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Global Well-Posedness for the Compressible Nematic Liquid Crystal Flows

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Abstract: In this paper, we prove the unique existence of global strong solutions and decay estimates for the simplified Ericksen–Leslie system describing compressible nematic liquid crystal flows in $\mathbb{R}^N$, $3 \le N \le 7$. Firstly, we rewrite the system in Lagrange coordinates, and secondly, we prove the global well-posedness for the transformed system, which is the main task in this paper. The proof is based on the maximal L_p-L_q regularity and the L_p-L_q decay estimates to the linearized problem.

Keywords: compressible Navier–Stokes equations; global strong solutions; Ericksen–Leslie system; liquid crystals

MSC: 35Q35; 76A15; 76N10

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1. Introduction

Nematic liquid crystals are aggregates of elongated, rod-like molecules that possess the same orientational order (cf. [1]). A continuum theory for the hydrodynamics of nematic liquid crystals was developed by Ericksen [2] and Leslie [3] in the 1960s. In this paper, we consider the following simplified Ericksen–Leslie system modeling compressible nematic liquid crystal flows in the N dimensional Euclidean space $\mathbb{R}^N$, $3 \le N \le 7$.

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0 & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ \rho(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) - \operatorname{Div}(\mathbf{S}(\mathbf{v}) - P(\rho)\mathbf{I}) & \\ \quad = -\eta \operatorname{Div}\left(\nabla \mathbf{d} \odot \nabla \mathbf{d} - \dfrac{1}{2}|\nabla \mathbf{d}|^2 \mathbf{I}\right) & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ \partial_t \mathbf{d} + (\mathbf{v}\cdot\nabla)\mathbf{d} = \zeta(\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}) & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ (\rho, \mathbf{v}, \mathbf{d})|_{t=0} = (\rho_* + \rho_0, \mathbf{v}_0, \mathbf{d}_* + \mathbf{d}_0) & \text{in } \mathbb{R}^N. \end{cases} \tag{1}$$

Here, $\partial_t = \partial/\partial t$, t is the time variable, $\rho = \rho(x,t)$, $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ is the density function of the fluid, $\mathbf{v}(x,t) = (v_1(x,t), \dots, v_N(x,t))^T$ is the fluid velocity, where M^T denotes the transposed M, $\mathbf{d} = (d_1(x,t), \dots, d_N(x,t))^T$ is the macroscopic average of the nematic liquid crystal orientation field, and $P(\rho)$ is the pressure satisfying a C^∞ function defined on $(0,\infty)$ and $P'(\rho_*) > 0$, where ρ_* is a positive constant describing the mass density of the liquid crystal flows in $\mathbb{R}^N$. For the vector of functions $\mathbf{u}$, we set $\operatorname{div}\mathbf{u} = \sum_{j=1}^N \partial_j u_j$, and also for $N \times N$ matrix field $\mathbf{A}$ with $(j,k)^{\text{th}}$ components A_{jk}, the quantity $\operatorname{Div}\mathbf{A}$ is an N-vector with j^{th} component $\sum_{k=1}^N \partial_k A_{jk}$, where $\partial_k = \partial/\partial x_k$. The tensor $\mathbf{S}(\mathbf{u})$ is

$$\mathbf{S}(\mathbf{u}) = \mu \mathbf{D}(\mathbf{u}) + (\nu - \mu)\operatorname{div}\mathbf{u}\mathbf{I},$$

where μ and ν are the viscosity coefficients satisfying $\mu, \nu > 0$, $\mathbf{D}(\mathbf{u}) = \nabla \mathbf{u} + (\nabla \mathbf{u})^T$ is the deformation tensor, $\mathbf{I}$ is the $N \times N$ identity matrix. Moreover, η and ζ are positive constants describing the competition between kinetic and potential energy and the microscopic elastic relaxation time, respectively; $\nabla\mathbf{d}\odot\nabla\mathbf{d} = (\nabla\mathbf{d})^T\nabla\mathbf{d}$, $|\nabla\mathbf{d}|^2 = \sum_{i,j=1}^N(\partial_i d_j)^2$, and $\mathbf{d}_*$ is a constant vector.

The system (1) is a simplified version, but still retains most of the interesting mathematical properties of the original Ericksen–Leslie system; see [4–6] for more discussions on the relations between the two models. The simplified Ericksen–Leslie system modeling the motion of incompressible nematic liquid crystals was first derived by Lin [7]. Since the nonlinear term $|\nabla\mathbf{d}|^2\mathbf{d}$ with the restriction $|\mathbf{d}| = 1$ causes mathematical difficulties, Lin [7] introduced the Ginzburg–Landau approximation of the simplified Ericksen–Leslie system, namely, $|\nabla\mathbf{d}|^2\mathbf{d}$ in the third equation of (1) is replaced by the Ginzburg–Landau energy functional $\nabla(|\mathbf{d}|^2-1)^2/\epsilon^2$ or more general smooth and bounded functions. Consequently, Lin and Liu [5] proved the global existence of weak solutions in the two-dimensional case and the three-dimensional case.

In the past several decades, there are many results on the analysis of (1) by overcoming the difficulty induced by the nonlinear term $|\nabla\mathbf{d}|^2\mathbf{d}$. For the incompressible case, Li and Wang [8] considered the problem in a three-dimensional bounded smooth domain and obtained a global strong solution with small data in certain Besov spaces. Hineman and Wang [9] proved the global well-posedness in $\mathbb{R}^3$ with small initial data in the space of uniformly locally L^3-integrable functions. Wang [10] established the global well-posedness in $\mathbb{R}^N$ for small initial data $(\mathbf{v}_0, \mathbf{d}_0)$ belonging to $\mathrm{BMO}^{-1}\times\mathrm{BMO}$ with $\operatorname{div}\mathbf{v}_0 = 0$, which is a invariant space with respect to parabolic scaling associated with the system for the incompressible nematic liquid crystal flows. Schonbek and Shibata [11] obtained the global well-posedness and decay properties in $\mathbb{R}^N$ for small initial data by using the maximal L_p-L_q regularity and L_p-L_q decay estimates for the Stokes and heat equations, which is the same as our motivation.

For the compressible case, Ding, Lin, Wang, and Wen [12] obtained the existence and uniqueness of global strong solutions in dimension one. Later, this result about the classical solution was improved in the presence of a vacuum by Ding, Lin, Wang, and Wen [13]. For the multi-dimensional case, Huang, Wang, and Wen [14] proved the local existence of unique strong solutions for the initial and initial-boundary value problem provided that initial data were sufficiently regular. Later, Huang, Wang, and Wen [15] showed the global well-posedness and L_p decay estimates for $1 \le p \le 6$ with initial condition close to a constant state in H^2 norm. In L_2 framework, Gao, Tao, and Yao [16], Xu, Zhang, Wu [17], and Xiong, Wang, and Wang [18] obtained the existence of a unique global solution to the Cauchy problem and optimal time-decay rates when initial data is a small perturbation near a steady state in $H^3(\mathbb{R}^3)\cap L_1(\mathbb{R}^3)$, $H^2(\mathbb{R}^3)\cap L_1(\mathbb{R}^3)$, and $H^3(\mathbb{R}^3)\cap\dot{B}^{-s}_{2,\infty}(\mathbb{R}^3)$ for $0\le s\le 5/2$, respectively. Schade and Shibata [19] proved the existence of local in time strong solutions in a uniform $W_q^{3-1/q}$ domain and global in time strong solution for small initial data in a bounded domain. In particular, they constructed local in time solutions $\rho = \rho_* + \omega$, $\mathbf{v}$, and $\mathbf{d} = \mathbf{d}_* + \mathbf{n}$ in the following maximal L_p-L_q regularity class:

$$\omega \in H^1_p((0,T), L_q(\mathbb{R}^N)) \cap L_p((0,T), H^1_q(\mathbb{R}^N)),$$
$$(\mathbf{v},\mathbf{n}) \in H^1_p((0,T), L_q(\mathbb{R}^N)^{2N}) \cap L_p((0,T), H^2_q(\mathbb{R}^N)^{2N})$$

with certain p and q.

Motivated by [11,19], we improve the existence result obtained by [19] in the whole space and establish the decay estimates by the maximal L_p-L_q regularity and L_p-L_q decay estimates of solutions to linearized equations. The spirit to use both of them are the same as in [11], but the idea of how to use them is different, and we think that our approach here gives a general framework to prove the global well-posedness for small initial data of quasilinear parabolic equations in unbounded domains. To explain our idea more precisely, we separate problem (1) into the linear part and nonlinear part by the Lagrangian transformation as follows:

$$\begin{cases} \partial_t \theta + \rho_* \operatorname{div} \mathbf{u} = f(\theta, \mathbf{u}) & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ \rho_* \partial_t \mathbf{u} - \operatorname{Div}(\mathbf{S}(\mathbf{u}) - P'(\rho_*)\theta \mathbf{I}) = \mathbf{g}(\theta, \mathbf{u}, \mathbf{k}) & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ (\theta, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{v}_0) & \text{in } \mathbb{R}^N, \end{cases} \tag{2}$$

$$\begin{cases} \partial_t \mathbf{k} - \zeta \Delta \mathbf{k} = \mathbf{h}(\mathbf{u}, \mathbf{k}) & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ \mathbf{d}|_{t=0} = \mathbf{d}_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{3}$$

where θ, $\mathbf{u}$, and $\mathbf{k}$ are the density, the fluid velocity, and the macroscopic average of the nematic liquid crystal orientation field in Lagrange coordinates, respectively, and nonlinear terms $f(\rho, \mathbf{v})$, $\mathbf{g}(\rho, \mathbf{v}, \mathbf{k})$, and $\mathbf{h}(\mathbf{v}, \mathbf{k})$ are detailed in Section 2 below. Note that the linear operators for (2) and (3) are the same as the compressible Navier–Stokes equations and the heat equation, respectively. In particular, we write (2) as $\partial_t u - Au = f$ and $u|_{t=0} = u_0$ symbolically, where $u = (\theta, \mathbf{u})$, $f = (f(\theta, \mathbf{u}), \mathbf{g}(\theta, \mathbf{u}))$, $u_0 = (\rho_0, \mathbf{v}_0)$ and A is a closed linear operator with domain $D(A) = H^1_q \times H^2_q$. We decompose a solution $u = u_1 + u_2$, where u_1 satisfies the time shifted equations: $\partial_t u_1 + \lambda_0 u_1 - Au_1 = f$ and $u_1|_{t=0} = 0$ with some large number λ_0 and u_2 satisfies compensation equations: $\partial_t u_2 - Au_2 = \lambda_0 u_1$ and $u_2|_{t=0} = u_0$. For the time-shifted equations, we use the maximal L_p-$D(A)$ regularity and we see that u_1 has the same decay properties as f. On the other hand, by Duhamel's principle, the solution of the compensation equations is written by $u_2 = e^{At}u_0 + \lambda_0 \int_0^t e^{A(t-s)} u_1(s)\, ds$. To estimate $\int_0^{t-1} e^{A(t-s)} u_1(s)\, ds$ we use L_p-L_q decay estimates of continuous analytic semigroup $\{e^{At}\}_{t \ge 0}$ associated with the operator A for $t > 1$, and to estimate $\int_{t-1}^t e^{A(t-s)} u_1(s)\, ds$ we use a standard estimate: $\|e^{A(t-s)} u_1(s)\|_{D(A)} \le C\|u_1(s)\|_{D(A)}$ for $0 < t - s < 1$, where $\|\cdot\|_{D(A)}$ denotes a domain norm. For the later part, what $u_1(t) \in D(A)$ for $t > 0$ is a key observation. Note that if we apply Duhamel's principle to the original equations directly, we need to use L_q-L_q decay estimate of semigroup $\{e^{At}\}_{t \ge 0}$ for $0 < t < 1$ proved in [20]. In this case, we can not choose $2 < p < \infty$ such that $\int_{t-1}^t e^{A(t-s)} u(s)\, ds$ is bounded. Therefore, we use a standard estimate of the semigroup.

Before stating the main result of this paper, we summarize several symbols and functional spaces used throughout the paper. $\mathbb{N}$ and $\mathbb{R}$ denote the sets of all natural numbers and real numbers, respectively. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let p' be the dual exponent of p defined by $p' = p/(p-1)$ for $1 < p < \infty$. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$, we write $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and $\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N}$ with $x = (x_1, \ldots, x_N)$. For N-vector of functions $\mathbf{u}$, we set $\nabla^2 \mathbf{u} = (\partial_x^\alpha u_1, \ldots, \partial_x^\alpha u_N)$ with $|\alpha| = 2$. For any $1 \le p, q \le \infty$, $L_q(\mathbb{R}^N)$, $H^m_q(\mathbb{R}^N)$ and $B^s_{q,p}(\mathbb{R}^N)$ denote the usual Lebesgue space, Sobolev space and Besov space, while $\|\cdot\|_{L_q(\mathbb{R}^N)}$, $\|\cdot\|_{H^m_q(\mathbb{R}^N)}$ and $\|\cdot\|_{B^s_{q,p}(\mathbb{R}^N)}$ denote their norms, respectively. We set $H^0_q(\mathbb{R}^N) = L_q(\mathbb{R}^N)$ and $H^s_q(\mathbb{R}^N) = B^s_{q,q}(\mathbb{R}^N)$. $C^\infty(\mathbb{R}^N)$ denotes the set of all C^∞ functions defined on $\mathbb{R}^N$. $L_p((a,b), X)$ and $H^m_p((a,b), X)$ denote the standard Lebesgue space and Sobolev space of X-valued functions defined on an interval (a,b), respectively. The d-product space of X is defined by X^d, while its norm is denoted by $\|\cdot\|_X$ instead of $\|\cdot\|_{X^d}$ for the sake of simplicity. Set

$$H^{m,\ell}_q(\mathbb{R}^N) = \{(f, \mathbf{g}) \mid f \in H^m_q(\mathbb{R}^N),\ \ \mathbf{g} \in H^\ell_q(\mathbb{R}^N)^N\},$$
$$\|(f, \mathbf{g})\|_{H^{m,\ell}_q(\mathbb{R}^N)} = \|f\|_{H^m_q(\mathbb{R}^N)} + \|\mathbf{g}\|_{H^\ell_q(\mathbb{R}^N)}.$$

The values of constant C may change from line to line. We use small boldface letters, e.g., $\mathbf{f}$ to denote vector-valued functions and capital boldface letters, e.g., $\mathbf{A}$ to denote matrix-valued functions, respectively.

The following theorem is the main result of this paper.

Theorem 1. *Let $3 \leq N \leq 7$ and $0 < T < \infty$. Let $2 < p < \infty$, $2 < q_1 < N < q_2 < \infty$, $b > 0$ be numbers such that*

$$q_1 \leq 4, \quad \frac{1}{q_1} = \frac{1}{N} + \frac{1}{q_2}, \quad \frac{N}{2q_1} < \frac{1}{p'} < b < \frac{N}{2q_1} + \frac{1}{2} - \frac{1}{p}.$$

Then, there exists a small number $\epsilon > 0$ such that for any initial data $\rho_0 \in \bigcap_{q=q_1,q_2} H^1_q(\mathbb{R}^N) \cap L_{q_1/2}(\mathbb{R}^N)$, $(\mathbf{v}_0, \mathbf{d}_0) \in \bigcap_{q=q_1,q_2} B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)^{2N} \cap L_{q_1/2}(\mathbb{R}^N)^{2N}$ with

$$\mathcal{I} := \sum_{q=q_1,q_2} (\|\rho_0\|_{H^1_q(\mathbb{R}^N)} + \|(\mathbf{v}_0, \mathbf{d}_0)\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)}) + \|(\rho_0, \mathbf{v}_0, \mathbf{d}_0)\|_{L_{q_1/2}(\mathbb{R}^N)} < \epsilon^2,$$

problem (1) admits unique solutions $\rho = \rho_* + \omega$, $\mathbf{v}$, and $\mathbf{d} = \mathbf{d}_* + \mathbf{n}$ with

$$\begin{aligned} &\omega \in H^1_p((0,T), L_q(\mathbb{R}^N)) \cap L_p((0,T), H^1_q(\mathbb{R}^N)), \\ &\mathbf{v} \in H^1_p((0,T), L_q(\mathbb{R}^N)^N) \cap L_p((0,T), H^2_q(\mathbb{R}^N)^N), \\ &\mathbf{n} \in H^1_p((0,T), L_q(\mathbb{R}^N)^N) \cap L_p((0,T), H^2_q(\mathbb{R}^N)^N) \end{aligned} \tag{4}$$

for $q = q_1$, and q_2 satisfying the estimate

$$\mathcal{E}(\omega, \mathbf{v}, \mathbf{n})(T) \leq \epsilon. \tag{5}$$

Here, we have set

$$\begin{aligned} &\mathcal{E}(\omega, \mathbf{v}, \mathbf{n})(T) \\ &= \| <t>^b \nabla\omega\|_{L_p((0,T),L_{q_1}(\mathbb{R}^N))} + \| <t>^b \nabla(\mathbf{v}, \mathbf{n})\|_{L_p((0,T),H^1_{q_1}(\mathbb{R}^N))} \\ &\quad + \| <t>^b \omega\|_{L_p((0,T),H^1_{q_2}(\mathbb{R}^N))} + \| <t>^b (\mathbf{v}, \mathbf{n})\|_{L_p((0,T),H^2_{q_2}(\mathbb{R}^N))} \\ &\quad + \| <t>^{N/2q_1} (\omega, \mathbf{v}, \mathbf{n})\|_{L_\infty((0,T),L_{q_1}(\mathbb{R}^N))} \\ &\quad + \| <t>^b (\omega, \mathbf{v}, \mathbf{n})\|_{L_\infty((0,T),L_{q_2}(\mathbb{R}^N))} \\ &\quad + \| <t>^b \nabla\mathbf{n}\|_{L_\infty((0,T),L_{q_1}(\mathbb{R}^N))} + \| <t>^b \mathbf{n}\|_{L_\infty((0,T),H^1_\infty(\mathbb{R}^N))} \\ &\quad + \sum_{q=q_1,q_2} \Big(\| <t>^b \partial_t\omega\|_{L_p((0,T),L_q(\mathbb{R}^N))} + \| <t>^b \partial_t(\mathbf{v}, \mathbf{n})\|_{L_p((0,T),L_q(\mathbb{R}^N))} \Big) \end{aligned}$$

with $<t> = (1+t^2)^{1/2}$.

Remark 1. (1) *$T > 0$ is taken arbitrarily and ϵ is chosen independent of T, therefore, Theorem 1 yields the global well-posedness for (1).*
(2) *Choosing $q_1 = 2 + \delta$ and $p = 4$ for small δ if $N = 3$, we can obtain the decay rate b satisfying $3/4 < b < (8+\delta)/4(2+\delta)$, for instance.*
(3) *Physically, it is natural to treat $\mathbf{d}$ satisfies the constraint $|\mathbf{d}| = 1$. We can show this condition in the same way as in ([19], Proposition 1.3). In fact, if $(\mathbf{v}, \mathbf{d})$ is the solution obtained in Theorem 1, we see that $\mathbf{v} \in L_\infty((0,T), L_\infty(\mathbb{R}^N))$ and $\mathbf{n} \in L_\infty((0,T), H^1_\infty(\mathbb{R}^N))$ by $B^{2(1-1/p)}_{q,p}(\mathbb{R}^N) \subset H^1_q(\mathbb{R}^N)$ provided by $p > 2$, (87), and Lemma 1 below. Thus, we can verify the uniqueness of the solution to the parabolic convection–reaction–diffusion equations with homogeneous initial data for $t \geq 0$. Therefore, if we assume that $|\mathbf{d}_0 + \mathbf{d}_*|^2 = 1$ and $|\mathbf{d}_*| = 1$, then $|\mathbf{d}| = 1$ for all $t \geq 0$.*

This paper is organized as follows: Section 2 introduce the Lagrange transformation and the key theorem, which is the global well-posedness for the system in Lagrange coordinates. In Section 3, we consider estimates of nonlinear terms as preparation for analyzing time-shifted equations and applying the contraction mapping principle below. In Section 4, we consider a priori estimates for the linearized problems with the help of the maximal L_p-L_q regularity and the L_p-L_q decay estimates. Section 5 proves the key theorem for equations with Lagrangian description. Section 6 proves the main theorem by using the key theorem proved in Section 5.

2. Lagrangian Formulation

In order to eliminate $\mathbf{v} \cdot \nabla \rho$ from the first equation of (1) and treat (1) in the maximal L_p-L_q regularity class, we reduce the problem by using the Lagrangian transformation. Let velocity fields $\mathbf{u}(\xi, t)$ and $\mathbf{v}(x, t)$ be known as vectors of functions of Lagrange coordinates ξ and Euler coordinates x of the same fluid particle, respectively. In this case, the connection between the Lagrange coordinate and the Euler coordinate is written in the form:

$$x = \xi + \int_0^t \mathbf{u}(\xi, s)\, ds \equiv \mathbf{X}_t(\xi) \tag{6}$$

for $0 < t < T$. In order to ensure the inverse transformation of $\mathbf{X}_t(\xi)$, we assume that

$$\int_0^T \|\nabla_\xi \mathbf{u}(\cdot, s)\|_{L_\infty(\mathbb{R}^N)}\, ds < \sigma \tag{7}$$

with sufficiently small $\sigma \in (0, 1)$, where $\nabla_\xi = \partial/\partial\xi$. By (6), we have

$$\frac{\partial x}{\partial \xi} = \nabla_\xi \mathbf{X}_t = \mathbf{I} + \int_0^t \nabla_\xi \mathbf{u}(\xi, s)\, ds,$$

and then by (7), there exists the inverse of $\nabla_\xi \mathbf{X}_t$ such that

$$\frac{\partial \xi}{\partial x} = (\nabla_\xi \mathbf{X}_t)^{-1} = \mathbf{I} + \mathbf{V}_0(\mathbf{K_u}),$$

where $\mathbf{V}_0(\mathbf{K_u})$ is an $N \times N$ matrix of C^∞ functions with respect to $\mathbf{K_u} = \int_0^t \nabla_\xi \mathbf{u}(\xi, s)\, ds$ defined on $|\mathbf{K_u}| < \sigma$ satisfying $\mathbf{V}_0(0) = 0$.

By the chain rule, we have the gradient, divergence, and deformation tensor in Lagrange coordinates as follows:

$$\begin{aligned}
\nabla_x &= (\mathbf{I} + \mathbf{V}_0(\mathbf{K_u}))\nabla_\xi, \\
\operatorname{div}_x \mathbf{v} &= \operatorname{div}_\xi \mathbf{u} + \mathbf{V}_{\operatorname{div}}(\mathbf{K_u})\nabla_\xi \mathbf{u}, \\
&\quad \mathbf{V}_{\operatorname{div}}(\mathbf{K_u})\nabla_\xi \mathbf{u} = \mathbf{V}_0(\mathbf{K_u}) : \nabla_\xi \mathbf{u}, \\
\mathbf{D}_x(\mathbf{v}) &= \mathbf{D}_\xi(\mathbf{u}) + \mathbf{V_D}(\mathbf{K_u})\nabla_\xi \mathbf{u}, \\
&\quad \mathbf{V_D}(\mathbf{K_u})\nabla_\xi \mathbf{u} = \mathbf{V}_0(\mathbf{K_u})\nabla_\xi \mathbf{u} + (\mathbf{V}_0(\mathbf{K_u})\nabla_\xi \mathbf{u})^T, \\
\operatorname{Div}_x \mathbf{A} &= \operatorname{Div}_\xi \mathbf{A} + \mathbf{V}_{\operatorname{Div}}(\mathbf{K_u})\nabla_\xi \mathbf{A}, \\
&\quad \mathbf{V}_{\operatorname{Div}}(\mathbf{K_u})\nabla_\xi \mathbf{A} = (\mathbf{V}_0(\mathbf{K_u})\nabla_\xi \mid \mathbf{A}),
\end{aligned} \tag{8}$$

where $\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^N A_{ij} B_{ij}$ and i^{th} component $(\mathbf{B}\nabla_\xi \mid \mathbf{A})_i = \sum_{j,k=1}^N B_{jk}\partial_k A_{ij}$ for $N \times N$ matrices $\mathbf{A}$ and $\mathbf{B}$ with $(i,j)^{\text{th}}$ components A_{ij} and B_{ij}, respectively. Assume that $\rho(x,t) = \rho_* + \omega(x,t)$, $\mathbf{u}(x,t)$, and $\mathbf{d}(x,t) = \mathbf{d}_* + \mathbf{n}(x,t)$ satisfy (1) in the Euler coordinate. Setting $\omega(\mathbf{X}_t(\xi), t) = \theta(\xi, t)$, $\mathbf{v}(\mathbf{X}_t(\xi), t) = \mathbf{u}(\xi, t)$, $\mathbf{n}(\mathbf{X}_t(\xi), t) = \mathbf{k}(\xi, t)$ and using (8), $(\theta, \mathbf{u}, \mathbf{k})$ satisfies the following systems:

$$\begin{cases}
\partial_t \theta + \rho_* \operatorname{div} \mathbf{u} = f(\theta, \mathbf{u}) & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\
\rho_* \partial_t \mathbf{u} - \operatorname{Div}(\mathbf{S}(\mathbf{u}) - P'(\rho_*)\theta \mathbf{I}) = \mathbf{g}(\theta, \mathbf{u}, \mathbf{k}) & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\
\partial_t \mathbf{k} - \zeta \Delta \mathbf{k} = \mathbf{h}(\mathbf{u}, \mathbf{k}) & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\
(\theta, \mathbf{u}, \mathbf{k})|_{t=0} = (\rho_0, \mathbf{v}_0, \mathbf{d}_0) & \text{in } \mathbb{R}^N,
\end{cases} \tag{9}$$

where

$$\begin{aligned}
f(\theta,\mathbf{u}) &= -\theta\,\mathrm{div}\,\mathbf{u} - (\rho_*+\theta)\mathbf{V}_{\mathrm{div}}(\mathbf{K}_\mathbf{u})\nabla\mathbf{u},\\
\mathbf{g}(\theta,\mathbf{u},\mathbf{k}) &= -\theta\partial_t\mathbf{u} + \mathbf{V}_1(\mathbf{K}_\mathbf{u})\nabla^2\mathbf{u} + \mathbf{V}_2(\mathbf{K}_\mathbf{u})\int_0^t \nabla^2\mathbf{u}\,ds\nabla\mathbf{u}\\
&\quad - (P'(\rho_*+\theta)-P'(\rho_*))\nabla\theta - P'(\rho_*+\theta)\mathbf{V}_0(\mathbf{K}_\mathbf{u})\nabla\theta\\
&\quad - \eta\{(\nabla\mathbf{k})^T\Delta\mathbf{k} + \mathbf{V}_3(\mathbf{K}_\mathbf{u})(\nabla^2\mathbf{k})\nabla\mathbf{k} + \mathbf{V}_4(\mathbf{K}_\mathbf{u})\int_0^t \nabla^2\mathbf{u}\,ds(\nabla\mathbf{k})^2\},\\
\mathbf{h}(\mathbf{u},\mathbf{k}) &= \zeta\{\mathbf{V}_5(\mathbf{K}_\mathbf{u})\nabla^2\mathbf{k} + \mathbf{V}_6(\mathbf{K}_\mathbf{u})\int_0^t \nabla^2\mathbf{u}\,ds\nabla\mathbf{k}\\
&\quad + |\nabla\mathbf{k}|^2(\mathbf{k}+\mathbf{d}_*) + (|\mathbf{V}_0(\mathbf{K}_\mathbf{u})\nabla\mathbf{k}|^2 + (\nabla\mathbf{k} : \mathbf{V}_0(\mathbf{K}_\mathbf{u})\nabla\mathbf{k}))(\mathbf{k}+\mathbf{d}_*)\}
\end{aligned}$$

with $\mathbf{V}_j(\mathbf{K})$ $(j=1,\ldots,6)$ are some matrices of C^∞ functions with respect to matrix $\mathbf{K}$ for $|\mathbf{K}|<\sigma$.

In order to prove Theorem 1, we first show the following theorem concerning the global well-posedness of the system in Lagrange coordinates.

Theorem 2. *Let $3\le N\le 7$ and $0<T<\infty$. Let $2<p<\infty$, $2<q_1<N<q_2<\infty$, $b>0$ be numbers such that*

$$q_1\le 4,\quad \frac{1}{q_1}=\frac{1}{N}+\frac{1}{q_2},\quad \frac{N}{2q_1}<\frac{1}{p'}<b<\frac{N}{2q_1}+\frac{1}{2}-\frac{1}{p}. \tag{10}$$

Then, there exists a small number $\epsilon>0$ such that for any initial data $\rho_0\in\bigcap_{q=q_1,q_2}H^1_q(\mathbb{R}^N)\cap L_{q_1/2}(\mathbb{R}^N)$, $(\mathbf{v}_0,\mathbf{d}_0)\in\bigcap_{q=q_1,q_2}B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)^{2N}\cap L_{q_1/2}(\mathbb{R}^N)^{2N}$ with

$$\mathcal{I}:=\sum_{q=q_1,q_2}(\|\rho_0\|_{H^1_q(\mathbb{R}^N)}+\|(\mathbf{v}_0,\mathbf{d}_0)\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)})+\|(\rho_0,\mathbf{v}_0,\mathbf{d}_0)\|_{L_{q_1/2}(\mathbb{R}^N)}<\epsilon^2, \tag{11}$$

problem (9) admits a unique solution $(\theta,\mathbf{u},\mathbf{k})$ with

$$\theta\in H^1_p((0,T),H^1_q(\mathbb{R}^N)),\quad \mathbf{u}\in H^1_p((0,T),L_q(\mathbb{R}^N)^N)\cap L_p((0,T),H^2_q(\mathbb{R}^N)^N)$$
$$\mathbf{k}\in H^1_p((0,T),L_q(\mathbb{R}^N)^N)\cap L_p((0,T),H^2_q(\mathbb{R}^N)^N)$$

for $q=q_1$, and q_2 satisfying the estimate

$$\mathcal{N}(\theta,\mathbf{u},\mathbf{k})(T)\le\epsilon.$$

Here, we have set

$$\mathcal{N}(\theta,\mathbf{u},\mathbf{k})(T)=\mathcal{E}(\theta,\mathbf{u},\mathbf{k})(T)+\sum_{q=q_1,q_2}\|<t>^b\partial_t\nabla\theta\|_{L_p((0,T),L_q(\mathbb{R}^N))} \tag{12}$$

with $<t>=(1+t^2)^{1/2}$.

Remark 2. (1) *Since nonlinear terms have products of derivatives of $\mathbf{k}$ (e.g., $(\nabla\mathbf{k})^T\Delta\mathbf{k}$), we need to estimate $\|\nabla\mathbf{k}\|_{L_\infty((0,T),L_q(\mathbb{R}^N))}$ for $q=q_1,\infty$. In order to estimate these norms in a short time interval, we use Sobolev's embedding properties provided by $p>2$ and $(N/2q_2+1/2)p'<1$, which implies that $N/2q_1<1/p'$. According to [19], the local well-posedness is also obtained under these conditions. For details, see Section 4.2 below.*
(2) *In order to get a priori estimates, we use the L_{q_1}-$L_{q_1/2}$ and L_{q_2}-$L_{q_1/2}$ decay estimates of semigroup associated with the homogeneous compressible Navier–Stokes equations, therefore, we assume $1<q_1/2\le 2$. By this condition and $N<q_2$, we also have $1/4\le 1/q_1=1/q_2+1/N<2/N$, which implies that $N\le 7$. For details see Section 4.1.2 in Section 4.1 below.*

3. Estimates of Nonlinear Terms

Let $X_T^i (i = 1, 2, 3)$ be underlying spaces for linearized equations corresponding to (9), which is defined by

$$
\begin{aligned}
X_T^1 &= \Big\{\theta \in \bigcap_{q=q_1,q_2} H_p^1((0,T), H_q^1(\mathbb{R}^N)) \mid \theta|_{t=0} = \rho_0, \sup_{t\in(0,T)} \|\theta(\cdot,t)\|_{L_\infty(\mathbb{R}^N)} \le \rho_*/4\Big\}, \\
X_T^2 &= \Big\{\mathbf{u} \in \bigcap_{q=q_1,q_2} H_p^1((0,T), L_q(\mathbb{R}^N)^N) \cap L_p((0,T), H_q^2(\mathbb{R}^N)^N) \mid \\
&\qquad\qquad \mathbf{u}|_{t=0} = \mathbf{v}_0, \int_0^T \|\nabla \mathbf{u}(\cdot,s)\|_{L_\infty(\mathbb{R}^N)}\,ds \le \sigma\Big\}, \\
X_T^3 &= \Big\{\mathbf{k} \in \bigcap_{q=q_1,q_2} H_p^1((0,T), L_q(\mathbb{R}^N)^N) \cap L_p((0,T), H_q^2(\mathbb{R}^N)^N) \mid \mathbf{k}|_{t=0} = \mathbf{d}_0\Big\}.
\end{aligned}
\tag{13}
$$

In this section, we consider the necessary estimates of the nonlinear terms: $f(\theta,\mathbf{u})$, $\mathbf{g}(\theta,\mathbf{u},\mathbf{k})$, and $\mathbf{h}(\mathbf{u},\mathbf{k})$ for $(\theta,\mathbf{u},\mathbf{k}) \in X_T^1 \times X_T^2 \times X_T^3$ and difference: $f(\theta_1,\mathbf{u}_1) - f(\theta_2,\mathbf{u}_2)$, $\mathbf{g}(\theta_1,\mathbf{u}_1,\mathbf{k}_1) - \mathbf{g}(\theta_2,\mathbf{u}_2,\mathbf{k}_2)$, and $\mathbf{h}(\mathbf{u}_1,\mathbf{k}_1) - \mathbf{h}(\mathbf{u}_2,\mathbf{k}_2)$ for $(\theta_i,\mathbf{u}_i,\mathbf{k}_i) \in X_T^1 \times X_T^2 \times X_T^3$ $(i = 1,2)$ to prove the global well-posedness. For this purpose, we review estimates of matrices $\mathbf{V}_{\rm div}(\mathbf{K_u})$ and $\mathbf{V}_j(\mathbf{K_u})$ $(j = 0,1,\ldots,6)$ proved in [21]. For notational simplicity, we write $\|f\|_{L_q(\mathbb{R}^N)} = \|f\|_{L_q}$, $\|f\|_{H_q^1(\mathbb{R}^N)} = \|f\|_{H_q^1}$, $\|f\|_{L_\infty((0,T),X)} = \|f\|_{L_\infty(X)}$, and $\| < t >^b f\|_{L_p((0,T),X)} = \|f\|_{L_{p,b}(X)}$ for the Banach space X. Recall that $\mathbf{K_u} = \int_0^t \nabla\mathbf{u}\,ds$ for $\mathbf{u} \in X_T^2$, then

$$
|\mathbf{K_u}(\xi,t)| \le \sigma \tag{14}
$$

for any $(\xi,t) \in \mathbb{R}^N \times (0,T)$. Moreover, by Hölder inequality and the condition $bp' > 1$,

$$
\|\mathbf{K_u}\|_{L_\infty(X)} \le \int_0^T \|\nabla\mathbf{u}(\cdot,t)\|_X\,dt \le C\|\nabla\mathbf{u}\|_{L_{p,b}(X)} \tag{15}
$$

for $X \in \{L_q, H_q^1\}$ with $q = q_1$ and q_2. Note that $\mathbf{V}_{\rm div}(0) = 0$ and $\mathbf{V}_j(0) = 0$ $(j = 0,1,\ldots,6)$, by (14) and (15), we have

$$
\begin{aligned}
&\|\mathbf{V}_{\rm div}(\mathbf{K_u})\|_{L_\infty(H_q^1)} \le C\|\mathbf{K_u}\|_{L_\infty(H_q^1)} \le C\|\nabla\mathbf{u}\|_{L_{p,b}(H_q^1)}, \\
&\|\mathbf{V}_j(\mathbf{K_u})\|_{L_\infty(L_q)} \le C\|\mathbf{K_u}\|_{L_\infty(L_q)} \le C\|\nabla\mathbf{u}\|_{L_{p,b}(L_q)}, \\
&\|\mathbf{V}_j(\mathbf{K_u})\|_{L_\infty(L_\infty)} \le C\|\mathbf{K_u}\|_{L_\infty(H_{q_2}^1)} \le C\|\nabla\mathbf{u}\|_{L_{p,b}(H_{q_2}^1)}
\end{aligned}
\tag{16}
$$

for $q = q_1, q_2$ and $j = 0,1,\ldots,6$. We also have estimates of difference:

$$
\begin{aligned}
&\|\mathbf{V}_{\rm div}(\mathbf{K}_{\mathbf{u}_1}) - \mathbf{V}_{\rm div}(\mathbf{K}_{\mathbf{u}_2})\|_{L_\infty(H_q^1)} \\
&\quad \le C(\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_{p,b}(H_q^1)} + \sum_{i=1,2} \|\nabla\mathbf{u}_i\|_{L_{p,b}(H_q^1)} \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_{p,b}(H_{q_2}^1)}), \\
&\|\mathbf{V}_j(\mathbf{K}_{\mathbf{u}_1}) - \mathbf{V}_j(\mathbf{K}_{\mathbf{u}_2})\|_{L_\infty(L_q)} \le C\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_{p,b}(L_q)}, \\
&\|\mathbf{V}_j(\mathbf{K}_{\mathbf{u}_1}) - \mathbf{V}_j(\mathbf{K}_{\mathbf{u}_2})\|_{L_\infty(L_\infty)} \le C\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_{p,b}(H_{q_2}^1)}
\end{aligned}
\tag{17}
$$

for $q = q_1, q_2$ and $i = 0,1,\ldots,6$.

3.1. Estimates of $f(\theta, \mathbf{u})$

According to [21], $f(\theta, \mathbf{u})$ and difference $f(\theta_1, \mathbf{u}_1) - f(\theta_2, \mathbf{u}_2)$ satisfy

$$\begin{aligned}
&\|f(\theta, \mathbf{u})\|_{L_{p,b}(H^1_q)} \\
&\quad \leq C\mathcal{N}(\theta, \mathbf{u}, \mathbf{k})(\mathcal{I} + \mathcal{N}(\theta, \mathbf{u}, \mathbf{k}))(1 + \mathcal{N}(\theta, \mathbf{u}, \mathbf{k})), \\
&\|f(\theta_1, \mathbf{u}_1) - f(\theta_2, \mathbf{u}_2)\|_{L_{p,b}(H^1_q)} \\
&\quad \leq C\mathcal{N}((\theta_1, \mathbf{u}_1, \mathbf{k}_1) - (\theta_2, \mathbf{u}_2, \mathbf{k}_2)) \\
&\quad \times \left(\mathcal{I} + \sum_{i=1,2} \mathcal{N}(\theta_i, \mathbf{u}_i, \mathbf{k}_i)\right)\left(1 + \sum_{i=1,2} \mathcal{N}(\theta_i, \mathbf{u}_i, \mathbf{k}_i)\right)(1 + \mathcal{N}(\theta_1, \mathbf{u}_1, \mathbf{k}_1))
\end{aligned} \tag{18}$$

for $q = q_1/2$, q_1, and q_2.

3.2. Estimates of $\mathbf{g}(\theta, \mathbf{u}, \mathbf{k})$

Set

$$\mathbf{g}(\theta, \mathbf{u}, \mathbf{k}) = \mathbf{g}_1(\theta, \mathbf{u}) + \mathbf{g}_2(\mathbf{u}, \mathbf{k}),$$

where

$$\begin{aligned}
\mathbf{g}_1(\theta, \mathbf{u}) &= -\theta\partial_t\mathbf{u} + \mathbf{V}_1(\mathbf{K}_\mathbf{u})\nabla^2\mathbf{u} + \mathbf{V}_2(\mathbf{K}_\mathbf{u})\int_0^t \nabla^2\mathbf{u}\, ds\nabla\mathbf{u} \\
&\quad - (P'(\rho_* + \theta) - P'(\rho_*))\nabla\theta - P'(\rho_* + \theta)\mathbf{V}_0(\mathbf{K}_\mathbf{u})\nabla\theta, \\
\mathbf{g}_2(\mathbf{u}, \mathbf{k}) &= -\eta\{(\nabla\mathbf{k})^T\Delta\mathbf{k} + \mathbf{V}_3(\mathbf{K}_\mathbf{u})(\nabla^2\mathbf{k})\nabla\mathbf{k} + \mathbf{V}_4(\mathbf{K}_\mathbf{u})\int_0^t \nabla^2\mathbf{u}\, ds(\nabla\mathbf{k})^2\}.
\end{aligned}$$

Due to [21], $\mathbf{g}_1(\theta, \mathbf{u})$ and difference $\mathbf{g}_1(\theta_1, \mathbf{u}_1) - \mathbf{g}_1(\theta_2, \mathbf{u}_2)$ satisfy

$$\begin{aligned}
&\|\mathbf{g}_1(\theta, \mathbf{u})\|_{L_{p,b}(L_q)} \\
&\quad \leq C\mathcal{N}(\theta, \mathbf{u}, \mathbf{k})(\mathcal{I} + \mathcal{N}(\theta, \mathbf{u}, \mathbf{k})), \\
&\|\mathbf{g}_1(\theta_1, \mathbf{u}_1) - \mathbf{g}_1(\theta_2, \mathbf{u}_2)\|_{L_{p,b}(L_q)} \\
&\quad \leq C\mathcal{N}((\theta_1, \mathbf{u}_1, \mathbf{k}_1) - (\theta_2, \mathbf{u}_2, \mathbf{k}_2))\left(\mathcal{I} + \sum_{i=1,2} \mathcal{N}(\theta_i, \mathbf{u}_i, \mathbf{k}_i)\right)\left(1 + \sum_{i=1,2} \mathcal{N}(\theta_i, \mathbf{u}_i, \mathbf{k}_i)\right)
\end{aligned} \tag{19}$$

for $q = q_1/2$, q_1, and q_2.

Now let us consider $\|\mathbf{g}_2(\mathbf{u}, \mathbf{k})\|_{L_{p,b}(L_{q_1/2})}$ and $\|\mathbf{g}_2(\mathbf{u}_1, \mathbf{k}_1) - \mathbf{g}_2(\mathbf{u}_2, \mathbf{k}_2)\|_{L_{p,b}(L_{q_1/2})}$ by using the following estimates:

$$\begin{aligned}
\|fg\|_{L_{p,b}(L_{q_1/2})} &\leq \|f\|_{L_\infty(L_{q_1})}\|g\|_{L_{p,b}(L_{q_1})}, \\
\|fgh\|_{L_{p,b}(L_{q_1/2})} &\leq \|f\|_{L_\infty(L_\infty)}\|g\|_{L_\infty(L_{q_1})}\|h\|_{L_{p,b}(L_{q_1})}.
\end{aligned} \tag{20}$$

By (14), (16), and (20), we have

$$\begin{aligned}
&\|\mathbf{g}_2(\mathbf{u}, \mathbf{k})\|_{L_{p,b}(L_{q_1/2})} \\
&\quad \leq C(\|\nabla\mathbf{k}\|_{L_\infty(L_{q_1})}\|\nabla^2\mathbf{k}\|_{L_{p,b}(L_{q_1})} + \|\nabla^2\mathbf{u}\|_{L_{p,b}(L_{q_1})}\|\nabla\mathbf{k}\|_{L_{p,b}(L_{q_1})}\|\nabla\mathbf{k}\|_{L_\infty(L_\infty)}).
\end{aligned} \tag{21}$$

In order to estimate difference $\mathbf{g}_2(\mathbf{u}_1, \mathbf{k}_1) - \mathbf{g}_2(\mathbf{u}_2, \mathbf{k}_2)$, we write

$$\|\mathbf{g}_2(\mathbf{u}_1, \mathbf{k}_1) - \mathbf{g}_2(\mathbf{u}_2, \mathbf{k}_2)\|_{L_{p,b}(L_{q_1/2})} \leq \sum_{j=1}^3 E^j_{g,q_1/2}, \tag{22}$$

where

$$E^1_{g,q_1/2} = \|(\nabla \mathbf{k}_1)^T \Delta \mathbf{k}_1 - (\nabla \mathbf{k}_2)^T \Delta \mathbf{k}_2\|_{L_{p,b}(L_{q_1/2})},$$
$$E^2_{g,q_1/2} = \|\mathbf{V}_3(\mathbf{K}_{\mathbf{u}_1})(\nabla^2 \mathbf{k}_1)\nabla \mathbf{k}_1 - \mathbf{V}_3(\mathbf{K}_{\mathbf{u}_2})(\nabla^2 \mathbf{k}_2)\nabla \mathbf{k}_2\|_{L_{p,b}(L_{q_1/2})},$$
$$E^3_{g,q_1/2} = \|\mathbf{V}_4(\mathbf{K}_{\mathbf{u}_1}) \int_0^t \nabla^2 \mathbf{u}_1\, ds(\nabla \mathbf{k}_1)^2 - \mathbf{V}_4(\mathbf{K}_{\mathbf{u}_2}) \int_0^t \nabla^2 \mathbf{u}_2\, ds(\nabla \mathbf{k}_2)^2\|_{L_{p,b}(L_{q_1/2})}.$$

By (14), (16), (17), and (20), we have

$$\begin{aligned}
E^1_{g,q_1/2} &\le C(\|\nabla(\mathbf{k}_1 - \mathbf{k}_2)\|_{L_\infty(L_{q_1})}\|\Delta \mathbf{k}_1\|_{L_{p,b}(L_{q_1})} + \|\nabla \mathbf{k}_2\|_{L_\infty(L_{q_1})}\|\Delta(\mathbf{k}_1 - \mathbf{k}_2)\|_{L_{p,b}(L_{q_1})}),\\
E^2_{g,q_1/2} &\le C(\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_{p,b}(L_{q_1})}\|\nabla^2 \mathbf{k}_1\|_{L_{p,b}(L_{q_1})}\|\nabla \mathbf{k}_1\|_{L_\infty(L_\infty)}\\
&\quad + \|\nabla^2(\mathbf{k}_1 - \mathbf{k}_2)\|_{L_{p,b}(L_{q_1})}\|\nabla \mathbf{k}_1\|_{L_\infty(L_{q_1})} + \|\nabla^2 \mathbf{k}_2\|_{L_{p,b}(L_{q_1})}\|\nabla(\mathbf{k}_1 - \mathbf{k}_2)\|_{L_\infty(L_{q_1})}),\\
E^3_{g,q_1/2} &\le C\{\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_{p,b}(L_{q_1})}\|\nabla^2 \mathbf{u}_1\|_{L_{p,b}(L_{q_1})}\|\nabla \mathbf{k}_1\|^2_{L_\infty(L_\infty)}\\
&\quad + \|\nabla^2(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_{p,b}(L_{q_1})}\|\nabla \mathbf{k}_1\|_{L_{p,b}(L_{q_1})}\|\nabla \mathbf{k}_1\|_{L_\infty(L_\infty)}\\
&\quad + \|\nabla^2 \mathbf{u}_2\|_{L_{p,b}(L_{q_1})}\|\nabla(\mathbf{k}_1 - \mathbf{k}_2)\|_{L_{p,b}(L_{q_1})}(\|\nabla \mathbf{k}_1\|_{L_\infty(L_\infty)} + \|\nabla \mathbf{k}_2\|_{L_\infty(L_\infty)})\}.
\end{aligned}$$

We next consider $\|\mathbf{g}_2(\mathbf{u}, \mathbf{k})\|_{L_{p,b}(L_q)}$ and $\|\mathbf{g}_2(\mathbf{u}_1, \mathbf{k}_1) - \mathbf{g}_2(\mathbf{u}_2, \mathbf{k}_2)\|_{L_{p,b}(L_q)}$ for $q = q_1$ and q_2. If terms do not include $\int_0^t \nabla^2 \mathbf{u}\, ds$, we use

$$\begin{aligned}
\|fg\|_{L_{p,b}(L_q)} &\le \|f\|_{L_\infty(L_\infty)}\|g\|_{L_{p,b}(L_q)},\\
\|fgh\|_{L_{p,b}(L_q)} &\le \|f\|_{L_\infty(L_\infty)}\|g\|_{L_\infty(L_\infty)}\|h\|_{L_{p,b}(L_q)},
\end{aligned} \tag{23}$$

if terms include $\int_0^t \nabla^2 \mathbf{u}\, ds$, we use

$$\begin{aligned}
\left\|\left(\int_0^t \nabla^2 \mathbf{u}\, ds\right) f\right\|_{L_{p,b}(L_q)} &\le C\|\nabla^2 \mathbf{v}\|_{L_{p,b}(L_q)}\|f\|_{L_{p,b}(H^1_{q_2})},\\
\left\|\left(\int_0^t \nabla^2 \mathbf{u}\, ds\right) fg\right\|_{L_{p,b}(L_q)} &\le C\|\nabla^2 \mathbf{v}\|_{L_{p,b}(L_q)}\|f\|_{L_{p,b}(H^1_{q_2})}\|g\|_{L_\infty(L_\infty)}.
\end{aligned} \tag{24}$$

By (14), (16), (23), and (24), we have

$$\begin{aligned}
&\|\mathbf{g}_2(\mathbf{u}, \mathbf{k})\|_{L_{p,b}(L_q)}\\
&\quad \le C(\|\nabla \mathbf{k}\|_{L_\infty(L_\infty)}\|\nabla^2 \mathbf{k}\|_{L_{p,b}(L_q)} + \|\nabla^2 \mathbf{u}\|_{L_{p,b}(L_q)}\|\nabla \mathbf{k}\|_{L_{p,b}(H^1_{q_2})}\|\nabla \mathbf{k}\|_{L_\infty(L_\infty)}).
\end{aligned} \tag{25}$$

Writing

$$\|\mathbf{g}_2(\mathbf{u}_1, \mathbf{k}_1) - \mathbf{g}_2(\mathbf{u}_2, \mathbf{k}_2)\|_{L_{p,b}(L_q)} \le \sum_{j=1}^3 E^j_{g,q}, \tag{26}$$

by (14), (16), (17), (23), and (24), we have

$$\begin{aligned}
E^1_{g,q} &\le C(\|\nabla(\mathbf{k}_1 - \mathbf{k}_2)\|_{L_\infty(L_\infty)}\|\Delta \mathbf{k}_1\|_{L_{p,b}(L_q)} + \|\nabla \mathbf{k}_2\|_{L_\infty(L_\infty)}\|\Delta(\mathbf{k}_1 - \mathbf{k}_2)\|_{L_{p,b}(L_q)}),\\
E^2_{g,q} &\le C(\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_{p,b}(H^1_{q_2})}\|\nabla^2 \mathbf{k}_1\|_{L_{p,b}(L_q)}\|\nabla \mathbf{k}_1\|_{L_\infty(L_\infty)}\\
&\quad + \|\nabla^2(\mathbf{k}_1 - \mathbf{k}_2)\|_{L_{p,b}(L_q)}\|\nabla \mathbf{k}_1\|_{L_\infty(L_\infty)} + \|\nabla^2 \mathbf{k}_2\|_{L_{p,b}(L_q)}\|\nabla(\mathbf{k}_1 - \mathbf{k}_2)\|_{L_\infty(L_\infty)}),\\
E^3_{g,q} &\le C\{\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_{p,b}(H^1_{q_2})}\|\nabla^2 \mathbf{u}_1\|_{L_{p,b}(L_q)}\|\nabla \mathbf{k}_1\|_{L_{p,b}(H^1_{q_2})}\|\nabla \mathbf{k}_1\|_{L_\infty(L_\infty)}\\
&\quad + \|\nabla^2(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_{p,b}(L_q)}\|\nabla \mathbf{k}_1\|_{L_{p,b}(H^1_{q_2})}\|\nabla \mathbf{k}_1\|_{L_\infty(L_\infty)}\\
&\quad + \|\nabla^2 \mathbf{u}_2\|_{L_{p,b}(L_q)}\|\nabla(\mathbf{k}_1 - \mathbf{k}_2)\|_{L_{p,b}(H^1_{q_2})}(\|\nabla \mathbf{k}_1\|_{L_\infty(L_\infty)} + \|\nabla \mathbf{k}_2\|_{L_\infty(L_\infty)})\}.
\end{aligned}$$

Thus, by (19), (21), (22), (25), and (26), $\mathbf{g}(\theta, \mathbf{u}, \mathbf{k})$ and difference $\mathbf{g}(\theta_1, \mathbf{u}_1, \mathbf{k}_1) - \mathbf{g}(\theta_2, \mathbf{u}_2, \mathbf{k}_2)$ satisfy

$$\begin{aligned} &\|\mathbf{g}(\theta, \mathbf{u}, \mathbf{k})\|_{L_{p,b}(L_q)} \\ &\quad \leq C\mathcal{N}(\theta, \mathbf{u}, \mathbf{k})(\mathcal{I} + \mathcal{N}(\theta, \mathbf{u}, \mathbf{k}))(1 + \mathcal{N}(\theta, \mathbf{u}, \mathbf{k})), \\ &\|\mathbf{g}(\theta_1, \mathbf{u}_1, \mathbf{k}_1) - \mathbf{g}(\theta_2, \mathbf{u}_2, \mathbf{k}_2)\|_{L_{p,b}(L_q)} \\ &\quad \leq C\mathcal{N}((\theta_1, \mathbf{u}_1, \mathbf{k}_1) - (\theta_2, \mathbf{u}_2, \mathbf{k}_2)) \\ &\quad \times \left(\mathcal{I} + \sum_{i=1,2} \mathcal{N}(\theta_i, \mathbf{u}_i, \mathbf{k}_i)\right)\left(1 + \sum_{i=1,2} \mathcal{N}(\theta_i, \mathbf{u}_i, \mathbf{k}_i)\right)(1 + \mathcal{N}(\theta_1, \mathbf{u}_1, \mathbf{k}_1)) \end{aligned} \tag{27}$$

for $q = q_1/2$, q_1, and q_2.

3.3. Estimates of $\mathbf{h}(\mathbf{u}, \mathbf{k})$

Recall that

$$\begin{aligned} \mathbf{h}(\mathbf{u}, \mathbf{k}) = \zeta\{&\mathbf{V}_5(\mathbf{K}_\mathbf{u})\nabla^2\mathbf{k} + \mathbf{V}_6(\mathbf{K}_\mathbf{u}) \int_0^t \nabla^2\mathbf{u}\, ds \nabla\mathbf{k} \\ &+ |\nabla\mathbf{k}|^2(\mathbf{k} + \mathbf{d}_*) + (|\mathbf{V}_0(\mathbf{K}_\mathbf{u})\nabla\mathbf{k}|^2 + (\nabla\mathbf{k} : \mathbf{V}_0(\mathbf{K}_\mathbf{u})\nabla\mathbf{k}))(\mathbf{k} + \mathbf{d}_*)\}. \end{aligned}$$

By the same calculation as in Section 3.2, we have

$$\begin{aligned} &\|\mathbf{h}(\mathbf{u}, \mathbf{k})\|_{L_{p,b}(L_{q_1/2})} \\ &\quad \leq C\{\|\nabla\mathbf{u}\|_{L_{p,b}(L_{q_1})}\|\nabla^2\mathbf{k}\|_{L_{p,b}(L_{q_1})} + \|\nabla^2\mathbf{u}\|_{L_{p,b}(L_{q_1})}\|\nabla\mathbf{k}\|_{L_{p,b}(L_{q_1})} \\ &\quad + \|\nabla\mathbf{k}\|_{L_{p,b}(L_{q_1})}\|\nabla\mathbf{k}\|_{L_\infty(L_{q_1})}(\|\mathbf{k}\|_{L_\infty(L_\infty)} + 1)\}, \end{aligned} \tag{28}$$

$$\begin{aligned} &\|\mathbf{h}(\mathbf{u}_1, \mathbf{k}_1) - \mathbf{h}(\mathbf{u}_2, \mathbf{k}_2)\|_{L_{p,b}(L_{q_1/2})} \\ &\leq C\{\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_{p,b}(L_{q_1})}\|\nabla^2\mathbf{k}_1\|_{L_{p,b}(L_{q_1})} + \|\nabla\mathbf{u}_2\|_{L_{p,b}(L_{q_1})}\|\nabla^2(\mathbf{k}_1 - \mathbf{k}_2)\|_{L_{p,b}(L_{q_1})} \\ &\quad + \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_{p,b}(H^1_{q_2})}\|\nabla^2\mathbf{u}_1\|_{L_{p,b}(L_{q_1})}\|\nabla\mathbf{k}_1\|_{L_{p,b}(L_{q_1})} \\ &\quad + \|\nabla^2(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_{p,b}(L_{q_1})}\|\nabla\mathbf{k}_1\|_{L_{p,b}(L_{q_1})} + \|\nabla^2\mathbf{u}_2\|_{L_{p,b}(L_{q_1})}\|\nabla(\mathbf{k}_1 - \mathbf{k}_2)\|_{L_{p,b}(L_{q_1})} \\ &\quad + \|\nabla(\mathbf{k}_1 - \mathbf{k}_2)\|_{L_{p,b}(L_{q_1})}(\|\mathbf{k}_1\|_{L_\infty(L_\infty)} + 1)(\|\nabla\mathbf{k}_1\|_{L_\infty(L_{q_1})} + \|\nabla\mathbf{k}_2\|_{L_\infty(L_{q_1})}) \\ &\quad + \|\mathbf{k}_1 - \mathbf{k}_2\|_{L_{p,b}(H^1_{q_2})}\|\nabla\mathbf{k}_2\|^2_{L_\infty(L_{q_1})} \\ &\quad + \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{L_{p,b}(L_{q_1})}(\|\mathbf{k}_1\|_{L_\infty(L_\infty)} + 1)\|\nabla\mathbf{k}_1\|_{L_{p,b}(H^1_{q_2})} \\ &\quad\quad \times (\|\nabla\mathbf{k}_1\|_{L_\infty(L_{q_1})} + \|\nabla\mathbf{k}_2\|_{L_\infty(L_{q_1})})\}. \end{aligned} \tag{29}$$

Moreover, $\|\mathbf{h}(\mathbf{u}, \mathbf{k})\|_{L_{p,b}(L_q)}$ and $\|\mathbf{h}(\mathbf{u}_1, \mathbf{k}_1) - (\mathbf{u}_2, \mathbf{k}_2)\|_{L_{p,b}(L_q)}$ with $q = q_1$, q_2 satisfy the following estimates:

$$\begin{aligned} &\|\mathbf{h}(\mathbf{u}, \mathbf{k})\|_{L_{p,b}(L_q)} \\ &\quad \leq C\{\|\nabla\mathbf{u}\|_{L_{p,b}(H^1_{q_2})}\|\nabla^2\mathbf{k}\|_{L_{p,b}(L_q)} + \|\nabla^2\mathbf{u}\|_{L_{p,b}(L_q)}\|\nabla\mathbf{k}\|_{L_{p,b}(H^1_{q_2})} \\ &\quad + \|\nabla\mathbf{k}\|_{L_{p,b}(L_q)}\|\nabla\mathbf{k}\|_{L_\infty(L_\infty)}(\|\mathbf{k}\|_{L_\infty(L_\infty)} + 1)\}, \end{aligned} \tag{30}$$

$$
\begin{aligned}
&\|\mathbf{h}(\mathbf{u}_1,\mathbf{k}_1)-\mathbf{h}(\mathbf{u}_2,\mathbf{k}_2)\|_{L_{p,b}(L_q)} \\
&\quad\le C\{\|\nabla(\mathbf{u}_1-\mathbf{u}_2)\|_{L_{p,b}(H^1_{q_2})}\|\nabla^2\mathbf{k}_1\|_{L_{p,b}(L_q)}+\|\nabla\mathbf{u}_2\|_{L_{p,b}(H^1_{q_2})}\|\nabla^2(\mathbf{k}_1-\mathbf{k}_2)\|_{L_{p,b}(L_q)} \\
&\quad+\|\nabla(\mathbf{u}_1-\mathbf{u}_2)\|_{L_{p,b}(H^1_{q_2})}\|\nabla^2\mathbf{u}_1\|_{L_{p,b}(L_q)}\|\nabla\mathbf{k}_1\|_{L_{p,b}(H^1_{q_2})} \\
&\quad+\|\nabla^2(\mathbf{u}_1-\mathbf{u}_2)\|_{L_{p,b}(L_q)}\|\nabla\mathbf{k}_1\|_{L_{p,b}(H^1_{q_2})}+\|\nabla^2\mathbf{u}_2\|_{L_{p,b}(L_q)}\|\nabla(\mathbf{k}_1-\mathbf{k}_2)\|_{L_{p,b}(H^1_{q_2})} \\
&\quad+\|\nabla(\mathbf{k}_1-\mathbf{k}_2)\|_{L_{p,b}(L_q)}(\|\mathbf{k}_1\|_{L_\infty(L_\infty)}+1)(\|\nabla\mathbf{k}_1\|_{L_\infty(L_\infty)}+\|\nabla\mathbf{k}_2\|_{L_\infty(L_\infty)}) \\
&\quad+\|\mathbf{k}_1-\mathbf{k}_2\|_{L_{p,b}(H^1_{q_2})}\|\nabla\mathbf{k}_2\|_{L_\infty(L_\infty)}\|\nabla\mathbf{k}_2\|_{L_\infty(L_q)} \\
&\quad+\|\nabla(\mathbf{u}_1-\mathbf{u}_2)\|_{L_{p,b}(L_q)}(\|\mathbf{k}_1\|_{L_\infty(L_\infty)}+1)\|\nabla\mathbf{k}_1\|_{L_{p,b}(H^1_{q_2})} \\
&\qquad\times(\|\nabla\mathbf{k}_1\|_{L_\infty(L_\infty)}+\|\nabla\mathbf{k}_2\|_{L_\infty(L_\infty)})\}.
\end{aligned}
\tag{31}
$$

Therefore, by (28), (29), (30), and (31), $\mathbf{h}(\mathbf{u},\mathbf{k})$ and difference $\mathbf{h}(\mathbf{u}_1,\mathbf{k}_1)-\mathbf{h}(\mathbf{u}_2,\mathbf{k}_2)$ satisfy the following estimate

$$
\begin{aligned}
&\|\mathbf{h}(\mathbf{u},\mathbf{k})\|_{L_{p,b}(L_q)}\le C\mathcal{N}(\theta,\mathbf{u},\mathbf{k})^2(1+\mathcal{N}(\theta,\mathbf{u},\mathbf{k})), \\
&\|\mathbf{h}(\mathbf{u}_1,\mathbf{k}_1)-\mathbf{h}(\mathbf{u}_2,\mathbf{k}_2)\|_{L_{p,b}(L_q)} \\
&\quad\le C\mathcal{N}((\theta_1,\mathbf{u}_1,\mathbf{k}_1)-(\theta_2,\mathbf{u}_2,\mathbf{k}_2)) \\
&\quad\times\sum_{i=1,2}\mathcal{N}(\theta_i,\mathbf{u}_i,\mathbf{k}_i)\left(1+\sum_{i=1,2}\mathcal{N}(\theta_i,\mathbf{u}_i,\mathbf{k}_i)\right)(1+\mathcal{N}(\theta_1,\mathbf{u}_1,\mathbf{k}_1))
\end{aligned}
\tag{32}
$$

for $q=q_1/2$, q_1, and q_2.

4. A Priori Estimates for Linearized Problems

Let ϵ be a small positive number and let $\mathcal{N}(\rho,\mathbf{v},\mathbf{d})(T)$ be the norm defined in (12). Let

$$\mathcal{I}_{T,\epsilon}=\{(\rho,\mathbf{v},\mathbf{d})\in X^1_T\times X^2_T\times X^3_T\mid\mathcal{N}(\rho,\mathbf{v},\mathbf{d})(T)\le\epsilon\}.$$

Given $(\rho,\mathbf{v},\mathbf{d})\in\mathcal{I}_{T,\epsilon}$, let $(\theta,\mathbf{u},\mathbf{k})$ be a solution to equations:

$$
\begin{cases}
\partial_t\theta+\rho_*\operatorname{div}\mathbf{u}=f(\rho,\mathbf{v}) & \text{in } \mathbb{R}^N \text{ for } t\in(0,T), \\
\rho_*\partial_t\mathbf{u}-\operatorname{Div}(\mathbf{S}(\mathbf{u})-P'(\rho_*)\theta\mathbf{I})=\mathbf{g}(\rho,\mathbf{v},\mathbf{d}) & \text{in } \mathbb{R}^N \text{ for } t\in(0,T), \\
(\theta,\mathbf{u})|_{t=0}=(\rho_0,\mathbf{v}_0) & \text{in } \mathbb{R}^N,
\end{cases}
\tag{33}
$$

$$
\begin{cases}
\partial_t\mathbf{k}-\zeta\Delta\mathbf{k}=\mathbf{h}(\mathbf{v},\mathbf{d}) & \text{in } \mathbb{R}^N \text{ for } t\in(0,T), \\
\mathbf{k}|_{t=0}=\mathbf{d}_0 & \text{in } \mathbb{R}^N.
\end{cases}
\tag{34}
$$

Now we shall prove the following inequality:

$$\mathcal{N}(\theta,\mathbf{u},\mathbf{k})(T)\le C(\epsilon^2+\epsilon^3+\epsilon^4). \tag{35}$$

For this purpose, we divide $\mathcal{N}(\theta, \mathbf{u}, \mathbf{k})(T)$ into two terms: $\mathcal{N}(\theta, \mathbf{u}, \mathbf{k})(T) = \mathcal{N}_1(\theta, \mathbf{u}, \mathbf{k})(T) + \mathcal{N}_2(\mathbf{k})(T)$, where

$$\begin{aligned}
\mathcal{N}_1(\theta, \mathbf{u}, \mathbf{k})(T) = &\| < t >^b \nabla\theta\|_{L_p((0,T),L_{q_1}(\mathbb{R}^N))} + \| < t >^b \nabla(\mathbf{u}, \mathbf{k})\|_{L_p((0,T),H^1_{q_1}(\mathbb{R}^N))} \\
&+ \| < t >^b \theta\|_{L_p((0,T),H^1_{q_2}(\mathbb{R}^N))} + \| < t >^b (\mathbf{u}, \mathbf{k})\|_{L_p((0,T),H^2_{q_2}(\mathbb{R}^N))} \\
&+ \| < t >^{N/(2q_1)} (\theta, \mathbf{u}, \mathbf{k})\|_{L_\infty((0,T),L_{q_1}(\mathbb{R}^N))} \\
&+ \| < t >^b (\theta, \mathbf{u}, \mathbf{k})\|_{L_\infty((0,T),L_{q_2}(\mathbb{R}^N))} \\
&+ \sum_{q=q_1,q_2} \Big(\| < t >^b \partial_t\theta\|_{L_p((0,T),H^1_q(\mathbb{R}^N))} + \| < t >^b \partial_t(\mathbf{u}, \mathbf{k})\|_{L_p((0,T),L_q(\mathbb{R}^N))} \Big) \\
\mathcal{N}_2(\mathbf{k})(T) = &\| < t >^b \nabla\mathbf{k}\|_{L_\infty((0,T),L_{q_1}(\mathbb{R}^N))} + \| < t >^b \mathbf{k}\|_{L_\infty((0,T),H^1_\infty(\mathbb{R}^N))}.
\end{aligned}$$

4.1. Estimates of $\mathcal{N}_1(\theta, \mathbf{u}, \mathbf{k})(T)$

In this subsection, we prove $\mathcal{N}_1(\theta, \mathbf{u}, \mathbf{k})(T) \leq C(\epsilon^2 + \epsilon^3 + \epsilon^4)$. To obtain this inequality, we decompose solutions $(\theta, \mathbf{u})$ to (33) by $(\theta, \mathbf{u}) = (\theta_1, \mathbf{u}_1) + (\theta_2, \mathbf{u}_2)$ and $\mathbf{k}$ to (34) by $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$, where $(\theta_1, \mathbf{u}_1)$ and $\mathbf{k}_1$ satisfy time-shifted equations:

$$\begin{cases} \partial_t\theta_1 + \lambda_0\theta_1 + \rho_* \operatorname{div} \mathbf{u}_1 = f(\rho, \mathbf{v}) & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\ \rho_*\partial_t\mathbf{u}_1 + \lambda_0\mathbf{u}_1 - \operatorname{Div}(\mathbf{S}(\mathbf{u}_1) - P'(\rho_*)\theta_1\mathbf{I}) = \mathbf{g}(\rho, \mathbf{v}, \mathbf{d}) & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\ (\theta_1, \mathbf{u}_1)|_{t=0} = (0, 0) & \text{in } \mathbb{R}^N, \end{cases} \tag{36}$$

$$\begin{cases} \partial_t\mathbf{k}_1 + \mathbf{k}_1 - \zeta\Delta\mathbf{k}_1 = \mathbf{h}(\mathbf{v}, \mathbf{d}) & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\ \mathbf{k}_1|_{t=0} = 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{37}$$

$(\theta_2, \mathbf{u}_2)$ and $\mathbf{k}_2$ satisfy compensation equations:

$$\begin{cases} \partial_t\theta_2 + \rho_* \operatorname{div} \mathbf{u}_2 = \lambda_0\theta_1 & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\ \rho_*\partial_t\mathbf{u}_2 - \operatorname{Div}(\mathbf{S}(\mathbf{u}_2) - P'(\rho_*)\theta_2\mathbf{I}) = \lambda_0\mathbf{u}_1 & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\ (\theta_2, \mathbf{u}_2)|_{t=0} = (\rho_0, \mathbf{v}_0) & \text{in } \mathbb{R}^N, \end{cases} \tag{38}$$

$$\begin{cases} \partial_t\mathbf{k}_2 - \zeta\Delta\mathbf{k}_2 = \mathbf{k}_1 & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\ \mathbf{k}_2|_{t=0} = \mathbf{d}_0 & \text{in } \mathbb{R}^N. \end{cases} \tag{39}$$

4.1.1. Analysis of Time Shifted Equations

We consider the following linearized problems for (36) and (37):

$$\begin{cases} \partial_t\theta + \lambda_0\theta + \rho_* \operatorname{div} \mathbf{u} = f & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\ \rho_*\partial_t\mathbf{u} + \lambda_0\mathbf{u} - \operatorname{Div}(\mathbf{S}(\mathbf{u}) - P'(\rho_*)\theta\mathbf{I}) = \mathbf{g} & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\ (\theta, \mathbf{u})|_{t=0} = (0, 0) & \text{in } \mathbb{R}^N, \end{cases} \tag{40}$$

$$\begin{cases} \partial_t\mathbf{k} + \mathbf{k} - \zeta\Delta\mathbf{k} = \mathbf{h} & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\ \mathbf{k}|_{t=0} = 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{41}$$

By the existence of $\mathcal{R}$-bounded solution operators for the resolvent problems corresponding to (40) and (41), we have the following theorem.

Theorem 3. *Let $1 < p, q < \infty$. Let $b \geq 0$. Then, there exists a constant $\lambda_0 > 0$ such that the following assertions hold:*

(1) *For any* $< t >^b (f, \mathbf{g}) \in L_p((0,T), H^{1,0}_q(\mathbb{R}^N))$, *problem* (40) *admits unique solutions* $\theta \in H^1_p((0,T), H^1_q(\mathbb{R}^N))$ *and* $\mathbf{u} \in H^1_p((0,T), L_q(\mathbb{R}^N)^N) \cap L_p((0,T), H^2_q(\mathbb{R}^N)^N)$ *possessing the estimate*

$$\begin{aligned}&\| < t >^b \partial_t(\theta, \mathbf{u})\|_{L_p((0,T),H^{1,0}_q(\mathbb{R}^N))} + \| < t >^b (\theta, \mathbf{u})\|_{L_p((0,T),H^{1,2}_q(\mathbb{R}^N))}\\&\le C\| < t >^b (f, \mathbf{g})\|_{L_p((0,T),H^{1,0}_q(\mathbb{R}^N))}.\end{aligned} \tag{42}$$

(2) *For any* $< t >^b \mathbf{h} \in L_p((0,T), L_q(\mathbb{R}^N)^N)$, *problem* (41) *admits a unique solution* $\mathbf{k} \in H^1_p((0,T), L_q(\mathbb{R}^N)^N) \cap L_p((0,T), H^2_q(\mathbb{R}^N)^N)$ *possessing the estimate*

$$\begin{aligned}&\| < t >^b \partial_t\mathbf{k}\|_{L_p((0,T),L_q(\mathbb{R}^N))} + \| < t >^b \mathbf{k}\|_{L_p((0,T),H^2_q(\mathbb{R}^N))}\\&\le C\| < t >^b \mathbf{h}\|_{L_p((0,T),L_q(\mathbb{R}^N))}.\end{aligned} \tag{43}$$

Proof. (1) Firstly, we consider the case $b = 0$. Let f_0 and $\mathbf{g}_0$ be the zero extension of f and $\mathbf{g}$ outside of $(0,T)$. By the existence of $\mathcal{R}$-bounded solution operators for the resolvent problem corresponding to (40) proved in ([22], Theorem 2.5), we see that there exists $\lambda_0 > 0$ such that the following system

$$\begin{cases}\partial_t\theta + \lambda_0\theta + \rho_* \operatorname{div}\mathbf{u} = f_0 & \text{in } \mathbb{R}^N \text{ for } t \in \mathbb{R},\\ \rho_*\partial_t\mathbf{u} + \lambda_0\mathbf{u} - \operatorname{Div}(\mathbf{S}(\mathbf{u}) - P'(\rho_*)\theta\mathbf{I}) = \mathbf{g}_0 & \text{in } \mathbb{R}^N \text{ for } t \in \mathbb{R}\end{cases}$$

has unique solutions $\theta \in H^1_p(\mathbb{R}, H^1_q(\mathbb{R}^N))$ and, $\mathbf{u} \in \times H^1_p(\mathbb{R}, L_q(\mathbb{R}^N)^N) \cap L_p(\mathbb{R}, H^2_q(\mathbb{R}^N)^N)$ satisfying

$$\begin{aligned}&\|\partial_t(\theta, \mathbf{u})\|_{L_p(\mathbb{R},H^{1,0}_q(\mathbb{R}^N))} + \|(\theta, \mathbf{u})\|_{L_p(\mathbb{R},H^{1,2}_q(\mathbb{R}^N))}\\&\quad\le C\|(f_0, \mathbf{g}_0)\|_{L_p(\mathbb{R},H^{1,0}_q(\mathbb{R}^N))} = C\|(f, \mathbf{g})\|_{L_p((0,T),H^{1,0}_q(\mathbb{R}^N))}.\end{aligned}$$

Moreover, for any $\gamma > \lambda_0$, we have

$$\begin{aligned}\gamma\|(\theta, \mathbf{u})\|_{L_p((-\infty,0],L_q(\mathbb{R}^N))} &\le \gamma\|e^{-\gamma t}(\theta, \mathbf{u})\|_{L_p((-\infty,0],L_q(\mathbb{R}^N))} \le \gamma\|e^{-\gamma t}(\theta, \mathbf{u})\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))}\\&\le C\|e^{-\gamma t}(f_0, \mathbf{g}_0)\|_{L_p(\mathbb{R},H^{1,0}_q(\mathbb{R}^N))} = C\|(f, \mathbf{g})\|_{L_p((0,T),H^{1,0}_q(\mathbb{R}^N))},\end{aligned}$$

where C is a constant independent of γ. Thus, letting $\gamma \to \infty$ yields that $(\theta, \mathbf{u})$ vanishes for $t \le 0$. In particular, we have $(\theta, \mathbf{u})|_{t=0} = (0, 0)$.

Secondly, we consider the case $b \in (0, 1]$. Multiplying $< t >^b$ to (40) and setting $< t >^b \theta = \rho$ and $< t >^b \mathbf{u} = \mathbf{v}$, we have

$$\begin{cases}\partial_t\rho + \lambda_0\rho + \rho_* \operatorname{div}\mathbf{v} =< t >^b f + (\partial_t < t >^b)\theta & \text{in } \mathbb{R}^N \text{ for } t \in (0,T),\\ \rho_*\partial_t\mathbf{v} + \lambda_0\mathbf{v} - \operatorname{Div}(\mathbf{S}(\mathbf{v}) - P'(\rho_*)\rho\mathbf{I}) & \\ \quad =< t >^b \mathbf{g} + (\partial_t < t >^b)\mathbf{u} & \text{in } \mathbb{R}^N \text{ for } t \in (0,T),\\ (\rho, \mathbf{v})|_{t=0} = (0, 0) & \text{in } \mathbb{R}^N.\end{cases}$$

Noting that $|\partial_t(< t >^b)| \le 1$ and applying (42) for the case $b = 0$ yields that

$$\begin{aligned}&\| < t >^b \partial_t(\theta, \mathbf{u})\|_{L_p((0,T),H^{1,0}_q(\mathbb{R}^N))} + \| < t >^b (\theta, \mathbf{u})\|_{L_p((0,T),H^{1,2}_q(\mathbb{R}^N))}\\&\le \|\partial_t(\rho, \mathbf{v})\|_{L_p((0,T),H^{1,0}_q(\mathbb{R}^N))} + \|(\rho, \mathbf{v})\|_{L_p((0,T),H^{1,2}_q(\mathbb{R}^N))} + \|(\theta, \mathbf{u})\|_{L_p((0,T),H^{1,0}_q(\mathbb{R}^N))}\\&\le C\| < t >^b (f, \mathbf{g})\|_{L_p((0,T),H^{1,0}_q(\mathbb{R}^N))}.\end{aligned}$$

Finally, if $b > 1$, the repeated use of the argument above yield estimates (42) for any $b > 0$.

(2) Let $\mathbf{h}_0$ be the zero extension of $\mathbf{h}$ outside of $(0,T)$. By the results concerning the $\mathcal{R}$-bounded solution operators to the heat equations proved in [23], we see that the following two estimates:

$$\|\partial_t \mathbf{k}\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))} + \|\mathbf{k}\|_{L_p(\mathbb{R},H_q^2(\mathbb{R}^N))} \leq C\|\mathbf{h}_0\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))},$$
$$\gamma\|e^{-\gamma t}\mathbf{k}\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))} \leq C\|e^{-\gamma t}\mathbf{h}_0\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))}$$

for any $\gamma > 0$, where C is a constant independent of γ. Thus, employing the same method as in (1), we have (43), which completes the proof of Theorem 3. $\square$

Applying Theorem 3 to (36) and (37), we have

$$\begin{aligned}
&\| < t >^b \partial_t(\theta_1, \mathbf{u}_1)\|_{L_p((0,T),H_q^{1,0}(\mathbb{R}^N))} + \| < t >^b (\theta_1, \mathbf{u}_1)\|_{L_p((0,T),H_q^{1,2}(\mathbb{R}^N))} \\
&\quad + \| < t >^b \partial_t \mathbf{k}_1\|_{L_p((0,T),L_q(\mathbb{R}^N))} + \| < t >^b \mathbf{k}_1\|_{L_p((0,T),H_q^2(\mathbb{R}^N))} \\
&\leq C(\| < t >^b (f(\rho,\mathbf{v}), \mathbf{g}(\rho,\mathbf{v},\mathbf{d}))\|_{L_p((0,T),H_q^{1,0}(\mathbb{R}^N))} \\
&\quad + \| < t >^b \mathbf{h}(\mathbf{v},\mathbf{d})\|_{L_p((0,T),L_q(\mathbb{R}^N))})
\end{aligned} \tag{44}$$

for $q = q_1/2$, q_1, and q_2. In order to estimate the right-hand side of (44), we recall that $\sum_{q=q_1,q_2} \|\rho_0\|_{H_q^1(\mathbb{R}^N)} \leq \epsilon^2$, $\mathcal{N}(\rho,\mathbf{v},\mathbf{d}) \leq \epsilon$, (18), (27), and (32). Then we have

$$\begin{aligned}
&\| < t >^b \partial_t(\theta_1, \mathbf{u}_1)\|_{L_p((0,T),H_q^{1,0}(\mathbb{R}^N))} + \| < t >^b (\theta_1, \mathbf{u}_1)\|_{L_p((0,T),H_q^{1,2}(\mathbb{R}^N))} \\
&\quad + \| < t >^b \partial_t \mathbf{k}_1\|_{L_p((0,T),L_q(\mathbb{R}^N))} + \| < t >^b \mathbf{k}_1\|_{L_p((0,T),H_q^2(\mathbb{R}^N))} \\
&\leq C(\epsilon^2 + \epsilon^3 + \epsilon^4)
\end{aligned} \tag{45}$$

for $q = q_1/2$, q_1, and q_2. Moreover, by the trace method of the real interpolation theorem,

$$\begin{aligned}
&\| < t >^b (\theta_1, \mathbf{u}_1)\|_{L_\infty((0,T),H_q^{1,0}(\mathbb{R}^N))} + \| < t >^b \mathbf{k}_1\|_{L_\infty((0,T),L_q(\mathbb{R}^N))} \\
&\leq C(\| < t >^b \partial_t(\theta_1, \mathbf{u}_1)\|_{L_p((0,T),H_q^{1,0}(\mathbb{R}^N))} + \| < t >^b (\theta_1, \mathbf{u}_1)\|_{L_p((0,T),H_q^{1,2}(\mathbb{R}^N))} \\
&\quad + \| < t >^b \partial_t \mathbf{k}_1\|_{L_p((0,T),L_q(\mathbb{R}^N))} + \| < t >^b \mathbf{k}_1\|_{L_p((0,T),H_q^2(\mathbb{R}^N))}),
\end{aligned}$$

which combined with (45) yields that

$$\| < t >^b (\theta_1, \mathbf{u}_1)\|_{L_\infty((0,T),H_q^{1,0}(\mathbb{R}^N))} + \| < t >^b \mathbf{k}_1\|_{L_\infty((0,T),L_q(\mathbb{R}^N))} \leq C(\epsilon^2 + \epsilon^3 + \epsilon^4) \tag{46}$$

for q_1 and q_2. Summing up, by (45) and (46), we have

$$\begin{aligned}
&\sum_{q=q_1/2,q_1,q_2} (\| < t >^b \partial_t(\theta_1, \mathbf{u}_1)\|_{L_p((0,T),H_q^{1,0}(\mathbb{R}^N))} + \| < t >^b (\theta_1, \mathbf{u}_1)\|_{L_p((0,T),H_q^{1,2}(\mathbb{R}^N))} \\
&\quad + \| < t >^b \partial_t \mathbf{k}_1\|_{L_p((0,T),L_q(\mathbb{R}^N))} + \| < t >^b \mathbf{k}_1\|_{L_p((0,T),H_q^2(\mathbb{R}^N))}) \\
&\quad + \sum_{q=q_1,q_2} \| < t >^b (\theta_1, \mathbf{u}_1, \mathbf{k}_1)\|_{L_\infty((0,T),L_q(\mathbb{R}^N))} \\
&\leq C(\epsilon^2 + \epsilon^3 + \epsilon^4).
\end{aligned} \tag{47}$$

4.1.2. Analysis of Compensation Equations for $(\theta, \mathbf{u})$

Let us consider problem (38). The existence of $\mathcal{R}$-bounded solution operators proved in ([22], Theorem 2.5) implies generation of continuous analytic semigroup $\{S(t)\}_{t\geq 0}$ on $H^{1,0}_p(\mathbb{R}^N)$ associated with the following homogeneous problem:

$$\begin{cases} \partial_t\theta + \rho_* \operatorname{div} \mathbf{u} = 0 & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ \rho_*\partial_t\mathbf{u} - \operatorname{Div}(\mathbf{S}(\mathbf{u}) - P'(\rho_*)\theta\mathbf{I}) = 0 & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ (\theta, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{v}_0) & \text{in } \mathbb{R}^N. \end{cases} \tag{48}$$

Applying Duhamel's principle to (38) furnishes that $(\theta_2, \mathbf{u}_2) = (\theta_2^1, \mathbf{u}_2^1) + (\theta_2^2, \mathbf{u}_2^2)$, where

$$(\theta_2^1, \mathbf{u}_2^1) = S(t)(\rho_0, \mathbf{v}_0), \;\; (\theta_2^2, \mathbf{u}_2^2) = \lambda_0 \int_0^t S(t-s)(\theta_1, \mathbf{u}_1)(\cdot, s)\, ds. \tag{49}$$

To get estimates of $(\theta_2^1, \mathbf{u}_2^1)$ and $(\theta_2^2, \mathbf{u}_2^2)$, we use the decay estimates for $(\theta, \mathbf{u}) = S(t)(f, \mathbf{g})$, which follow from ([20], Theorem 2.3 and 2.4):

$$\begin{aligned} &\|(\theta, \mathbf{u})\|_{L_p(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}[(f, \mathbf{g})]_{p,q}, \\ &\|\nabla(\theta, \mathbf{u})\|_{L_p(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}[(f, \mathbf{g})]_{p,q}, \\ &\|\nabla^2\mathbf{u}\|_{L_p(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-1}[(f, \mathbf{g})]_{p,q} \end{aligned} \tag{50}$$

for $t > 1$ and $1 < q \leq 2 \leq p < \infty$. Here, $[f, \mathbf{g}]_{p,q} = \|(f, \mathbf{g})\|_{H^{1,0}_p(\mathbb{R}^N)} + \|(f, \mathbf{g})\|_{L_q(\mathbb{R}^N)}$. Moreover, we use the following standard estimates for continuous analytic semigroup:

$$\begin{aligned} &\|(\theta, \mathbf{u})\|_{H^{1,2}_p(\mathbb{R}^N)} \leq C\|(f, \mathbf{g})\|_{H^{1,2}_p(\mathbb{R}^N)} \quad \text{for } (f, \mathbf{g}) \in H^{1,2}_p(\mathbb{R}^N), \\ &\|(\theta, \mathbf{u})\|_{H^{1,0}_p(\mathbb{R}^N)} \leq C\|(f, \mathbf{g})\|_{H^{1,0}_p(\mathbb{R}^N)} \quad \text{for } (f, \mathbf{g}) \in H^{1,0}_p(\mathbb{R}^N) \end{aligned} \tag{51}$$

for $0 < t < 2$.

Estimates of $(\theta_2^1, \mathbf{u}_2^1)$

Firstly, we consider the case $1 < t < T$. Using (50) with $(p, q) = (q_1, q_1/2), (q_2, q_1/2)$ and noting that $1/q_1 = 1/q_2 + 1/N$, we have

$$\begin{aligned} &\|(\theta_2^1, \mathbf{u}_2^1)\|_{L_{q_1}(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2q_1}}[(\rho_0, \mathbf{v}_0)]_{q_1,q_1/2}, \\ &\|(\theta_2^1, \mathbf{u}_2^1)\|_{L_{q_2}(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2}(\frac{2}{q_1}-\frac{1}{q_2})}[(\rho_0, \mathbf{v}_0)]_{q_2,q_1/2} = t^{-\frac{N}{2q_1}-\frac{1}{2}}[(\rho_0, \mathbf{v}_0)]_{q_2,q_1/2}, \\ &\|\nabla(\theta_2^1, \mathbf{u}_2^1)\|_{L_{q_1}(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2q_1}-\frac{1}{2}}[(\rho_0, \mathbf{v}_0)]_{q_1,q_1/2}, \\ &\|\nabla(\theta_2^1, \mathbf{u}_2^1)\|_{L_{q_2}(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2}(\frac{2}{q_1}-\frac{1}{q_2})-\frac{1}{2}}[(\rho_0, \mathbf{v}_0)]_{q_2,q_1/2} = t^{-\frac{N}{2q_1}-1}[(\rho_0, \mathbf{v}_0)]_{q_2,q_1/2}, \\ &\|\nabla^2\mathbf{u}_2^1\|_{L_{q_1}(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2q_1}-1}[(\rho_0, \mathbf{v}_0)]_{q_1,q_1/2}, \\ &\|\nabla^2\mathbf{u}_2^1\|_{L_{q_2}(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2}(\frac{2}{q_1}-\frac{1}{q_2})-1}[(\rho_0, \mathbf{v}_0)]_{q_2,q_1/2} = t^{-\frac{N}{2q_1}-\frac{3}{2}}[(\rho_0, \mathbf{v}_0)]_{q_2,q_1/2}. \end{aligned} \tag{52}$$

Noting that all decay rates obtained in (52) except for $\|(\theta_2^1, \mathbf{u}_2^1)\|_{L_{q_1}(\mathbb{R}^N)}$ is greater or equal to $N/2q_1 + 1/2$ and using the condition $1 < (N/2q_1 + 1/2 - b)p$ in (10), we have

$$\begin{aligned} &\| < t >^b \nabla(\theta_2^1, \mathbf{u}_2^1)\|_{L_p((1,T),H^{0,1}_{q_1}(\mathbb{R}^N))} + \| < t >^b (\theta_2^1, \mathbf{u}_2^1)\|_{L_p((1,T),H^{1,2}_{q_2}(\mathbb{R}^N))} \\ &+ \;\; \| < t >^{N/(2q_1)} (\theta, \mathbf{u})\|_{L_\infty((1,T),L_{q_1}(\mathbb{R}^N))} + \| < t >^b (\theta, \mathbf{u})\|_{L_\infty((1,T),L_{q_2}(\mathbb{R}^N))} \\ &\leq C \sum_{q=q_1,q_2} [(\rho_0, \mathbf{v}_0)]_{q,q_1/2} \leq C\mathcal{I}, \end{aligned} \tag{53}$$

where $\mathcal{I}$ is defined in (11).

Secondly, we consider the case $0 < t < \min(1, T)$. Since $(\theta_2^1, \mathbf{u}_2^1)$ satisfies (48), we infer from ([24], Theorem 2.7) that

$$\begin{aligned}&\sum_{q=q_1,q_2} \| < t >^b (\theta_2^1, \mathbf{u}_2^1)\|_{L_p((0,1),H_q^{1,2}(\mathbb{R}^N))}\\ &\leq C \sum_{q=q_1,q_2} \|(\rho_0, \mathbf{v}_0)\|_{H_q^1(\mathbb{R}^N)\times B_{q,p}^{2(1-1/p)}(\mathbb{R}^N)}\\ &\leq C\mathcal{I}.\end{aligned} \tag{54}$$

Moreover, by (51), we have

$$\begin{aligned}&\| < t >^{\frac{N}{2q_1}} (\theta_2^1, \mathbf{u}_2^1)\|_{L_\infty((0,1),L_{q_1}(\mathbb{R}^N))} + \| < t >^b (\theta_2^1, \mathbf{u}_2^1)\|_{L_\infty((0,1),L_{q_2}(\mathbb{R}^N))}\\ &\leq C \sum_{q=q_1,q_2} \|(\rho_0, \mathbf{v}_0)\|_{H_q^{1,0}(\mathbb{R}^N)}\\ &\leq C\mathcal{I}.\end{aligned} \tag{55}$$

Then, (53), (54), and (55) give us

$$\begin{aligned}&\| < t >^b \nabla(\theta_2^1, \mathbf{u}_2^1)\|_{L_p((0,T),H_{q_1}^{0,1}(\mathbb{R}^N))} + \| < t >^b (\theta_2^1, \mathbf{u}_2^1)\|_{L_p((0,T),H_{q_2}^{1,2}(\mathbb{R}^N))}\\ &\quad + \| < t >^{\frac{N}{2q_1}} (\theta_2^1, \mathbf{u}_2^1)\|_{L_\infty((0,T),L_{q_1}(\mathbb{R}^N))} + \| < t >^b (\theta_2^1, \mathbf{u}_2^1)\|_{L_\infty((0,T),L_{q_2}(\mathbb{R}^N))}\\ &\leq C\mathcal{I}.\end{aligned} \tag{56}$$

Furthermore, we can obtain estimates of time derivatives $\partial_t(\theta_2^1, \mathbf{u}_2^1)$ by using equations of $(\theta_2^1, \mathbf{u}_2^1)$. In fact, by (48) and (56), we have

$$\begin{aligned}&\sum_{q=q_1,q_2} \| < t >^b \partial_t(\theta_2^1, \mathbf{u}_2^1)\|_{L_p((0,T),H_q^{1,0}(\mathbb{R}^N))}\\ &\leq C \sum_{q=q_1,q_2} (\| < t >^b \nabla\theta_2^1\|_{L_p((0,T),L_q(\mathbb{R}^N))} + \| < t >^b \nabla\mathbf{u}_2^1\|_{L_p((0,T),H_q^1(\mathbb{R}^N))})\\ &\leq C\mathcal{I}.\end{aligned} \tag{57}$$

<u>Estimates of $(\theta_2^2, \mathbf{u}_2^2)$</u>

Let

$$[[(\theta_1, \mathbf{u}_1)(\cdot, s)]] = \|(\theta_1, \mathbf{u}_1)(\cdot, s)\|_{L_{q_1/2}(\mathbb{R}^N)} + \sum_{q=q_1,q_2} \|(\theta_1, \mathbf{u}_1)(\cdot, s)\|_{H_q^{1,2}(\mathbb{R}^N)}.$$

Setting

$$\tilde{\mathcal{N}}(\theta_1, \mathbf{u}_1)(T) = \left(\int_0^T (< t >^b [[(\theta_1, \mathbf{u}_1)(\cdot, t)]])^p \, dt\right)^{1/p}$$

and using (47), we have

$$\tilde{\mathcal{N}}(\theta_1, \mathbf{u}_1)(T) \leq C(\epsilon^2 + \epsilon^3 + \epsilon^4). \tag{58}$$

In what follows, we estimate $(\theta_2^2, \mathbf{u}_2^2)$ with the help of $\tilde{\mathcal{N}}(\theta_1, \mathbf{u}_1)$. Note that

$$(\theta_2^2, \mathbf{u}_2^2) = \lambda_0 \int_0^t S(t-s)(\theta_1, \mathbf{u}_1)(\cdot, s) \, ds \tag{59}$$

satisfies the linearized problem:

$$\begin{cases}\partial_t\theta_2^2+\rho_*\operatorname{div}\mathbf{u}_2=\lambda_0\theta_1 & \text{in } \mathbb{R}^N \text{ for } t\in(0,T),\\ \rho_*\partial_t\mathbf{u}_2^2-\operatorname{Div}(\mu\mathbf{D}(\mathbf{u}_2^2)+(\nu-\mu)\operatorname{div}\mathbf{u}_2^2\mathbf{I}-P'(\rho_*)\theta_2^2\mathbf{I}) & \\ \quad=\lambda_0\mathbf{u}_1 & \text{in } \mathbb{R}^N \text{ for } t\in(0,T),\\ (\theta_2^2,\mathbf{u}_2^2)|_{t=0}=(0,0) & \text{in } \mathbb{R}^N.\end{cases} \tag{60}$$

Firstly, we consider the decay estimates of spatial derivatives of $(\theta_2^2,\mathbf{u}_2^2)$. Set $(\theta_3,\mathbf{u}_3)=(\nabla\theta_2^2,\bar\nabla^1\nabla\mathbf{u}_2^2)$ when $q=q_1$ and $(\theta_3,\mathbf{u}_3)=(\bar\nabla^1\theta_2^2,\bar\nabla^2\mathbf{u}_2^2)$ when $q=q_2$. Here, $\bar\nabla^m f=(\partial_x^\alpha f\mid |\alpha|\le m)$. Let us consider the case $2\le t\le T$. In this case, we decompose

$$\begin{aligned}&\|(\theta_3,\mathbf{u}_3)(\cdot,t)\|_{L_q(\mathbb{R}^N)}\\ &\le C\Big(\int_0^{t/2}+\int_{t/2}^{t-1}+\int_{t-1}^t\Big)\|(\nabla,\bar\nabla^1\nabla)\text{ or }(\bar\nabla^1,\bar\nabla^2)S(t-s)(\lambda_0\theta_1,\lambda_0\mathbf{u}_1)(\cdot,s)\|_{L_q(\mathbb{R}^N)}\,ds\\ &=: I_q(t)+II_q(t)+III_q(t).\end{aligned}$$

We shall consider estimates of $I_q(t)$, $II_q(t)$, and $III_q(t)$ by (50). Setting $\ell=N/2q_1+1/2$, we see that all the decay rates used below are greater than or equal to ℓ. In fact, by (10) and (50) with $(p,q)=(q_1,q_1/2),(q_2,q_1/2)$, we have the following decay rates:

$$\begin{aligned}&\frac{N}{2}\Big(\frac{2}{q_1}-\frac{1}{q_1}\Big)+\frac{j}{2}=\frac{N}{2q_1}+\frac{j}{2}\ge\ell\ \ (j=1,2),\\ &\frac{N}{2}\Big(\frac{2}{q_1}-\frac{1}{q_2}\Big)+\frac{j}{2}=\frac{N}{2}\Big(\frac{2}{q_1}-\frac{1}{q_1}+\frac{1}{N}\Big)+\frac{j}{2}=\frac{N}{2q_1}+\frac{1}{2}+\frac{j}{2}\ge\ell\ \ (j=0,1,2).\end{aligned}$$

Using (50) with $(p,q)=(q,q_1/2)$ and Hölder's inequality, we have

$$\begin{aligned}I_q(t)&\le C\int_0^{t/2}(t-s)^{-\ell}[[(\theta_1,\mathbf{u}_1)(\cdot,s)]]\,ds\\ &\le C(t/2)^{-\ell}\Big(\int_0^\infty <s>^{-p'b}\,ds\Big)^{1/p'}\Big(\int_0^T\Big(<s>^b[[(\theta_1,\mathbf{u}_1)(\cdot,s)]]\Big)^p ds\Big)^{1/p}\\ &\le Ct^{-\ell}\tilde{\mathcal{N}}(\theta_1,\mathbf{u}_1)(T).\end{aligned}$$

Since $1-(\ell-b)p<0$, we have

$$\begin{aligned}\int_2^T\Big(<t>^b I_q(t)\Big)^p dt&\le C\int_2^T<t>^{-(\ell-b)p}\,dt\,\tilde{\mathcal{N}}(\theta_1,\mathbf{u}_1)(T)^p\\ &\le C\tilde{\mathcal{N}}(\theta_1,\mathbf{u}_1)(T)^p.\end{aligned}\tag{61}$$

Using (50) with $(p,q)=(q,q_1/2)$, $<t>^b\le C<s>^b$ for $t/2<s<t-1$, and Hölder's inequality, we have

$$\begin{aligned}&<t>^b II_q(t)\\ &\le C\Big(\int_{t/2}^{t-1}(t-s)^{-\ell}\,ds\Big)^{1/p'}\Big(\int_{t/2}^{t-1}(t-s)^{-\ell}\Big(<s>^b[[(\theta_1,\mathbf{u}_1)(\cdot,s)]]\Big)^p ds\Big)^{1/p}.\end{aligned}$$

By Fubini's theorem, we have

$$\begin{aligned}\int_2^T\Big(<t>^b II_q(t)\Big)^p dt&\le C\int_1^T\int_{s+1}^{2s}(t-s)^{-\ell}\,dt\Big(<s>^b[[(\theta_1,\mathbf{u}_1)(\cdot,s)]]\Big)^p ds\\ &\le C\tilde{\mathcal{N}}(\theta_1,\mathbf{u}_1)(T)^p.\end{aligned}\tag{62}$$

By (51), we have

$$III_q(t) \leq C \int_{t-1}^{t} \|(\theta_1, \mathbf{u}_1)(\cdot, s)\|_{H_q^{1,2}(\mathbb{R}^N)}\, ds \leq C \int_{t-1}^{t} [[(\theta_1, \mathbf{u}_1)(\cdot, s)]]\, ds.$$

Employing the same method as the estimate of $II_q(t)$, we have

$$\int_2^T \Big(< t >^b III_q(t) \Big)^p dt \leq C\tilde{\mathcal{N}}(\theta_1, \mathbf{u}_1)(T)^p. \tag{63}$$

Combining (61), (62), and (63), we have

$$\int_2^T \Big(< t >^b \|(\theta_3, \mathbf{u}_3)(\cdot, t)\|_{L_q(\mathbb{R}^N)} \Big)^p dt \leq C\tilde{\mathcal{N}}(\theta_1, \mathbf{u}_1)(T)^p. \tag{64}$$

In the case that $0 < t < \min(2, T)$, by the same method as the estimate of $III_q(t)$, we have

$$\int_0^{\min(2,T)} \Big(< t >^b \|(\theta_3, \mathbf{u}_3)(\cdot, t)\|_{L_q(\mathbb{R}^N)} \Big)^p dt \leq C\tilde{\mathcal{N}}(\theta_1, \mathbf{u}_1)(T)^p,$$

which combined (64), we have

$$\int_0^T \Big(< t >^b \|(\theta_3, \mathbf{u}_3)(\cdot, t)\|_{L_q(\mathbb{R}^N)} \Big)^p dt \leq C\tilde{\mathcal{N}}(\theta_1, \mathbf{u}_1)(T)^p,$$

namely,

$$\begin{aligned} &\| < t >^b \nabla(\theta_2^2, \mathbf{u}_2^2)\|_{L_p((0,T),L_{q_1}(\mathbb{R}^N))} + \| < t >^b \nabla^2 \mathbf{u}_2^2\|_{L_p((0,T),L_{q_1}(\mathbb{R}^N))} \\ &\quad + \| < t >^b (\theta_2^2, \mathbf{u}_2^2)\|_{L_p((0,T),H_{q_2}^{1,2}(\mathbb{R}^N))} \\ &\leq C\tilde{\mathcal{N}}(\theta_1, \mathbf{u}_1)(T). \end{aligned} \tag{65}$$

In the same way as estimates of $(\theta_2^1, \mathbf{u}_2^1)$, using Equations (60), we can obtain estimates of time derivatives $\partial_t(\theta_2^2, \mathbf{u}_2^2)$ as follows:

$$\sum_{q=q_1,q_2} \| < t >^b \partial_t(\theta_2^2, \mathbf{u}_2^2)\|_{L_p((0,T),H_q^{1,0}(\mathbb{R}^N))} \leq C\tilde{\mathcal{N}}(\theta_1, \mathbf{u}_1)(T). \tag{66}$$

Secondly, we consider estimates of $(\theta_2^2, \mathbf{u}_2^2)$ in L_∞ in time L_q in space setting for $q = q_1$ and q_2. Since we can obtain the case $q = q_2$ by the similar calculation as the case $q = q_1$, we only verify the case $q = q_1$, namely, we consider the estimate of $\| < t >^{N/(2q_1)} (\theta_2^2, \mathbf{u}_2^2)\|_{L_\infty((0,T),L_{q_1}(\mathbb{R}^N))}$. In the case that $2 < t < T$, we divide three parts as follows:

$$\begin{aligned} &\|(\theta_2^2, \mathbf{u}_2^2)(\cdot, t)\|_{L_{q_1}(\mathbb{R}^N)} \\ &\quad \leq C\Big(\int_0^{t/2} + \int_{t/2}^{t-1} + \int_{t-1}^{t} \Big) \|S(t-s)(\lambda_0\theta_1, \lambda_0\mathbf{u}_1)(\cdot, s)\|_{L_{q_1}(\mathbb{R}^N)}\, ds \\ &\quad =: I_{q_1,0}(t) + II_{q_1,0}(t) + III_{q_1,0}(t). \end{aligned}$$

Using (50) with $(p,q) = (q_1, q_1/2)$ and noting that $1 - bp' < 0$ and $t - s << s >$ if $t/2 < s < t-1$, we have estimates of $I_{q_1,0}(t)$ and $II_{q_1,0}(t)$ as follows:

$$\begin{aligned}
I_{q_1,0}(t) &\leq C\int_0^{t/2} (t-s)^{-N/(2q_1)} \|(\theta_1, \mathbf{u}_1)(\cdot,s)\|_{L_{q_1/2}(\mathbb{R}^N)}\,ds \\
&\leq C(t/2)^{-N/(2q_1)} \left(\int_0^{t/2} < s >^{-bp'}\,ds\right)^{1/p'} \\
&\quad\times \left(\int_0^{t/2} \left(< s >^b \|(\theta_1, \mathbf{u}_1)(\cdot,s)\|_{L_{q_1/2}(\mathbb{R}^N)}\right)^p ds\right)^{1/p} \\
&\leq Ct^{-N/(2q_1)}\tilde{\mathcal{N}}(\theta_1, \mathbf{u}_1)(T).
\end{aligned} \tag{67}$$

$$\begin{aligned}
II_{q_1,0}(t) &\leq C\int_{t/2}^{t-1} (t-s)^{-N/(2q_1)} \|(\theta_1, \mathbf{u}_1)(\cdot,s)\|_{L_{q_1/2}(\mathbb{R}^N)}\,ds \\
&\leq C\left(\int_{t/2}^{t-1} \left((t-s)^{-N/(2q_1)} < s >^{-b}\right)^{p'} ds\right)^{1/p'} \\
&\quad\times \left(\int_{t/2}^{t-1} \left(< s >^b \|(\theta_1, \mathbf{u}_1)(\cdot,s)\|_{L_{q_1/2}(\mathbb{R}^N)}\right)^p ds\right)^{1/p} \\
&\leq C < t/2 >^{-N/(2q_1)} \left(\int_{t/2}^{t-1} \left((t-s)^{-N/(2q_1)} < s >^{N/(2q_1)-b}\right)^{p'} ds\right)^{1/p'} \tilde{\mathcal{N}}(\theta_1, \mathbf{u}_1)(T) \\
&\leq C < t >^{-N/(2q_1)} \left(\int_{t/2}^{t-1} (t-s)^{-bp'}\,ds\right)^{1/p'} \tilde{\mathcal{N}}(\theta_1, \mathbf{u}_1)(T). \\
&\leq C < t >^{-N/(2q_1)} \tilde{\mathcal{N}}(\theta_1, \mathbf{u}_1)(T).
\end{aligned} \tag{68}$$

Using (51) and noting that $N/(2q_1) < b$ and $< t > \leq C < s >$ if $1 < t-1 < s < t$, we have

$$\begin{aligned}
III_{q_1,0}(t) &\leq C\int_{t-1}^{t} \|(\theta_1, \mathbf{u}_1)(\cdot,s)\|_{H^{1,2}_{q_1}(\mathbb{R}^N)}\,ds \\
&\leq C < t >^{-N/(2q_1)} \left(\int_{t-1}^{t} < s >^{-(b-N/(2q_1))p'}\,ds\right)^{1/p'} \\
&\quad\times \left(\int_{t-1}^{t} \left(< s >^b \|(\theta_1, \mathbf{u}_1)(\cdot,s)\|_{H^{1,2}_{q_1}(\mathbb{R}^N)}\right)^p ds\right)^{1/p} \\
&\leq C < t >^{-N/(2q_1)} \left(\int_{t-1}^{t} ds\right)^{1/p'} \tilde{\mathcal{N}}(\theta_1, \mathbf{u}_1)(T) \\
&\leq C < t >^{-N/(2q_1)} \tilde{\mathcal{N}}(\theta_1, \mathbf{u}_1)(T).
\end{aligned} \tag{69}$$

In the case that $0 < t < \min(2,T)$, by (51), we have

$$\| < t >^{N/(2q_1)} (\theta_2^2, \mathbf{u}_2^2)\|_{L_\infty((0,\min(2,T)),L_{q_1}(\mathbb{R}^N))} \leq C\| < t >^b (\theta_1, \mathbf{u}_1)\|_{L_\infty((0,T),H^{1,0}_{q_1}(\mathbb{R}^N))},$$

which combined with (67), (68), and (69) yields that

$$\begin{aligned}
&\| < t >^{N/(2q_1)} (\theta_2^2, \mathbf{u}_2^2)\|_{L_\infty((0,T),L_{q_1})} \\
&\leq C(\| < t >^b (\theta_1, \mathbf{u}_1)\|_{L_\infty((0,T),H^{1,0}_{q_1}(\mathbb{R}^N))} + \tilde{\mathcal{N}}(\theta_1, \mathbf{u}_1)(T)).
\end{aligned} \tag{70}$$

Similarly, we have

$$\begin{aligned}
&\| < t >^{b} (\theta_2^2, \mathbf{u}_2^2)\|_{L_\infty((0,T),L_{q_2})} \\
&\leq C(\| < t >^b (\theta_1, \mathbf{u}_1)\|_{L_\infty((0,T),H^{1,0}_{q_2}(\mathbb{R}^N))} + \tilde{\mathcal{N}}(\theta_1, \mathbf{u}_1)(T)).
\end{aligned} \tag{71}$$

4.1.3. Analysis of Compensation Equations for $\mathbf{k}$

Let $\{T(t)\}_{t\geq 0}$ be continuous analytic semigroup on $L_p(\mathbb{R}^N)$ associated with the heat equations:

$$\partial_t \mathbf{d} - \zeta\Delta\mathbf{d} = 0 \text{ in } \mathbb{R}^N \text{ for } t > 0, \ \ \mathbf{d}|_{t=0} = \mathbf{f} \text{ in } \mathbb{R}^N. \tag{72}$$

Recall that $\{T(t)\}_{t\geq 0}$ has the following L_p-L_q decay estimates:

$$\|\partial_t^k \nabla^j T(t)\mathbf{f}\|_{L_p(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{j}{2}-k}\|\mathbf{f}\|_{L_q(\mathbb{R}^N)} \tag{73}$$

for $t > 0$, $1 \leq q \leq p \leq \infty$, k and j are non-negative integers. Moreover, $\{T(t)\}_{t\geq 0}$ has the following standard estimate for continuous analytic semigroup:

$$\begin{aligned} &\|T(t)\mathbf{f}\|_{H^2_p(\mathbb{R}^N)} \leq C\|\mathbf{f}\|_{H^2_p(\mathbb{R}^N)} && \text{for } \mathbf{f} \in H^2_p(\mathbb{R}^N), \\ &\|T(t)\mathbf{f}\|_{L_p(\mathbb{R}^N)} \leq C\|\mathbf{f}\|_{L_p(\mathbb{R}^N)} && \text{for } \mathbf{f} \in L_p(\mathbb{R}^N) \end{aligned} \tag{74}$$

for $0 < t < 2$. Now we consider (39). By Duhamel's principle, we write $\mathbf{k}_2$ as $\mathbf{k}_2 = \mathbf{k}_2^1 + \mathbf{k}_2^2$, where

$$\mathbf{k}_2^1 = T(t)\mathbf{d}_0, \ \ \mathbf{k}_2^2 = \lambda_0\int_0^t T(t-s)\mathbf{k}_1(\cdot, s)\,ds. \tag{75}$$

Firstly, we consider estimates of $\mathbf{k}_2^1$. Using (73) with $(p,q) = (q_1, q_1/2), (q_2, q_1/2)$ if $1 < t < T$, (74) and the maximal L_p-L_q regularity if $0 < t < \min(1,T)$, namely, we use the fact that if $\mathbf{d}$ satisfies (72), the following estimate holds:

$$\|\partial_t\mathbf{d}\|_{L_p((0,\min(1,T)),L_q(\mathbb{R}^N))} + \|\mathbf{d}\|_{L_p((0,\min(1,T)),H^2_q(\mathbb{R}^N))} \leq C\|\mathbf{f}\|_{B^{2(1-1/p)}_{q,p}}$$

with constants C (cf. ([19], Theorem 2.2(2))), then we have

$$\begin{aligned} &\| < t >^b \nabla\mathbf{k}_2^1\|_{L_p((0,T),H^1_{q_1}(\mathbb{R}^N))} + \| < t >^b \mathbf{k}_2^1\|_{L_p((0,T),H^2_{q_2}(\mathbb{R}^N))} \\ &+ \| < t >^{\frac{N}{2q_1}} \mathbf{k}_2^1\|_{L_\infty((0,T),L_{q_1}(\mathbb{R}^N))} + \| < t >^b \mathbf{k}_2^1\|_{L_\infty((0,T),L_{q_2}(\mathbb{R}^N))} \\ &\leq C\sum_{q=q_1,q_2}(\|\mathbf{d}_0\|_{L_{q_1/2}(\mathbb{R}^N)} + \|\mathbf{d}_0\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)}) \leq C\mathcal{I}. \end{aligned} \tag{76}$$

Secondly, we consider estimates of $\mathbf{k}_2^2$. Let

$$[[\mathbf{k}_1(\cdot,s)]] = \|\mathbf{k}_1(\cdot,s)\|_{L_{q_1/2}(\mathbb{R}^N)} + \sum_{q=q_1,q_2}\|\mathbf{k}_1(\cdot,s)\|_{H^2_q(\mathbb{R}^N)}.$$

Setting

$$\tilde{\mathcal{N}}(\mathbf{k}_1)(T) = \left(\int_0^T (< t >^b [[\mathbf{k}_1(\cdot,t)]])^p\,dt\right)^{1/p}$$

and using (47), we have

$$\tilde{\mathcal{N}}(\mathbf{k}_1)(T) \leq C(\epsilon^2 + \epsilon^3 + \epsilon^4). \tag{77}$$

Employing the same calculation as estimates of $(\theta_2^2, \mathbf{u}_2^2)$ and using (73) and (74), we have

$$\begin{aligned} &\| < t >^b \nabla\mathbf{k}_2^2\|_{L_p((0,T),H^1_{q_1}(\mathbb{R}^N))} + \| < t >^b \mathbf{k}_2^2\|_{L_p((0,T),H^2_{q_2}(\mathbb{R}^N))} \\ &+ \ \ \| < t >^{N/(2q_1)} \mathbf{k}_2^2\|_{L_\infty((0,T),L_{q_1})} + \| < t >^b \mathbf{k}_2^2\|_{L_\infty((0,T),L_{q_1})} \\ &\leq C\sum_{q=q_1,q_2}(\| < t >^b \mathbf{k}_1\|_{L_\infty((0,T),L_q(\mathbb{R}^N))} + \tilde{\mathcal{N}}(\mathbf{k}_1)(T)). \end{aligned} \tag{78}$$

Summing up, by (11), (56), (57), (58), (65), (66), (70), (71), (76), (77), and (78), we have

$$\mathcal{N}_1(\theta_2, \mathbf{u}_2, \mathbf{k}_2)(T) \leq C(\epsilon^2 + \epsilon^2 + \epsilon^4),$$

which combined (47) yields that

$$\mathcal{N}_1(\theta, \mathbf{u}, \mathbf{k})(T) \le C(\epsilon^2 + \epsilon^2 + \epsilon^4). \tag{79}$$

4.2. Estimates of $\mathcal{N}_2(\mathbf{k})(T)$

Recall that $\mathbf{k}$ is a solution to the equations:

$$\begin{cases} \partial_t \mathbf{k} - \zeta \Delta \mathbf{k} = \mathbf{h}(\mathbf{v}, \mathbf{d}) & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\ \mathbf{k}|_{t=0} = \mathbf{d}_0 & \text{in } \mathbb{R}^N \end{cases} \tag{80}$$

for given $(\mathbf{v}, \mathbf{d}) \in X_T^2 \times X_T^3$. In this subsection, we prove

$$\mathcal{N}_2(\mathbf{k})(T) = \| < t >^b \nabla \mathbf{k} \|_{L_\infty((0,T), L_{q_1}(\mathbb{R}^N))} + \| < t >^b \mathbf{k} \|_{L_\infty((0,T), H^1_\infty(\mathbb{R}^N))} \le C(\epsilon^2 + \epsilon^3).$$

Firstly, we consider the case $2 \le t \le T$ by using L_p-L_q decay estimates for the heat semigroup (73). By Duhamel's principle, we write $\mathbf{k}$ as

$$\mathbf{k} = T(t)\mathbf{d}_0 + \int_0^t T(t-s)\mathbf{h}(\mathbf{v}, \mathbf{d})(\cdot, s)\, ds. \tag{81}$$

Since $\nabla T(t)\mathbf{d}_0$ and $\bar{\nabla}^1 T(t)\mathbf{d}_0$ can be estimated by (73) with $(p, q) = (q_1, q_1/2)$ and $(p, q) = (\infty, q_1/2)$, respectively, we only consider the second term of (81) below. Set $\tilde{\mathbf{k}} = \int_0^t T(t-s)\mathbf{h}(\mathbf{v}, \mathbf{d})(\cdot, s)\, ds$. Let $\mathbf{k}_3 = \nabla \tilde{\mathbf{k}}$ in $L_{q_1}(\mathbb{R}^N)$ and $\mathbf{k}_3 = \bar{\nabla}^1 \tilde{\mathbf{k}}$ in $L_\infty(\mathbb{R}^N)$. To estimate $\mathbf{k}_3$, we divide three parts as follows:

$$\begin{aligned} &\|\mathbf{k}_3(\cdot, t)\|_{L_q(\mathbb{R}^N)} \\ &\le C\left(\int_0^{t/2} + \int_{t/2}^{t-1} + \int_{t-1}^t\right) \|(\nabla \text{ or } \bar{\nabla}^1) T(t-s)\mathbf{h}(\mathbf{v}, \mathbf{d})(\cdot, s)\|_{L_q(\mathbb{R}^N)}\, ds \\ &=: I_{q,0}(t) + II_{q,0}(t) + III_{q,0}(t) \end{aligned}$$

for q_1 and ∞. Employing the similar calculation as (67) and (68), we have

$$\begin{aligned} I_{q,0}(t) &\le Ct^{-b} \| < t >^b \mathbf{h}(\mathbf{v}, \mathbf{d})\|_{L_p((0,T), L_{q_1/2}(\mathbb{R}^N))}, \\ II_{q,0}(t) &\le C < t >^{-b} \| < t >^b \mathbf{h}(\mathbf{v}, \mathbf{d})\|_{L_p((0,T), L_{q_1/2}(\mathbb{R}^N))}. \end{aligned} \tag{82}$$

Using (73) with $(p, q) = (\infty, q_2)$ and noting that $1 - \left(\frac{N}{2q_2} + \frac{1}{2}\right)p' > 0$ provided by $p > 2$ and $< t >^b \le C < s >^b$ for $t - 1 < s < t$, we have

$$\begin{aligned} III_{\infty,0}(t) &\le C \int_{t-1}^t (t-s)^{-(N/2q_2 + 1/2)} \|\mathbf{h}(\mathbf{v}, \mathbf{d})(\cdot, s)\|_{L_{q_2}(\mathbb{R}^N)}\, ds \\ &= C \int_{t-1}^t (t-s)^{-(N/2q_2 + 1/2)} < s >^{-b} < s >^b \|\mathbf{h}(\mathbf{v}, \mathbf{d})(\cdot, s)\|_{L_{q_2}(\mathbb{R}^N)}\, ds \\ &\le C < t >^{-b} \left(\int_{t-1}^t (t-s)^{-(N/2q_2 + 1/2)p'}\, ds\right)^{1/p'} \| < t >^b \mathbf{h}(\mathbf{v}, \mathbf{d})\|_{L_p((0,T), L_{q_2}(\mathbb{R}^N))} \\ &\le C < t >^{-b} \| < t >^b \mathbf{h}(\mathbf{v}, \mathbf{d})\|_{L_p((0,T), L_{q_2}(\mathbb{R}^N))}. \end{aligned} \tag{83}$$

Using (73) with $(p, q) = (q_1, q_1)$ and noting that $1 - p'/2 > 0$ provided by $p > 2$, we have

$$\begin{aligned} III_{q_1,0}(t) &\leq C \int_{t-1}^{t} (t-s)^{-1/2} \|\mathbf{h}(\mathbf{v}, \mathbf{d})(\cdot, s)\|_{L_{q_1}(\mathbb{R}^N)}\, ds \\ &\leq C < t >^{-b} \left(\int_{t-1}^{t} (t-s)^{-p'/2}\, ds \right)^{1/p'} \| < t >^{b} \mathbf{h}(\mathbf{v}, \mathbf{d})\|_{L_p((0,T), L_{q_1}(\mathbb{R}^N))} \\ &\leq C < t >^{-b} \| < t >^{b} \mathbf{h}(\mathbf{v}, \mathbf{d})\|_{L_p((0,T), L_{q_1}(\mathbb{R}^N))}. \end{aligned} \tag{84}$$

Combining (32), (82), (83), and (84), we have

$$\sup_{2<t<T} < t >^{b} \|\mathbf{k}_3(\cdot, t)\|_{L_q(\mathbb{R}^N)} \leq C(\epsilon^2 + \epsilon^3) \tag{85}$$

for $q = q_1$ and ∞.

Secondly, we consider the case $0 < t < \min(2, T)$ by using the following lemma proved in ([11], Lemma 1).

Lemma 1. *Let $u \in H^1_p((0, T), L_q(\mathbb{R}^N)) \cap L_p((0, T), H^2_q(\mathbb{R}^N))$ with $1 < p, q < \infty$ and $T > 0$. Then,*

$$\begin{aligned} &\sup_{t \in (0,T)} \|u(\cdot, t)\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)} \\ &\qquad \leq C(\|u(\cdot, 0)\|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)} + \|u\|_{L_p((0,T), H^2_q(\mathbb{R}^N))} + \|\partial_t u\|_{L_p((0,T), L_q(\mathbb{R}^N))}), \end{aligned}$$

where C is a constant independent of T.

Since $2(1 - 1/p) > 1$ as follows from $p > 2$, we have $B^{2(1-1/p)}_{q,p}(\mathbb{R}^N) \subset H^1_q(\mathbb{R}^N)$, so that by Lemma 1, the maximal L_p-L_q regularity with finite times interval for the heat equations proved in ([19], Theorem 2.2(2)), and (30) with $b = 0$, we have

$$\begin{aligned} &\sup_{0<t<\min(2,T)} < t >^{b} \|\mathbf{k}(\cdot, t)\|_{H^1_{q_1}(\mathbb{R}^N)} \\ &\leq C(\|\mathbf{d}_0\|_{B^{2(1-1/p)}_{q_1,p}(\mathbb{R}^N)} + \|\mathbf{k}\|_{L_p((0,\min(2,T)), H^2_{q_1}(\mathbb{R}^N))} + \|\partial_t \mathbf{k}\|_{L_p((0,\min(2,T)), L_{q_1}(\mathbb{R}^N))}) \\ &\leq C(\|\mathbf{d}_0\|_{B^{2(1-1/p)}_{q_1,p}(\mathbb{R}^N)} + \|\mathbf{h}(\mathbf{v}, \mathbf{d})\|_{L_p((0,\min(2,T)), L_{q_1}(\mathbb{R}^N))}) \\ &\leq C(\epsilon^2 + \epsilon^3). \end{aligned} \tag{86}$$

Moreover, since we can choose a small number δ such that $N/q_2 + 1 + \delta < 2(1 - 1/p)$ provided by $(N/2q_2 + 1/2)p' < 1$, by Sobolev' embedding theorem we have

$$\|\mathbf{k}(\cdot, t)\|_{H^1_\infty(\mathbb{R}^N)} \leq C\|\mathbf{k}(\cdot, t)\|_{H^{N/q_2+1+\delta}_{q_2}(\mathbb{R}^N)} \leq C\|\mathbf{k}(\cdot, t)\|_{B^{2(1-1/p)}_{q_2,p}(\mathbb{R}^N)}. \tag{87}$$

Then employing the same calculation as (86) yields that

$$\sup_{0<t<\min(2,T)} < t >^{b} \|\mathbf{k}(\cdot, t)\|_{H^1_\infty(\mathbb{R}^N)} \leq C(\epsilon^2 + \epsilon^3). \tag{88}$$

Summing up, by (85), (86), and (88), we have

$$\mathcal{N}_2(\mathbf{k})(T) \leq C(\epsilon^2 + \epsilon^3),$$

which combined with (79) yields that (35).

5. A Proof of Theorem 2

In this section, we prove Theorem 2. By (35), choosing $\epsilon > 0$ small that $C(\epsilon + \epsilon^2 + \epsilon^3) < 1$, we have $\mathcal{N}(\theta, \mathbf{u}, \mathbf{k})(T) \le \epsilon$. In particular, $\| < t >^{N/2q_1} (\theta, \mathbf{u}, \mathbf{k})\|_{L_\infty((0,T), L_{q_1}(\mathbb{R}^N))} \le \epsilon$ implies that $(\theta, \mathbf{u}, \mathbf{k}) \in L_p((0,T), L_{q_1}(\mathbb{R}^N))$ by $q_1 < N$ and $p > 2$. Moreover, by $\theta = \rho_0 + \int_0^t \partial_s \theta \, ds$, Sobolev's inequality: $\|f\|_{L_\infty(\mathbb{R}^N)} \le C\|f\|_{H^1_{q_2}(\mathbb{R}^N)}$ $(q_2 > N)$, Hölder inequality, and the condition $bp' > 1$, we have

$$
\begin{aligned}
&\|\theta\|_{L_\infty((0,T), L_\infty(\mathbb{R}^N))} \\
&\le C\Big(\|\rho_0\|_{H^1_{q_2}(\mathbb{R}^N)} + \int_0^T \|\partial_t \theta(\cdot, t)\|_{H^1_{q_2}(\mathbb{R}^N)} \, dt \Big) \\
&\le C\Big\{ \|\rho_0\|_{H^1_{q_2}(\mathbb{R}^N)} + \Big(\int_0^\infty < t >^{-p'b} \, dt \Big)^{1/p'} \| < t >^b \partial_t \theta\|_{L_p((0,T), H^1_{q_2}(\mathbb{R}^N))} \Big\} \\
&\le C(\|\rho_0\|_{H^1_{q_2}(\mathbb{R}^N)} + \| < t >^b \partial_t \theta\|_{L_p((0,T), H^1_{q_2}(\mathbb{R}^N))}) \\
&\le C(\epsilon^2 + \epsilon^3 + \epsilon^4).
\end{aligned}
$$

Choosing $\epsilon > 0$ so small that $C(\epsilon^2 + \epsilon^3 + \epsilon^4) \le \rho_*/4$, we have $\|\theta\|_{L_\infty((0,T), L_\infty(\mathbb{R}^N))} \le \rho_*/4$. Furthermore, by Hölder inequality and $1 - bp' < 0$, we have

$$
\begin{aligned}
\int_0^T \|\nabla \mathbf{u}(\cdot, s)\|_{L_\infty(\mathbb{R}^N)} \, ds &\le \Big(\int_0^\infty < s >^{-bp'} \, ds \Big)^{1/p'} \| < t >^b \nabla \mathbf{u}\|_{L_p((0,T), H^1_{q_2}(\mathbb{R}^N)} \\
&\le C(\epsilon^2 + \epsilon^3 + \epsilon^4).
\end{aligned}
$$

Choosing $\epsilon > 0$ so small that $C(\epsilon^2 + \epsilon^3 + \epsilon^4) \le \sigma$, we have $\int_0^T \|\nabla \mathbf{u}(\cdot, s)\|_{L_\infty(\mathbb{R}^N)} \, ds \le \sigma$. Thus, we have $(\theta, \mathbf{u}, \mathbf{k}) \in \mathcal{I}_{T,\epsilon}$. Therefore, we define a map Φ acting on $(\rho, \mathbf{v}, \mathbf{d}) \in \mathcal{I}_{T,\epsilon}$ by $\Phi(\rho, \mathbf{v}, \mathbf{d}) = (\theta, \mathbf{u}, \mathbf{k})$, and then Φ is the map from $\mathcal{I}_{T,\epsilon}$ into itself.

Let $(\rho_i, \mathbf{v}_i, \mathbf{d}_i) \in \mathcal{I}_{T,\epsilon}$. Setting $(\theta, \mathbf{u}, \mathbf{k}) = (\theta_1, \mathbf{u}_1, \mathbf{k}_1) - (\theta_2, \mathbf{u}_2, \mathbf{k}_2) = \Phi(\rho_1, \mathbf{v}_1, \mathbf{d}_1) - \Phi(\rho_2, \mathbf{v}_2, \mathbf{d}_2)$, $f = f(\rho_1, \mathbf{v}_1) - f(\rho_2, \mathbf{v}_2)$, $\mathbf{g} = \mathbf{g}(\rho_1, \mathbf{v}_1, \mathbf{d}_1) - \mathbf{g}(\rho_2, \mathbf{v}_2, \mathbf{d}_2)$, and $\mathbf{h} = \mathbf{h}(\mathbf{v}_1, \mathbf{d}_1) - \mathbf{h}(\mathbf{v}_2, \mathbf{d}_2)$, by (33) and (34), we see that $(\theta, \mathbf{u}, \mathbf{k})$ is a solution to the following system:

$$
\begin{cases}
\partial_t \theta + \rho_* \operatorname{div} \mathbf{u} = f & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\
\rho_* \partial_t \mathbf{u} - \operatorname{Div} (\mathbf{S}(\mathbf{u}) - P'(\rho_*)\theta \mathbf{I}) = \mathbf{g} & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\
\partial_t \mathbf{k} - \zeta \Delta \mathbf{k} = \mathbf{h} & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\
(\theta, \mathbf{u}, \mathbf{k})|_{t=0} = (0,0,0) & \text{in } \mathbb{R}^N.
\end{cases}
$$

Employing the same argument as in the proof of (35) and using (18), (27), and (32), we have

$$
\mathcal{N}(\Phi(\rho_1, \mathbf{v}_1, \mathbf{d}_1) - \Phi(\rho_2, \mathbf{v}_2, \mathbf{d}_2))(T) \le C(\epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4)\mathcal{N}((\rho_1, \mathbf{v}_1, \mathbf{d}_1) - (\rho_2, \mathbf{v}_2, \mathbf{d}_2))(T)
$$

with some C independent of T and ϵ. Therefore, choosing $\epsilon > 0$ so small that $C(\epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4) < 1$, we see that Φ is a contraction map on $\mathcal{I}_{T,\epsilon}$, and therefore Φ has a unique fixed point $(\rho, \mathbf{v}, \mathbf{d}) \in \mathcal{I}_{T,\epsilon}$ which solves (9) uniquely by the contraction mapping principle. This completes the proof of Theorem 2.

6. Proof of Theorem 1

In this section, we prove Theorem 1 by (2). Assume that p, q_1, q_2, b, and initial data $(\rho_0, \mathbf{v}_0, \mathbf{d}_0)$ satisfy the same condition (10) and (11) as in Theorem 2, respectively. As was mentioned in ([21], Section 2), Theorem 2 implies that the Lagrange transformation $x = \mathbf{X}_t(\xi)$ given by (6) is a $C^{1+\omega}$ $(\omega \in (0, 1/2))$ diffeomorphism on $\mathbb{R}^N$ for any $t \in (0,T)$. In particular, since $\|\mathbf{K}_\mathbf{u}\|_{L_\infty(\mathbb{R}^N)} \le \sigma < 1$ provided by Theorem 2, choosing σ smaller if necessary, we may assume that $C^{-1} \le \det(\mathbf{I} + \mathbf{K}_\mathbf{u}) \le C$ with some constant C for any $(\xi, t) \in \mathbb{R}^N \times (0, T)$, where we have set $\mathbf{K}_{\mathbf{u}} = \int_0^t \nabla \mathbf{u}(\xi, s)\,ds$. Let $\xi = \mathbf{X}_t^{-1}(x)$ be an inverse map of $x = \mathbf{X}_t(\xi)$ and let $\omega(x, t) = \theta(\mathbf{X}_t^{-1}(x), t)$, $\mathbf{v}(x, t) = \mathbf{u}(\mathbf{X}_t^{-1}(x), t)$, and $\mathbf{n}(x, t) = \mathbf{k}(\mathbf{X}_t^{-1}(x), t)$. From now, we verify $\mathcal{E}(\omega, \mathbf{v}, \mathbf{n})(T)$ is estimated by $\mathcal{N}(\theta, \mathbf{u}, \mathbf{k})(T)$. Noting that $dx = \det(\mathbf{I} + \mathbf{K}_{\mathbf{u}})\,d\xi$, we have

$$\|(\omega, \mathbf{v}, \mathbf{n})\|_{L_q(\mathbb{R}^N)} + \|\mathbf{n}\|_{L_\infty(\mathbb{R}^N)} \leq C\|(\theta, \mathbf{u}, \mathbf{k})\|_{L_q(\mathbb{R}^N)} + \|\mathbf{k}\|_{L_\infty(\mathbb{R}^N)}$$

for $q = q_1$ and q_2. By the chain rule, we have

$$\begin{aligned}
&\|\nabla(\omega, \mathbf{v}, \mathbf{n})\|_{L_q(\mathbb{R}^N)} + \|\nabla \mathbf{n}\|_{L_\infty(\mathbb{R}^N)} \\
&\quad \leq C(1 - \|\mathbf{K}_{\mathbf{u}}\|_{L_\infty(\mathbb{R}^N)})^{-1}(\|\nabla(\theta, \mathbf{u}, \mathbf{k})\|_{L_q(\mathbb{R}^N)} + \|\nabla \mathbf{k}\|_{L_\infty(\mathbb{R}^N)}) \\
&\quad \leq C(\|\nabla(\theta, \mathbf{u}, \mathbf{k})\|_{L_q(\mathbb{R}^N)} + \|\nabla \mathbf{k}\|_{L_\infty(\mathbb{R}^N)}), \\
&\|\nabla^2(\mathbf{v}, \mathbf{n})\|_{L_q(\mathbb{R}^N)} \\
&\quad \leq C\{(1 - \|\mathbf{K}_{\mathbf{u}}\|_{L_\infty(\mathbb{R}^N)})^{-2}\|\nabla^2(\mathbf{u}, \mathbf{k})\|_{L_q(\mathbb{R}^N)} \\
&\quad\quad + (1 - \|\mathbf{K}_{\mathbf{u}}\|_{L_\infty(\mathbb{R}^N)})^{-1}\|\nabla \mathbf{K}_{\mathbf{u}}\|_{L_q(\mathbb{R}^N)}\|\nabla(\mathbf{u}, \mathbf{k})\|_{L_\infty(\mathbb{R}^N)}\},
\end{aligned}$$

which combined with (15) and $\|\nabla(\mathbf{u}, \mathbf{k})\|_{L_\infty(\mathbb{R}^N)} \leq C\|\nabla(\mathbf{u}, \mathbf{k})\|_{H^1_{q_2}(\mathbb{R}^N)}$ yields that

$$\begin{aligned}
&\sum_{q=q_1,q_2} \| < t >^b \nabla(\omega, \mathbf{v}, \mathbf{n})\|_{L_p((0,T),L_q(\mathbb{R}^N))} \leq C \sum_{q=q_1,q_2} \| < t >^b \nabla(\theta, \mathbf{u}, \mathbf{k})\|_{L_p((0,T),L_q(\mathbb{R}^N))}, \\
&\| < t >^b (\omega, \mathbf{v}, \mathbf{n})\|_{L_p((0,T),L_{q_2}(\mathbb{R}^N))} \leq C\| < t >^b (\theta, \mathbf{u}, \mathbf{k})\|_{L_p((0,T),L_{q_2}(\mathbb{R}^N))},
\end{aligned}$$

$$\begin{aligned}
&\| < t >^{N/2q_1} (\omega, \mathbf{v}, \mathbf{n})\|_{L_\infty((0,T),L_{q_1}(\mathbb{R}^N))} + \| < t >^b (\omega, \mathbf{v}, \mathbf{n})\|_{L_\infty((0,T),L_{q_2}(\mathbb{R}^N))} \\
&\quad \leq C\| < t >^{N/2q_1} (\theta, \mathbf{u}, \mathbf{k})\|_{L_\infty((0,T),L_{q_1}(\mathbb{R}^N))} + \| < t >^b (\theta, \mathbf{u}, \mathbf{k})\|_{L_\infty((0,T),L_{q_2}(\mathbb{R}^N))},
\end{aligned}$$

$$\begin{aligned}
&\| < t >^b \nabla \mathbf{n}\|_{L_\infty((0,T),L_{q_1}(\mathbb{R}^N))} \leq C\| < t >^b \nabla \mathbf{k}\|_{L_\infty((0,T),L_{q_1}(\mathbb{R}^N))}, \\
&\| < t >^b \nabla \mathbf{n}\|_{L_\infty((0,T),L_\infty(\mathbb{R}^N))} \leq C\| < t >^b \nabla \mathbf{k}\|_{L_\infty((0,T),L_\infty(\mathbb{R}^N))}, \\
&\| < t >^b \mathbf{n}\|_{L_\infty((0,T),L_\infty(\mathbb{R}^N))} \leq C\| < t >^b \mathbf{k}\|_{L_\infty((0,T),L_\infty(\mathbb{R}^N))},
\end{aligned}$$

$$\begin{aligned}
&\sum_{q=q_1,q_2} \| < t >^b \nabla^2(\mathbf{v}, \mathbf{n})\|_{L_p((0,T),L_q(\mathbb{R}^N))} \\
&\quad \leq C \sum_{q=q_1,q_2} (\| < t >^b \nabla^2(\mathbf{u}, \mathbf{k})\|_{L_p((0,T),L_q(\mathbb{R}^N))} \\
&\quad\quad + \| < t >^b \nabla^2 \mathbf{u}\|_{L_p((0,T),L_q(\mathbb{R}^N))}\| < t >^b \nabla(\mathbf{u}, \mathbf{k})\|_{L_p((0,T),H^1_{q_2}(\mathbb{R}^N))}).
\end{aligned}$$

Noting that $\partial_t(\theta, \mathbf{u}, \mathbf{k})(\xi, t) = \partial_t(\omega, \mathbf{v}, \mathbf{n})(x, t) + \mathbf{u} \cdot \nabla(\omega, \mathbf{v}, \mathbf{n})(x, t)$, we have

$$\begin{aligned}
&\|\partial_t(\omega, \mathbf{v}, \mathbf{n})\|_{L_q(\mathbb{R}^N)} \\
&\quad \leq C(\|\partial_t(\theta, \mathbf{u}, \mathbf{k})\|_{L_q(\mathbb{R}^N)} + \|\mathbf{u}\|_{L_\infty(\mathbb{R}^N)}\|\nabla\theta\|_{L_q(\mathbb{R}^N)} + \|\mathbf{u}\|_{L_q(\mathbb{R}^N)}\|\nabla(\mathbf{u}, \mathbf{k})\|_{L_\infty}).
\end{aligned} \tag{89}$$

By $\theta = \rho_0 + \int_0^t \partial_s\theta\,ds$, Hölder inequality, and the condition $bp' > 1$, we have

$$\|\nabla\theta\|_{L_\infty((0,T),L_q(\mathbb{R}^N))} \leq \|\nabla\rho_0\|_{L_q(\mathbb{R}^N)} + C\| < t >^b \partial_t\theta\|_{L_p((0,T),H^1_q(\mathbb{R}^N))},$$

which combined with (89) yields that

$$\sum_{q=q_1,q_2} \| < t >^b \partial_t(\omega, \mathbf{v}, \mathbf{n})\|_{L_p((0,T),L_q(\mathbb{R}^N))}$$
$$\leq C \sum_{q=q_1,q_2} \{\| < t >^b \partial_t(\theta, \mathbf{u}, \mathbf{k})\|_{L_p((0,T),L_q(\mathbb{R}^N))}$$
$$+ \| < t >^b \mathbf{u}\|_{L_p((0,T),H^1_{q_2}(\mathbb{R}^N))}(\|\nabla\rho_0\|_{L_q(\mathbb{R}^N)} + \| < t >^b \partial_t\theta\|_{L_p((0,T),H^1_q(\mathbb{R}^N))})$$
$$+ (\| < t >^{N/2q_1} \mathbf{u}\|_{L_\infty((0,T),L_{q_1}(\mathbb{R}^N))} + \| < t >^b \mathbf{u}\|_{L_\infty((0,T),L_{q_2}(\mathbb{R}^N))})$$
$$\times \| < t >^b \nabla(\mathbf{u}, \mathbf{k})\|_{L_p((0,T),H^1_{q_2}(\mathbb{R}^N))}\}.$$

Summing up, we have

$$\mathcal{E}(\omega, \mathbf{v}, \mathbf{n})(T) \leq C\mathcal{N}(\theta, \mathbf{u}, \mathbf{k})(T).$$

Using (35) and choosing $\epsilon > 0$ smaller if necessary, we have

$$\mathcal{E}(\omega, \mathbf{v}, \mathbf{n})(T) \leq \epsilon,$$

which implies that (1) has solutions, $\rho = \rho_* + \omega$, $\mathbf{v}$, and $\mathbf{d} = \mathbf{d}_* + \mathbf{n}$ satisfying (4) and (5). Moreover, the uniqueness of solutions also follows from Theorem 2, which completes the proof of Theorem 1.

7. Concluding Remarks

In this paper, we proved global well-posedness for the simplified Ericksen–Leslie system in the maximal L_p-L_q regularity class. We provided a general framework to prove the global well-posedness for small initial data of quasilinear parabolic or hyperbolic–parabolic equations in $\mathbb{R}^N$. This approach can be extended to boundary value problems with a non-homologous boundary condition (cf. [25]).

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