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Perov Fixed-Point Results on F-Contraction Mappings Equipped with Binary Relation

Fahim Ud Din [1], Muhammad Din [1], Umar Ishtiaq [2,*] and Salvatore Sessa [3,*]

[1] Abdus Salam School of Mathematical Sciences, Government College University, Lahore 54600, Pakistan
[2] Office of Research, Innovation and Commercialization, University of Management and Technology, Lahore 54782, Pakistan
[3] Dipartimento di Architettura, Università Dinapoli Federico II, Via Toledo 403, 80121 Napoli, Italy
* Correspondence: umar.ishtiaq@umt.edu.pk (U.I.); sessa@unina.it (S.S.)

Abstract: The purpose of this article is to discuss some new aspects of the vector-valued metric space. The idea of an arbitrary binary relation along with the well-known F contraction is used to demonstrate the existence of fixed points in the context of a complete vector-valued metric space for both single- and multi-valued mappings. Utilizing the idea of binary relation, and with the help of F contraction, this work extends and complements some of the very recently established Perov-type fixed-point results in the literature. Furthermore, this work includes examples to justify the validity of the given results. During the discussion, it was found that some of the renowned metrical results proven by several authors using different binary relations, such as partial order, pre-order, transitive relation, tolerance, strict order and symmetric closure, can be weakened by using an arbitrary binary relation.

Keywords: Perov fixed point; ordered theoretic Perov fixed point; $\mathcal{F}$ contraction

MSC: 47H10; 54H25

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1. Introduction

As an effective tool, the classical Banach contraction principle [1] is not only widely used in different fields of mathematics such as ordinary differential equations, partial differential equations, integral equations, optimization, and variational analysis, but it has been a useful tool in other subjects such as economics, game theory, and biology as well. This theorem provides the existence and uniqueness of the fixed point of a self-map, satisfying the contraction condition defined on a complete metric space. Since its outset, this classical result has been revived in different forms and shapes. Scientists created different approaches to extend, complement, and generalize this result, such as the Ciric theorem, Caristi theorem, Boyd–Wong theorem, and Browder–Kirk theorem. In this regard, Perov [2] made a very elegant attempt to broaden this result to mappings defined on product spaces. Many researchers have shown keen interest in this context and have made some very interesting contributions to metric fixed point theory. For example, Abbas et al. [3] investigated the fixed points of Perov-type contractive mappings on a set endowed with a graphic structure, and Filip and Petrusel [4] explored fixed-point theorems on a set endowed with vector-valued metric. Many other researchers, namely Cretkovic and Rakocevic [5], Altun et al. [6], Ilic et al. [7], and Vetro and Radenovic [8], discussed this result under various circumstances to obtain fixed points. Another interesting result was proven by Wardowski [9], who initiated the notion of F contraction to prove fixed-point theorems, which is a real generalization of the Banach fixed-point theorem. This result was extended by I. Altun and M. Olgun [10] to investigate fixed-point results for Perov-type F contractions.

Recently, Alam and Imdad [11] extended the Banach fixed-point theorem to a complete metric space endowed with a binary relation and discussed a more generalized way to obtain fixed points. Almalki et al. [12] initiated the notion of a vector-valued metric space enriched with a binary relation. In this work, we extend the Perov fixed-point theorem for single-valued and multi-valued mappings using F contraction in the framework of a generalized metric space equipped with a binary relation. We have used a weaker contractive inequality that only holds for those elements which are related under the binary relation instead of the entire space.

2. Preliminaries

Through out this paper, we represent $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{R}_{\geq 0}$, $\mathbb{R}_t$, and $\mathbb{R}_{t_{>k}}$ with the set of all natural numbers, a set of non-negative integers, real numbers, non-negative real numbers, real matrices of the order $t \times 1$, and real matrices of the order $t \times 1$ with entries greater than k, respectively. We denote $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}_t$ as $\boldsymbol{u} = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ and $\boldsymbol{v} = (\omega_1, \omega_2, \ldots, \omega_t)$ and set $\boldsymbol{u} \leq \boldsymbol{v}$ (or $\boldsymbol{u} < \boldsymbol{v}$) if and only if $\lambda_i \leq \omega_i$ (or $\lambda_i < \omega_i$), where $\forall\ i = 1, 2, \cdots, t$.

Perov [2] introduced the concept of generalized metric space, also known as vector-valued metric space, in the following way:

Definition 1 (See [2])**.** *A mapping $\rho : M \times M \to \mathbb{R}_t$ is said to be a vector-valued metric on M if the following properties are satisfied for all $p, q, r \in M$:*
$M_1 : \rho(p, q) \geq \mathbf{0}$;
$M_2 : \rho(p, q) = \mathbf{0} \Leftrightarrow p = q$;
$M_3 : \rho(p, q) = \rho(q, p)$;
$M_4 : \rho(p, q) \leq \rho(p, r) + \rho(r, q)$.
Here, $\mathbf{0}$ is the zero matrix of the order $t \times 1$. Thus, the pair (M, ρ), where M be any non-empty set and ρ may be a vector-valued metric on M, is called a generalized metric space or vector-valued metric space.

The notions of a Cauchy sequence, convergent sequence, and completeness for vector-valued metric spaces are alike to those for usual metric spaces. In this paper, we symbolize the set of all square matrices of the order $t \times t$ with $\mathbb{M}_t(\mathbb{R}_{\geq 0})$, where all entries are greater than or equal to zero, the null matrix of the order $t \times t$ with $\mathbf{0}_t \in \mathbb{M}_t(\mathbb{R}_{\geq 0})$, and the identity matrix with I_t. Notice that for any matrix $S \in \mathbb{M}_t(\mathbb{R}_{\geq 0})$, we have $S^0 = I_t$.

The concept of a matrix convergent to zero is elaborated upon by the following definition and example:

Definition 2 (See [2])**.** *A matrix $S \in \mathbb{M}_t(\mathbb{R}_{\geq 0})$ is called a matrix convergent to zero if $S^n \to \mathbf{0_t}$ as $n \to \infty$.*

Example 1. *If each element of $\{b_1, b_2, \ldots, b_t\}$ is less than 1, then*

$$B := \begin{pmatrix} b_1 & 0 & \ldots & 0 \\ 0 & b_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_t \end{pmatrix}_{t \times t}$$

is convergent to zero in $\mathbb{M}_t(\mathbb{R}_{\geq 0})$.

Here, we present some equivalent conditions of matrices convergent to zero from Petrusel [4]:

Proposition 1 (See [4])**.** *Let $S \in \mathbb{M}_t(\mathbb{R}_{\geq 0})$. Then, the following are true:*

1. *S is convergent to zero;*

2. *All eigenvalues of S belong to an open unit disc, where $|\lambda| < 1$ (i.e., every $\lambda \in \mathbb{C}$ with* $\det(S - \lambda I_t) = 0$ *is such that* $|\lambda| < 1$*);*
3. *$det(I_t - S) \neq 0$, and*

$$(I_t - S)^{-1} = I_t + S + \cdots + S^n + \cdots,$$

4. *$S^n \boldsymbol{u} \to \mathbf{0}$ as $n \to \infty$, where $\quad \forall \boldsymbol{u} \in \mathbb{R}_t$.*

Example 2 (See [6])**.** *Let $m + n \geq 1$ and $p + q \geq 1$. Then, the matrix $C = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$ is not convergent to zero in $\mathbb{M}_2(\mathbb{R}_{\geq 0})$.*

Now, let us discuss some concepts related to binary relations:

Definition 3 (See [13])**.** *Let M be a non-empty set. Then, the Cartesian product on M is defined as follows:*

$$M^2 = \{(m, n) : m, n \in M\}.$$

All subsets of M^2 are known as the binary relations on M.

Let R be any subset of M^2. Then, notice that for each pair $m, n \in M$, there are two possibilities: either $(m, n) \in R$ or $(m, n) \notin R$:

1. For $(m, n) \in R$, we mean that m relates to n under R.
2. For $(m, n) \notin R$, we mean that m does not relates to n under R.

M^2 and ϕ are called the trivial binary relations of M^2. The binary relation M^2 is called the universal relation or full relation, while ϕ is called an empty binary relation. Another useful binary relation on M is the equality relation, identity relation, or diagonal relation and is defined as

$$\Delta_M := \{(m, m) : m \in M\}.$$

Throughout this paper, we use R for a non-empty binary relation, but for our convenience, we write it as binary relation instead of writing a non-empty binary relation. Alam and Imdad [11] presented the notion of R-comparative elements in the following manner:

Definition 4 (See [11])**.** *R is a binary relation defined on M. For $m, n \in M$, if either $(m, n) \in R$ or $(n, m) \in R$, then these elements are called R-comparative elements and denoted as $[m, n] \in R$.*

Using suitable conditions, a binary relation can be classified into various types. Some of the well-known binary relations along with some important properties can be found in [13]. The following well-known proposition states that every universal relation is a complete equivalence relation:

Proposition 2 (See [13])**.** *Let $\mathcal{R}$ be the full binary relation (universal relation) defined on a non-empty set X. Then, $\mathcal{R}$ is a complete equivalence relation.*

Almaliki et al. [12] extended Proposition 2.3, presented in [11], in the following manner:

Proposition 3 (See [12])**.** *Let (X, ρ) be a generalized metric space endowed with a binary relation $\mathcal{R}$, and let $T : X \to X$ be a mapping. In addition, assume that $A \in \mathbb{M}_m(\mathbb{R}_+)$ is a matrix convergent to zero. Then, the following contractive conditions are equivalent:*

(1) $\rho(Ta, Tb) \leq A\rho(a, b), \forall a, b \in X$ with $(a, b) \in \mathcal{R}$;

(2) $\rho(Ta, Tb) \leq A\rho(a, b), \forall a, b \in X$ with $[a, b] \in \mathcal{R}$.

Proof. It is trivial that if (2) holds, then (1) exists. Now, we show that the existence of (1) implies the existence of (2).

Suppose (1) holds and $a, b \in X$ with $[a, b] \in \mathcal{R}$, which is either $(a, b) \in \mathcal{R}$ or $(b, a) \in \mathcal{R}$.
If $(a, b) \in \mathcal{R}$, then (2) holds directly from (1).
If $(b, a) \in \mathcal{R}$, by using M_3, we obtain

$$\rho(Ta, Tb) = \rho(Tb, Ta) \leq A\rho(b, a) = A\rho(a, b).$$

This implies that (2) holds. □

The concept of d self-closedness for an arbitrary binary relation defined on a metric space (X, d) as presented by Alam and Imdad in [11] was recently extended by Almaliki et al. [12] in the following way:

Definition 5 (See [12])**.** *Let (X, ρ) be a generalized metric space endowed with a binary relation $\mathcal{R}$. Then, $\mathcal{R}$ is said to be ρ-self-closed if for each $\mathcal{R}$-preserving sequence (a_n) with a limit point $a \in X$, there exists a subsequence (a_{n_k}) of (a_n) with $[a_{n_k}, a] \in \mathcal{R}$, $\forall k \in \mathbb{N}$.*

Example 3. *Let $X = \mathbb{R}$, equipped with a generalized metric ρ, be defined by*

$$\rho(a, b) = \begin{bmatrix} |a - b| \\ |a - b| \end{bmatrix}$$

and a binary relation $\mathcal{R}$ be defined by $\mathcal{R} = \{(a, b) : a, b \geq 0\}$. Consider an arbitrary $\mathcal{R}$-preserving sequence (a_n) with a limit point $a \in X$, where $\lim_{n\to\infty} \rho(a_n, a) = \overline{0}$ and $a_n \mathcal{R} a_{n+1}$ for each $n \in \mathbb{N}$. With the $\mathcal{R}$-preserving nature of (a_n), we obtain $a_n, a_{n+1} \geq 0$ for each $n \in \mathbb{N}$. Additionally, the fact that $\lim_{n\to\infty} \rho(a_n, a) = \overline{0}$ implies $a \geq 0$, since $a_n \geq 0$ for each $n \in \mathbb{N}$. Thus, $a_n, a \geq 0$ for each $n \in \mathbb{N}$; that is, $a_n \mathcal{R} a$ for each $n \in \mathbb{N}$. Hence, we say that there exists a subsequence (a_{n_k}) of (a_n) with $[a_{n_k}, a] \in \mathcal{R}$ $\forall k \in \mathbb{N}$.

Definition 6 (See [12])**.** *Let (M, ρ) be a vector-valued metric space equipped with a binary relation R. Then, the following are true:*

1. *The inverse relation of R is defined as*

$$R^{-1} = \{(m, n) \in M^2 : (n, m) \in R\}.$$

2. *The symmetric closure of R is defined as $R^s = R \cup R^{-1}$.*

Almalki et al. [12] defined the notion of an R-preserving sequence that is Γ-closed, ρ-self-closed, and R-directed in a vector-valued metric space in the following way:

Definition 7 (See [12])**.** *Let (M, ρ) be a vector-valued metric space equipped with a binary relation R. Then, a sequence $(c_n) \subseteq M$ is called an R-preserving sequence if*

$$(c_n, c_{n+1}) \in R, \forall n \in \mathbb{Z}.$$

Definition 8 (See [12])**.** *Let (M, ρ) be a vector-valued metric space equipped with a binary relation R. Then, an R-preserving sequence (c_n) with a limit $c^* \in M$ is called ρ-self-closed if there exists a subsequence (c_{n_k}) of (c_n) such that $[c_{n_k}, c^*] \in R$, $\forall k \in \mathbb{Z}$.*

Lemma 1 (See [12])**.** *Let (M, ρ) be a vector-valued metric space equipped with a binary relation R, and let $\Gamma : M \to M$ be a mapping. Then, R^s is Γ-closed whenever R is Γ-closed.*

Definition 9 (See [12])**.** *Let (M, ρ) be a vector-valued metric space equipped with binary relation R. A subset E of M is called R-directed if for each $m, n \in E$, $\exists l \in M$ such that $(m, l) \in R$ with $(n, l) \in R$.*

The notion of a path between two points of a set endowed with a binary relation in a vector-valued metric space was given in [12] in the following way:

Definition 10 (See [12]). *Let (M, ρ) be a vector-valued metric space equipped with a binary relation R. Then, for $C_R(m, n) \neq \phi, \forall m, n \in M$, we mean that for every pair $m, n \in M$, there exists a finite subset $\{l_1, l_2, \ldots, l_{k+1}\}$ of M such that the following are true:*

1. *$l_1 = m$ and $l_{k+1} = n$;*
2. *$(l_i, l_{i+1}) \in R$, $\forall i = 1, 2, \cdots, k$.*

In this case, the finite subset $\{l_1, l_2, \ldots, l_{k+1}\}$, is called a path in R from m to n of a length k.

Almalki et al. [12] initiated the notion of a vector-valued metric space enriched with a binary relation in the following fashion, which is the generalized form of the result presented by Perov [2]:

Theorem 1 (See [12]). *Let (X, ρ) be a complete vector-valued metric space endowed with a binary relation $\mathcal{R}$ and $T : X \to X$ be a mapping. Suppose the following:*

(1) There exists $a \in X$ such that $(a, Ta) \in \mathcal{R}$;

(2) $\mathcal{R}$ is T-closed; that is, for each $a, b \in X$ with $(a, b) \in \mathcal{R}$, we have $(Ta, Tb) \in \mathcal{R}$;

(3) Either T is continuous or $\mathcal{R}$ is ρ-self-closed;

(4) There exists a matrix $A \in \mathbb{M}_m(\mathbb{R}_+)$ convergent to zero such that

$$\rho(Ta, Tb) \leq A\rho(a, b), \quad \forall a, b \in X \quad \textit{with} \quad (a, b) \in \mathcal{R}.$$

Then, T has a fixed point;

(5) Furthermore, if $C_{\mathcal{R}}(a, b) \neq \phi, \forall a, b \in X$, then T has a unique fixed point.

Wardowski [9] initiated the notion of F contraction and defined F contraction as follows:

Definition 11 (See [9]). *Let $F : \mathbb{R}_{>0} \to \mathbb{R}$ be a mapping satisfying the following properties:*

F_1: F is strictly increasing; in other words, for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}_{>0}$, we have

$$\boldsymbol{u} < \boldsymbol{v} \Rightarrow F(\boldsymbol{u}) < F(\boldsymbol{v}),$$

F_2: For each sequence $\boldsymbol{u}_n$ of $\mathbb{R}_{>0}$, we have

$$\lim_{n\to\infty} \boldsymbol{u}_n = 0 \Leftrightarrow \lim_{n\to\infty} F(\boldsymbol{u}_n) = -\infty$$

F_3: There exists $\lambda \in (0, 1)$ such that $\lim_{\boldsymbol{u}_n \to 0^+} \boldsymbol{u}_n^{\lambda} F(\boldsymbol{u}_n) = 0$.

The set of all functions F satisfying (F_1)-(F_3) is denoted as $\mathcal{F}$:

Definition 12 (See [9]). *Let Γ be a self-mapping on a metric space (M, ρ). Then, Γ is said to be an F contraction if $F \in \mathcal{F}$ and there exists $\xi > 0$ such that $\forall p, q \in M$ with $\rho(\Gamma p, \Gamma q) > 0$:*

$$\xi + F[\rho(\Gamma p, \Gamma q)] \leq F[\rho(p, q)]. \tag{1}$$

Definition 13 (See [9]). *From F_2 and the inequality in Equation (1), we note that every F contraction Γ is also a contractive mapping, where*

$$\rho(\Gamma p, \Gamma q) < \rho(p, q), \quad \forall p, q \in M, \Gamma p \neq \Gamma q. \tag{2}$$

Thus, every F contraction Γ is a continuous mapping.

The main fixed-point theorem of Wardowski [9] is given as follows:

Theorem 2 (See [9]). *Let (M, ρ) be a complete metric space and $\Gamma : M \to M$ be an F contraction mapping. Then, Γ has a unique fixed point.*

Ishak Altun et al. in [10] used the concept of an F contraction in a vector-valued metric space in the following way:

Definition 14 (See [10]). *Let $F : \mathbb{R}_{t_{>0}} \to \mathbb{R}_t$ be a function which satisfies the following conditions:*

F_1: *F is strictly increasing; in other words, $\forall \boldsymbol{u} = (\mathbf{u}_i)_{i=1}^t, \boldsymbol{v} = (\mathbf{v}_i)_{i=1}^t \in \mathbb{R}_{t_{>0}}$, where*

$$\boldsymbol{u} < \boldsymbol{v} \Rightarrow F(\boldsymbol{u}) < F(\boldsymbol{v}),$$

F_2: *For each sequence $\{\boldsymbol{u}_n\} = \left(\mathbf{u}_1^{(n)}, \mathbf{u}_2^{(n)}, \ldots, \mathbf{u}_t^{(n)}\right)$ of $\mathbb{R}_{t_{>0}}$, we have*

$$\lim_{n\to\infty} \mathbf{u}_i^{(n)} = 0 \Leftrightarrow \lim_{n\to\infty} \mathbf{v}_i^{(n)} = -\infty$$

for every $i = 1, \cdots, t$, where $F\left[\left(\mathbf{u}_1^{(n)}, \mathbf{u}_2^{(n)}, \ldots, \mathbf{u}_t^{(n)}\right)\right] = \left(\mathbf{v}_1^{(n)}, \mathbf{v}_2^{(n)}, \ldots, \mathbf{v}_t^{(n)}\right)$;

F_3: *There exists $\lambda \in (0,1)$ such that $\lim_{\mathbf{u}_i \to 0^+} \mathbf{u}_i^\lambda \mathbf{v}_i = 0$, $\forall i = 1, \cdots, t$, where*

$$F[(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_t)] = (\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_t).$$

Here, $\mathbb{R}_{t_{>0}}$ is the set of all $t \times 1$ real matrices with positive entries. Then, the set of all functions F satisfying (F_1)–(F_3) is denoted as $\mathcal{F}^t$.

Example 4. *Let $F : \mathbb{R}_{2_{>0}} \to \mathbb{R}_2$ be a function defined by*

$$F(m_1, m_2) = (\ln m_1, \ln m_2)$$

Then, $F \in \mathcal{F}^2$.

Example 5. *Suppose $F : \mathbb{R}_{3_{>0}} \to \mathbb{R}_3$ is a function defined by*

$$F(m_1, m_2, m_3) = (\ln m_1, m_2 + \ln m_2, \frac{-1 + m_1}{\sqrt{m_3}})$$

Then, $F \in \mathcal{F}^3$.

By considering the class $\mathcal{F}^t$, Ishak Altun et al. in [10] introduced the concept of Perov-type F contraction in the following manner:

Definition 15 (See [10]). *Let (M, ρ) be a vector-valued metric space and Γ be a self-mapping on M. If there exist $F \in \mathcal{F}^t$ and $\xi = \left(\xi^{(i)}\right)_{i=1}^t \in \mathbb{R}_{t_{>0}}$ such that*

$$\xi + F[\rho(\Gamma p, \Gamma q)] \leq F[\rho(p, q)], \quad \forall p, q \in M, \rho(\Gamma p, \Gamma q) > \mathbf{0}, \tag{3}$$

then Γ is called a Perov-type F contraction.

By using the idea of a Perov-type F contraction, Ishak Altun et al. in [10] gave their fixed-point theorem as follows:

Theorem 3 (See [10]). *Let (M, ρ) be a complete vector-valued metric space and Γ be a Perov-type F contraction. Then, the mapping Γ has a unique fixed point.*

Before going into our main results, we introduce some important definitions and a lemma with a proof, which would be important toward proving our results on F contraction in terms of binary relation:

Definition 16. *Let (M, ρ) be a vector-valued metric space equipped with a binary relation R. Then, a self-mapping Γ on M, is called a theoretic-order Perov-type F contraction if there exist $\xi = \left(\xi^{(i)}\right)_{i=1}^{t} \in \mathbb{R}_{t_{>0}}$ and $F \in \mathcal{F}^t$ such that*

$$\xi + F[\rho(\Gamma p, \Gamma q)] \leq F[\rho(p, q)], \tag{4}$$

where $\forall (p, q) \in R$ with $\rho(\Gamma p, \Gamma q) > \mathbf{0}$.

Lemma 2. *Let (M, ρ) be a vector-valued metric space equipped with a binary relation R, $\Gamma : M \to M$ be a mapping, $\xi \in \mathbb{R}_{t_{>0}}$, and $F \in \mathcal{F}^t$. Then, the following conditions are equivalent (whenever $\rho(\Gamma p, \Gamma q) > \mathbf{0}$):*

1. $\xi + F[\rho(\Gamma p, \Gamma q)] \leq F[\rho(p, q)]$ *with* $(p, q) \in R$;
2. $\xi + F[\rho(\Gamma p, \Gamma q)] \leq F[\rho(p, q)]$ *with* $[p, q] \in R$.

Proof. $(1) \Rightarrow (2)$:

This implication is trivial.

$(2) \Rightarrow (1)$:

Suppose (2) holds with $[p, q] \in R$. Then, if $(p, q) \in R$, (2) directly implies (1). Otherwise, $(q, p) \in R$, and then by the metric property M_3, we have

$$\xi + F[\rho(\Gamma q, \Gamma p))] = \xi + F[\rho(\Gamma p, \Gamma q))] \leq F[\rho(p, q)] = F[\rho(q, p)]$$

which implies the truth of (1). □

We refer to the set of all fixed points of Γ in M as $fix_M(\Gamma)$, the collection of all paths from m to n (where $m, n \in M$) in R as $C_R(m, n)$ and $\mathbf{Y}$ as a subset of M defined as

$$\mathbf{Y} := \{m \in M : (m, \Gamma m) \in R\}.$$

$CL_\rho(M)$ denotes the class of all non-empty closed subsets of M with a metric ρ.

Definition 17. *The pair $(R : \Gamma)$ between a binary relation R and a self-mapping Γ over a vector-valued metric space (M, ρ) is called a* compound structure *if the following conditions hold:*

(1): $\mathbf{Y} := \{c \in M : (m, \Gamma c) \in R\} \neq \phi$;
(2): *R is Γ-closed;*
(3): *Either R is ρ-self-closed or Γ is continuous.*

Definition 18. *Let M, ρ, and R have the usual meanings. Suppose $\Gamma : M \to CL_\rho(M)$ is a multi-valued mapping. Then, R defined on M is called Γ-closed if $\forall p, q \in M$:*

$$(p, q) \in R \Rightarrow (u, v) \in R, \quad \forall u \in \Gamma p, \forall v \in \Gamma q.$$

Definition 19. *Let (M, ρ) be a vector-valued metric space equipped with a binary relation R and $CL_\rho(M)$ denote the class of all non-empty subsets of M. Then, the pair $(R : \Gamma)$ is called a compound structure for multi-valued mappings if the following conditions are satisfied:*

1. *R is Γ-closed;*
2. $\mathbf{Y} := \{c \in M : \exists u \in \Gamma(c) \quad \text{such} \quad \text{that} \quad (c, u) \in R\} \neq \phi$;
3. *R is strongly ρ-self-closed; that is, for each sequence (c_n) in M with $(c_n, c_{n+1}) \in R$ for all $n \in \mathbb{N}$ and $c_n \xrightarrow{\rho} c$, we have $(c_n, c) \in R$ for all $n \geq k$, where k is some positive integer.*

3. Main Theorem

Now, we present our first result for an F contraction in a complete metric space endowed with a binary relation:

Theorem 4. *Let (M, ρ) be a complete metric space equipped with a binary relation R and Γ be a self-mapping. Suppose the following:*

1. *The pair $(R : \Gamma)$ is a compound structure;*
2. *$\forall (p, q) \in R$ with $\rho(\Gamma p, \Gamma q) > 0$ such that*

$$\xi + F[\rho(\Gamma p, \Gamma q)] \leq F[\rho(p, q)]$$

where $\xi \in \mathbb{R}_{>0}$ and $F \in \mathcal{F}$, then Γ has a fixed point;

3. *Furthermore, if $C_R(p, q) \neq \phi, \forall p, q \in M$,*

then Γ has a unique fixed point.

Proof. Suppose that $c_0 \in \mathbf{Y}$ is any element of M. Then, we define a Picard iterative sequence $\{c_n\}$ as

$$c_0, c_1 = \Gamma c_0, c_2 = \Gamma c_1 = \Gamma^2 c_0, \cdots, c_n = \Gamma c_{n-1} = \Gamma^n c_0, \cdots.$$

Then, by using the definition of $\mathbf{Y}$, $(c_0, c_1) = (c_0, \Gamma c_0) \in R$.

Under Assumption 1, R is Γ-closed. Thus, we have

$$(\Gamma c_0, \Gamma^2 c_0), (\Gamma^2 c_0, \Gamma^3 c_0), (\Gamma^3 c_0, \Gamma^4 c_0), \cdots, (\Gamma^n c_0, \Gamma^{n+1} c_0), \cdots \in R,$$

or

$$(c_1, c_2), (c_2, c_3), (c_3, c_4), \cdots, (c_n, c_{n+1}), \cdots \in R.$$

This shows that the sequence $\{c_n\}$ is an R-preserving sequence.

If for some $n_0 \in \mathbb{N}, c_{n_0+1} = c_{n_0}$, then $c_{n_0+1} = \Gamma c_{n_0} = c_{n_0}$. Thus, Γ has a fixed point. Otherwise, $c_{n+1} \neq c_n$ for all $n \in \mathbb{N}$, and therefore $\rho(\Gamma c_n, \Gamma c_{n+1}) > 0, \forall n \in \mathbb{N}$.

Thus, under Assumption 2, we obtain

$$\xi + F[\rho(c_n, c_{n+1})] = \xi + F[\rho(\Gamma c_{n-1}, \Gamma c_n)] \leq F[\rho((c_{n-1}, c_n)]$$

or

$$F[\rho(c_n, c_{n+1})] \leq F[\rho(\Gamma c_{n-1}, \Gamma c_n)] - \xi.$$

By applying the same procedure, finally we get

$$F[\rho(c_n, c_{n+1})] \leq F[\rho(c_0, c_1] - n\xi. \tag{5}$$

By applying a limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} F[\rho(c_n, c_{n+1})] = -\infty.$$

With F_2, we obtain

$$\lim_{n\to\infty} \rho(c_n, c_{n+1}) = 0^+. \tag{6}$$

With F_3, there exists $\lambda \in (0, 1)$ such that

$$\lim_{n\to\infty} [\rho(c_n, c_{n+1})]^\lambda F[\rho(c_n, c_{n+1})] = 0. \tag{7}$$

Now, from the inequality in Equation (5), we obtain

$$[\rho(c_n, c_{n+1})]^\lambda F[\rho(c_n, c_{n+1})] - [\rho(c_n, c_{n+1})]^\lambda F[\rho(c_0, c_1] \leq -n[\rho(c_n, c_{n+1})]^\lambda \xi \leq 0.$$

By taking the limit as $n \to \infty$, we find the following from the inequality in Equations (6) and (7):

$$\lim_{n\to\infty} n[\rho(c_n, c_{n+1})]^\lambda = 0$$

Thus, there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, and we have

$$\lim_{n\to\infty} n[\rho(c_n, c_{n+1})]^{\lambda} \leq 1.$$

We can then rearrange this to form

$$\rho(c_n, c_{n+1}) \leq \frac{1}{n^{\frac{1}{\lambda}}}. \tag{8}$$

In order to show a Cauchy sequence, we take $m > n \geq n_0$ and use a triangular inequality and the inequality in Equation (8) to obtain

$$\begin{aligned} \rho(x_n, x_m) &\leq \rho(c_n, c_{n+1}) + \rho(c_{n+1}, c_{n+2}) + \cdots + \rho(c_{m-1}, c_m) \\ &= \sum_{j=n}^{m-1} \rho(c_j, c_{j+1}) \\ &\leq \sum_{j=1}^{\infty} \rho(c_j, c_{j+1}) \\ &\leq \sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{\lambda}}}. \end{aligned}$$

As the series $\sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{\lambda}}} \to 0$, $\rho(c_n, c_m) \to 0$. Hence, $\{c_n\}$ is a Cauchy sequence, and by the completeness of M, there exists $c^* \in M$ such that $c_n \to c^*$.

Now, we will show that c^* is a fixed point of Γ. From F_1, and using Assumption 2, we have $\forall (p,q) \in R$ with $\rho(\Gamma p, \Gamma q) > 0$:

$$\rho(\Gamma p, \Gamma q) < \rho(p, q) \tag{9}$$

Now, under Assumption 1, if Γ is continuous, then

$$c_{n+1} = \Gamma c_n \xrightarrow{\rho} \Gamma c^*$$

which means

$$c^* = \lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \Gamma c_n = \Gamma c^* \Rightarrow \Gamma c^* = c^*.$$

Therefore, c^* is a fixed point of Γ.

Otherwise, in line with Assumption 1, if R is ρ-self-closed, then as $\{c_n\}$ is an R-preserving sequence with $c_n \to c^*$, there exists a subsequence $\{c_{n_k}\}$ of $\{c_n\}$ with $[c_{n_k}, c^*] \in R$ and $\rho(\Gamma c_{n_k}, \Gamma c^*) > 0, \forall k \in \mathbb{N}$.

Thus, under Lemma 2 (for $m = 1$) and the inequality in Equation (9), for $[c_{n_k}, c^*] \in R$, we obtain

$$\rho(c_{n_k+1}, \Gamma c^*) = \rho(\Gamma c_{n_k}, \Gamma c^*) < \rho(c_{n_k}, c^*) \to 0 \quad as \quad k \to \infty.$$

which yields

$$\lim_{k\to\infty} c_{n_k+1} = \Gamma(c^*).$$

Hence, we obtain

$$\Gamma(c^*) = \lim_{k\to\infty} c_{n_k+1} = \lim_{n\to\infty} c_n = c^*.$$

which shows that c^* is a fixed point of Γ.

Now, for uniqueness, suppose Assumption 3 holds. Thus, for $p, q \in fix_M(\Gamma)$ with $\Gamma p = p \neq q = \Gamma q$ (i.e., $\rho(\Gamma p, \Gamma q) > 0$), there exists a path (say $\{m_1, m_2, \ldots, m_{l+1}\}$) of a length l in R^s such that

$$m_1 = p, \quad m_{l+1} = q, \quad [m_i, m_{i+1}] \in R, \quad \forall i = 1, 2, \cdots, l. \tag{10}$$

Now, from the triangular inequality and Equations (8)–(10), we have

$$\begin{aligned}\rho(p,q) &= \rho(\Gamma p,\Gamma q)\\ &= \rho(\Gamma m_1,\Gamma m_{l+1})\\ &\leq \sum_{j=1}^{l}\rho(\Gamma m_j,\Gamma m_{j+1})\\ &< \sum_{j=1}^{l}\rho(m_j,m_{j+1})\\ &\leq \sum_{j=1}^{\infty}\rho(m_j,m_{j+1})\\ &\leq \sum_{j=1}^{\infty}\frac{1}{j^{\frac{1}{\lambda}}} \quad\longrightarrow 0.\end{aligned}$$

Thus, $p = q$. Hence, the fixed point of Γ is unique in M: □

Remark 1. *Note that if R is a complete order or M is an R-directed set, then $C_R(p,q) \neq \phi \ \forall p, q \in M$.*

Proof. If R is a complete order, then each $p,q \in M$ is R-comparative (i.e., $[p,q] \in R \ \forall p,q \in M$), which implies that $\{p,q\}$ is a path from p to q of a length of 1 in R. Hence, $C_R(p,q) \neq \phi \ \forall p,q \in M$.

If M is an R-directed set, then for each $p,q \in M$, there exists $r \in M$ such that $(p,r) \in R$ as well as $(q,r) \in R$. This shows that for each $p,q \in M$, we have a path $\{p,q,r\}$ from p to q of a length of 2 in R. Hence, $C_R(p,q)$ is non-empty for each $p,q \in M$. □

Corollary 1. *If hypotheses (1) and (2) in Theorem 4 are true with either R as a complete order or M as an R-directed set. Then, Γ has a unique fixed point.*

Example 6. *Let $M = A \cup B$, where $A = \{\frac{1}{n^2} : n \in \mathbb{N}\} \cup \{0\}$ and $B = \mathbb{N}$ with a usual metric defined by $\rho(p,q) = |p-q|, \forall p,q \in M$. Define a binary relation as $R := \Delta_M \cup \{(p,q) \in M^2 : p,q \in A \quad with \quad p < q\}$ and a self-mapping Γ on M as*

$$\Gamma(m) = \begin{cases} m_{n+1} & if\, m = m_n = \frac{1}{n^2} \\ m & if\, m \in \mathbb{Z}\setminus\{1\} \end{cases}$$

Clearly, for each $c_0 \in \mathbb{Z}\setminus\{1\}$, $\Gamma c_0 = c_0$. Therefore, $(c_0,\Gamma c_0) \in R$, and thus $\mathbf{Y} \neq \phi$. R is also Γ-closed because if $(p,q) \in R$, then $p = q$ or $p,q \in A$ when $p < q$. In either case, $(\Gamma p,\Gamma q) \in R$. Additionally, it is not difficult to show that Γ is continuous (and R is ρ-self-closed). Hence, the pair (R,Γ) is a compound structure.

Now, we can take $F \in \mathcal{F}$ (with $\lambda = \frac{2}{3}$ and $\xi = \ln 2$) as

$$F(\sigma) = \begin{cases} \frac{\ln\sigma}{\sqrt{\sigma}} & if \quad 0 < \sigma < e^2 \\ \sigma - e^2 + \frac{2}{e} & if \quad \sigma \geq e^2 \end{cases}$$

Now, we have to show that $\forall (p,q) \in R$, with $\rho(\Gamma p,\Gamma q) > 0$ implying

$$\xi + F[\rho(\Gamma p,\Gamma q)] \leq F[\rho(p,q)].$$

$$\Leftrightarrow p,q \in A \quad with \quad p < q \Longrightarrow \ln 2 + F[\rho(\Gamma p,\Gamma q)] \leq F[\rho(p,q)].$$

$$\Leftrightarrow p,q \in A \quad with \quad p < q \Longrightarrow \ln 2 + \frac{\ln|\Gamma p - \Gamma q|}{\sqrt{|\Gamma p - \Gamma q|}} \leq \frac{\ln|p-q|}{\sqrt{|p-q|}}$$

$$\Leftrightarrow p, q \in A \quad with \quad p < q \Longrightarrow |\Gamma p - \Gamma q|^{\frac{1}{\sqrt{|\Gamma p - \Gamma q|}}} |p - q|^{\frac{-1}{\sqrt{|p-q|}}} \leq \frac{1}{2} \tag{11}$$

From Example 2.3 in [10], we conclude that the inequality in Equation (11) holds. Thus, all assumptions in Theorem 4 are satisfied, and hence Γ has a fixed point in M.

In addition, for each distinct pair $p, q \in \mathbb{Z} \setminus \{1\} \subseteq M$, we have $\rho(\Gamma p, \Gamma q) = \rho(p, q) > 0$, but

$$\xi + F[\rho(\Gamma p, \Gamma q)] > F[\rho(p, q)],$$

for each $\xi > 0$, and $F \in \mathcal{F}$.

Therefore, the main theorem of Wardowski [9] is not applicable here.

Assumption 3 of Theorem 4 is not satisfied because for $3, 4 \in M$, $C_R(3, 4) = \phi$. Therefore, the fixed point of Γ may not be unique.

Now, we state the fixed point results for an F contraction for single- as well as multi-valued mappings over vector-valued metric spaces:

Theorem 5. *Let (M, ρ) be a complete vector-valued metric space equipped with a binary relation R and Γ be a theoretic-order Perov-type F contraction such that the pair $(R : \Gamma)$ is a compound structure. Then, Γ has a fixed point.*

Moreover, Γ has a unique fixed point if $C_R(p, q) \neq \phi, \forall p, q \in M$.

Proof. Let $c_0 \in \mathbf{Y} \subseteq M$ be any element. Define a sequence $\{c_n\}$ as $c_n = \Gamma^n c_0$. Then, as $(c_0, c_1) = (c_0, \Gamma c_0) \in R$, by the definition of Γ-closedness, we have

$$(\Gamma c_0, \Gamma^2 c_0), (\Gamma^2 c_0, \Gamma^3 c_0), (\Gamma^3 c_0, \Gamma^4 c_0), \cdots, (\Gamma^n c_0, \Gamma^{n+1} c_0), \cdots \in R$$

In other words, we have

$$(c_1, c_2), (c_2, c_3), (c_3, c_4), \cdots, (c_n, c_{n+1}), \cdots \in R.$$

which shows that $\{c_n\}$ is an R-preserving sequence.

If for some $n_0 \in \mathbb{N}, c_{n_0+1} = c_{n_0}$, then $c_{n_0+1} = \Gamma c_{n_0} = c_{n_0}$ (i.e., Γ has a fixed point). Otherwise, $x_{n+1} \neq c_n$ for all $n \in \mathbb{N}$, and thus $\rho(\Gamma c_n, \Gamma c_{n+1}) > \mathbf{0}, \forall n \in \mathbb{N}$. Suppose that

$$\rho(c_n, c_{n+1}) = \left(\mathbf{u}_n^{(1)}, \mathbf{u}_n^{(2)}, \cdots, \mathbf{u}_n^{(t)}\right)$$

and

$$F[\rho(c_n, c_{n+1})] = \left(\mathbf{v}_n^{(1)}, \mathbf{v}_n^{(2)}, \cdots, \mathbf{v}_n^{(t)}\right).$$

As $\rho(\Gamma c_n, \Gamma c_{n+1}) > \mathbf{0}$, by definition of Γ, there exist $\xi = \left(\xi^{(i)}\right)_{i=1}^t \in \mathbb{R}_{t>0}$ and $F \in \mathcal{F}^t$, such that

$$\xi + F[\rho(c_n, c_{n+1})] = \xi + F[\rho(\Gamma c_{n-1}, \Gamma c_n)] \leq F[\rho((c_{n-1}, c_n)]$$

or

$$F[\rho(c_n, c_{n+1})] \leq F[\rho(\Gamma c_{n-1}, \Gamma c_n)] - \xi.$$

Thus, we obtain

$$F[\rho(c_n, c_{n+1})] \leq F[\rho(c_0, c_1)] - n\xi.$$

In component form, we have

$$\left(\mathbf{v}_n^{(i)}\right)_{i=1}^t \leq \left(\mathbf{v}_0^{(i)}\right)_{i=1}^t - n\left(\xi^{(i)}\right)_{i=1}^t$$

$$\mathbf{v}_n^{(i)} \leq \mathbf{v}_0^{(i)} - n\xi^{(i)}, \quad \forall i = 1, 2, \cdots, t. \tag{12}$$

By letting $n \to \infty$, we have

$$\lim_{n\to\infty} \mathbf{v}_n^{(i)} = -\infty, \quad \forall i = 1, 2, \cdots, t.$$

By using F_2, we have

$$\lim_{n\to\infty} \mathbf{u}_n^{(i)} = 0^+, \quad \forall i = 1, 2, \cdots, t. \tag{13}$$

Now, with F_3, there exists $\lambda \in (0, 1)$ such that

$$\lim_{n\to\infty} \left[\mathbf{u}_n^{(i)}\right]^\lambda \mathbf{v}_n^{(i)} = 0, \quad \forall i = 1, 2, \cdots, t. \tag{14}$$

From Equation (12), we have for all $n \in \mathbb{N}$ and $i = 1, 2, \cdots, t$ the following:

$$\mathbf{v}_n^{(i)} \leq \mathbf{v}_0^{(i)} - n\xi^{(i)}.$$

By multiplying both sides by $\left[\mathbf{u}_n^{(i)}\right]^\lambda$, we obtain

$$\left[\mathbf{u}_n^{(i)}\right]^\lambda \mathbf{v}_n^{(i)} - \left[\mathbf{u}_n^{(i)}\right]^\lambda \mathbf{v}_0^{(i)} \leq -n\left[\mathbf{u}_n^{(i)}\right]^\lambda \xi^{(i)}, \quad \forall i = 1, 2, \cdots, t.$$

By taking the limit to be $n \to \infty$, we find the following from Equations (13) and (14):

$$\lim_{n\to\infty} n\left[\mathbf{u}_n^{(i)}\right]^\lambda = 0, \quad \forall i = 1, 2, \cdots, t.$$

Therefore, by the definition of the limit, for $\epsilon = 1$, $\exists N^{(i)} \in \mathbb{N}$ such that $\forall n \geq N^{(i)}$, we have

$$n\left[\mathbf{u}_n^{(i)}\right]^\lambda \leq 1, \quad \forall i = 1, 2, \cdots, t.$$

Consider $n_0 = \max\{N^{(i)} : i = 1, 2, \cdots, t\}$. Then, for all $n \geq n_0$, we have

$$n\left[\mathbf{u}_n^{(i)}\right]^\lambda \leq 1, \quad \forall i = 1, 2, \cdots, t.$$

$$\Rightarrow \mathbf{u}_n^{(i)} \leq \frac{1}{n^{\frac{1}{\lambda}}}, \quad \forall i = 1, 2, \cdots, t. \tag{15}$$

In order to show that $\{c_n\}$ is a Cauchy sequence, by taking $m > n \geq n_0$, a triangular inequality, and the inequality in Equation (15), we have

$$\begin{aligned}
\rho(c_n, c_m) &\leq \rho(c_n, c_{n+1}) + \rho(c_{n+1}, c_{n+2}) + \cdots + \rho(c_{m-1}, c_m) \\
&= \left(\mathbf{u}_n^{(i)}\right)_{i=1}^t + \left(\mathbf{u}_{n+1}^{(i)}\right)_{i=1}^t + \cdots + \left(\mathbf{u}_{m-1}^{(i)}\right)_{i=1}^t \\
&= \left(\sum_{j=n}^{m-1} \mathbf{u}_j^{(i)}\right)_{i=1}^t \\
&\leq \left(\sum_{j=1}^{\infty} \mathbf{u}_j^{(i)}\right)_{i=1}^t \\
&\leq \left(\sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{\lambda}}}\right)_{i=1}^t.
\end{aligned}$$

As the series $\sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{\lambda}}}$ is convergent to 0, then $\rho(c_n, c_m) \to \mathbf{0}$. Therefore, $\{c_n\}$ is a Cauchy sequence. Hence, by the completeness of M, there exists $c^* \in M$ such that $c_n \to c^*$ as $n \to \infty$.

Using definition of Γ and F_1, we have $\forall (p, q) \in R$ with $\rho(\Gamma p, \Gamma q) > \mathbf{0}$:

$$\rho(\Gamma p, \Gamma q) < \rho(p, q) \tag{16}$$

From the definition of $(R : \Gamma)$, if Γ is continuous, then we have

$$c_{n+1} = \Gamma c_n \xrightarrow{\rho} \Gamma c^*,$$

such that

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \Gamma c_n = \Gamma c^* \Rightarrow \Gamma c^* = c^*.$$

Then, Γ has a fixed point.

Otherwise, if R is ρ-self-closed, then as $\{c_n\}$ is an R-preserving sequence with $c_n \to c^*$, there exists a subsequence $\{c_{n_k}\}$ of $\{c_n\}$ with $[c_{n_k}, c^*] \in R \quad and \quad \rho(\Gamma c_{n_k}, \Gamma c^*) > \mathbf{0}, \forall k \in \mathbb{N}$.

Thus, according to Lemma 2 and the inequality in Equation (16), for $[c_{n_k}, c^*] \in R$, we obtain

$$\rho(c_{n_k+1}, \Gamma c^*) = \rho(\Gamma c_{n_k}, \Gamma c^*) < \rho(c_{n_k}, c^*) \to \mathbf{0} \quad as \quad k \to \infty.$$

which yields

$$\lim_{k\to\infty} c_{n_k+1} = \Gamma(c^*).$$

Hence, we have

$$\Gamma(c^*) = \lim_{k\to\infty} c_{n_k+1} = \lim_{n\to\infty} c_n = c^*.$$

which shows that c^* is a fixed point of Γ.

For the sake of uniqueness, we assume that $C_R(p, q) \neq \phi, \forall p, q \in M$; that is, we can find a path between every pair of M. Suppose, on the contrary, that $p, q \in fix_M(\Gamma)$ with $\Gamma p = p \neq q = \Gamma q$ (i.e., $\rho(\Gamma p, \Gamma q) > \mathbf{0}$).

Then, for $p, q \in fix_M(\Gamma) \subseteq M$, there exists a path $\{m_1, m_2, \cdots, m_{l+1}\}$ such that the following are true:

1. $m_1 = p, \quad m_{l+1} = q$;
2. $[m_i, m_{i+1}] \in R, \forall i = 1, 2, \cdots, l$.

As we found (for example, $\rho(m_j, m_{j+1}) = (\gamma_j^{(i)})_{i=1}^t$) with the triangular inequality, and from Equations (15) and (16), we have

$$\begin{aligned}
\rho(p, q) &= \rho(\Gamma p, \Gamma q) = \rho(\Gamma m_1, \Gamma m_{l+1}) \\
&\leq \sum_{j=1}^{l} \rho(\Gamma m_j, \Gamma m_{j+1}) < \sum_{j=1}^{l} \rho(m_j, m_{j+1}) \\
&= \sum_{j=1}^{l} \left(\gamma_j^{(i)}\right)_{i=1}^{t} = \left(\sum_{j=1}^{l} \gamma_j^{(i)}\right)_{i=1}^{t} \\
&\leq \left(\sum_{j=1}^{\infty} \gamma_j^{(i)}\right)_{i=1}^{t} \leq \left(\sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{\lambda}}}\right)_{i=1}^{t} \quad \longrightarrow \mathbf{0}.
\end{aligned}$$

Thus, $p = q$, which is a contradiction. Hence, the fixed point of T is unique. $\square$

Example 7. *Let $M = A \cup B$, where $A = \{\frac{1}{n^2} : n \in \mathbb{N}\} \cup \{0\}$ and $B = \{2, 3, 4, 5\}$ and with the vector-valued metric defined as $\rho(p, q) = (|p - q|, |p - q|), \forall p, q \in M$. Define a binary relation as $R := \Delta_M \cup \{(p, q) \in M^2 : p, q \in A \quad with \quad p < q\}$ and the self-mapping Γ on M as*

$$\Gamma(m) = \begin{cases} m_{n+1} & \text{if } m = m_n = \frac{1}{n^2}; \\ m & \text{if } m \in \{0, 2, 3, 4, 5\}. \end{cases}$$

Clearly, for each $c_0 \in \{0, 2, 3, 4, 5\}$, $\Gamma c_0 = c_0$. Therefore, $(c_0, \Gamma c_0) \in R$, and thus $\mathbf{Y} \neq \phi$. R is also Γ-closed because if $(p, q) \in R$, then $p = q$ or $p, q \in A$ with $p < q$. In either case, $(\Gamma p, \Gamma q) \in R$. Additionally, it is not difficult to show that Γ is continuous (and R is ρ-self-closed). Hence, the pair (R, Γ) is a compound structure.

Now, we can take $F \in \mathcal{F}^2$ (with $\lambda = \frac{2}{3}$ and $\xi = (\ln 2, 1)$) as

$$F(m_1, m_2) = \begin{cases} \left(\frac{\ln m_1}{\sqrt{m_1}}, \frac{-1}{\sqrt{m_2}}\right) & \text{if} \quad m_1 \leq e; \\ \left(\frac{m_1}{e\sqrt{e}}, \frac{-1}{\sqrt{m_2}}\right) & \text{if} \quad m_1 > e. \end{cases}$$

Now, in order to show that Γ is a theoretic-order Perov-type F contraction, we have to show that $\forall (p, q) \in R$ with $\rho(\Gamma p, \Gamma q) > \mathbf{0}$ implies

$$\xi + F[\rho(\Gamma p, \Gamma q)] \leq F[\rho(p, q)].$$

$$\Leftrightarrow p, q \in A \quad with \quad p < q \Longrightarrow (\ln 2, 1) + F[\rho(\Gamma p, \Gamma q)] \leq F[\rho(p, q)].$$

$$\Leftrightarrow p, q \in A \quad with \quad p < q \Longrightarrow (\ln 2, 1) + \left(\frac{\ln|\Gamma p - \Gamma q|}{\sqrt{|\Gamma p - \Gamma q|}}, \frac{-1}{\sqrt{|\Gamma p - \Gamma q|}}\right) \leq \left(\frac{\ln|p - q|}{\sqrt{|p - q|}}, \frac{-1}{\sqrt{|p - q|}}\right)$$

$$\Leftrightarrow p, q \in A \quad with \quad p < q \Longrightarrow \left(\ln 2 + \frac{\ln|\Gamma p - \Gamma q|}{\sqrt{|\Gamma p - \Gamma q|}}, 1 - \frac{1}{\sqrt{|\Gamma p - \Gamma q|}}\right) \leq \left(\frac{\ln|p - q|}{\sqrt{|p - q|}}, \frac{-1}{\sqrt{|p - q|}}\right)$$

$$\Leftrightarrow p, q \in A \quad with \quad p < q \Longrightarrow \ln 2 + \frac{\ln|\Gamma p - \Gamma q|}{\sqrt{|\Gamma p - \Gamma q|}} \leq \frac{\ln|p - q|}{\sqrt{|p - q|}} \quad \& \quad 1 - \frac{1}{\sqrt{|\Gamma p - \Gamma q|}} \leq \frac{-1}{\sqrt{|p - q|}}$$

$$\Leftrightarrow p, q \in A \quad with \quad p < q \Longrightarrow \begin{cases} |\Gamma p - \Gamma q|^{\frac{1}{\sqrt{|\Gamma p - \Gamma q|}}} |p - q|^{\frac{-1}{\sqrt{|p - q|}}} \leq \frac{1}{2} \\ \& \\ \frac{1}{\sqrt{|\Gamma p - \Gamma q|}} - \frac{1}{\sqrt{|p - q|}} \geq 1. \end{cases} \tag{17}$$

Now, suppose $p = \frac{1}{u^2}$ and $q = \frac{1}{v^2}$ with $u > v$. Then, clearly $p, q \in A$ with $p < q$, and also

$$
\begin{aligned}
|\Gamma p-\Gamma q|^{\frac{1}{\sqrt{|\Gamma p-\Gamma q|}}}|p-q|^{\frac{-1}{\sqrt{|p-q|}}} &= |\Gamma q-\Gamma p|^{\frac{1}{\sqrt{|\Gamma q-\Gamma p|}}}|q-p|^{\frac{-1}{\sqrt{|q-p|}}} \\
&= \left(\frac{1}{(v+1)^2}-\frac{1}{(u+1)^2}\right)^{\frac{1}{\sqrt{\frac{1}{(v+1)^2}-\frac{1}{(u+1)^2}}}}\left(\frac{1}{v^2}-\frac{1}{u^2}\right)^{\frac{-1}{\sqrt{\frac{1}{v^2}-\frac{1}{u^2}}}} \\
&= \left(\frac{(u+1)^2-(v+1)^2}{(v+1)^2(u+1)^2}\right)^{\frac{1}{\sqrt{\frac{(u+1)^2-(v+1)^2}{(v+1)^2(u+1)^2}}}}\left(\frac{u^2-v^2}{v^2u^2}\right)^{\frac{-1}{\sqrt{\frac{u^2-v^2}{v^2u^2}}}} \\
&= \left(\frac{(u+1)^2-(v+1)^2}{(v+1)^2(u+1)^2}\right)^{\frac{(v+1)(u+1)}{\sqrt{(u+1)^2-(v+1)^2}}}\left(\frac{u^2-v^2}{v^2u^2}\cdot\frac{u+v+2}{u+v+2}\right)^{\frac{-vu}{\sqrt{u^2-v^2}}} \\
&= \left(\frac{(u+1)^2-(v+1)^2}{(v+1)^2(u+1)^2}\right)^{\frac{(v+1)(u+1)}{\sqrt{(u+1)^2-(v+1)^2}}}\times \\
&\quad \left(\frac{(u+1)^2-(v+1)^2}{(v+1)^2(u+1)^2}\cdot\frac{(u+v)(v+1)^2(u+1)^2}{(u+v+2)v^2u^2}\right)^{\frac{-vu}{\sqrt{u^2-v^2}}} \\
&= \left(\frac{(u+1)^2-(v+1)^2}{(v+1)^2(u+1)^2}\right)^{\frac{(v+1)(u+1)}{\sqrt{(u+1)^2-(v+1)^2}}-\frac{vu}{\sqrt{u^2-v^2}}}\times \\
&\quad \left(\frac{(u+v+2)v^2u^2}{(u+v)^2(v+1)^2(u+1)^2}\right)^{\frac{vu}{\sqrt{u^2-v^2}}}.
\end{aligned}
$$

However, as $\frac{(u+1)^2-(v+1)^2}{(v+1)^2(u+1)^2}\leq\frac{1}{2}$, $\frac{(v+1)(u+1)}{\sqrt{(u+1)^2-(v+1)^2}}-\frac{vu}{\sqrt{u^2-v^2}}\geq 1$ *and* $(\frac{(u+v+2)v^2u^2}{(u+v)^2(v+1)^2(u+1)^2}<1$. *Thus, we have*

$$|\Gamma p-\Gamma q|^{\frac{1}{\sqrt{|\Gamma p-\Gamma q|}}}|p-q|^{\frac{-1}{\sqrt{|p-q|}}}\leq\frac{1}{2}$$

and

$$\frac{1}{\sqrt{|\Gamma p-\Gamma q|}}-\frac{1}{\sqrt{|p-q|}}\geq 1.$$

Hence, Equation (17) is satisfied.

Now, if $p=0$ *and* $q=\frac{1}{v^2}$, *then clearly* $p,q\in A$ *with* $p<q$ *and*

$$
\begin{aligned}
|\Gamma p-\Gamma q|^{\frac{1}{\sqrt{|\Gamma p-\Gamma q|}}}|p-q|^{\frac{-1}{\sqrt{|p-q|}}} &= \left|\frac{1}{(v+1)^2}\right|^{\frac{1}{\sqrt{\frac{1}{(v+1)^2}}}}\cdot\left|\frac{1}{v^2}\right|^{\frac{-1}{\sqrt{\frac{1}{v^2}}}} \\
&= \frac{v^{2v}}{(v+1)^{2(v+1)}} \\
&= \frac{v^{2(v+1)}}{(v+1)^{2(v+1)}}\cdot\frac{1}{v^2} \\
&= \left(\frac{v}{v+1}\right)^{2(v+1)}\cdot\frac{1}{v^2} \\
&\leq \frac{1}{2}.
\end{aligned}
$$

In addition, for $p=0$ *and* $q=\frac{1}{v^2}$, *clearly,* $p,q\in A$ *with* $p<q$. *Therefore, we have*

$$
\begin{aligned}
\frac{1}{\sqrt{|\Gamma p-\Gamma q|}}-\frac{1}{\sqrt{|p-q|}} &= \frac{1}{\sqrt{\frac{1}{(v+1)^2}}}-\frac{1}{\sqrt{\frac{1}{v^2}}} \\
&= (v+1)-v \\
&= 1.
\end{aligned}
$$

Hence, Equation (17) is satisfied. Thus, Γ is a theoretic-order Perov-type F contraction such that the pair $(R:\Gamma)$ is a compound structure. Therefore, under Theorem 5, Γ has a fixed point in M.

Additionally, as for $3,4 \in M$, we have $\rho(\Gamma 3, \Gamma 4) = \rho(3,4) = (1,1) > \mathbf{0}$, but

$$\xi + F[\rho(\Gamma p, \Gamma q)] > F[\rho(p,q)],$$

for each $\xi \in \mathbb{R}_{t_{>0}}$, and $F \in \mathcal{F}^t$.

Therefore, the main theorem of Ishak Altun et al. [10] is not applicable here.

The last assumption of Theorem 5 is not satisfied, because for $4,5 \in M$, $C_R(4,5) = \phi$. Thus, the fixed point of Γ may not be unique.

Theorem 6. *Let (M,ρ) be a complete vector-valued metric space equipped with any binary relation R and $\Gamma : M \to CL_\rho(M)$ be a multi-valued mapping. Suppose the following:*

1. *the pair $(R:\Gamma)$ is a compound structure for multi-valued mappings;*
2. *For each $(p,q) \in R$ and $u \in \Gamma p$, $\exists v \in \Gamma q$ such that*

$$\xi + F[\rho(u,v)] \leq F[\rho(p,q)], \tag{18}$$

where $\xi \in \mathbb{R}_{t_{>0}}$ and $F \in \mathcal{F}^t$.

Then, Γ has a fixed point.

Proof. Let $c_0 \in \mathbf{Y} \subseteq M$ be any element. Then, there exists $c_1 \in \Gamma c_0$ such that $(c_0, c_1) \in R$. Now, under Assumption 2, for $(c_0,c_1) \in R$ and $c_1 \in \Gamma c_0$, there exists $c_2 \in \Gamma c_1$ such that

$$\xi + F[\rho(c_1,c_2)] \leq F[\rho(c_0,c_1)],$$

which implies

$$F[\rho(c_1,c_2)] \leq F[\rho(c_0,c_1)] - \xi \tag{19}$$

As R is Γ-closed, then $(c_1,c_2) \in R$, and again, under assumption 2, for $(c_1,c_2) \in R$ and $c_2 \in \Gamma c_1$, there exists $c_3 \in \Gamma c_2$ such that

$$\xi + F[\rho(c_2,c_3)] \leq F[\rho(c_1,c_2)],$$

and (c_2,c_3), which implies by the inequality in Equation (19) that

$$F[\rho(c_2,c_3)] \leq F[\rho(c_1,c_2)] - \xi \leq F[\rho(c_0,c_1)] - 2\xi.$$

By continuing in this way, we obtain a sequence $\{c_n\}$ defined as $c_n \in \Gamma c_{n-1}$, $\forall n \in \mathbb{N}$ such that $(c_n, c_{n+1}) \in R$ (i.e., $\{c_n\}$ is R-preserving) and

$$F[\rho(c_n,c_{n+1})] \leq F[\rho(c_0,c_1)] - n\xi \tag{20}$$

By letting $\forall n \in \mathbb{N}$, we obtain

$$\rho(c_n,c_{n+1}) = \left(\mathbf{u}_n^{(1)}, \mathbf{u}_n^{(2)}, \cdots, \mathbf{u}_n^{(t)}\right)$$

and

$$F[\rho(c_n,c_{n+1})] = \left(\mathbf{v}_n^{(1)}, \mathbf{v}_n^{(2)}, \cdots, \mathbf{v}_n^{(t)}\right).$$

Through the same steps as those in Theorem 5, we have $\forall n \geq n_0$ and

$$\mathbf{u}_n^{(i)} \leq \frac{1}{n^{\frac{1}{\lambda}}}, \quad \forall i = 1,2,\cdots,t. \tag{21}$$

Now, as in the inequality in Equation (21), a triangular inequality, and for $m > n \geq n_0$, we obtain

$$\begin{aligned}\rho(c_n, c_m) &\leq \rho(c_n, c_{n+1}) + \rho(c_{n+1}, c_{n+2}) + \cdots + \rho(c_{m-1}, c_m)\\ &= \left(\mathbf{u}_n^{(i)}\right)_{i=1}^{t} + \left(\mathbf{u}_{n+1}^{(i)}\right)_{i=1}^{t} + \cdots + \left(\mathbf{u}_{m-1}^{(i)}\right)_{i=1}^{t}\\ &= \left(\sum_{j=n}^{m-1} \mathbf{u}_j^{(i)}\right)_{i=1}^{t}\\ &\leq \left(\sum_{j=1}^{\infty} \mathbf{u}_j^{(i)}\right)_{i=1}^{t}\\ &\leq \left(\sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{\lambda}}}\right)_{i=1}^{t} \longrightarrow \mathbf{0}.\end{aligned}$$

Therefore, $\{c_n\}$ is a Cauchy sequence. By using the completeness of M, we find $c \in M$ such that $c_n \to c$. By using F_1 and Assumption 2, for $(p, q) \in R$, we find that

$$\rho(u, v) \leq \rho(p, q), \quad where \quad u \in \Gamma p, v \in \Gamma q. \tag{22}$$

As R is strongly ρ-self-closed, we obtain $(c_n, c) \in R$, $\forall n \geq N_0$, where N_0 is any natural number. Thus, under Assumption 2 and the inequality in Equation (22), for each $(c_n, c) \in R, \forall n \geq N_0$ and $c_{n+1} \in \Gamma c_n$, there exists $c^* \in \Gamma c$ such that

$$\rho(c_{n+1}, c^*) \leq \rho(c_n, c) \to \mathbf{0} \quad as \quad n \to \infty.$$

However, we also have

$$c^* = \lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} c_n = c.$$

Hence, $c \in \Gamma c$. $\square$

4. Conclusions

In this manuscript, a new generalization for the classical Perov fixed-point theorem is given. The proposed results are for both single-valued as well as multi-valued mappings. In a complete vector-valued metric space, the well-known F contraction is endowed with an arbitrary binary relation to attain fixed-point results. A relatively weaker contractive condition is used compared with those in the recent literature, as in this work, the contractive condition is required to hold only for those points which are related to each other under some particular binary relation and not for the entire space. This work also reveals that instead of using different binary relations, one can choose an arbitrary binary relation to obtain some of the famous and well-known fixed-point results.

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