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Symmetries and Solutions for a Class of Advective Reaction-Diffusion Systems with a Special Reaction Term

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Abstract: This paper is devoted to apply the Lie methods to a class of reaction diffusion advection systems of two interacting species u and v with two arbitrary constitutive functions f and g. The reaction term appearing in the equation for the species v is a logistic function of Lotka-Volterra type. Once obtained the Lie algebra for any form of f and g a Lie classification is carried out. Interesting reduced systems are derived admitting wide classes of exact solutions.

Keywords: reaction-diffusion-advection equations; symmetries; exact solutions; Lotka-Volterra funtctions

MSC: 35K57; 35B06; 35C06; 35K40; 35Q92; 92D25

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1. Introduction

In this paper we study, in the framework of the symmetry groups, the following class of reaction diffusion advection systems

$$\begin{cases} u_t = D_x(f(u)u_x) + g(u,v,u_x), \\ v_t = a_1 v(1 - a_2 u - a_3 v), \end{cases} \tag{1}$$

with $a_1 a_2 a_3 \neq 0$, $f(u) \neq 0$, and f, g analytic functions with respect to their arguments, and where the operator D_x is the total derivative with respect to x.

The reaction term of the second equation is specialized with a two variables logistic function of the Lotka-Volterra type. This special form can be found in [1–3].

In [1–3] Volterra proposed an ODE system as a model to explain the phenomenon of the simultaneous increase of predatory fish and the decrease of prey fish in the Adriatic Sea where, during the First World War, fishing was largely suspended. At the same time, the equation system studied by Volterra was derived (independently) by Lotka [4] as a mathematical model describing a hypothetical chemical reaction with chemical concentrations oscillating. That ODE system, usually called Lotka–Volterra model, is the simplest model of predator–prey interactions. In the past decades, reaction-diffusion-advection (hereafter RDA) equations have been frequently studied as standard models to address problems related to spatial ecology and evolution. These studies originated a wide production of scientific papers. In [5] the system of two RDA interacting equations with linear advection and diffusion is considered and features of solutions with periodic boundary conditions are discussed. A review describing RDA models for the ecological effects and evolution of dispersal together with mathematical methods for analyzing them is shown in [6]. In [7–9], more specific cases are introduced in the framework of populations dynamics. In [10] a coupled Lotka-Volterra reaction-diffusion model is considered and studied in the framework of Lie methods. A special class of the aforesaid system can be found in [11]. In [12–15] several reaction diffusion systems with reaction functions of Volterra type have been analyzed by mean the symmetry analysis methods. Finally all known results about Lie and conditional symmetries of the diffusive Lotka–Volterra system are summarized in the recent review [16].

The systems (1) belong to the more general class

$$\begin{cases} u_t = D_x(f(u)u_x) + g(u,v,u_x), \\ \\ v_t = h(u,v). \end{cases} \tag{2}$$

These systems characterize the mathematical models of a wide class of two interacting populations u and v as for instance *Proteus mirabilis* bacterial colonies or mosquitos as *Aedes Aegypt* [17–20], *Anopheles* [21,22] .

The first equation, derived from the balance equation for the species u, is a quite general RDA equation where we assumed the diffusion coefficient f depending only on u instead on u and v [23–26]. Moreover if the species u does not feel external *stimuli* (usually water currents or wind) the function g could be independent of the advection u_x. For instance, the *swarming* cells in the model for *proteus mirabilis* colonies.

The second equation, concerned with the species v, is assumed to feel neither diffusion nor advection respectively, as for instance, for *swinging cells* or the so called *Aquatic populations.*

In this paper we continue our analysis in the framework of the classical Lie symmetry methods application to the class (2), see [18,27–30]. We remind that Lie symmetry approach, in general, offers a methodological way to look for solutions of differential equations, that make it a milestone in this field.

To the best of our knowledge, RDA systems of type (1) have not been previously studied. The existence of symmetries, i.e. of continuous groups of transformations leaving the system invariant, is very useful in this study. It allows to find exact solutions. Such solutions in general are not solutions of a specific real problem (with boundary conditions, initial conditions,...) but are often used as test suggestions for the validation of numerical methods devoted to solve specific real problems. Moreover it is worth noticing that carrying out a symmetry analysis of the models (1) could involve several parameters. They give rise to different cases or sub-cases or to special forms of constitutive functions which, quite frequently, have particular significance in the biological process studied. We can be stress that once the form of constitutive functions is more specialized then some additional specific symmetry methods as potential symmetries [31,32] or non classical symmetries can be easier applied [10,12,33–36].

The plan of this paper is as it follows. In Section 2 we recall some preliminaries devoted to the derivations of the determining system for symmetries of the general class (2) and we classify the system (1) with respect the arbitrary functions $f(u)$ and $g(u,v,u_x)$. At this aim we specialize the determining system of class (2) by assuming $h(u,v)$ of the Lotka -Volterra type and we get extensions of the Principal Lie Algebra $L_{\mathcal{P}}$. Reductions to an ODE system are performed in Section 3 and wide classes of exact solutions are derived. Conclusions are shown in Section 4 .

2. Symmetries

2.1. Preliminaries

In this subsection we recall some results concerned with the system (2) obtained in our previous paper [30] .

A symmetry infinitesimal operator for the class (2) has the form

$$X = \xi^1(x,t,u,v)\partial_x + \xi^2(x,t,u,v)\partial_t + \eta^1(x,t,u,v)\partial_u + \eta^2(x,t,u,v)\partial_v. \tag{3}$$

As the second equation of system (2) is of lower order with respect to the first one, we need to consider also its differential consequences up to second order, that is

$$\begin{cases} u_t = D_x(f(u)u_x) + g(u,v,u_x), \\ v_t = h(u,v), \\ v_{tt} = D_t(h(u,v)), \\ v_{tx} = D_x(h(u,v)), \end{cases} \tag{4}$$

where the operators D_t is the total derivative with respect to t. The coordinates ξ^1, ξ^2, η^1, and η^2 are derived from the following invariance conditions

$$\begin{cases} X^{(2)}(-u_t + D_x(f(u)u_x) + g(u,v,u_x))\,|_{(4)} = 0, \\ X^{(2)}(-v_t + h(u,v))\,|_{(4)} = 0, \\ X^{(2)}(-v_{tt} + D_t(h(u,v)))\,|_{(4)} = 0, \\ X^{(2)}(-v_{tx} + D_x(h(u,v)))\,|_{(4)} = 0. \end{cases} \tag{5}$$

The operators $X^{(2)}$ is

$$X^{(2)} = X + \zeta^1_t \frac{\partial}{\partial_{u_t}} + \zeta^1_x \frac{\partial}{\partial_{u_x}} + \zeta^1_{xx} \frac{\partial}{\partial_{u_{xx}}} + \zeta^2_t \frac{\partial}{\partial_{v_t}} + \zeta^2_{tt} \frac{\partial}{\partial_{v_{tt}}} + \zeta^2_{tx} \frac{\partial}{\partial_{v_{tx}}}. \tag{6}$$

As usual the expressions of the coordinates ζ^1_t, ζ^1_x, ζ^1_{xx}, ζ^2_t, ζ^2_{tt}, ζ^2_{tx} are written as (see e.g., [37–44])

$$\zeta^1_t = D_t(\eta^1) - u_t D_t(\xi^1) - u_x D_t(\xi^2), \tag{7}$$
$$\zeta^1_x = D_x(\eta^1) - u_t D_x(\xi^1) - u_x D_x(\xi^2), \tag{8}$$
$$\zeta^1_{xx} = D_x(\zeta^1_x) - u_{xt} D_x(\xi^1) - u_{xx} D_x(\xi^2), \tag{9}$$
$$\zeta^2_t = D_t(\eta^2) - v_t D_t(\xi^1) - v_x D_t(\xi^2), \tag{10}$$
$$\zeta^2_{tt} = D_t(\zeta^2_t) - v_{tt} D_t(\xi^1) - v_{tx} D_t(\xi^2), \tag{11}$$
$$\zeta^2_{tx} = D_x(\zeta^2_t) - v_{tt} D_x(\xi^1) - v_{tx} D_x(\xi^2). \tag{12}$$

After having solved only the equations of the determining system derived from (5) that do not depend on the arbitrary elements f, g, and h we have obtained the following restrictions on the coordinates ξ^1, ξ^2, η^1, and η^2

$$\xi^1 = \alpha(x), \quad \xi^2 = \beta(t), \quad \eta^1 = \phi(x,t,u), \quad \eta^2 = \psi(x,t,v), \tag{13}$$

that must satisfy the following remaining determining equations where the arbitrary elements f, g, and h appear

$$(2\alpha' - \beta')f - \phi f_u = 0, \tag{14}$$
$$\begin{aligned}&(f'g + u_x^2 f'^2)\phi - \left[\phi_{uu}u_x^2 - (\alpha'' - 2\phi_{xu})u_x + \phi_{xx}\right]f^2 - \psi g_v f - \phi g_u f + \\ &\quad (\phi_u - 2\alpha')gf + \phi_t f - \phi u_x^2 f'' f - (2\phi_x + \phi_u u_x)u_x f' f + \\ &\quad ((\alpha' - \phi_u)u_x - \phi_x)g_{u_x} f = 0,\end{aligned} \tag{15}$$
$$(\psi_v - \beta')h - \phi h_u - \psi h_v + \psi_t = 0. \tag{16}$$

From where it is easy to ascertain that for f, g and h arbitrary the solutions of (14)–(16) is

$$\alpha(x) = c_1,\ \beta(t) = c_2,\ \phi(t,x,u) = 0,\ \psi(t,x,v) = 0,$$

with c_1, c_2 arbitrary constants, then the principal Lie Algebra $L_{\mathcal{P}}$ [45] is spanned by the time translations and space translations

$$X_1 = \partial_t, \qquad X_2 = \partial_x.$$

2.2. Symmetries Classification

In order to determine the infinitesimal symmetry operators of system (1) we must discuss the Equations (14)–(16) in the unknowns α, β, ϕ, and ψ, taking into account that $h(u,v) = a_1 v(1 - a_2 u - a_3 v)$ (with $a_1 a_2 a_3 \neq 0$). Then the Equation (16) becomes

$$(\psi_v - \beta')a_1 v(1 - a_2 u - a_3 v) + a_1 a_2 v\phi - (a_1 - a_1 a_2 u - 2a_2 a_3 v)\psi + \psi_t = 0. \tag{17}$$

Deriving twice with respect to u we deduce that $\phi(t,x,u)$ must be linear in u then

$$\phi(t,x,u) = \phi_0(t,x) + \phi_1(t,x)u$$

and the condition (17) can be written as

$$(\psi_v - \beta')a_1 v(1 - a_3 v) + a_1 a_2 v\phi_0 - (a_1 - 2a_2 a_3 v)\psi + \psi_t + a_1 a_2\big[(\beta' - \psi_v)v + v\phi_1 + \psi\big]u = 0.$$

From this equation, as no functions depend on u, we derive

$$\phi(t,x,u) = -u\beta' + \frac{a_1\beta' + \beta''}{a_1 a_2}, \tag{18}$$

$$\psi(x,t,v) = -v\beta'. \tag{19}$$

By substituting in the condition (14), we get

$$a_1 a_2(2\alpha' - \beta')f - \big(a_1\beta'(1 - a_2 u) + \beta''\big)f_u = 0, \tag{20}$$

then it must be

$$\alpha(x) = c_1 x + c_2,$$

with c_1, c_2 arbitrary constants, and

$$a_1 a_2(2c_1 - \beta')f - \big(a_1\beta'(1 - a_2 u) + \beta''\big)f_u = 0. \tag{21}$$

In order to have extensions of the principal Lie algebra, we observe that the function $\beta(t)$ can not be constant, and we need to consider different cases depending on f.

1. $f(u) = f_0 = const.$
   In this case from (21) we have $\beta(t) = 2c_1 t + c_3$ (with c_3 arbitrary constant) and, after having substituted in (15), in order to obtain additional generators, it must be

$$g(u,v,u_x) = (1 - a_2 u)^2 G(\omega_1, \omega_2), \tag{22}$$

   with $\omega_1 = \frac{v}{1 - ua_2}$ and $\omega_2 = \frac{u_x^2}{(1 - ua_2)^3}$. We get the additional generator

$$X_3 = 2a_2 t\partial_t + a_2 x\partial_x + 2(1 - a_2 u)\partial_u - 2a_2 v\partial_v. \tag{23}$$

2. $f(u) = f_0 + uf_1$ with f_0, f_1 constitutive constants.
   In this case from (21) we have $c_1 = 0$ and two different cases depending on the constant f_1.

(a) If $f_1 \neq -f_0 a_2$, then $\beta(t) = c_3 e^{-\frac{a_1(a_2 f_0 + f_1)}{f_1} t} + c_4$, with c_3, c_4 arbitrary constants, and, after having substituted in (15), in order to have additional generators, it must be

$$g(u, v, u_x) = (f_0 + u f_1)^2 \left(\frac{a_1(a_2 f_0 + f_1)}{f_1^2 (f_0 + u f_1)} + G(\omega_1, \omega_2) \right), \tag{24}$$

with $\omega_1 = \frac{v}{u f_1 + f_0}$ and $\omega_2 = \frac{u_x}{u f_1 + f_0}$. The additional generator is

$$X_3 = e^{-\frac{a_1(a_2 f_0 + f_1)}{f_1} t} \left(f_1^2 \partial_t + a_1(a_2 f_0 + f_1)(f_0 + u f_1)\partial_u + a_1 f_1 (a_2 f_0 + f_1) v \partial_v \right). \tag{25}$$

(b) If $f_1 = -f_0 a_2$, that is $f(u) = f_0(1 - u a_2)$, then $\beta(t) = c_3 t + c_4$, with c_3, c_4 arbitrary constants, and, after having substituted in (15), in order to have additional generators, it must be

$$g(u, v, u_x) = (1 - a_2 u)^2 G(\omega_1, \omega_2), \tag{26}$$

with $\omega_1 = \frac{v}{1 - u a_2}$ and $\omega_2 = \frac{u_x}{1 - u a_2}$. The additional generator is

$$X_3 = a_2 t \partial_t + (1 - u a_2)\partial_u - a_2 v \partial_v. \tag{27}$$

3. $f(u) = (f_0 + f_1 u)^{f_2}$, where f_0, f_1, f_2 are constitutive constants with $f_2 \neq 0, 1$.
In this case from (21) in order to have additional generators, it must be $f_1 = -a_2 f_0$ (that is $f(u) = f_0(1 - u a_2)^{f_2}$) and $\beta(t) = \frac{2 c_1 t}{1 - f_2} + c_3$, with c_3 arbitrary constant. After having substituted in (15), in order to have additional generators, it must be

$$g(u, v, u_x) = (1 - a_2 u)^2 G(\omega_1, \omega_2), \tag{28}$$

with $\omega_1 = \frac{v}{1 - u a_2}$ and $\omega_2 = (1 - u a_2)^{f_2 - 3} u_x^2$. Then in this case we find the following additional generator

$$X_3 = 2 a_2 t \partial_t + a_2(1 - f_2) x \partial_x + 2(1 - u a_2)\partial_u - 2 a_2 v \partial_v. \tag{29}$$

We can summarize the previous results in only two cases A and B shown in the Table 1, because it is possible to obtain the case 1 from the case 3 by removing the condition $f_2 \neq 0$, while the case 2(b) comes from the case 3 by removing the condition $f_2 \neq 1$.

Table 1. Cases with additional generators.

Principal Lie algebra: $X_1 = \partial_t$, $X_2 = \partial_x$.	
	Extension case A
$f(u)$	$f_0 + f_1 u$, $f_1 \neq -a_2 f_0$, $f_1 \neq 0$
$g(u, v, u_x)$	$f^2(u) \left(\frac{a_1(a_2 f_0 + f_1)}{f_1^2 (f_0 + u f_1)} + G^A(\omega_1, \omega_2) \right)$, $\omega_1 = \frac{v}{f_0 + f_1 u}$, $\omega_2 = \frac{u_x}{f_0 + f_1 u}$.
X_3^A	$e^{-\frac{a_1(a_2 f_0 + f_1)}{f_1} t} \left(f_1^2 \partial_t + a_1(a_2 f_0 + f_1)(f_0 + u f_1)\partial_u + a_1 f_1 (a_2 f_0 + f_1) v \partial_v \right)$
	Extension case B
$f(u)$	$f_0(1 - a_2 u)^{f_2}$, $f_0 \neq 0$
$g(u, v, u_x)$	$(1 - a_2 u)^2 G^B(\omega_1, \omega_2)$, $\omega_1 = \frac{v}{1 - u a_2}$, $\omega_2 = (1 - u a_2)^{f_2 - 3} u_x^2$
X_3^B	$2 a_2 t \partial_t + a_2(1 - f_2) x \partial_x + 2(1 - u a_2)\partial_u - 2 a_2 v \partial_v$

3. Reduced Systems and Exact Solutions

In this section we reduce the system (1) to an ODE system to search for invariant solutions. We neglect stationary, spatially homogeneous and travelling wave solutions having in mind to give them a wide consideration in future researches. Then we consider only the forms of the functions $f(u)$ and $G(u, v, u_x)$ that allow additional generators with respect the principal Lie algebra. It is worthwhile stressing that the reductions will be done without any restriction on the general compatible form of the functions. We study the two obtained cases separately.

3.1. Reduced Systems and Exact Solutions of Case A

In the case A of the Table 1 the system (1) becomes

$$\begin{cases} u_t = (f_0 + uf_1)u_{xx} + f_1 u_x^2 + (f_0 + uf_1)^2 \left(\frac{a_1(a_2 f_0 + f_1)}{f_1^2 (f_0 + uf_1)} + G^A \left(\frac{v}{uf_1 + f_0}, \frac{u_x}{uf_1 + f_0} \right) \right), \\ v_t = a_1 v(1 - a_2 u - a_3 v). \end{cases} \tag{30}$$

In this case the commutator table of generators X_1, X_2, $X_3^{(A)}$ is given in Table 2, where the entry in row i and column j representing

$$[X_i, X_j] = X_i X_j - X_j X_i. \tag{31}$$

Table 2. Commutator table in the case A.

	X_1	X_2	$X_3^{(A)}$
X_1	0	0	$-\frac{a_1(a_2 f_0 + f_1)}{f_1} X_3^{(A)}$
X_2	0	0	0
$X_3^{(A)}$	$\frac{a_1(a_2 f_0 + f_1)}{f_1} X_3^{(A)}$	0	0

The invariants corresponding to the generator $cX_2 + X_3^A$ are

$$u = e^{\frac{a_1(a_2 f_0 + f_1)}{f_1} t} \frac{1}{f_1} U(\sigma) - \frac{f_0}{f_1}, \; v = e^{\frac{a_1(a_2 f_0 + f_1)}{f_1} t} V(\sigma), \; \sigma = x - c \frac{e^{\frac{a_1(a_2 f_0 + f_1)}{f_1} t}}{a_1(a_2 f_0 + f_1)}, \tag{32}$$

where U and V are solutions of the reduced system

$$\begin{cases} f_1 U U'' + f_1 U'^2 + f_1^2 U^2 G^A \left(\frac{V}{U}, \frac{U'}{f_1 U} \right) + cU' = 0, \\ a_1(a_2 U + a_3 f_1 V)V - cV' = 0. \end{cases} \tag{33}$$

If we consider $c = 0$ that is if we use only the generator X_3^A from the second equation we get

$$V = -\frac{a_2}{a_3 f_1} U, \tag{34}$$

and U will be a solution of the following second order equation

$$UU'' + U'^2 + f_1 U^2 \tilde{G}^A \left(\frac{U'}{U} \right) = 0, \tag{35}$$

where $\tilde{G}^A\left(\frac{U'}{U}\right) = G^A(\omega_1, \omega_2)\,|_{\omega_1=-\frac{a_2}{a_3 f_1}, \omega_2=\frac{U'}{f_1 U}}$.
By introducing the new variable

$$W(\sigma) = \frac{U'(\sigma)}{U(\sigma)}, \tag{36}$$

we reduce the Equation (35) to the following first order differential equation integrable in closed form

$$W' + 2W + f_1 \tilde{G}^A(W) = 0. \tag{37}$$

The solution is

$$\sigma = -\int \frac{dW}{2W + f_1 \tilde{G}^A(W)} + k, \tag{38}$$

with k arbitrary constant.
Particular solutions of the Equation (37) are the constant solutions

$$W(\sigma) = W_k, \tag{39}$$

where W_k are possible roots of the algebraic (transcendental) equation

$$2W_k + f_1 \tilde{G}^A(W_k) = 0. \tag{40}$$

Then, going back, we obtain

$$U(\sigma) = U_0 e^{W_k \sigma}, \quad V(\sigma) = -\frac{a_2}{a_3 f_1} U_0 e^{W_k \sigma}, \tag{41}$$

and then, by substituting in (32), we obtain solutions of the system (30).
We observe that in this case if we choose $f_0 = 0$ and

$$G^A(\omega_1, \omega_2) = -\frac{b_2}{f_1}\omega_1 - \frac{b_3}{f_1^2} + \frac{1}{f_1^2}\Gamma(\omega_2),$$

with b_2, b_3 additional constants and $\Gamma(\omega_2)$ arbitrary function, the system (1) becomes

$$\begin{cases} u_t = f_1 u u_{xx} + f_1 u_x^2 + a_1 u(1 - b_2 v - b_3 u) + u^2 \Gamma\left(\frac{u_x}{f_1 u}\right), \\ v_t = a_1 v(1 - a_2 u - a_3 v), \end{cases} \tag{42}$$

that is, in the first equation also it appears a reaction term of type Lotka-Volterra.
By considering $c = 0$, the invariants (32) become

$$u = e^{a_1 t} U(\sigma),\ v = e^{a_1 t} V(\sigma),\ \sigma = x, \tag{43}$$

and, by choosing $\Gamma\left(\frac{u_x}{f_1 u}\right) = \gamma_0 f_1 \frac{u_x}{f_1 u}$, the system (42) becomes

$$\begin{cases} f_1 U U'' + f_1 U'^2 - a_1 U(b_2 V + b_3 U) + \gamma_0 U U' = 0, \\ a_2 U + a_3 V = 0. \end{cases} \tag{44}$$

Then $V = -\frac{a_2}{a_3} U$ and U must be solution of

$$f_1 U U'' + f_1 U'^2 + a_1 U^2 \left(\frac{a_2 b_2}{a_3} - b_3\right) + \gamma_0 U U' = 0. \tag{45}$$

By introducing the new variable (36) we reduce it to the following first order equation

$$f_1 W' + 2f_1 W^2 + \gamma_0 W + a_1\left(\frac{a_2 b_2}{a_3} - b_3\right) = 0. \tag{46}$$

By introducing the quantity

$$k_0 = \frac{8f_1(a_1a_2b_2 - a_1a_3b_3) - a3\gamma_0^2}{16f_1^2 a_3}, \tag{47}$$

we get the following solutions of the Equation (46)

$$W(\sigma) = \sqrt{k_0}\tan(-2\sqrt{k_0}\sigma + W_0) - \frac{\gamma_0}{4f_1} \tag{48}$$

in the case $k_0 \geq 0$ or, if $k_0 < 0$,

$$W(\sigma) = \sqrt{k_0}\tanh(2\sqrt{k_0}\sigma + W_0) - \frac{\gamma_0}{4f_1}. \tag{49}$$

In both cases W_0 is an arbitrary constant.

3.2. Reduced Systems and Exact Solutions of Case B

In the case B of the Table 1 the system (1) becomes

$$\begin{cases} u_t = f_0(1-a_2u)^{f_2}u_{xx} - a_2f_0f_2(1-a_2u)^{f_2-1}u_x^2 + \\ \quad +(1-a_2u)^2 G^B\left(\dfrac{v}{1-ua_2}, (1-ua_2)^{f_2-3}u_x^2\right), \\ v_t = a_1v(1-a_2u-a_3v). \end{cases} \tag{50}$$

In this case the commutator table of generators X_1, X_2, $X_3^{(B)}$ is given in Table 3.

Table 3. Commutator table in the case B.

	X_1	X_2	$X_3^{(B)}$
X_1	0	0	$2a_2X_1$
X_2	0	0	$a_2(1-f_2)X_2$
$X_3^{(B)}$	$-2a_2X_1$	$-a_2(1-f_2)X_2$	0

We distinguish two cases.

1. If $f_2 \neq 1$, the invariants corresponding to generator X_3^B are

$$u = \frac{U(\sigma)}{t} + \frac{1}{a_2},\ v = \frac{V(\sigma)}{t},\ \sigma = x^2 t^{f_2-1}, \tag{51}$$

where U and V are solutions of the reduced system

$$\begin{cases} 2f_0(-a_2U)^{f_2}\left(2\sigma f_2U'^2 + 2\sigma UU'' + UU'\right) + \sigma UU'(1-f_2) + U^2 + \\ \quad +a_2^2U^3G^B\left(-\frac{V}{a_2U}, 4\sigma U'^2(-a_2U)^{f_2-3}\right) = 0, \\ \sigma(1-f_2)V' = a_1a_3V^2 - (1-a_1a_2U)V. \end{cases} \tag{52}$$

For $f_2 = 2$ we find particular solutions of the form

$$U(\sigma) = u_0\sigma,\ V(\sigma) = -\frac{a_2}{a_3}u_0\sigma, \tag{53}$$

provided that the function G^B is such that the equation

$$10f_0u_0 + G^B(\omega_1,\omega_2)\,|_{\omega_1=\frac{1}{a_3},\omega_2=-\frac{4u_0}{a_2}} = 0 \tag{54}$$

admits solution in u_0.

In the special case with $f_2 = 0$ ($f(u) = f_0$), the system (50) becomes

$$\begin{cases} u_t = f_0u_{xx} + (1-a_2u)^2G^B\left(\dfrac{v}{1-ua_2},(1-ua_2)^{-3}u_x^2\right), \\ v_t = a_1v(1-a_2u-a_3v), \end{cases} \tag{55}$$

and the invariants are

$$u = \frac{U(\sigma)}{t} + \frac{1}{a_2},\ v = \frac{V(\sigma)}{t},\ \sigma = \frac{x^2}{t}, \tag{56}$$

where U and V are solutions of the reduced system

$$\begin{cases} 2f_0(2\sigma UU'' + UU') + \sigma UU' + U^2 + a_2U^3G^B\left(-\frac{V}{a_2U},4\sigma U'^2(-a_2U)^{-3}\right) = 0, \\ \sigma V' = a_1a_3V^2 - (1-a_1a_2U)V. \end{cases} \tag{57}$$

2. If $f_2 = 1$, the invariants corresponding to generator $cX_2 + X_3^B$ are

$$u = \frac{U(\sigma)}{t} + \frac{1}{a_2},\ v = \frac{V(\sigma)}{t},\ \sigma = x - \frac{c}{2a_2}\ln t, \tag{58}$$

where U and V are solutions of the reduced system

$$\begin{cases} 2f_0a_2^2(UU'' + U'^2) - 2a_2U - 2a_2^3U^2G^B\left(-\dfrac{V}{a_2U},\dfrac{U'^2}{a_2^2U^2}\right) - cU' = 0, \\ 2a_2V(a_1a_2U + a_1a_3V - 1) - cV' = 0. \end{cases} \tag{59}$$

We observe that if $c \neq 0$ we find particular solutions of systems (59) of the form

$$U(\sigma) = U_0e^{-\frac{2a_2\sigma}{c}},\ V(\sigma) = -\frac{a_2}{a_3}U_0e^{-\frac{2a_2\sigma}{c}}, \tag{60}$$

provided that the function G^B is such that the equation

$$c^2G^B(\omega_1,\omega_2)\,|_{\omega_1=\frac{1}{a_3},\omega_2=\frac{4}{c^2}} - 8a_2f_0 = 0 \tag{61}$$

admits solutions in c.

4. Conclusions

We have taken in consideration the advection reaction diffusion system (1) that in general can describe the dispersal dynamics of two interacting u and v where the equation describing the evolution of v has the form

$$v_t = a_1v(1-a_2u-a_3v), \tag{62}$$

that is the form of reaction diffusion function is assumed to be of a Lotka -Volterra type. In this case the two dimensional principal Lie Algebra is extended by one in two cases: A, B (see Table 1). We like to stress that both cases show interesting specializations for the arbitrary constitutive functions $f(u)$ and $g(u, v, u_x)$. These specializations contain not only several numerical parameters but also two arbitrary functions G^A and G^B whose independent variables are specialized in agreement with that one of the two cases being under consideration. These parameters as well as the arbitrary functions G^i $(i = A, B)$ still appear in the reduced systems. Moreover we stress that the power form of f, that allows the extension of $L_{\mathcal{P}}$, is in agreement with suggestions of kinetic gas theory. Apart the novelty and the mathematical interest, it is not negligible that, in our results, many arbitrary constitutive numerical constants and two classes of arbitrary constitutive functions are involved. This allows the biologists to make several choices in order to write the mathematical model in a way that could fit closely the real phenomenon. Finally wide classes of exact solutions are still obtained in a quite general form as they still depend on arbitrary constants and on not completely specialized functions G^A or G^B.

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