mathematics

Article

A Weighted Solution Concept under Replicated Behavior

Yu-Hsien Liao

Department of Applied Mathematics, National Pingtung University, Pingtung 900, Taiwan;
twincos@ms25.hinet.net; Tel.:+886-8-766-3800

Abstract: In the framework of traditional transferable-utility (TU) models, the participants are either entirely involved or not involved in interactive processes with some other participants. Based on the distribution notion of the equal allocation of non-separable costs (EANSC), all participants first receive their marginal contributions and further distribute the remaining utilities equally. In real-world situations, however, participants might adopt different participation levels to participate. Moreover, participants might represent coalitions of different scales; participants might have corresponding influences under different situations. Thus, in this paper we propose a generalization of the EANSC by considering weights and replicated notions under conditions of multi-choice behavior simultaneously. In order to dissect the mathematical accuracy and the applied rationality of this expanded EANSC, a specific reduction is introduced to present an axiomatic result and a dynamic process, respectively.

Keywords: the EANSC; replicated notion; weight

MSC: 91A12; 91A40; 91B06; 91B16

check for updates

Citation: Liao, Y.-H. A Weighted Solution Concept under Replicated Behavior. *Mathematics* **2023**, *11*, 150. https://doi.org/10.3390/math11010150

Academic Editor: Manuel Alberto M. Ferreira

Received: 20 October 2022
Revised: 22 December 2022
Accepted: 24 December 2022
Published: 28 December 2022

1. Introduction

In transferable-utility (TU) models, the participants are either entirely involved or not involved in the operation with some other participants. In multi-choice TU models, participants can operate with different activity levels. Van den Nouweland et al. [1] reported several applied situations related to multi-choice TU models, such as a large architecture plan with a deadline and an amercement for each day if this deadline is missed. The completion date relies upon the efforts that all the participants put into the plan: the more they apply themselves, the earlier the completion of the plan. This situation gives rise to a multi-choice model. The worth of a coalition in which each participant operates at a certain working level is defined as the negative value of the amercement which needs to be paid in association with the date of completion of the plan when every participant makes a relative effort. Based on the notion of replicated behavior in multi-choice models, Calvo and Santos [2] reported that the extended Shapley value [3], due to the findings of van den Nouweland et al. [1], corresponds with the solution concept defined by Moulin [4]. Later, Hwang and Liao [5] adopted axiomatic results to analyze this extended Shapley value. However, Hwang and Liao [6] applied the replicated notion to introduce an extended core and achieved comparable results.

Consistency, initially investigated by Harsanyi [7] via the designation of bilateral equilibrium, is a critical characteristic in the axiomatic processes for solutions. The notion behind this type of consistency is as follows. In a given model, participants might expand their expectations of the model and be willing to permit the calculation of its payments to rely upon these expectations. A solution concept is said to be consistent if it affords duplicate payments to participants in the initial model, as it does for participants in the hypothetical reduced situation. Hence, consistency could be regarded as a qualification of reconciliation's internal "robustness." Consistency has been examined in relation to various concerns by employing *reduced models* of analysis, such as bankruptcy and taxation, negotiation, cost distribution, fair assignments of indivisible goods, resource allocation, and other criteria. Different definitions of a reduction have been introduced, depending on how the participants outside of the sub-coalition should be handed in. The *equal allocation of non-separable costs (EANSC)* (Ransmeier [8]) is an advanced allocation rule. Moulin [9] utilized a particular reduction and a corresponding concept of consistency to demonstrate that the EANSC is a consistent and reasonable allocating concept. The criteria proposed by van den Brink and van der Lann [10], Driessen and Funaki [11], Ju and Wettstein [12], Moulin [9], as well as other criteria, have shown similar effects.

The influence arising from units might vary, relying on various objective and subjective elements in real-world situations, such as the weight of the electoral district carried by a member of congress, the dedication arising from a member of a company, or the varying negotiating abilities of the members of a business team. Furthermore, a lack of symmetry might be generated when different haggling abilities for distinct units are modeled. In accordance with the previous observations, one may expect the resource to be shared in proportion to *weights* determined by the units and its energy classes. Weights emerge naturally in resource-allocation scenarios. For instance, one may be concerned about resource distribution among funding strategies. Therefore, the weights may be provided for the profitability of the various options in all plans. Liao et al. [13] developed the concept of the *weighted allocation of non-separable costs (WANSC)*.

Based on the discussion presented, our research was motivated by the question of whether the WANSC and its related results could be extended to multi-choice TU models by applying replicated behavior.

The study was aimed at responding to this question. The three main results were as follows.

1. We have introduced a novel generalization of the EANSC, the *replicated WANSC*, by taking weights and replicated behavior into consideration in multi-choice TU models. We propose that the replicated WANSC of a multi-choice model corresponds to the WANSC of the corresponding "replicated" TU model.
2. A solution concept can be given an *axiomatic justification*. Inspired by Moulin [9], Hart and Mas-Colell [14] and Liao et al. [15], a specific reduction is defined to characterize the replicated WANSC.
3. Moreover, *dynamic processes* lead the participants to that solution, starting from an arbitrarily efficient payoff vector. In the framework of multi-choice TU models, we aimed to introduce a dynamic process leading to the replicated WANSC via the application of reduced models.

2. Replicated WANSC

Let $\mathbb{U}$ be the universe of all potential participants, for example, the collection of people on the Earth. Any $p \in \mathbb{U}$ is said to be an element, for example, a person on the Earth. For $p \in \mathbb{U}$, $\Delta_p = \{0, 1, \cdots, \delta_p\}$ can be regarded as the participation level collection of participant p, where 0 indicates that they are not participating. Suppose that $\Xi \subseteq \mathbb{U} \setminus \{\emptyset\}$ is the grand collection of total participating elements of a specific interactive process, for example, all people in a country. For $\Xi \subseteq \mathbb{U} \setminus \{\emptyset\}$, let $\Delta^\Xi = \prod_{p\in\Xi} \Delta_p$ be the product set of the level collections for participants Ξ. We denote the zero vector under $\mathbb{R}^\Xi$ as 0_Ξ.

A **multi-choice TU model** is denoted by (Ξ, δ, v), where $\Xi \neq \emptyset$ is a finite collection of participants, $\delta = (\delta_p)_{p\in\Xi}$ is the vector that shows the number of participation levels for each participant, and $v : \Delta^\Xi \to \mathbb{R}$ is a function which assigns to each participating vector $\eta = (\eta_p)_{p\in\Xi} \in \Delta^\Xi$ the value that the participants can gain if each participant p operates at participation level $\eta_p \in \Delta_p$ with $v(0_\Xi) = 0$. In multi-choice TU models, different participants may take on different levels of participation, and the value that a coalition can gain relies upon the participation level at which each participant in the coalition has decided to operate. We denote the class of all multi-choice TU models as MTM.

Let $(\Xi, \delta, v) \in MTM$. A **unit level payoff vector** of (Ξ, δ, v) is a vector $x = (x_p)_{p\in\Xi}$ where x_p indicates the **unit payoff** that participant p obtains for each $p \in \Xi$; hence, $\delta_p \cdot x_p$ is the **accumulation payoff** that participant p obtains at (Ξ, δ, v). For simplicity, one could write "payoff vector" instead of "unit level payoff vector".A **solution** on MTM is a mapping ψ appointing to every $(\Xi, \delta, v) \in MTM$ an element

$$\psi(\Xi, \delta, v) = \big(\psi_p(\Xi, \delta, v)\big)_{p\in\Xi} \in \mathbb{R}^{\Xi}.$$

For convenience, one could stipulate that $\psi_p(\Xi, \delta, v) = 0$ for each $p \in \Xi$ with $\delta_p = 0$.

Given that $(\Xi, \delta, v) \in MTM$, $\eta \in \Delta^{\Xi}$ and $H \subseteq \Xi$, $|H|$ is the number of elements in H and $\eta_H \in \mathbb{R}^H$ is the restriction of η to H, $\eta(H) = \sum_{p\in H} \eta_p$, and $N(\eta) = \{k \in \Xi \mid \eta_k \neq 0\}$. Letting $\eta, \kappa \in \mathbb{R}^{\Xi}$, we define $\kappa \leq \eta$ if $\kappa_p \leq \eta_p$ for all $p \in \Xi$.

Let $(\Xi, \delta, v) \in MTM$. We say that a function $w : \Xi \to \mathbb{R}^+$ is a **weight function** if w is a non-negative mapping. Furthermore, we define $\|H\|_w = \sum_{p\in H} w(p)$ for all weight functions w and for all $H \subseteq \Xi$. Under different conditions, participants in Ξ could be appointed different weights via weight functions. Weights could be considered as *a priori measures of importance*; they are adopted to represent considerations that are not identified by means of the characteristic mapping process. For instance, in apportioning travel expenses among specific sites, weights could be regarded as the number of days consumed at each one (cf., Shapley [3]).

A **standard TU model** is a pair (Ξ, V) where $\Xi \neq \emptyset$ is a finite coalition of participants and V is a function $V : 2^{\Xi} \longrightarrow \mathbb{R}$ with $V(\emptyset) = 0$. We denote the class of all standard TU models as TM. A **solution** on TM is a map Ψ which appoints to every $(\Xi, V) \in TM$ an element $\Psi(\Xi, V)$ of $\mathbb{R}^{\Xi}$.

Given that $(\Xi, \delta, v) \in MTM$, let Ξ^{δ} be a collection of **replicated participants** as follows:

$$\Xi^{\delta} = \bigcup_{p\in\Xi} \Xi^{\delta}_p,$$

where for all $p \in N(\delta)$, $\Xi^{\delta}_p = \{p_1, \ldots, p_{\delta_p}\}$ and for all $p \notin N(\delta)$, $\Xi^{\delta}_p = \emptyset$. Now, for any $H \subseteq \Xi^{\delta}$, we define the action vector $\Lambda^H \in \Delta^{\Xi}$ as follows: for all $p \in \Xi$,

$$\Lambda^H_p = |H \cap \Xi^{\delta}_p|.$$

Then, we define the **replicated TU mode** $\big(\Xi^{\delta}, V^r\big)$ as follows: for all $H \subseteq \Xi^{\delta}$,

$$V^r(H) = v(\Lambda^H).$$

Definition 1. *The* **WANSC** Φ^w *value is defined by*

$$\Phi^w_p(\Xi, V) = \Big[V(\Xi) - V(\Xi \setminus \{p\})\Big] + \frac{w(p)}{\|\Xi\|_w} \cdot \Big[V(\Xi) - \sum_{q\in\Xi} [V(\Xi) - V(\Xi \setminus \{q\})]\Big] \tag{1}$$

for each $(\Xi, \delta, v) \in MTM$, for each weight function w, and for each $p \in \Xi$. Based on the allocation concept expressed by Φ^w, participants first partake in marginal contributions, and then distribute the rest of the utility proportionally according to the weights. Furthermore, the **replicated WANSC** *ϕ^w is defined by*

$$\phi^w_p(\Xi, \delta, v) = \Phi^w_q\big(\Xi^{\delta}, V^r\big) = \frac{1}{\delta_p} \cdot \sum_{t\in\Xi^{\delta}_p} \Phi^w_t\big(\Xi^{\delta}, V^r\big) \tag{2}$$

for each $(\Xi, \delta, v) \in MTM$, for each weight function w, for each $p \in \Xi$, and for each $q \in \Xi^{\delta}_p$, where $\big(\Xi^{\delta}, V^r\big)$ is the corresponding replicated TU model (without the loss of generality, one could suppose that $N(\delta) = \Xi$). Equations (1) and (2) assert that the replicated WANSC of a multi-choice TU model is the WANSC of a corresponding replicated TU model.

Next, a numerical example is provided. Let $(\Xi, \delta, v) \in MTM$, with the participant coalition $\Xi = \{i, j\}$ and participation vector $\delta = (2, 1)$. Define $v(2, 1) = 7$, $v(1, 0) = -3$, $v(1, 1) = 4$, $v(0, 1) = -1$, $v(2, 0) = -3$, and $v(0, 0) = 0$ as the indicators of the efficacy that the participants can produce by means of an interactive process. Depending on definitions presented in Section 1, $N^i = \{i_1, i_2\}$, $N^j = \{j_1\}$ and $N^{(2,1)} = \{i_1, i_2, j_1\}$. Furthermore, $V^r(\{i_1\}) = V^r(\{i_2\}) = v(1,0)$, $V^r(\{i_1, j_1\}) = V^r(\{i_2, j_1\}) = v(1,1)$, $V^r(\{i_1, i_2\}) = v(2,0)$, $V^r(\{i_1, i_2, j_1\}) = v(2,1)$, $V^r(\{j_1\}) = v(0,1)$, and $V^r(\emptyset) = v(0,0)$. Assume that $w(i_1) = w(i_2) = 1$ and $w(j_1) = 3$. According to Definition 1, $\phi^w_i(\Xi, \delta, v) = \Phi^w_{i_1}(\Xi, V^r) = \Phi^w_{i_2}(\Xi, V^r) = 1.6$, and $\phi^w_j(\Xi, \delta, v) = \Phi^w_{j_1}(\Xi, V^r) = 3.8$.

3. Axiomatic Result

Based on a reduced model, several axiomatic results of the replicated WANSC are provided in this section.

Given a solution ψ, $(\Xi, \delta, v) \in MTM$, and $K \subseteq \Xi$, the **reduced model** $\left(K, \delta_K, v^\psi_K\right)$ with respect to ψ and K is defined as follows: for all $\eta \in \Delta^K$,

$$v^\psi_K(\eta) = \begin{cases} 0 & \text{, if } \eta = 0_K, \\ v(\eta, \delta_{\Xi \setminus K}) - \sum\limits_{p \in \Xi \setminus K} \delta_p \cdot \psi_p(\Xi, \delta, v) & \text{, otherwise.} \end{cases}$$

The *consistency* requirement can be stated as follows. For every coalition of participants in a model, one can define a "reduced model" among them by predicting the amounts remaining after the rest of the participants are rendered the payoffs provided via a solution ψ. ψ is *consistent* if it always begets payoffs coincident with those in the initial model when applied to every reduced model. Formally, a solution ψ satisfies the requirement of **consistency (CON)** if $\psi_p\left(K, \delta_K, v^\psi_K\right) = \psi_p(\Xi, \delta, v)$ for each $(\Xi, \delta, v) \in MTM$, for each $K \subseteq \Xi$, and for each $p \in K$.

Remark 1.

1. *Moulin [9] defined the reduction of the TM as follows. Given solution Ψ, $(\Xi, V) \in TM$ and $K \subseteq \Xi$, the* **standard reduced mode** $\left(K, (V)^\Psi_K\right)$ *is defined as follows: for each $H \subseteq K$,*

$$(V)^\Psi_K(H) = \begin{cases} 0 & \text{, if } H = \emptyset, \\ V\big(H \cup (\Xi \setminus K)\big) - \sum\limits_{t \in \Xi \setminus K} \Psi_t(\Xi, V) & \text{, otherwise.} \end{cases}$$

   Moulin [9] considered the standard reduction $\left(K, (V)^\Psi_K\right)$ as that in which every coalition in the subgroup K can achieve payoffs to its participants only if they are consistent with the initial payoffs to "whole" group of participants in $\Xi \setminus K$.
2. *Ψ satisfies* **M-consistency (MCON)** *if $\Psi_p(\Xi, V) = \Psi_p\left(K, (V)^\Psi_K\right)$ for each $(\Xi, V) \in TM$, for each $K \subseteq \Xi$, and for each $p \in K$. Moulin [9] showed that the EANSC satisfies MCON.*
3. *Let $(\Xi, \delta, v) \in MTM$, $K \subseteq \Xi$, and ψ be a solution on MTM. In the reduction $\left(K, \delta_K, v^\psi_K\right)$, the participants in K are gathered to perform an operation. If every participant in K did not participate in the operation, the value that could be achieved is zero. If some participants in K exhibit nonzero participation levels in the operation, the participants in $\Xi \setminus K$ would be asked to make all-out effort. Clearly, the reduction defined in this paper is a multi-choice generalization of the Moulin reduction.*

Let ψ be a solution in the MTM and Ψ be a solution in the TM. Suppose that for all $(\Xi, \delta, v) \in MTM$ and the corresponding replicated TU models $\left(\Xi^\delta, V^r\right)$, for all $p \in \Xi$ and for all $q \in \Xi^\delta_p$,

$$\psi_p(\Xi, \delta, v) = \Psi_q\left(\Xi^\delta, V^r\right) = \frac{1}{\delta_p} \cdot \sum_{t \in \Xi^\delta_p} \Psi_t\left(\Xi^\delta, V^r\right). \tag{3}$$

In fact, "the corresponding replicated TU model of the reduction of a multi-choice model" coincides with "the standard reduction of the corresponding replicated TU model of a multi-choice model", i.e., the order of the "reduction" and "replication" does not matter.

Lemma 1. *Let $(\Xi,\delta,v)\in MTM$ and $\big(\Xi^\delta,V^r\big)$ be the corresponding replicated TU model. Let ψ be a solution on MTM and Ψ be a solution on TM. Suppose that ψ and Ψ satisfy Equation (3). Let $K\subseteq\Xi$ and $\bigcup_{p\in K}\Xi^\delta_p=K^\delta=K^{\delta_K}$. Then*

$$\big(K^{\delta_K},(v^\psi_K)^r_{TU}\big)=\big(K^\delta,(V^r)^\Psi_{K^\delta}\big), \tag{4}$$

where $\big(K^{\delta_K},(v^\psi_K)^r_{TU}\big)$ is the corresponding replicated TU model of $\big(K,\delta_K,v^\psi_K\big)$. Furthermore, for all $p\in N(\delta_K)$ and for all $p_q\in\Xi^\delta_p$,

$$\psi_p\big(K,\delta_K,v^\psi_K\big)=\Psi_{p_q}\big(K^\delta,(V^r)^\Psi_{K^\delta}\big)=\frac{1}{\delta_p}\cdot\sum_{p_k\in\Xi^\delta_p}\Psi_{p_k}\big(K^\delta,(V^r)^\Psi_{K^\delta}\big). \tag{5}$$

Proof. To verify Equation (4), let $K\subseteq\Xi$ and $\bigcup_{p\in K}\Xi^\delta_p=K^\delta=K^{\delta_K}$. Let $H\subseteq K^\delta$. Two situations can be distinguished:

- **Situation 1:** if $H=K^\delta$, then

$$\begin{aligned}(v^\psi_K)^r_{TU}(K^\delta)&=v^\psi_K(\Lambda^{K^\delta})\\&=v^\psi_K(\delta_K)\\&=v(\delta)-\sum_{p\in\Xi\setminus K}\delta_p\cdot\psi_p(\Xi,\delta,v)\\&=V^r(\Xi^\delta)-\sum_{p\in\Xi\setminus K}\delta_p\cdot[\tfrac{1}{\delta_p}\sum_{p_k\in\Xi^\delta_p}\Psi_{p_k}(\Xi^\delta,V^r)]\\&=V^r(\Xi^\delta)-\sum_{t\in\Xi^\delta\setminus K^\delta}\Psi_t(\Xi^\delta,V^r)\\&=(V^r)^\Psi_{K^\delta}(K^\delta).\end{aligned}$$

- **Situation 2:** if $H\subseteq K^\delta, H\neq K^\delta$, then

$$\begin{aligned}(v^\psi_K)^r_{TU}(H)&=v^\psi_K(\Lambda^H)\\&=v(\Lambda^H,\delta_{\Xi\setminus K})-\sum_{p\in\Xi\setminus K}\delta_p\cdot\psi_p(\Xi,\delta,v)\\&=V^r(H\cup(\Xi^\delta\setminus K^\delta))-\sum_{p\in\Xi\setminus K}\delta_p\cdot[\tfrac{1}{\delta_p}\sum_{p_k\in\Xi^\delta_p}\Psi_{p_k}(\Xi^\delta,V^r)]\\&=V^r(H\cup(\Xi^\delta\setminus K^\delta))-\sum_{t\in\Xi^\delta\setminus K^\delta}\Psi_t(\Xi^\delta,V^r)\\&=(V^r)^\Psi_{K^\delta}(H).\end{aligned}$$

Finally, Equation (5) follows via Equations (3) and (4). □

Lemma 2. *The replicated WANSC ϕ^w is consistent.*

Proof. This lemma immediately follows via Definition 1 and Lemma 1 since Φ^w satisfies MCON. □

To characterize the replicated WANSC, one can make use of some more properties. Let ψ be a solution in the MTM. ψ satisfies the requirement of **efficiency (EFF)** if for each $(\Xi,\delta,v)\in MTM$, $\sum_{p\in\Xi}\delta_p\cdot\psi_p(\Xi,\delta,v)=v(\delta)$. ψ satisfies the requirement of **weak efficiency (WEFF)** if for each $(\Xi,\delta,v)\in MTM$ with $|\Xi|=1$, ψ satisfies the requirement of EFF under (Ξ,δ,v). ψ satisfies the **weighted standard for two-person models (WSTM)** if for each $(\Xi,\delta,v)\in MTM$ with $|\Xi|=2$, $\psi_p(\Xi,\delta,v)=\phi^w_p(\Xi,\delta,v)$.

EFF asserts that all participants allot the total utility in its entirety. WEFF simply states that one-person models should be solved efficiently. WSTM is an analogue of the two-person standard according to the work of Hart and Mas-Colell [14]. WSTM asserts that all participants allot the total utility via the solution ϕ^w under two-person models. Based on Definition 1, the replicated WANSC satisfies EFF and WSTM absolutely.

Remark 2. *It is easy to observe that a solution ψ satisfies WEFF if ψ satisfies WSTM and CON. Indeed, for each $(\{p\}, \delta_p, v) \in MTM$, $\delta_p \cdot \psi(\{p\}, \delta_p, v) = v(\delta_p) = \delta_p \cdot \phi^w(\{p\}, \delta_p, v)$. The concept of this proof is similar to that expressed by Hart and Mas-Colell [14], page 599.*

Inspired by Hart and Mas-Colell [14], the replicated WANSC can be characterized by consistency and weighted two-person standardness.

Lemma 3. *A solution ψ satisfies EFF in the MTM if it satisfies WEFF and CON.*

Proof. Let $(\Xi, \delta, v) \in MTM$, and ψ be a solution in the MTM satisfying WEFF and CON. It is trivial if $|\Xi| = 1$ via WEFF. Assume that $|\Xi| \geq 2$. Let $q \in \Xi$, considering the reduction $\big(\{q\}, \delta_q, v^{\psi}_{\{q\}}\big)$ of (Ξ, δ, v). Based on the definition of $v^{\psi}_{\{q\}}$,

$$v^{\psi}_{\{q\}}(\delta_q) = v(\delta) - \sum_{p \in \Xi \setminus \{q\}} \delta_p \cdot \psi_p(\Xi, \delta, v).$$

Since ψ satisfies CON,

$$\psi_q(\Xi, \delta, v) = \psi_q\big(\{q\}, \delta_q, v^{\psi}_{\{q\}}\big).$$

Then, based on WEFF,

$$\psi_q(\Xi, \delta, v) = v^{\psi}_{\{q\}}(\delta_q).$$

Hence, $\sum_{p \in \Xi} \delta_p \cdot \psi_p(\Xi, \delta, v) = v(\delta)$, i.e., ψ satisfies EFF. □

Theorem 1. *A solution ψ on MTM satisfies WSTM and CON if and only if $\psi = \Phi^w$.*

Proof. Clearly, Φ^w satisfies WSTM. Furthermore, Φ^w satisfies CON according to Lemma 2.

In order to represent uniqueness, suppose that ψ satisfies WSTM and CON. Let $(\Xi, \delta, v) \in MTM$. If $|\Xi| \leq 2$, then, based on Remark 2 and WSTM of ψ, $\psi(\Xi, \delta, v) = \Phi^w(\Xi, \delta, v)$. The condition $|\Xi| > 2$: Let $p \in \Xi$ and $K = \{p, k\}$ for some $k \in \Xi \setminus \{p\}$. Then,

$$
\begin{aligned}
& \psi_p(\Xi,\delta,v) - \phi^w_p(\Xi,\delta,v) \\
= \ & \psi_p(K,\delta_K,v^{\psi}_K) - \phi^w_p(K,\delta_K,v^{\phi^w}_K) \ \Big(\text{by CON of } \psi\Big) \\
= \ & \phi^w_p(K,\delta_K,v^{\psi}_K) - \phi^w_p(K,\delta_K,v^{\phi^w}_K) \ \Big(\text{by WSTM of } \psi\Big) \\
= \ & \tfrac{1}{\delta_p} \sum_{p_q \in \Xi^\delta_p} \Phi^w_{p_q}(K^\delta,(V^r)^{\Psi}_{K^\delta}) - \tfrac{1}{\delta_p} \sum_{p_q \in \Xi^\delta_p} \Phi^w_{p_q}(K^\delta,(V^r)^{\Phi^w}_{K^\delta}) \ \Big(\text{by Lemma 1}\Big) \\
= \ & \tfrac{1}{\delta_p} \sum_{p_q \in \Xi^\delta_p} \Big[[(V^r)^{\Psi}_{K^\delta}(K^\delta) - (V^r)^{\Psi}_{K^\delta}(K^\delta \setminus \{p_q\})] \\
& \qquad + \tfrac{w(p_q)}{\|K^\delta\|_w} \cdot [(V^r)^{\Psi}_{K^\delta}(K^\delta) - \sum_{t \in K^\delta} [(V^r)^{\Psi}_{K^\delta}(K^\delta) - (V^r)^{\Psi}_{K^\delta}(K^\delta \setminus \{t\})]] \Big] \\
& -\tfrac{1}{\delta_p} \sum_{p_q \in \Xi^\delta_p} \Big[[(V^r)^{\Phi^w}_{K^\delta}(K^\delta) - (V^r)^{\Phi^w}_{K^\delta}(K^\delta \setminus \{k_l\})] \\
& \qquad + \tfrac{w(p_q)}{\|K^\delta\|_w} \cdot [(V^r)^{\Phi^w}_{K^\delta}(K^\delta) - \sum_{t \in K^\delta} [(V^r)^{\Phi^w}_{K^\delta}(K^\delta) - (V^r)^{\Phi^w}_{K^\delta}(K^\delta \setminus \{t\})]] \Big] \\
& \Big(\text{based on the definition of } \Phi^w \Big) \\
= \ & \tfrac{1}{\delta_p} \sum_{p_q \in \Xi^\delta_p} \tfrac{w(p_q)}{\|K^\delta\|_w} \Big[(V^r)^{\Psi}_{K^\delta}(K^\delta) - (V^r)^{\Phi^w}_{K^\delta}(K^\delta) \Big]. \\
& \Big(\text{based on the definitions of } (V^r)^{\Psi}_{K^\delta} \text{ and } (V^r)^{\Phi^w}_{K^\delta} \Big) \\
= \ & \tfrac{\|\Xi^\delta_p\|_w}{\delta_p \cdot \|K^\delta\|_w} \cdot \Big[(V^r)^{\Psi}_{K^\delta}(K^\delta) - (V^r)^{\Phi^w}_{K^\delta}(K^\delta) \Big].
\end{aligned}
$$

Similarly,

$$
\psi_k(\Xi,\delta,v) - \phi^w_k(\Xi,\delta,v) = \frac{\|\Xi^\delta_k\|_w}{\delta_k \cdot \|K^\delta\|_w} \cdot \Big[(V^r)^{\Psi}_{K^\delta}(K^\delta) - (V^r)^{\Phi^w}_{K^\delta}(K^\delta) \Big].
$$

Clearly, ψ and Φ^w satisfy EFF via Lemma 3 and Remark 2. Based on the EFF of ψ and ϕ^w,

$$
\begin{aligned}
\big[\psi_p(\Xi,\delta,v) - \phi^w_p(\Xi,\delta,v)\big] \cdot \sum_{k \in \Xi} \tfrac{\delta_k}{\|\Xi^\delta_k\|_w} &= \tfrac{\delta_p}{\|\Xi^\delta_p\|_w} \sum_{k \in \Xi} \big[\psi_k(\Xi,\delta,v) - \phi^w_k(\Xi,\delta,v)\big] \\
&= \tfrac{\delta_p}{\|\Xi^\delta_p\|_w} \big[v(\delta) - v(\delta)\big] \\
&= 0.
\end{aligned}
$$

Hence, $\psi_p(\Xi,\delta,v) = \phi^w_p(\Xi,\delta,v)$ for all $p \in \Xi$. $\square$

4. Dynamic Approach

Under standard TU models, Maschler and Owen [16] introduced an X-reduced model to represent dynamic processes that lead participants to the Shapley value [3] as follows.

- Let $(\Xi, V) \in TM$. A **payoff vector** of (Ξ, V) is a vector $(X_p)_{p\in\Xi} \in \mathbb{R}^\Xi$ where X_p denotes the payoff to the participant p for all $p \in \Xi$. A payoff vector $X \in \mathbb{R}^\Xi$ is **efficient** under (Ξ, V) if $\sum_{p\in\Xi} X_p = V(\Xi)$. Define $E(\Xi,V) = \{X \in \mathbb{R}^\Xi | X \text{ as efficient under } (\Xi,V)\}$. Let $(\Xi, V) \in TM$, $K \subseteq \Xi$, and $X \in E(\Xi,V)$. The X**-dependent standard reduced mode** $(K,(V)^X_K)$ is defined as follows: for each $H \subseteq K$,

$$
(V)^X_K(H) = \begin{cases} V(\Xi) - \sum_{p \in \Xi \setminus K} X_p & \text{, if } H = K, \\ (V)^{\Phi^w}_K(H) & \text{, otherwise.} \end{cases}
$$

In the following, the X-dependent standard reduced model is extended to multi-choice TU models.

- Let $(\Xi,\delta,v) \in MTM$. A vector $x \in \mathbb{R}^{\Xi}$ is **efficient** under (Ξ,δ,v) if $\sum_{p\in\Xi}\delta_p x_p = v(\delta)$. Define $E(\Xi,\delta,v) = \{x \in \mathbb{R}^{\Xi} | x \text{ as efficient under } (\Xi,\delta,v)\}$. Let $(\Xi,\delta,v) \in MTM$, $K \subseteq \Xi$, and $x \in E(\Xi,\delta,v)$. The x-**dependent reduced model** (K,δ_K,v^x_K) is defined as follows: for each $\eta \in \Delta^S$,

$$v^x_K(\eta) = \begin{cases} v(\delta) - \sum\limits_{p\in\Xi\setminus K} \delta_p \cdot x_p & \text{, if } \eta = \delta_K, \\ v_K^{\phi^w}(\eta) & \text{, otherwise.} \end{cases}$$

Lemma 4. *Let $(\Xi,\delta,v) \in MTM$, $x \in E(\Xi,\delta,v)$, and (Ξ^δ, V^r) be the corresponding replicated TU model. Furthermore, let $X \in E(\Xi^\delta, V^r)$ with $X_{p_q} = x_p$ for all $p \in \Xi$ and for all $p_q \in \Xi^\delta_p$. Let $K \subseteq \Xi$ and $\bigcup_{p\in K}\Xi^\delta_p = K^\delta = K^{\delta_K}$. Then,*

$$\left(K^{\delta_K}, (v^x_K)^r_{TU}\right) = \left(K^\delta, (V^r)^X_{K^\delta}\right),$$

where $(K^{\delta_K}, (v^x_K)^r_{TU})$ is the corresponding replicated TU model of (K,δ_K,v^x_K). That is, $(K^\delta, (V^r)^X_{K^\delta})$ be the corresponding replicated TU model of (K,δ_K,v^x_K).

Proof. The proof is similar to Lemma 1; therefore, we omit it. □

Similarly to Maschler and Owen [16], Hwang and Liao [17] presented a dynamic process leading to the EANSC. First, we recall the results of Hwang and Liao [17].

Given that $(\Xi, V) \in TM$ with $|\Xi| \geq 3$ and $X \in E(\Xi,V)$, define the *TU-correction function* $F : E(\Xi,V) \to E(\Xi,V)$ as follows: for each $p \in \Xi$,

$$F_p(X) = X_p + \lambda \cdot \sum_{q\in\Xi\setminus\{p\}} \left(\Phi^w_p(\{p,q\},(V)^X_{\{p,q\}}) - X_p\right). \tag{6}$$

Consider the dynamic sequences $\{X^q\}^\infty_{q=1}$; for each $q \in \mathbb{N}$,

$$X^0 = X,\ X^1 = F(X^0),\ X^2 = F(X^1), \cdots,\ X^q = F(X^{q-1}).$$

Theorem 2 (Hwang and Liao [17]). *If $0 < \lambda < \frac{2}{|\Xi|}$, then for each $X \in E(\Xi,V)$, the above dynamic sequence $\{X^q\}^\infty_{q=1}$ converges to $\Phi^w(\Xi,V)$.*

Based on Lemma 1, it is known that the replicated WANSC ϕ^w of a multi-choice TU model is the WANSC Φ^w of a corresponding replicated TU model. In order to provide a relative dynamic result for ϕ^w, it is reasonable that the TU-correction function could be extended to the multi-choice TU models as follows. Let $(\Xi,\delta,v) \in MTM$ with $|\Xi| \geq 3$ and $x \in E(\Xi,\delta,v)$. One can define the *correction function* f: $E(\Xi,\delta,v) \to E(\Xi,\delta,v)$ as follows: for each $p \in \Xi$,

$$f_p(x) = x_p + \lambda \cdot \sum_{q\in\Xi\setminus\{p\}} \frac{\delta_p + \delta_q}{2}\left(\phi^w_p(\{p,q\},\delta_{\{p,q\}},v^x_{\{p,q\}}) - x_p\right). \tag{7}$$

Based on the definition of the x-dependent reduced model and the EFF of Φ^w,

$$\delta_p\phi^w_p(\{p,q\},\delta_{\{p,q\}},v^x_{\{p,q\}}) + \delta_q\phi^w_q(\{p,q\},\delta_{\{p,q\}},v^x_{\{p,q\}}) = \delta_p x_p + \delta_q x_q, \tag{8}$$

and it is easy to verify that $f(x) \in E(\Xi,\delta,v)$ if x is EFF.

Consider the dynamic sequences $\{x^q\}^\infty_{q=1}$; for each $q \in \mathbb{N}$,

$$x^0 = x,\ x^1 = f(x^0),\ x^2 = f(x^1), \cdots,\ x^q = f(x^{q-1}).$$

Based on the notion of the replication and Lemma 1, it is clear that the number of participation levels for each participant in a multi-choice TU model can be treated as the number of replicated participants in a corresponding replicated TU model, and the replicated WANSC of a multi-choice TU model is the WANSC of a corresponding replicated TU model. Thus, it is reasonable that Theorem 2 can be extended to express the dynamic result for the replicated WANSC as follows.

Theorem 3. *Given that $(\Xi, \delta, v) \in MTM$ with $|\Xi| \geq 3$, if $0 < \lambda < \frac{2}{\delta(\Xi)}$, then for each $x \in E(\Xi, \delta, v)$, the above dynamic sequence $\{x^q\}_{q=1}^{\infty}$ converges geometrically to $\phi^w(\Xi, \delta, v)$.*

Proof. Let $(\Xi, \delta, v) \in MTM$, $x \in E(\Xi, \delta, v)$, and (Ξ^δ, V^r) be the corresponding replicated TU model. Furthermore, let $X \in E(\Xi^\delta, V^r)$ with $X_{p_k} = x_p$ for all $p \in \Xi$ and for all $p_k \in \Xi^\delta_p$. Let $p, q \in \Xi$ and $\Xi^\delta_p \cup \Xi^\delta_q = \{p,q\}^\delta = \{p,q\}^{\delta_{\{p,q\}}}$. Let $p \in \Xi^\delta_p$ and $q \in \Xi^\delta_q$. We will show that

$$\begin{aligned} \Phi^w_p(\{p,q\}, \delta_{\{p,q\}}, v^x_{\{p,q\}}) - x_p &= \Phi^w_p(\{p,q\}^\delta, (v^x_{\{p,q\}})^r_{TU}) - x_p \\ &= \Phi^w_p(\{p,q\}^\delta, (V^r)^X_{\{p,q\}^\delta}) - x_p \\ &= \frac{2\delta_q}{\delta_p + \delta_q}\{\Phi^w_p(\{p,q\}, ((V^r)^X_{\{p,q\}^\delta})^X_{\{p,q\}}) - x_p\} \\ &= \frac{2\delta_q}{\delta_p + \delta_q}\{\Phi^w_p(\{p,q\}, (V^r)^X_{\{p,q\}}) - x_p\}. \end{aligned} \tag{9}$$

If Equation (9) holds, then Equation (7) becomes

$$\begin{aligned} f_p(x) &= x_p + \lambda \cdot \sum_{q \in \Xi \setminus \{p\}} \frac{\delta_p + \delta_q}{2}(\Phi^w_p(\{p,q\}, \delta_{\{p,q\}}, v^x_{\{p,q\}}) - x_p) \\ &= x_p + \lambda \cdot \sum_{q \in \Xi \setminus \{p\}} \delta_q \{\phi^w_p(\{p,q\}, (V^r)^X_{\{p,q\}}) - x_p\}. \end{aligned} \tag{10}$$

Consider the corresponding replicated TU model (Ξ^δ, V^r) and the TU-correction function $F : E(\Xi^\delta, V^r) \to E(\Xi^\delta, V^r)$ by

$$F_p(X) = X_p + \lambda \cdot \sum_{q \in \Xi^\delta \setminus \{p\}} (\Phi^w_p(\{p,q\}, (V)^X_{\{p,q\}}) - X_p). \tag{11}$$

It is easy to observe that $\Phi^w_p(\{p,q\}, (V^r)^X_{\{p,q\}}) = x_p$ if $p, q \in \Xi^\delta_p$. So, Equation (10) coincides with Equation (11). Therefore, Theorem 3 immediately follows via Theorem 2. It remains to prove Equation (9). The proof of Equation (9) can be generated via the following three steps.

Step 1: According to Lemma 4,

$$(\{p,q\}^\delta, (v^x_{\{p,q\}})^r_{TU}) = (\{p,q\}^\delta, (V^r)^X_{\{p,q\}^\delta}). \tag{12}$$

Based on Definition 1,

$$\phi^w_p(\{p,q\}, \delta_{\{p,q\}}, v^x_{\{p,q\}}) = \Phi^w_p(\{p,q\}^\delta, (v^x_{\{p,q\}})^r_{TU}). \tag{13}$$

Combining Equation (12) with Equation (13),

$$\phi^w_p(\{p,q\}, \delta_{\{p,q\}}, v^x_{\{p,q\}}) = \Phi^w_p(\{p,q\}^\delta, (V^r)^X_{\{p,q\}^\delta}). \tag{14}$$

Step 2: According to the MCON of Φ^w and the equal treatment property noted by Moulin (1985),

$$\Phi^w_p(\{p,q\}^\delta, (V^r)^X_{\{p,q\}^\delta}) = \Phi^w_p(\{p,q\}, ((V^r)^X_{\{p,q\}^\delta})^{\Phi^w}_{\{p,q\}}). \tag{15}$$

$$\Phi^w_q(\{p,q\}^\delta, (V^r)^X_{\{p,q\}^\delta}) = \Phi^w_q(\{p,q\}, ((V^r)^X_{\{p,q\}^\delta})^{\Phi^w}_{\{p,q\}}). \tag{16}$$

It is also straightforward to verify that

$$(\{p,q\},((V^r)^X_{\{p,q\}^\delta})^X_{\{p,q\}}) = (\{p,q\},(V^r)^X_{\{p,q\}}), \tag{17}$$

and

$$((V^r)^X_{\{p,q\}^\delta})^{\Phi^w}_{\{p,q\}}(\{k\}) = ((V^r)^X_{\{p,q\}^\delta})^X_{\{p,q\}}(\{k\}) \text{ for } k = p,q. \tag{18}$$

Based on Equations (17) and (18),

$$\begin{array}{rl} & \Phi^w_p(\{p,q\},((V^r)^X_{\{p,q\}^\delta})^{\Phi^w}_{\{p,q\}}) - \Phi^w_q(\{p,q\},((V^r)^X_{\{p,q\}^\delta})^{\Phi^w}_{\{p,q\}}) \\ = & \Phi^w_p(\{p,q\},((V^r)^X_{\{p,q\}^\delta})^X_{\{p,q\}}) - \Phi^w_q(\{p,q\},((V^r)^X_{\{p,q\}^\delta})^X_{\{p,q\}}) \\ = & \Phi^w_p(\{p,q\},(V^r)^X_{\{p,q\}}) - \Phi^w_q(\{p,q\},(V^r)^X_{\{p,q\}}). \end{array} \tag{19}$$

Based on Equations (14)–(16) and (19),

$$\begin{array}{rl} & \phi^w_p(\{p,q\},\delta_{\{p,q\}},v^x_{\{p,q\}}) - \phi^w_q(\{p,q\},\delta_{\{p,q\}},v^x_{\{p,q\}}) \\ = & \Phi^w_p(\{p,q\},(V^r)^X_{\{p,q\}}) - \Phi^w_q(\{p,q\},(V^r)^X_{\{p,q\}}). \end{array} \tag{20}$$

Based on Equations (8) and (20), it is easy to observe that

$$\begin{array}{rl} & \phi^w_p(\{p,q\},\delta_{\{p,q\}},v^x_{\{p,q\}}) \\ = & \frac{1}{\delta_p+\delta_q}[\delta_p x_p + \delta_q x_q + \delta_q \Phi^w_p(\{p,q\},(V^r)^X_{\{p,q\}}) - \delta_q \Phi^w_q(\{p,q\},(V^r)^X_{\{p,q\}})]. \end{array} \tag{21}$$

Step 3:

$$\begin{array}{rl} & \phi^w_p(\{p,q\},\delta_{\{p,q\}},v^x_{\{p,q\}}) - x_p \\ = & \frac{1}{\delta_p+\delta_q}[\delta_p x_p + \delta_q x_q + \delta_q \Phi^w_p(\{p,q\},(V^r)^X_{\{p,q\}}) - \delta_q \Phi^w_q(\{p,q\},(V^r)^X_{\{p,q\}})] - x_p \\ & \qquad (\text{ by Equation (21) }) \\ = & \frac{1}{\delta_p+\delta_q}[\delta_q x_q + \delta_q \Phi^w_p(\{p,q\},(V^r)^X_{\{p,q\}}) - \delta_q \Phi^w_q(\{p,q\},(V^r)^X_{\{p,q\}}) - \delta_q x_p] \\ = & \frac{\delta_q}{\delta_p+\delta_q}[x_q + \Phi^w_p(\{p,q\},(V^r)^X_{\{p,q\}}) - \Phi^w_q(\{p,q\},(V^r)^X_{\{p,q\}}) - x_p] \\ = & \frac{\delta_q}{\delta_p+\delta_q}[2(\Phi^w_p(\{p,q\},(V^r)^X_{\{p,q\}}) - X_p)] \\ & (\text{by } x_p = X_p, x_q = X_q, \Phi^w_p(\{p,q\},(V^r)^X_{\{p,q\}}) + \Phi^w_q(\{p,q\},(V^r)^X_{\{p,q\}}) = X_p + X_q\ ) \\ = & \frac{2\delta_q}{\delta_p+\delta_q}[\Phi^w_p(\{p,q\},(V^r)^X_{\{p,q\}}) - X_p]. \end{array}$$

□

5. Discussion and Conclusions

5.1. Discussion and Comparisons

Inspired by some existing outcomes related to solution concepts for standard TU models and multi-choice models, a different solution concept and its related outcomes are presented in this paper. One should compare the outcomes of this paper with the outcomes in the existing literature. Several significant differences can be noted as follows:

1. Traditional TU models have solely focused on non-participation or participation among all participants. However, as previously indicated, it is equitable for each participant to employ distinct levels of participation. Moreover, participants may have corresponding influences in various situations.
   - Therefore, in contrast from the solution concepts used in traditional TU models, the replicated WANSC was generated here to extend the WEANSC by employing weights and multi-choice considerations simultaneously.
   - The axiomatic techniques adopted in this paper are multi-choice analogues of the relative results introduced in the works of Moulin [9], Hart and Mas-Colell [14], and Liao et al. [13].

- The dynamic techniques presented by Hwang and Liao [17] and in this paper correspond exactly to the related techniques presented by Maschler and Owen [16]. However, the main differences between the dynamic outcomes used in this paper and those proposed by Maschler and Owen [16] are that the notion of the 'participation level' and 'weight' are not present in the work of Maschler and Owen [16]. The main differences between the dynamic outcomes used in this paper and those proposed by Hwang and Liao [17] are that our dynamic outcomes (the definition of the correction function and Theorem 3) are based on 'participation level vectors', and Hwang and Liao's [17] dynamic outcomes (the definition of the TU-correction function and Theorem 2) are based on 'participants and coalitions'.

2. Several extensions of the EANSC have been proposed under the considerations of the multi-choice TU model.
    - By applying the *maximal effects* of participants among all participation levels in multi-choice TU models, Liao [15] proposed the *maximal EANSC*. Unlike the present study [15], the notion of weights and replicated behavior are not present in the work of Liao [15].
    - Under multi-choice TU considerations, Hwang and Liao [18] proposed an extended EANSC to increase the effect associated with a specific participant when it is utilized at a specific participation level. In contrast with the work of Hwang and Liao [18], in this study we focused on the overall value of a specific participant by gathering the marginal effects of these participants among their participation levels. The other central difference is that the notion of weights and replicated behavior are not present in the work of Hwang and Liao [18].

5.2. Conclusions

In sum, the purpose of this study was to introduce a different solution concept by applying the notion of weights and replicated behavior in a model of multi-choice consideration. Our main findings were as follows.

1. A generalized analogue of the WANSC, the replicated WANSC, has been generalized to increase the global value for a specific participant by gathering the marginal effects of this participant at its participation level.
2. To convey the applied rationality and mathematical accuracy of the replicated WANSC, an axiomatic result has been presented.
3. The corrective notion and relevant dynamic process have defined to show that the replicated WANSC can be reached by participants who start from an arbitrary efficient payoff vector.

One question providing a motivation for further research arises from the results of this paper, which may be expressed as follows:

- To what extent could alternative solution ideas and their axiomatic characterizations be utilized to generalize the most efficient suited notions in utility-allocating situations?

To the best of our knowledge, these relatied concerns remain unresolved.

Funding: This research received no external funding.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The author is grateful to the editor, the associate editor, and the anonymous referees for very helpful suggestions and comments.

Conflicts of Interest: The author declares no conflict of interest.

References

1. van den Nouweland, A.; Tijs, S.; Potters, J.; Zarzuelo, J. Core and relative solution concepts for multi-choice games. *ZOR-Methods Oper. Res.* **1995**, *41*, 289–311. [CrossRef]
2. Calvo, E.; Santos, J.C. A value for multichoice games. *Math. Soc. Sci.* **2000**, *40*, 341–354. [CrossRef]
3. Shapley, L.S. A value for n-person Game. In *Distinctions Theory Games II*; Kuhn, H.W., Tucker, A.W., Eds.; Princeton University Press: Princeton, NJ, USA, 1953; pp. 307–317.
4. Moulin, H. On additive methods to share joint costs. *Jpn. Econ. Rev.* **1995**, *46*, 303–332. [CrossRef]
5. Hwang, Y.A.; Liao, Y.H. Potentializability and consistency for multi-choice solutions. *Span. Econ. Rev.* **2008**, *10*, 289–301. [CrossRef]
6. Hwang, Y.A.; Liao, Y.H. The unit-level-core for multi-choice games: The replicated core for TU games. *J. Glob. Optim.* **2010**, *47*, 161–171. [CrossRef]
7. Harsanyi, J.C. A bargaining model for the cooperative n-person TMame. In *Contributions to the Theory of Games IV (Annals of Mathematics Studies)*; Tucker, A.W., Luce, R.D., Eds.; Princeton University Press: Princeton, NJ, USA, 1959; Volume 40, pp. 325–355.
8. Ransmeier, J.S. *The Tennessee Valley Authority*; Vanderbilt University Press: Nashville, TN, USA, 1942.
9. Moulin, H. The separability axiom and equal-sharing methods. *J. Econ. Theory* **1985**, *36*, 120–148. [CrossRef]
10. van den Brink, R.; van der Lann, G. Axiomatizations of the normalized Banzhaf value and the Shapley value. *Soc. Choice Welf.* **1998**, *15*, 567–582. [CrossRef]
11. Driessen, T.S.H.; Funakp, Y. Coincidence of and collinearity between game theoretic solutions. *OR Spektrum* **1991**, *13*, 15–30. [CrossRef]
12. Ju, Y.; Wettstein, D. Implementing cooperative solution concepts: A generalized bidding approach. *Econ. Theory* **2009**, *39*, 307–330. [CrossRef]
13. Liao, Y.H.; Chung, L.Y.; Wu, P.H. The EANSC: A weighted extension and axiomatization. *Econ. Bull.* **2015**, *35*, 475–480.
14. Hart, S.; Mas-Colell, A. Potential, value and consistency. *Econometrica* **1989**, *57*, 589–614. [CrossRef]
15. Liao, Y.H. The maximal equal allocation of nonseparable costs on multi-choice games. *Econ. Bull.* **2008**, *3*, 1–8.
16. Maschler, M.; Owen, G. The consonant Shapley value for hyperplane games. *Int. J. Game Theory* **1989**, *18*, 389–407. [CrossRef]
17. Hwang, Y.A.; Liao, Y.H. Consistency and dynamic approach of indexes. *Soc. Choice Welf.* **2010**, *34*, 679–694. [CrossRef]
18. Hwang, Y.A.; Liao, Y.H. Note: Natural extensions of the equal allocation of non-separable costs. *Eng. Lett.* **2020**, *28*, 1325–1330.