x_n^G = \Pi_n y,$$

with approximate solution x_n and approximate operator A_n. The author originally established approximate operator A_n as a Galerkin approximation.

$$A_n^G := \Pi_n A \Pi_n.$$

Here, G stands for Galerkin, and $(\Pi_n)_{n\geq 1}$ is a sequence of bounded projections each one of finite rank, that is,

$$\Pi_n^2 = \Pi_n \ \text{ also } \ \Pi_n^* = \Pi_n.$$

Second, he utilized Kulkarni's approximation in the following manner:

$$A_n^K := \Pi_n A + A\Pi_n - \Pi_n A \Pi_n.$$

More recently, the authors of [15] introduced a novel projection approach for the following system of classical Cauchy integro-differential equations on $L^2([0,1],\mathbb{C})$ using shifted Legendre polynomials.

$$\begin{aligned} \varphi'(\tau) + \oint_0^1 \frac{\omega(\varsigma)}{\varsigma - \tau} d\varsigma &= h(\tau), \quad 0 \le \tau \le 1, \\ \omega'(\tau) + \oint_0^1 \frac{\varphi(\varsigma)}{\varsigma - \tau} d\varsigma &= k(\tau), \qquad 0 \le \tau \le 1, \\ \varphi(0) = 0, \ \ \omega(0) &= 0. \end{aligned}$$

This work presents a projection method for solving the following system of fractional Cauchy integro-differential equations on $L^2([0,1],\mathbb{C})$:

$$\begin{aligned} {}^cD^{\eta}_{0_+}\varphi(\tau) + \oint_0^1 \frac{\omega(\varsigma)}{\varsigma - \tau} d\varsigma &= h(\tau), \quad 0 \le \tau \le 1, \\ {}^cD^{\eta}_{0_+}\omega(\tau) + \oint_0^1 \frac{\varphi(\varsigma)}{\varsigma - \tau} d\varsigma &= k(\tau), \qquad 0 \le \tau \le 1, \\ \sum_{k=0}^{r-1} \frac{\tau^k}{k!}\varphi^{(k)}(0) = 0, \ \ \sum_{k=0}^{r-1} \frac{\tau^k}{k!}\omega^{(k)}(0) &= 0, \quad r-1 < \eta < r. \end{aligned}$$

We turn the problem into a system of two separate equations that looks like this:

$$\begin{cases} {}^cD^{\eta}_{0_+}\Omega + \mathcal{K}\Omega = H, \\ {}^cD^{\eta}_{0_+}\phi - \mathcal{K}\phi = K. \end{cases}$$

Our aim is employ Vieta–Lucas polynomials and present an approximation scheme to solve the above system. The solution is found for two different linear equation systems. The existence of a solution to the approximation equation is demonstrated, and an investigation of error analysis is presented. The theory is illustrated with numerical examples.

The remaining sections of this research paper are described in the following: The following section discusses some of the fundamental terms and theoretical concepts used in fractional theory. Section 3 discusses the system of fractional Cauchy integral equations. Section 4 discusses some key aspects of Vieta–Lucas polynomials and the development of the method. Section 5 improves the convergence of the approximate solution and estimates the error analysis. Section 6 explores some numerical examples.

2. Preliminaries

In this section, we begin by reviewing some of the basic terms and theoretical concepts used in fractional theory.

Let Γ indicate the fundamental Euler Gamma function in the analysis of fractional differential equations.

Definition 1. *The left-sided Riemann–Liouville fractional integral of order $\eta > 0$ of an integrable function $\omega : (0,\infty) \to \mathbb{R}$ is described by*

$$J^{\eta}_{0_+}\omega(\tau) := \frac{1}{\Gamma(\eta)} \int_0^{\tau} \frac{\omega(\varsigma)}{(\tau - \varsigma)^{1-\eta}} d\varsigma, \quad \tau > 0.$$

Remark 1. *The above integral can be represented in convolution form as follows:*

$$J^{\eta}_{0_+}\omega(\tau) = (\psi_\eta \star \omega)(\tau),$$

where

$$\psi_\eta(\tau) := \begin{cases} \tau^{\eta-1}/\Gamma(\eta), & \tau > 0, \\ 0, & \tau \leq 0. \end{cases}$$

Definition 2. *The left-sided Riemann–Liouville fractional derivative of order $\eta > 0$ of a continuous function $\omega : (0,\infty) \to \mathbb{R}$ is defined by*

$$D^\eta_{0_+}\omega(\tau) := \frac{1}{\Gamma(1-\eta)}\frac{d}{d\tau}\int_0^\tau \frac{\omega(\varsigma)}{(\tau-\varsigma)^\eta}d\varsigma.$$

Remark 2. *For $\eta > 0$, we have*

$$D^\eta_{0_+}\omega(\tau) = \frac{d}{d\tau}J^{1-\eta}_{0_+}\omega(\tau),$$

$$J^\eta_{0_+}D^\eta_{0_+}\Big(\omega(\tau)-\omega(0)\Big) = \omega(\tau)-\omega(0).$$

Definition 3. *For an absolutely continuous function ω, the Caputo fractional derivative of order $\eta > 0$ is defined by*

$${}^cD^\eta_{0_+}\omega(\tau) = J^{1-\eta}_{0_+}\frac{d}{d\tau}\omega(\tau) = \frac{1}{\Gamma(1-\eta)}\int_0^\tau (\tau-\varsigma)^{-\eta}\omega'(\varsigma)d\varsigma.$$

Remark 3. *For a continuous function ω, the relationship between the Caputo and Riemann–Liouville fractional derivatives is provided by*

$${}^cD^\eta_{0_+}\omega(\tau) = D^\eta_{0_+}(\omega(\tau)-\omega(0)), \quad \eta > 0.$$

In addition,

$${}^cD^\eta_{0_+}\omega(\tau) = D^\eta_{0_+}\Big(\omega(\tau) - \sum_{k=0}^{r-1}\frac{\tau^k}{k!}\omega^{(k)}(0)\Big), \quad \tau > 0,\ r-1 < \eta < r.$$

3. System of Fractional Cauchy Integro-Differential Equations

Let $\mathcal{H} := L^2([0,1],\mathbb{C})$ be the space of complex-valued Lebesgue square integrable functions on $[0,1]$. In this study, a new projection technique for solving a system of fractional Cauchy integro-differential equations using Vieta–Lucas polynomials in $\mathcal{H}$ is presented.

Consider the system of fractional Cauchy integro-differential equations of the following form:

$$\begin{aligned} {}^cD^\eta_{0_+}\varphi(\tau) + \fint_0^1 \frac{\omega(\varsigma)}{\varsigma-\tau}d\varsigma &= h(\tau), \quad 0 \leq \tau \leq 1, \\ {}^cD^\eta_{0_+}\omega(\tau) + \fint_0^1 \frac{\varphi(\varsigma)}{\varsigma-\tau}d\varsigma &= k(\tau), \quad 0 \leq \tau \leq 1, \\ \sum_{k=0}^{r-1}\frac{\tau^k}{k!}\varphi^{(k)}(0) = 0,\ \sum_{k=0}^{r-1}\frac{\tau^k}{k!}\omega^{(k)}(0) &= 0, \quad r-1 < \eta < r. \end{aligned}$$

The integral of each equation denotes the Cauchy principal value as follows:

$$\fint_0^1 \frac{\omega(\varsigma)}{\varsigma-\tau}d\varsigma = \lim_{\varepsilon\to 0}\bigg(\int_0^{\tau-\varepsilon}\frac{\omega(\varsigma)}{\varsigma-\tau}d\varsigma + \int_{\tau+\varepsilon}^1\frac{\omega(\varsigma)}{\varsigma-\tau}d\varsigma\bigg),$$
$$\fint_0^1 \frac{\varphi(\varsigma)}{\varsigma-\tau}d\varsigma = \lim_{\varepsilon\to 0}\bigg(\int_0^{\tau-\varepsilon}\frac{\varphi(\varsigma)}{\varsigma-\tau}d\varsigma + \int_{\tau+\varepsilon}^1\frac{\varphi(\varsigma)}{\varsigma-\tau}d\varsigma\bigg).$$

As in [16], let us consider the following transform:

$$\phi := \omega - \varphi, \quad H := k + h;$$

$$\Omega := \omega + \varphi, \quad K := k - h.$$

Lemma 1. *Problem (1) can be expressed in the following form:*

$$\begin{aligned} {}^{c}D^{\eta}_{0_+}\Omega(\tau) + \oint_0^1 \frac{\Omega(\varsigma)}{\varsigma - \tau} d\varsigma &= H(\tau), \quad 0 \le \tau \le 1, \\ {}^{c}D^{\eta}_{0_+}\phi(\tau) - \oint_0^1 \frac{\phi(\varsigma)}{\varsigma - \tau} d\varsigma &= K(\tau), \quad 0 \le \tau \le 1, \\ \sum_{k=0}^{r-1} \frac{\tau^k}{k!}\Omega^{(k)}(0) = 0, \ \sum_{k=0}^{r-1} \frac{\tau^k}{k!}\phi^{(k)}(0) &= 0, \ \ r-1 < \eta < r. \end{aligned}$$

Proof. In fact,

$$\omega = \frac{\Omega + \phi}{2}, \quad h = \frac{H - K}{2};$$

$$\varphi = \frac{\Omega - \phi}{2}, \quad k = \frac{H + K}{2}.$$

By substituting them into (1), we obtain

$${}^{c}D^{\eta}_{0_+}(\Omega - \phi)(\tau) + \oint_0^1 \frac{(\Omega + \phi)(\varsigma)}{\varsigma - \tau} d\varsigma = (H - K)(\tau), \tag{1}$$

$${}^{c}D^{\eta}_{0_+}(\Omega + \phi)(\tau) + \oint_0^1 \frac{(\Omega - \phi)(\varsigma)}{\varsigma - \tau} d\varsigma = (H + K)(\tau). \tag{2}$$

We derive (1) by adding Equations (1) and (2) together and then subtracting (1) from (2). □

System (1) can be rewritten in operator form as follows:

$$\begin{cases} {}^{c}D^{\eta}_{0_+}\Omega(\tau) + \mathcal{K}\Omega(\tau) = H(\tau), \\ {}^{c}D^{\eta}_{0_+}\phi(\tau) - \mathcal{K}\phi(\tau) = K(\tau), \end{cases}$$

where $\mathcal{K}$ is the Cauchy integral operator, i.e.,

$$\mathcal{K}\varphi(\tau) := \oint_0^1 \frac{\varphi(\varsigma)}{\varsigma - \tau} d\varsigma, \quad 0 \le \tau \le 1.$$

Following [8], operator $\mathcal{K}$ is bounded from $\mathcal{H}$ into itself.
Letting

$$\mathcal{D} := \left\{ \omega \in \mathcal{H} : \ \omega^{(k)} \in \mathcal{H}, \quad \sum_{k=0}^{r-1} \frac{\tau^k}{k!}\omega^{(k)}(0) = 0, \ \ r-1 < \eta < r \right\}.$$

It is well known that

$$J^{\eta}_{0_+}\,{}^{c}D^{\eta}_{0_+}\omega(\tau) = \omega(\tau) - \sum_{k=0}^{r-1} \frac{\tau^k}{k!}\omega^{(k)}(0).$$

So,

$$J^{\eta}_{0_+}\,{}^{c}D^{\eta}_{0_+}\omega(\tau) = \omega(\tau), \ \ \text{for all} \ \ \omega \in \mathcal{D}.$$

In addition, $J^{\eta}_{0_+} : \mathcal{H} \to \mathcal{D}$ is compact.

4. Vieta–Lucas Polynomials

In this section, we look at a class of orthogonal polynomials. These polynomials can be used to construct a new family of orthogonal polynomials known as Vieta–Lucas polynomials using recurrence relations and an analytical formula.

Vieta–Lucas polynomials $\mathcal{V}_n$ of degree $n \in \mathbb{N}$ are defined as follows:

$$\mathcal{V}_n(\tau) = 2\cos(n\theta), \quad \theta = \cos^{-1}\left(\frac{\tau}{2}\right), \ \theta \in [0, \pi] \ \text{ for all } \ |\tau| \le 2.$$

The following iterative formula can be used to generate polynomial $\mathcal{V}_n$:

$$\mathcal{V}_n(\tau) = \tau\mathcal{V}_{n-1}(\tau) - \mathcal{V}_{n-2}(\tau), \quad n = 2, 3, \ldots, \qquad \mathcal{V}_0(\tau) = 2, \qquad \mathcal{V}_1(\tau) = \tau.$$

Also, the explicit power series formula shown below can be used to calculate $\mathcal{V}_n$:

$$\mathcal{V}_n(\tau) = \sum_{i=0}^{\lceil \frac{n}{2} \rceil} (-1)^i \frac{n\Gamma(n-i)}{\Gamma(i+1)\Gamma(n+1-2i)} \tau^{n-2i}, \quad n = \{2, 3, \ldots\}.$$

In addition, $\mathcal{V}_n$ are orthogonal polynomials with respect to the integral shown below:

$$\langle \mathcal{V}_m, \mathcal{V}_n \rangle = \int_{-2}^{2} \frac{1}{\sqrt{4-\tau^2}} \mathcal{V}_m(\tau)\mathcal{V}_n(\tau)\, d\tau = \begin{cases} 0, & m \ne n \ne 0, \\ 4\pi, & m = n = 0, \\ 2\pi, & m = n \ne 0. \end{cases}$$

Let

$$\mathcal{V}_k^{\mathcal{S}}(\tau) = \mathcal{V}_k(4\tau - 2) = \mathcal{V}_{2k}(2\sqrt{\tau}), \ \ k = 0, 2, 3, \ldots$$

denote the corresponding normalized sequence. Additionally, $\mathcal{V}_k^{\mathcal{S}}$ are produced using the recurrence formula shown below:

$$\mathcal{V}_{k+1}^{\mathcal{S}}(\tau) = (4\tau - 2)\mathcal{V}_k^{\mathcal{S}}(\tau) - \mathcal{V}_{k-1}^{\mathcal{S}}(\tau), \quad k = 1, 2, \ldots,$$

with

$$\mathcal{V}_0^{\mathcal{S}}(\tau) = 2, \qquad \mathcal{V}_1^{\mathcal{S}}(\tau) = 4\tau - 2.$$

We note that

$$\mathcal{V}_p^{\mathcal{S}}(\tau) = 2p \sum_{j=0}^{p} (-1)^j \frac{4^{p-j}\Gamma(2p-j)}{\Gamma(j+1)\Gamma(2p-2j+1)} \tau^{p-j}, \quad p = \{2, 3, \ldots\}$$

and

$$\langle \mathcal{V}_k^{\mathcal{S}}, \mathcal{V}_j^{\mathcal{S}} \rangle_\omega = \int_0^1 \mathcal{V}_k^{\mathcal{S}}(\tau)\mathcal{V}_j^{\mathcal{S}}(\tau)\omega(\tau)\, d\tau = \begin{cases} 0, & k \ne j \ne 0, \\ 4\pi, & k = j = 0, \\ 2\pi, & k = j \ne 0, \end{cases}$$

where

$$\omega(\tau) = \frac{1}{\sqrt{\tau - \tau^2}}.$$

Letting

$$\Delta_j^{\mathcal{S}} := \begin{cases} \frac{\mathcal{V}_j^{\mathcal{S}}}{2\sqrt{\pi}}, & j = 0, \\ \\ \frac{\mathcal{V}_j^{\mathcal{S}}}{\sqrt{2\pi}}, & j \ne 0. \end{cases}$$

Now, we introduce the first six terms of $\Delta_j^{\mathcal{S}}$:

$$\begin{aligned}
\Delta_0^{\mathcal{S}}(s) &= \frac{1}{\sqrt{\pi}}, \\
\Delta_1^{\mathcal{S}}(s) &= \frac{\sqrt{2}(2s-1)}{\sqrt{\pi}}, \\
\Delta_2^{\mathcal{S}}(s) &= \frac{\sqrt{2}\left(8s^2-8s+1\right)}{\sqrt{\pi}}, \\
\Delta_3^{\mathcal{S}}(s) &= \frac{\sqrt{2}\left(32s^3-48s^2+18s-1\right)}{\sqrt{\pi}}, \\
\Delta_4^{\mathcal{S}}(s) &= \frac{\sqrt{2}\left(128s^4-256s^3+160s^2-32s+1\right)}{\sqrt{\pi}}, \\
\Delta_5^{\mathcal{S}}(s) &= \frac{\sqrt{2}\left(512s^5-1280s^4+1120s^3-400s^2+50s-1\right)}{\sqrt{\pi}}.
\end{aligned}$$

Let $\pi_n^{\mathcal{S}}$ be the chain of bounded finite rank orthogonal projections described by

$$\pi_n^{\mathcal{S}}\psi := \sum_{j=0}^{n-1}\left\langle \psi, \Delta_j^{\mathcal{S}}\right\rangle_{\omega}\Delta_j^{\mathcal{S}}, \text{ where } \left\langle \psi, \Delta_j^{\mathcal{S}}\right\rangle_{\omega} := \int_0^1 \omega(\sigma)\psi(\sigma)\Delta_j^{\mathcal{S}}(\sigma)d\sigma.$$

Denote with $\|\cdot\|$ the corresponding norm on $\mathcal{H}$. Thus,

$$\lim_{n\to\infty}\|\pi_n^{\mathcal{S}}\vartheta-\vartheta\| = 0, \text{ for all } \vartheta\in\mathcal{H}.$$

Let $\mathcal{H}_n$ represent the space covered by the first n-shifted Vieta–Lucas polynomials. It is obvious that $J_{0_+}^{\eta}(\mathcal{H}_n) = \mathcal{H}_{n+1}$.

We note that $\Omega_n, \phi_n \in \mathcal{D}\cap\mathcal{H}_{n+1}$. Thus, the system

$$\begin{cases}\Omega + J_{0_+}^{\eta}\mathcal{K}\Omega = J_{0_+}^{\eta}H, \\ \phi - J_{0_+}^{\eta}\mathcal{K}\phi = J_{0_+}^{\eta}K\end{cases}$$

is approximated by

$$\begin{cases}\Omega_n + J_{0_+}^{\eta}\pi_n^{\mathcal{S}}\mathcal{K}\Omega_n = J_{0_+}^{\eta}\pi_n^{\mathcal{S}}H, \\ \phi_n - J_{0_+}^{\eta}\pi_n^{\mathcal{S}}\mathcal{K}\phi_n = J_{0_+}^{\eta}\pi_n^{\mathcal{S}}K.\end{cases}$$

We assume that -1 and 1 are not eigenvalues of $J_{0_+}^{\eta}\mathcal{K}$. Thus, both operators $I + J_{0_+}^{\eta}\mathcal{K}$ and $I - J_{0_+}^{\eta}\mathcal{K}$ are invertible.

We recall that $J_{0_+}^{\eta}$ is compact and

$$\lim_{n\to\infty}\left\|\left(J_{0_+}^{\eta}\pi_n^{\mathcal{S}}\mathcal{K} - J_{0_+}^{\eta}\mathcal{K}\right)J_{0_+}^{\eta}\mathcal{K}\right\| = 0, \quad \lim_{n\to\infty}\left\|\left(J_{0_+}^{\eta}\pi_n^{\mathcal{S}}\mathcal{K} - J_{0_+}^{\eta}\mathcal{K}\right)J_{0_+}^{\eta}\pi_n^{\mathcal{S}}\mathcal{K}\right\| = 0.$$

Writing

$$\begin{cases}\Omega_n = \sum_{j=0}^{n} a_{n,j}\Delta_j^{\mathcal{S}}, \\ \phi_n = \sum_{j=0}^{n} b_{n,j}\Delta_j^{\mathcal{S}}.\end{cases}$$

We obtain $2n+2$ unknowns $a_{n,j}$ and $b_{n,j}$ by solving the two separate linear systems,

$$\begin{cases} \sum\limits_{j=0}^{n} a_{n,j}\left[D_{0_+}^{\eta}\Delta_j^{\mathcal{S}} + \pi_n^{\mathcal{S}}\mathcal{K}\Delta_j^{\mathcal{S}}\right] = \pi_n^{\mathcal{S}} J_{0_+}^{\eta} H, \;\text{ with }\; \sum\limits_{j=0}^{n} a_{n,j}\Delta_j^{\mathcal{S}}(0) = 0, \\ \sum\limits_{j=0}^{n} b_{n,j}\left[D_{0_+}^{\eta}\Delta_j^{\mathcal{S}} - \pi_n^{\mathcal{S}}\mathcal{K}\Delta_j^{\mathcal{S}}\right] = \pi_n^{\mathcal{S}} J_{0_+}^{\eta} K, \;\text{ with }\; \sum\limits_{j=0}^{n} b_{n,j}\Delta_j^{\mathcal{S}}(0) = 0. \end{cases}$$

As a result, two separate linear systems are produced,

$$\begin{cases} \sum\limits_{j=0}^{n} M_n(i,j)a_{n,j} = E_{n,i}, \;\; i = 0,\cdots,n, \\ \sum\limits_{j=0}^{n} \widehat{M}_n(i,j)b_{n,j} = \widehat{E}_{n,i}, \;\; i = 0,\cdots,n. \end{cases}$$

where, for $i = 0,\cdots,n-1$ and $j = 0,\cdots,n$,

$$\begin{aligned} M_n(i,j) &:= \left[\int_0^1 {}^cD_{0_+}^{\eta}\Delta_j^{\mathcal{S}}(s)\Delta_i^{\mathcal{S}}(s)\omega(s)ds + \int_0^1\left(\oint_0^1 \frac{\Delta_j^{\mathcal{S}}(\sigma)}{\sigma - s}d\sigma\right)\Delta_i^{\mathcal{S}}(s)\omega(s)ds\right], \\ \widehat{M}_n(i,j) &:= \left[\int_0^1 {}^cD_{0_+}^{\eta}\Delta_j^{\mathcal{S}}(s)\Delta_i^{\mathcal{S}}(s)\omega(s)ds - \int_0^1\left(\oint_0^1 \frac{\Delta_j^{\mathcal{S}}(\sigma)}{\sigma - s}d\sigma\right)\Delta_i^{\mathcal{S}}(s)\omega(s)ds\right], \\ M_n(n,j) &:= \Delta_j^{\mathcal{S}}(0), \quad \widehat{M}_n(n,j) := \Delta_j^{\mathcal{S}}(0), \\ E_n(i) &:= \int_0^1 H(s)^cD_{0_+}^{\eta}\Delta_i^{\mathcal{S}}(s)\omega(s)ds, \;\; \widehat{E}_n(i) := \int_0^1 K(s)^cD_{0_+}^{\eta}\Delta_i^{\mathcal{S}}(s)\omega(s)ds, \\ E_n(n) &:= 0, \;\; \widehat{E}_n(n) := 0. \end{aligned}$$

5. Convergence Analysis

We now show how the current method converges. To that end, consider $L_{\varpi}^2([0,1],\mathbb{C})$ to be the weighted space and $\|.\|_{\varpi}$ to be its norm.

Denote with I the identity operator. We recall that there exists $M > 0$ such that

$$\left\|(I - \pi_n^{\mathcal{S}})\psi\right\|_{\varpi} \le \frac{M}{12n^{3/2}}, \;\; \text{for all } \;\psi \in L_{\varpi}^2([0,1],\mathbb{C}).$$

Since $J_{0_+}^{\eta}\mathcal{K}$ is compact, according to [8], operators $(I + J_{0_+}^{\eta}\pi_n^{\mathcal{S}}\mathcal{K})^{-1}$ and $(I - J_{0_+}^{\eta}\pi_n^{\mathcal{S}}\mathcal{K})^{-1}$ exist for n large enough and are uniformly bounded with respect to n.

Theorem 1. *Assume that $k, h, \mathcal{K}\omega, \mathcal{K}\varphi \in L_{\varpi}^2([0,1],\mathbb{C})$. Then, there exist $M_1,\ M_2 > 0$ such that*

$$\|\Omega_n - \Omega\|_{\varpi} \le \frac{M_1}{12\, n^{3/2}},$$

and

$$\|\phi_n - \phi\|_{\varpi} \le \frac{M_2}{12\, n^{3/2}}.$$

Proof. In fact,

$$\begin{cases} \Omega_n + J_{0_+}^{\eta}\pi_n^{\mathcal{S}}\mathcal{K}\Omega_n = J_{0_+}^{\eta}\pi_n^{\mathcal{S}} H, \\ \phi_n - J_{0_+}^{\eta}\pi_n^{\mathcal{S}}\mathcal{K}\phi_n = J_{0_+}^{\eta}\pi_n^{\mathcal{S}} K. \end{cases}$$

Moreover,

$$\begin{aligned}\Omega-\Omega_n &= \left[\left(I+J^{\eta}_{0_+}\mathcal{K}\right)^{-1}J^{\eta}_{0_+}H-\left(I+J^{\eta}_{0_+}\pi^{\mathcal{S}}_n\mathcal{K}\right)^{-1}J^{\eta}_{0_+}\pi^{\mathcal{S}}_nH\right]\\&\quad+\left(I+J^{\eta}_{0_+}\pi^{\mathcal{S}}_n\mathcal{K}\right)^{-1}J^{\eta}_{0_+}H-\left(I+J^{\eta}_{0_+}\pi^{\mathcal{S}}_n\mathcal{K}\right)^{-1}J^{\eta}_{0_+}H\\&=\left(I+J^{\eta}_{0_+}\pi^{\mathcal{S}}_n\mathcal{K}\right)^{-1}J^{\eta}_{0_+}\left[\left(I-\pi^{\mathcal{S}}_n\right)\mathcal{K}\Omega+\left(I-\pi^{\mathcal{S}}_n\right)H\right].\end{aligned}$$

In addition,

$$\begin{aligned}\phi-\phi_n &= \left[\left(I-J^{\eta}_{0_+}\mathcal{K}\right)^{-1}J^{\eta}_{0_+}K-\left(I-J^{\eta}_{0_+}\pi^{\mathcal{S}}_n\mathcal{K}\right)^{-1}J^{\eta}_{0_+}\pi^{\mathcal{S}}_nK\right]\\&\quad+\left(I-J^{\eta}_{0_+}\pi^{\mathcal{S}}_n\mathcal{K}\right)^{-1}J^{\eta}_{0_+}K-\left(I-J^{\eta}_{0_+}\pi^{\mathcal{S}}_n\mathcal{K}\right)^{-1}J^{\eta}_{0_+}K\\&=\left(I-J^{\eta}_{0_+}\pi^{\mathcal{S}}_n\mathcal{K}\right)^{-1}J^{\eta}_{0_+}\left[\left(I-\pi^{\mathcal{S}}_n\right)\mathcal{K}\phi+\left(I-\pi^{\mathcal{S}}_n\right)K\right].\end{aligned}$$

Hence,

$$\Omega-\Omega_n=\left(I+J^{\eta}_{0_+}\pi^{\mathcal{S}}_n\mathcal{K}\right)^{-1}J^{\eta}_{0_+}\left[\left(I-\pi^{\mathcal{S}}_n\right)\mathcal{K}(\omega+\varphi)+\left(I-\pi^{\mathcal{S}}_n\right)(k+h)\right],$$

and thus

$$\phi_n-\phi=\left(I-J^{\eta}_{0_+}\pi^{\mathcal{S}}_n\mathcal{K}\right)^{-1}J^{\eta}_{0_+}\left[\left(I-\pi^{\mathcal{S}}_n\right)\mathcal{K}(\omega-\varphi)+\left(I-\pi^{\mathcal{S}}_n\right)(k-h)\right].$$

Letting

$$\begin{aligned}\delta &:= \left\|\left(I+J^{\eta}_{0_+}\pi^{\mathcal{S}}_n\mathcal{K}\right)^{-1}\right\|\left\|J^{\eta}_{0_+}\right\|,\\ \gamma &:= \left\|\left(I-J^{\eta}_{0_+}\pi^{\mathcal{S}}_n\mathcal{K}\right)^{-1}\right\|\left\|J^{\eta}_{0_+}\right\|,\end{aligned}$$

we obtain

$$\begin{aligned}\|\Omega_n-\Omega\|_{\varpi} &\le \delta\left[\left\|\left(I-\pi^{\mathcal{S}}_n\right)\mathcal{K}(\omega+\varphi)\right\|_{\varpi}+\left\|\left(I-\pi^{\mathcal{S}}_n\right)(k+h)\right\|_{\varpi}\right]\\&\le \delta\left[\left\|\left(I-\pi^{\mathcal{S}}_n\right)\mathcal{K}(\omega+\varphi)\right\|_{\varpi}+\left\|\left(I-\pi^{\mathcal{S}}_n\right)(k+h)\right\|_{\varpi}\right]\\&\le \delta\left[\frac{M_3}{12\,n^{3/2}}+\frac{M_4}{12\,n^{3/2}}\right],\end{aligned}$$

for some constants $M_3,\ M_4>0$. Moreover,

$$\begin{aligned}\|\phi_n-\phi\|_{\varpi} &= \left\|\left(I-J^{\eta}_{0_+}\pi^{\mathcal{S}}_n\mathcal{K}\right)^{-1}\left[J^{\eta}_{0_+}\left(I-\pi^{\mathcal{S}}_n\right)\mathcal{K}\phi+J^{\eta}_{0_+}\left(I-\pi^{\mathcal{S}}_n\right)K\right]\right\|_{\varpi}\\&\le \gamma\left[\left\|\left(I-\pi^{\mathcal{S}}_n\right)\mathcal{K}(\omega-\varphi)\right\|_{\varpi}+\left\|\left(I-\pi^{\mathcal{S}}_n\right)(k-h)\right\|_{\varpi}\right]\\&\le \gamma\left[\frac{M_5}{12\,n^{3/2}}+\frac{M_6}{12\,n^{3/2}}\right],\end{aligned}$$

for some constants $M_5,\ M_6>0$.

Letting

$$M_1:=\delta\max\{M_3,M_4\}\ \text{ and }\ M_2:=\gamma\max\{M_5,M_6\},$$

we obtain the desired results. □

6. Numerical Example

In this section, numerical experiments are established to illustrate the results presented in the previous section. In these numerical evaluations, the Maple programming language was implemented.

Example 1. *We study the fractional integro-differential system (1) in this example, which has the exact solution as follows:*

$$\varphi(\tau) = \frac{1}{2}\tau^3, \;\; \omega(\tau) = \frac{1}{2}\Big(\tau^3 - 2\tau^2\Big), \;\; \eta = \frac{1}{3}.$$

In this case, we obtain

$$\Omega(\tau) = \tau^3 - \tau^2, \;\; \phi(\tau) = -\tau^2.$$

$$\begin{aligned} H(\tau) \;\; &= \;\; -\frac{1}{120\Gamma\left(\frac{2}{3}\right)}\Big[-243\tau^{8/3} + 120\Gamma\left(\frac{2}{3}\right)\tau^3\ln(\tau) + 216\tau^{5/3} \\ &- \;\; 120\Gamma\left(\frac{2}{3}\right)\tau^3\ln(-\tau+1) - 120\Gamma\left(\frac{2}{3}\right)\tau^2\ln(\tau) + 120\Gamma\left(\frac{2}{3}\right)\tau^2\ln(-\tau+1) \\ &- \;\; 120\Gamma\left(\frac{2}{3}\right)\tau^2 + 60\tau + 20\Gamma\left(\frac{2}{3}\right)\Big] \end{aligned}$$

and

$$\begin{aligned} K(\tau) \;\; &= \;\; -\frac{1}{10\Gamma\left(\frac{2}{3}\right)}\Big[18\tau^{5/3} - 10\Gamma\left(\frac{2}{3}\right)\tau^2\ln(1-\tau) + 216\tau^{5/3} \\ &+ \;\; 10\Gamma\left(\frac{2}{3}\right)\tau^2\ln(\tau) - 10\Gamma\left(\frac{2}{3}\right)\tau - 5\Gamma\left(\frac{2}{3}\right)\Big]. \end{aligned}$$

To show the efficiency of this example, we shall offer some numerical testes. For example, for $n = 7$, unknowns $a_{7,0}\cdots a_{7,7}$ are as follows:

$$\begin{aligned} a_{7,0} \;\; &= \;\; -0.78334 \times 10^{-1}, \quad a_{7,1} = -0.39165 \times 10^{-1}, \\ a_{7,2} \;\; &= \;\; 0.78330 \times 10^{-1}, \quad a_{7,3} = 0.39165 \times 10^{-1}, \\ a_{7,4} \;\; &= \;\; -5.6631 \times 10^{-7}, \;\; a_{7,5} = -0.17120 \times 10^{-5}, \\ a_{7,6} \;\; &= \;\; -3.2531 \times 10^{-7}, \;\; a_{7,7} = -0.14416 \times 10^{-5}. \end{aligned}$$

Approximate solution Ω_7 is given by

$$\begin{aligned} \Omega_7(\tau) \;\; &= \;\; -0.16755 \times 10^{-3}\tau + 0.98870\tau^3 + 0.30814 \times 10^{-1}\tau^4 - 0.44449 \times 10^{-1}\tau^5 \\ &+ \;\; 0.32446 \times 10^{-1}\tau^6 - 0.94226 \times 10^{-2}\tau^7 - 0.99779\tau^2 - 0.21845 \times 10^{-5}. \end{aligned}$$

In addition,

$$\begin{aligned} b_{7,0} \;\; &= \;\; -0.46998, \quad b_{7,1} = -0.62666, \\ b_{7,2} \;\; &= \;\; -0.15668, \quad b_{7,3} = 0.30641 \times 10^{-5}, \\ b_{7,4} \;\; &= \;\; -0.28020 \times 10^{-5}, \quad b_{7,5} = 0.44393 \times 10^{-5}, \\ b_{7,6} \;\; &= \;\; -0.20660 \times 10^{-5}, \;\; b_{7,7} = 0.25847 \times 10^{-5}. \end{aligned}$$

Approximate solution ϕ_7 is provided by

$$\begin{aligned} \phi_7(\tau) \;\; &= \;\; 0.71009 \times 10^{-3}\tau + 0.29927 \times 10^{-1}\tau^3 - 1.0066\tau^2 - 0.11933 \times 10^{-4} \\ &- \;\; 0.71647 \times 10^{-1}\tau^4 + 0.93245 \times 10^{-1}\tau^5 - 0.62505\times 10^{-1}\tau^6 + 0.16894 \times 10^{-1}\tau^7. \end{aligned}$$

Table 1 shows the numerical findings obtained for Example 1 using the method we suggest.

Table 1. Numerical results for Example 1.

n	$\|\Omega - \Omega_n\|_\varpi$	$\|\phi - \phi_n\|_\varpi$
3	4.1428×10^{-5}	1.5520×10^{-4}
5	8.6994×10^{-6}	3.5924×10^{-5}
7	9.6542×10^{-8}	8.2544×10^{-8}
13	7.2543×10^{-13}	7.2547×10^{-12}
17	9.2541×10^{-15}	3.8531×10^{-14}

Figure 1 depicts the graphs of the achieved absolute errors for Example 1. Here, we note that the τ symbol in the figures refers to the τ-axis labels.

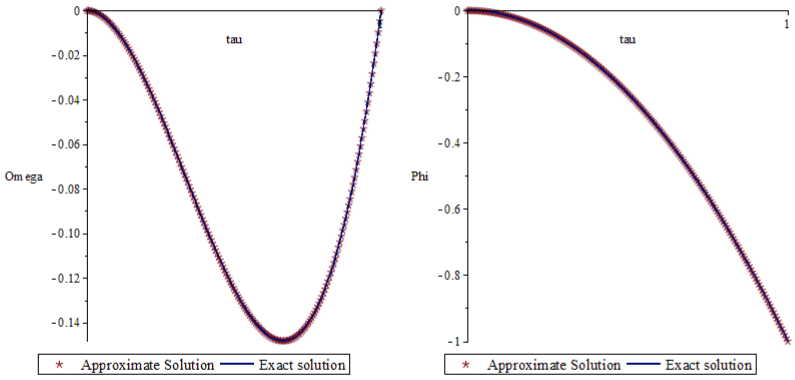

Figure 1. Comparison of exact solutions Ω and ϕ and approximate solutions Ω_n and ϕ_n, respectively, for the first example, where $n = 7$.

Example 2. *Various types of fractional integro-differential and integral systems can be studied and solved with the proposed method. As a second example for testing, we consider the system of fractional logarithmic integro-differential equations of the following form:*

$$\begin{aligned} {}^cD^{\eta}_{0_+}\varphi(\tau) + \int_0^1 \omega(\varsigma)\ln|\varsigma - \tau| d\varsigma &= h(\tau), \quad 0 \le \tau \le 1, \\ {}^cD^{\eta}_{0_+}\omega(\tau) + \int_0^1 \varphi(\varsigma)\ln|\varsigma - \tau| d\varsigma &= k(\tau), \qquad 0 \le \tau \le 1, \\ \sum_{k=0}^{r-1} \frac{\tau^k}{k!}\varphi^{(k)}(0) = 0, \ \sum_{k=0}^{r-1} \frac{\tau^k}{k!}\omega^{(k)}(0) &= 0, \ \ r-1 < \eta < r, \end{aligned}$$

which has the following exact solution:

$$\varphi(\tau) = -\frac{1}{2}\left(\tau^3 + \tau^4\right), \ \ \omega(\tau) = \frac{1}{2}\left(\tau^4 - \tau^3\right), \ \ \eta = \frac{1}{2}.$$

Here, we have

$$\Omega(\tau) = -\tau^3, \ \ \phi(\tau) = \tau^4.$$

We provide some numerical tests to demonstrate the efficacy of this illustration. If $n = 5$, then the values of $a_{5,0} \cdots a_{5,5}$ are as shown below:

$$\begin{aligned} a_{5,0} &= -0.39168, \quad a_{5,1} = -0.58757, \\ a_{5,2} &= -0.23503, \quad a_{5,3} = -0.39169 \times 10^{-1}, \\ a_{5,4} &= 0.15704 \times 10^{-5}, \quad a_{5,5} = 9.5589 \times 10^{-7}. \end{aligned}$$

Approximate solution Ω_5 is given by

$$\begin{aligned} \Omega_5(\tau) &= 0.15159 \times 10^{-3}\tau - 0.30425 \times 10^{-3}\tau^2 - 0.99947\tau^3 + 0.39048 \times 10^{-3}\tau^5 \\ &+ 0.68731 \times 10^{-5}. \end{aligned}$$

In addition,

$$\begin{aligned} b_{5,0} &= 0.34276, \quad b_{5,1} = 0.54834, \\ b_{5,2} &= 0.27418, \quad b_{5,3} = 0.78336 \times 10^{-1}, \\ b_{5,4} &= 0.97933 \times 10^{-2}, \quad b_{5,5} = 5.6341 \times 10^{-7}. \end{aligned}$$

Approximate solution ϕ_5 is provided by

$$\begin{aligned} \phi_5(\tau) &= -0.79785 \times 10^{-5}\tau + 0.99955\tau^4 + 0.27127 \times 10^{-3}\tau^3 - 0.23936 \times 10^{-4}\tau^2 \\ &+ 0.45188 \times 10^{-4} + 0.23015 \times 10^{-4}\tau^5. \end{aligned}$$

Figure 2 illustrates the numerical results produced for Example 2 using the indicated approach.

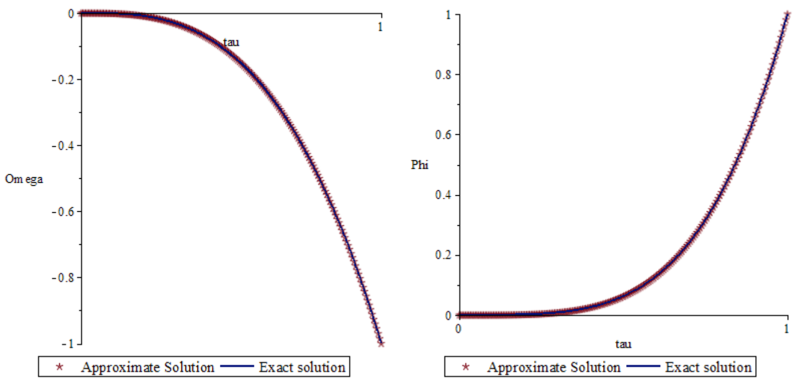

Figure 2. Comparison of exact solutions Ω and ϕ and approximate solutions Ω_n and ϕ_n, respectively, for the second example, where $n = 5$.

Example 3. *Now, we consider the above system of fractional logarithmic integro-differential equations, with the same exact solutions.*

$$\varphi(\tau) = -\frac{1}{2}\left(\tau^3 + \tau^4\right), \quad \omega(\tau) = \frac{1}{2}\left(\tau^4 - \tau^3\right).$$

In Table 2, we examine the influence of fractional order η on the approximate solutions.

Table 2. Influence of fractional order η on the approximate solutions.

η	$\|\Omega - \Omega_n\|_\varpi$	$\|\phi - \phi_n\|_\varpi$
0.3	1.0548×10^{-5}	3.1473×10^{-5}
0.4	1.2668×10^{-4}	1.0739×10^{-4}
0.6	7.9977×10^{-6}	3.2081×10^{-5}
0.7	8.4737×10^{-5}	4.7114×10^{-5}
0.8	9.8905×10^{-5}	9.2993×10^{-5}
1.3	9.1313×10^{-5}	9.7317×10^{-5}
1.6	8.4393×10^{-5}	2.3766×10^{-4}
2.4	5.1952×10^{-5}	7.0123×10^{-4}
3.7	4.0924×10^{-5}	1.4281×10^{-3}

7. Conclusions

The application of projection methods is extended in this study so that it can be applied to a fractional Cauchy singular integro-differential system. Shifted Vieta–Lucas polynomials are a class of orthogonal polynomials that serve as the foundation of this approach. The significance of this fractional Cauchy singular integro-differential system is evident in mathematical sciences issues, particularly in physics interactions. It has been observed that the fractional operator has a substantial impact on the development of the numerical results. Numerous kinds of fractional integro-differential and integral systems can be investigated with this method, and their solutions can be found.

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References

1. Xu, C.; ur Rahman, M.; Baleanu, D. On fractional-order symmetric oscillator with offset-boosting control. *Nonlinear Anal. Model. Control.* **2022** *27*, 994–1008. [CrossRef]
2. Eskandari, Z.; Avazzadeh, Z.; Khoshsiar Ghaziani, R.; Li, B. Dynamics and bifurcations of a discrete-time Lotka–Volterra model using nonstandard finite difference discretization method. *Math Meth. Appl. Sci.* **2022**. [CrossRef]
3. Qu, H.D.; Liu, X.; Lu, X.; ur Rahman, M.; She, Z.H. Neural network method for solving nonlinear fractional advection-diffusion equation with spatiotemporal variable-order. *Chaos Solitons Fractals* **2022**, *156*, 111856. [CrossRef]
4. Li, B.; Liang, H.; He, Q. Multiple and generic bifurcation analysis of a discrete Hindmarsh-Rose model. *Chaos Solitons Fractals* **2021**, *146*, 110856. [CrossRef]
5. Li, B.; Liang, H.; Shi, L.; He, Q. Complex dynamics of Kopel model with nonsymmetric response between oligopolists. *Chaos Solitons Fractals* **2022**, *156*, 111860. [CrossRef]
6. Mennouni, A. Two projection methods for skew-hermitian operator equations. *Math. Comput. Model.* **2012**, *55*, 1649–1654. [CrossRef]
7. Araour, M.; Mennouni, A. A New Procedures for Solving Two Classes of Fuzzy Singular Integro-Differential Equations: Airfoil Collocation Methods. *Int. J. Appl. Comput. Math.* **2022**, *8*, 35. [CrossRef]

8. Mennouni, A. A projection method for solving Cauchy singular integro-differential equations. *Appl. Math. Lett.* **2012**, *25*, 986–989. [CrossRef]
9. Mennouni, A. Airfoil polynomials for solving integro-differential equations with logarithmic kernel. *Appl. Math. Comput.* **2012**, *218*, 11947–11951. [CrossRef]
10. Tasci, D.; Yalcin, F. Vieta-Pell and Vieta-Pell-Lucas polynomials. *Adv. Differ. Equ.* **2013**, *2013*, 224. [CrossRef]
11. Izadi, M.; Yüzbaşı, Ş.; Ansari, K.J. Application of Vieta–Lucas Series to Solve a Class of Multi-Pantograph Delay Differential Equations with Singularity. *Symmetry* **2021**, *13*, **2370**. [CrossRef]
12. Heydari, M.H.; Avazzadeh, Z.; Razzaghi, M. Vieta-Lucas polynomials for the coupled nonlinear variable-order fractional Ginzburg-Landau equations. *Appl. Numer. Math.* **2021**, *165*, 442–458. [CrossRef]
13. Adel, M.; Assiri, T.A.; Khader, M.M.; Osman, M.S. Numerical simulation by using the spectral collocation optimization method associated with Vieta-Lucas polynomials for a fractional model of non-Newtonian fluid. *Results Phys.* **2022**, *41*, 105927. [CrossRef]
14. Agarwal, P.; El-Sayed, A.A. Vieta–Lucas polynomials for solving a fractional-order mathematical physics model. *Adv. Differ. Equ.* **2020**, *626*, 2020. [CrossRef]
15. Althubiti, S.; Mennouni, A. A Novel Projection Method for Cauchy-Type Systems of Singular Integro-Differential Equations. *Mathematics* **2022**, *10*, 2694. [CrossRef]
16. Mennouni, A. A new efficient strategy for solving the system of Cauchy integral equations via two projection methods. *Transylv. J. Math. Mech.* **2022**, *14*, 63–71.