mathematics

MDPI

Article

On Some Error Bounds for Milne's Formula in Fractional Calculus

Muhammad Aamir Ali [1], Zhiyue Zhang [1,*] and Michal Fečkan [2,3,*]

[1] Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China

[2] Department of Mathematical Analysis and Numerical Mathematics, Comenius University in Bratislava, Mlynská dolina, 842 48 Bratislava, Slovakia

[3] Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia

* Correspondence: 05298@njnu.edu.cn (Z.Z.); michal.feckan@fmph.uniba.sk (M.F.)

Abstract: In this paper, we found the error bounds for one of the open Newton–Cotes formulas, namely Milne's formula for differentiable convex functions in the framework of fractional and classical calculus. We also give some mathematical examples to show that the newly established bounds are valid for Milne's formula.

Keywords: open Newton–Cotes formulas; fractional calculus; convex functions

MSC: 26D10; 26D15; 26A51

check for updates

Citation: Ali, M.A., Zhang, Z., Fečkan, M. On Some Error Bounds for Milne's Formula in Fractional Calculus. *Mathematics* **2023**, *11*, 146. https://doi.org/10.3390/math11010146

Academic Editors: Ravi P. Agarwal and Bashir Ahmad

Received: 2 December 2022
Revised: 22 December 2022
Accepted: 25 December 2022
Published: 28 December 2022

1. Introduction

In recent years, many researchers have developed numerical integration formulas and found their error bounds using different techniques. To determine the error bounds of numerical integration formulas, mathematical inequalities are used, and the authors used various functions, such as convex functions, bounded functions, Lipschitzian functions, and so on. For example, some error bounds for trapezoidal and midpoint formulas of numerical integration using the convex functions were found in [1,2]. A number of papers have been published on the error bounds of Simpson's formula using the convex functions in different calculi, and some of these bounds can be found in [3–10]. Some error bounds for Newton's formula in numerical integration have also been established by using the convex functions in different calculi, and these bounds can be found in [11–15].

In open Newton–Cotes formulas, Milne's formula is very important and its error bounds for four times differentiable functions were found in [16]. Let $F : [\rho, \sigma] \to \mathbb{R}$ be four times differentiable functions over (ρ, σ) and $\left\| F^{(4)} \right\|_{\infty} = \sup_{x \in (\rho,\sigma)} \left| F^{(4)}(x) \right| < \infty$, then from the following inequality, we can find the error bound of Milne's formula:

$$\left| \frac{1}{3} \left[2F(\rho) - F\left(\frac{\rho + \sigma}{2} \right) + F(\sigma) \right] - \frac{1}{\sigma - \rho} \int_{\rho}^{\sigma} F(x)dx \right| \leq \frac{7(\sigma - \rho)^4}{23040} \left\| F^{(4)} \right\|_{\infty}.$$

Due to its proven applications in a variety of seemingly unrelated fields of science and industry, fractional calculus, which is the calculus of integrals and derivatives of any arbitrary real or complex order, has gained prominence and relevance during the past three decades. In fact, it provides several potentially useful techniques for resolving differential and integral equations, as well as a number of other issues involving unique mathematical physics functions, as well as their extensions and generalizations in one or more variables.

In this paper, we will use the well-known Riemann–Liouville fractional integrals that are given below:

Definition 1 ([17,18])**.** *Let $F \in L_1[\rho,\sigma]$. The Riemann–Liouville fractional integrals (RLFIs) $J^{\alpha}_{\rho+}F$ and $J^{\alpha}_{\sigma-}F$ of order $\alpha > 0$ with $\rho \geq 0$ are defined as follows:*

$$J^{\alpha}_{\rho+}F(x) = \frac{1}{\Gamma(\alpha)}\int_{\rho}^{x}(x-\kappa)^{\alpha-1}F(\kappa)d\kappa,\ x > \rho$$

and

$$J^{\alpha}_{\sigma-}F(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\sigma}(\kappa-x)^{\alpha-1}F(\kappa)d\kappa, x < \sigma,$$

respectively, where Γ is the well-known Gamma function.

Considering the importance of fractional calculus (see [19–23]), researchers have used Riemann–Liouville fractional integrals to develop a number of fractional integral inequalities that have been demonstrated to be very helpful in approximation theory. We can determine the boundaries of formulas used in numerical integration by employing inequalities, such as Hermite–Hadamard, Simpson's, the midpoint, Ostrowski's, and trapezoidal inequalities, etc. In [24], Sarikaya et al. proved some Hermite–Hadamard-type inequalities and trapezoidal-type inequalities for the first time using the Riemann–Liouville fractional integrals. The authors of [25] proved a Riemann–Liouville fractional version of Ostrowski's inequalities for differentiable functions. İşcan and Wu used harmonic convexity and proved Hermite–Hadamard-type inequalities in [26]. For more inequalities via fractional integrals, one can consult [27–30] and references therein.

Inspired by the ongoing studies, we give the fractional version of Milne's formula-type inequalities for differentiable convex functions and Riemann–Liouville fractional integrals. We also give some mathematical examples for newly established inequalities to show their validity. The main advantage of these inequalities is that they can be converted into classical inequalities for $\alpha = 1$ and we can also find some more results for the whole interval of α.

2. Main Results

In this section, we establish and prove some Milne's rule-type inequalities for differentiable convex functions. For this, we will prove the following lemma first.

Lemma 1. *Let $F : [\rho,\sigma] \to \mathbb{R}$ be a differentiable function over $[\rho,\sigma]$. If $F \in L_1[\rho,\sigma]$, then the following equality holds:*

$$\begin{aligned}
&\frac{1}{3}\left[2F\left(\frac{\rho+3\sigma}{4}\right) - F\left(\frac{\rho+\sigma}{2}\right) + 2F\left(\frac{3\rho+\sigma}{4}\right)\right] \qquad (1)\\
&-\frac{\Gamma(\alpha+1)}{2(\sigma-\rho)^{\alpha}}\left[J^{\alpha}_{\rho+}F(\sigma) + J^{\alpha}_{\sigma-}F(\rho)\right]\\
=\ &\frac{\sigma-\rho}{2}\Bigg[\int_0^{\frac{1}{4}}\kappa^{\alpha}\left[F'(\kappa\sigma+(1-\kappa)\rho) - F'(\kappa\rho+(1-\kappa)\sigma)\right]d\kappa\\
&+\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\kappa^{\alpha}-\frac{2}{3}\right)\left[F'(\kappa\sigma+(1-\kappa)\rho) - F'(\kappa\rho+(1-\kappa)\sigma)\right]d\kappa\\
&+\int_{\frac{1}{2}}^{\frac{3}{4}}\left(\kappa^{\alpha}-\frac{1}{3}\right)\left[F'(\kappa\sigma+(1-\kappa)\rho) - F'(\kappa\rho+(1-\kappa)\sigma)\right]d\kappa\\
&+\int_{\frac{3}{4}}^{1}(\kappa^{\alpha}-1)\left[F'(\kappa\sigma+(1-\kappa)\rho) - F'(\kappa\rho+(1-\kappa)\sigma)\right]d\kappa\Bigg].
\end{aligned}$$

Proof. From the fundamental rules of integration and Riemann–Liouville fractional integrals, we have

$$I_1 = (\sigma-\rho)\int_0^{\frac{1}{4}}\kappa^{\alpha}F'(\kappa\sigma+(1-\kappa)\rho)d\kappa \qquad (2)$$

$$
\begin{aligned}
&= \kappa^{\alpha} F(\kappa\sigma + (1-\kappa)\rho)\big|_0^{\frac{1}{4}} - \alpha \int_0^{\frac{1}{4}} \kappa^{\alpha-1} F(\kappa\sigma + (1-\kappa)\rho) d\kappa \\
&= \left(\frac{1}{4}\right)^{\alpha} F\left(\frac{3\rho+\sigma}{4}\right) - \alpha \int_0^{\frac{1}{4}} \kappa^{\alpha-1} F(\kappa\sigma + (1-\kappa)\rho) d\kappa,
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= (\sigma-\rho) \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\kappa^{\alpha} - \frac{2}{3}\right) F'(\kappa\sigma + (1-\kappa)\rho) d\kappa \qquad (3) \\
&= \left(\kappa^{\alpha} - \frac{2}{3}\right) F(\kappa\sigma + (1-\kappa)\rho)\bigg|_{\frac{1}{4}}^{\frac{1}{2}} - \alpha \int_{\frac{1}{4}}^{\frac{1}{2}} \kappa^{\alpha-1} F(\kappa\sigma + (1-\kappa)\rho) d\kappa \\
&= \left(\left(\frac{1}{2}\right)^{\alpha} - \frac{2}{3}\right) F\left(\frac{\rho+\sigma}{2}\right) - \left(\left(\frac{1}{4}\right)^{\alpha} - \frac{2}{3}\right) F\left(\frac{3\rho+\sigma}{4}\right) \\
&\quad -\alpha \int_{\frac{1}{4}}^{\frac{1}{2}} \kappa^{\alpha-1} F(\kappa\sigma + (1-\kappa)\rho) d\kappa,
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= (\sigma-\rho) \int_{\frac{1}{2}}^{\frac{3}{4}} \left(\kappa^{\alpha} - \frac{1}{3}\right) F'(\kappa\sigma + (1-\kappa)\rho) d\kappa \qquad (4) \\
&= \left(\kappa^{\alpha} - \frac{1}{3}\right) F(\kappa\sigma + (1-\kappa)\rho)\bigg|_{\frac{1}{2}}^{\frac{3}{4}} - \alpha \int_{\frac{1}{2}}^{\frac{3}{4}} \kappa^{\alpha-1} F(\kappa\sigma + (1-\kappa)\rho) d\kappa \\
&= \left(\left(\frac{3}{4}\right)^{\alpha} - \frac{1}{3}\right) F\left(\frac{\rho+3\sigma}{4}\right) - \left(\left(\frac{1}{2}\right)^{\alpha} - \frac{1}{3}\right) F\left(\frac{\rho+\sigma}{2}\right) \\
&\quad -\alpha \int_{\frac{1}{2}}^{\frac{3}{4}} \kappa^{\alpha-1} F(\kappa\sigma + (1-\kappa)\rho) d\kappa
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &= (\sigma-\rho) \int_{\frac{3}{4}}^{1} (\kappa^{\alpha} - 1) F'(\kappa\sigma + (1-\kappa)\rho) d\kappa \qquad (5) \\
&= (\kappa^{\alpha} - 1) F(\kappa\sigma + (1-\kappa)\rho)\big|_{\frac{3}{4}}^{1} - \alpha \int_{\frac{3}{4}}^{1} \kappa^{\alpha-1} F(\kappa\sigma + (1-\kappa)\rho) d\kappa \\
&= \left(1 - \left(\frac{3}{4}\right)^{\alpha}\right) F\left(\frac{\rho+3\sigma}{4}\right) - \alpha \int_{\frac{3}{4}}^{1} \kappa^{\alpha-1} F(\kappa\sigma + (1-\kappa)\rho) d\kappa.
\end{aligned}
$$

Thus, we get the following equality by adding (2)–(5)

$$
I_1 + I_2 + I_3 + I_4 = \frac{1}{3}\left[2F\left(\frac{\rho+3\sigma}{4}\right) - F\left(\frac{\rho+\sigma}{2}\right) + 2F\left(\frac{3\rho+\sigma}{4}\right)\right] - J_{\sigma-}^{\alpha} F(\rho). \qquad (6)
$$

Similarly, we have

$$
\begin{aligned}
I_5 + I_6 + I_7 + I_8 &= (\sigma-\rho)\left[\int_0^{\frac{1}{4}} -\kappa^{\alpha} F'(\kappa\rho + (1-\kappa)\sigma) d\kappa \right. \qquad (7) \\
&\quad + \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{2}{3} - \kappa^{\alpha}\right) F'(\kappa\rho + (1-\kappa)\sigma) d\kappa \\
&\quad + \int_{\frac{1}{2}}^{\frac{3}{4}} \left(\frac{1}{3} - \kappa^{\alpha}\right) F'(\kappa\rho + (1-\kappa)\sigma) d\kappa \\
&\quad \left. + \int_{\frac{3}{4}}^{1} (1 - \kappa^{\alpha}) F'(\kappa\rho + (1-\kappa)\sigma) d\kappa \right] \\
&= \frac{1}{3}\left[2F\left(\frac{\rho+3\sigma}{4}\right) - F\left(\frac{\rho+\sigma}{2}\right) + 2F\left(\frac{3\rho+\sigma}{4}\right)\right] - J_{\rho+}^{\alpha} F(\sigma).
\end{aligned}
$$

Hence, we get the required equality by adding (6) and (7). $\square$

Theorem 1. *If all conditions of Lemma 1 hold and $|F'|$ is a convex function, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2F\left(\frac{\rho+3\sigma}{4}\right)-F\left(\frac{\rho+\sigma}{2}\right)+2F\left(\frac{3\rho+\sigma}{4}\right)\right]\right. \\
&\left.-\frac{\Gamma(\alpha+1)}{2(\sigma-\rho)^{\alpha}}\left[J^{\alpha}_{\rho+}F(\sigma)+J^{\alpha}_{\sigma-}F(\rho)\right]\right| \\
\leq\ &\frac{(\sigma-\rho)}{2}\left[\frac{1}{4^{\alpha+1}(\alpha+1)}\left[|F'(\sigma)|+|F'(\rho)|\right]+A_1(\alpha)\left[|F'(\sigma)|+|F'(\rho)|\right]\right. \\
&\left.+A_2(\alpha)\left[|F'(\sigma)|+|F'(\rho)|\right]+\frac{1}{4^{\alpha+1}(\alpha+1)}\left[|F'(\sigma)|+|F'(\rho)|\right]\right],
\end{aligned} \tag{8}$$

where

$$A_1(\alpha)=\begin{cases}
-\dfrac{\left((\alpha+1)\cdot 2^{\alpha+1}-6\right)\cdot 4^{\alpha}+3\cdot 2^{\alpha}}{12(\alpha+1)\cdot 2^{\rho}\cdot 4^{\rho}}, & 0<\alpha\leq\dfrac{\ln\left(\frac{2}{3}\right)}{\ln\left(\frac{1}{4}\right)} \\
-\dfrac{\left(2(\alpha+1)\cdot 3^{\frac{1}{\alpha}}-\alpha\cdot 2^{\frac{1}{\alpha}+3}\right)\cdot 4^{\alpha}-3^{\frac{1}{\alpha}+1}}{12(\alpha+1)\cdot 3^{\frac{1}{\alpha}}\cdot 4^{\alpha}} & \\
-\dfrac{\left(2(\alpha+1)\cdot 3^{\frac{1}{\alpha}}-\alpha\cdot 2^{\frac{1}{\alpha}+2}\right)\cdot 2^{\alpha}-3^{\frac{1}{\alpha}+1}}{6(\alpha+1)\cdot 3^{\frac{1}{\alpha}}\cdot 2^{\alpha}}, & \dfrac{\ln\left(\frac{2}{3}\right)}{\ln\left(\frac{1}{4}\right)}<\alpha\leq\dfrac{\ln\left(\frac{2}{3}\right)}{\ln\left(\frac{1}{2}\right)} \\
\dfrac{\left((\alpha+1)\cdot 2^{\alpha+1}-6\right)\cdot 4^{\alpha}+3\cdot 2^{\alpha}}{12(\alpha+1)\cdot 2^{\alpha}\cdot 4^{\alpha}}, & \alpha>\dfrac{\ln\left(\frac{2}{3}\right)}{\ln\left(\frac{1}{2}\right)}
\end{cases}$$

$$A_2(\alpha)=\begin{cases}
\frac{3^{\alpha+1}}{4^{\alpha+1}(\alpha+1)}-\frac{1}{2^{\alpha+1}(\alpha+1)}-\frac{1}{12}, & 0<\alpha\leq\dfrac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{1}{2}\right)} \\
\dfrac{3^{\frac{1}{\alpha}+1}-\left((\alpha+1)\cdot 3^{\frac{1}{\alpha}}-2\alpha\right)\cdot 2^{\alpha}}{6(\alpha+1)\cdot 3^{\frac{1}{\alpha}}\cdot 2^{\alpha}} & \\
-\dfrac{\left(3(\alpha+1)\cdot 3^{\frac{1}{\alpha}}-4\alpha\right)\cdot 4^{\alpha}-3^{\alpha+\frac{1}{\alpha}+2}}{12(\alpha+1)\cdot 3^{\frac{1}{\alpha}}\cdot 4^{\alpha}}, & \dfrac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{1}{2}\right)}<\alpha\leq\dfrac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{3}{4}\right)} \\
\frac{1}{2^{\alpha+1}(\alpha+1)}-\frac{3^{\alpha+1}}{4^{\alpha+1}(\alpha+1)}+\frac{1}{12}, & \alpha>\dfrac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{3}{4}\right)}.
\end{cases}$$

Proof. Taking modulus of equality (1), we have

$$\begin{aligned}
&\left|\frac{1}{3}\left[2F\left(\frac{\rho+3\sigma}{4}\right)-F\left(\frac{\rho+\sigma}{2}\right)+2F\left(\frac{3\rho+\sigma}{4}\right)\right]\right. \\
&\left.-\frac{\Gamma(\alpha+1)}{2(\sigma-\rho)^{\alpha}}\left[J^{\alpha}_{\rho+}F(\sigma)+J^{\alpha}_{\sigma-}F(\rho)\right]\right| \\
\leq\ &\frac{(\sigma-\rho)}{2}\left[\int_0^{\frac{1}{4}}\kappa^{\alpha}|F'(\kappa\sigma+(1-\kappa)\rho)|d\kappa+\int_0^{\frac{1}{4}}\kappa^{\alpha}|F'(\kappa\rho+(1-\kappa)\sigma)|d\kappa\right. \\
&+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\kappa^{\alpha}-\frac{2}{3}\right||F'(\kappa\sigma+(1-\kappa)\rho)|d\kappa+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\kappa^{\alpha}-\frac{2}{3}\right||F'(\kappa\rho+(1-\kappa)\sigma)|d\kappa \\
&+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\kappa^{\alpha}-\frac{1}{3}\right||F'(\kappa\sigma+(1-\kappa)\rho)|d\kappa+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\kappa^{\alpha}-\frac{1}{3}\right||F'(\kappa\rho+(1-\kappa)\sigma)|d\kappa \\
&\left.+\int_{\frac{3}{4}}^{1}|\kappa^{\alpha}-1||F'(\kappa\sigma+(1-\kappa)\rho)|d\kappa+\int_{\frac{3}{4}}^{1}|\kappa^{\alpha}-1||F'(\kappa\rho+(1-\kappa)\sigma)|d\kappa\right].
\end{aligned}$$

Now, using the convexity of $|F'|$ and the fact that for $\alpha \in (0,1]$ and $\kappa_1, \kappa_2 \in [0,1]$,

$$|\kappa_1^\alpha - \kappa_2^\alpha| \leq |\kappa_1 - \kappa_2|^\alpha,$$

$$\begin{aligned}
&\left|\frac{1}{3}\left[2F\left(\frac{\rho+3\sigma}{4}\right) - F\left(\frac{\rho+\sigma}{2}\right) + 2F\left(\frac{3\rho+\sigma}{4}\right)\right]\right. \\
&\left. -\frac{\Gamma(\alpha+1)}{2(\sigma-\rho)^\alpha}\left[J_{\rho+}^\alpha F(\sigma) + J_{\sigma-}^\alpha F(\rho)\right]\right| \\
\leq\;& \frac{(\sigma-\rho)}{2}\left[\int_0^{\frac{1}{4}} \kappa^\alpha |F'(\kappa\sigma + (1-\kappa)\rho)| d\kappa + \int_0^{\frac{1}{4}} \kappa^\alpha |F'(\kappa\rho + (1-\kappa)\sigma)| d\kappa\right. \\
&+ \int_{\frac{1}{4}}^{\frac{1}{2}} \left|\kappa^\alpha - \frac{2}{3}\right| |F'(\kappa\sigma + (1-\kappa)\rho)| d\kappa + \int_{\frac{1}{4}}^{\frac{1}{2}} \left|\kappa^\alpha - \frac{2}{3}\right| |F'(\kappa\rho + (1-\kappa)\sigma)| d\kappa \\
&+ \int_{\frac{1}{2}}^{\frac{3}{4}} \left|\kappa^\alpha - \frac{1}{3}\right| |F'(\kappa\sigma + (1-\kappa)\rho)| d\kappa + \int_{\frac{1}{2}}^{\frac{3}{4}} \left|\kappa^\alpha - \frac{1}{3}\right| |F'(\kappa\rho + (1-\kappa)\sigma)| d\kappa \\
&\left.+ \int_{\frac{3}{4}}^{1} (1-\kappa)^\alpha |F'(\kappa\sigma + (1-\kappa)\rho)| d\kappa + \int_{\frac{3}{4}}^{1} (1-\kappa)^\alpha |F'(\kappa\rho + (1-\kappa)\sigma)| d\kappa\right] \\
\leq\;& \frac{(\sigma-\rho)}{2}\left[\int_0^{\frac{1}{4}} \kappa^\alpha \left[|F'(\sigma)| + |F'(\rho)|\right] d\kappa + \int_{\frac{1}{4}}^{\frac{1}{2}} \left|\kappa^\alpha - \frac{2}{3}\right| \left[|F'(\sigma)| + |F'(\rho)|\right] d\kappa\right. \\
&\left.+ \int_{\frac{1}{2}}^{\frac{3}{4}} \left|\kappa^\alpha - \frac{1}{3}\right| \left[|F'(\sigma)| + |F'(\rho)|\right] d\kappa + \int_{\frac{3}{4}}^{1} (1-\kappa)^\alpha \left[|F'(\sigma)| + |F'(\rho)|\right] d\kappa\right] \\
=\;& \frac{(\sigma-\rho)}{2}\left[\frac{1}{4^{\alpha+1}(\alpha+1)}\left[|F'(\sigma)| + |F'(\rho)|\right] + A_1(\alpha)\left[|F'(\sigma)| + |F'(\rho)|\right]\right. \\
&\left.+ A_2(\alpha)\left[|F'(\sigma)| + |F'(\rho)|\right] + \frac{1}{4^{\alpha+1}(\alpha+1)}\left[|F'(\sigma)| + |F'(\rho)|\right]\right].
\end{aligned}$$

Thus, the proof is completed. □

Corollary 1. *For $\alpha = 1$ in Theorem 1, we have the following Milne's rule type inequality:*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2F\left(\frac{\rho+3\sigma}{4}\right) - F\left(\frac{\rho+\sigma}{2}\right) + 2F\left(\frac{3\rho+\sigma}{4}\right)\right] - \frac{1}{\sigma-\rho}\int_\rho^\sigma F(x)dx\right| \\
\leq\;& \frac{5(\sigma-\rho)}{48}\left[|F'(\rho)| + |F'(\sigma)|\right].
\end{aligned}$$

This inequality helps us find the error bound of Milne's rule.

Theorem 2. *If all conditions in Lemma 1 hold and $|F'|^q$, $q > 1$ is a convex function, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2F\left(\frac{\rho+3\sigma}{4}\right) - F\left(\frac{\rho+\sigma}{2}\right) + 2F\left(\frac{3\rho+\sigma}{4}\right)\right]\right. \\
&\left. -\frac{\Gamma(\alpha+1)}{2(\sigma-\rho)^\alpha}\left[J_{\rho+}^\alpha F(\sigma) + J_{\sigma-}^\alpha F(\rho)\right]\right| \\
\leq\;& \frac{\sigma-\rho}{2}\left[2\left(\frac{1}{4^{\alpha p+1}(\alpha p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{|F'(\sigma)|^q + 7|F'(\rho)|^q}{32}\right)^{\frac{1}{q}}\right.\right. \\
&\left.+\left(\frac{7|F'(\sigma)|^q + |F'(\rho)|^q}{32}\right)^{\frac{1}{q}}\right\}
\end{aligned} \tag{9}$$

$$
\begin{aligned}
&+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\kappa^{\alpha}-\frac{2}{3}\right|^{p}d\kappa\right)^{\frac{1}{p}}\left\{\left(\frac{3|F'(\sigma)|^{q}+5|F'(\rho)|^{q}}{32}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{5|F'(\sigma)|^{q}+3|F'(\rho)|^{q}}{32}\right)^{\frac{1}{q}}\right\}\\
&+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\kappa^{\alpha}-\frac{1}{3}\right|^{p}d\kappa\right)^{\frac{1}{p}}\left\{\left(\frac{5|F'(\sigma)|^{q}+3|F'(\rho)|^{q}}{32}\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\frac{3|F'(\sigma)|^{q}+5|F'(\rho)|^{q}}{32}\right)^{\frac{1}{q}}\right\}\right].
\end{aligned}
$$

where $p^{-1}+q^{-1}=1$.

Proof. Taking modulus of equality (1) and using Hölder's inequality, we have

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2F\left(\frac{\rho+3\sigma}{4}\right)-F\left(\frac{\rho+\sigma}{2}\right)+2F\left(\frac{3\rho+\sigma}{4}\right)\right]\right.\\
&\left.-\frac{\Gamma(\alpha+1)}{2(\sigma-\rho)^{\alpha}}\left[J_{\rho+}^{\alpha}F(\sigma)+J_{\sigma-}^{\alpha}F(\rho)\right]\right|\\
\leq\;&\frac{(\sigma-\rho)}{2}\left[\int_0^{\frac{1}{4}}\kappa^{\alpha}\left|F'(\kappa\sigma+(1-\kappa)\rho)\right|d\kappa+\int_0^{\frac{1}{4}}\kappa^{\alpha}\left|F'(\kappa\rho+(1-\kappa)\sigma)\right|d\kappa\right.\\
&+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\kappa^{\alpha}-\frac{2}{3}\right|\left|F'(\kappa\sigma+(1-\kappa)\rho)\right|d\kappa+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\kappa^{\alpha}-\frac{2}{3}\right|\left|F'(\kappa\rho+(1-\kappa)\sigma)\right|d\kappa\\
&+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\kappa^{\alpha}-\frac{1}{3}\right|\left|F'(\kappa\sigma+(1-\kappa)\rho)\right|d\kappa+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\kappa^{\alpha}-\frac{1}{3}\right|\left|F'(\kappa\rho+(1-\kappa)\sigma)\right|d\kappa\\
&\left.+\int_{\frac{3}{4}}^{1}|\kappa^{\alpha}-1|\left|F'(\kappa\sigma+(1-\kappa)\rho)\right|d\kappa+\int_{\frac{3}{4}}^{1}|\kappa^{\alpha}-1|\left|F'(\kappa\rho+(1-\kappa)\sigma)\right|d\kappa\right]\\
\leq\;&\frac{\sigma-\rho}{2}\left[\left(\int_0^{\frac{1}{4}}\kappa^{\alpha p}d\kappa\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{4}}\left|F'(\kappa\sigma+(1-\kappa)\rho)\right|^{q}d\kappa\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^{\frac{1}{4}}\kappa^{\alpha p}d\kappa\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{4}}\left|F'(\kappa\rho+(1-\kappa)\sigma)\right|^{q}d\kappa\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\kappa^{\alpha}-\frac{2}{3}\right|^{p}d\kappa\right)^{\frac{1}{p}}\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|F'(\kappa\sigma+(1-\kappa)\rho)\right|^{q}d\kappa\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\kappa^{\alpha}-\frac{2}{3}\right|^{p}d\kappa\right)^{\frac{1}{p}}\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|F'(\kappa\rho+(1-\kappa)\sigma)\right|^{q}d\kappa\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\kappa^{\alpha}-\frac{1}{3}\right|^{p}d\kappa\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|F'(\kappa\sigma+(1-\kappa)\rho)\right|^{q}d\kappa\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\kappa^{\alpha}-\frac{1}{3}\right|^{p}d\kappa\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|F'(\kappa\rho+(1-\kappa)\sigma)\right|^{q}d\kappa\right)^{\frac{1}{q}}\\
&\left.+\left(\int_{\frac{3}{4}}^{1}(1-\kappa)^{\alpha p}\right)^{\frac{1}{p}}\left(\int_{\frac{3}{4}}^{1}\left|F'(\kappa\rho+(1-\kappa)\sigma)\right|^{q}d\kappa\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Now, using convexity of $|F'|^q$, $q > 1$, we have

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2F\left(\frac{\rho+3\sigma}{4}\right)-F\left(\frac{\rho+\sigma}{2}\right)+2F\left(\frac{3\rho+\sigma}{4}\right)\right]\right.\\
&\left.-\frac{\Gamma(\alpha+1)}{2(\sigma-\rho)^{\alpha}}\left[J^{\alpha}_{\rho+}F(\sigma)+J^{\alpha}_{\sigma-}F(\rho)\right]\right|\\
\leq\ & \frac{\sigma-\rho}{2}\left[\left(\int_0^{\frac{1}{4}}\kappa^{\alpha p}d\kappa\right)^{\frac{1}{p}}\left\{\left(\int_0^{\frac{1}{4}}\left[\kappa\left|F'(\sigma)\right|^q+(1-\kappa)\left|F'(\rho)\right|^q\right]d\kappa\right)^{\frac{1}{q}}\right.\right.\\
&\left.+\left(\int_0^{\frac{1}{4}}\left[\kappa\left|F'(\rho)\right|^q+(1-\kappa)\left|F'(\sigma)\right|^q\right]d\kappa\right)^{\frac{1}{q}}\right\}\\
&+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\kappa^{\alpha}-\frac{2}{3}\right|^p d\kappa\right)^{\frac{1}{p}}\left\{\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left[\kappa\left|F'(\sigma)\right|^q+(1-\kappa)\left|F'(\rho)\right|^q\right]d\kappa\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left[\kappa\left|F'(\rho)\right|^q+(1-\kappa)\left|F'(\sigma)\right|^q\right]d\kappa\right)^{\frac{1}{q}}\right\}\\
&+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\kappa^{\alpha}-\frac{1}{3}\right|^p d\kappa\right)^{\frac{1}{p}}\left\{\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left[\kappa\left|F'(\sigma)\right|^q+(1-\kappa)\left|F'(\rho)\right|^q\right]d\kappa\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left[\kappa\left|F'(\rho)\right|^q+(1-\kappa)\left|F'(\sigma)\right|^q\right]d\kappa\right)^{\frac{1}{q}}\right\}\\
&+\left(\int_{\frac{3}{4}}^{1}(1-\kappa)^{\alpha p}\right)^{\frac{1}{p}}\left\{\left(\int_{\frac{3}{4}}^{1}\left[\kappa\left|F'(\sigma)\right|^q+(1-\kappa)\left|F'(\rho)\right|^q\right]d\kappa\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\int_{\frac{3}{4}}^{1}\left[\kappa\left|F'(\rho)\right|^q+(1-\kappa)\left|F'(\sigma)\right|^q\right]d\kappa\right)^{\frac{1}{q}}\right\}\right]\\
=\ & \frac{\sigma-\rho}{2}\left[\left(\frac{1}{4^{\alpha p+1}(\alpha p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{|F'(\sigma)|^q+7|F'(\rho)|^q}{32}\right)^{\frac{1}{q}}\right.\right.\\
&\left.+\left(\frac{7|F'(\sigma)|^q+|F'(\rho)|^q}{32}\right)^{\frac{1}{q}}\right\}\\
&+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\kappa^{\alpha}-\frac{2}{3}\right|^p d\kappa\right)^{\frac{1}{p}}\left\{\left(\frac{3|F'(\sigma)|^q+5|F'(\rho)|^q}{32}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{5|F'(\sigma)|^q+3|F'(\rho)|^q}{32}\right)^{\frac{1}{q}}\right\}\\
&+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\kappa^{\alpha}-\frac{1}{3}\right|^p d\kappa\right)^{\frac{1}{p}}\left\{\left(\frac{5|F'(\sigma)|^q+3|F'(\rho)|^q}{32}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{3|F'(\sigma)|^q+5|F'(\rho)|^q}{32}\right)^{\frac{1}{q}}\right\}\\
&+\left(\frac{1}{4^{\alpha p+1}(\alpha p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{7|F'(\sigma)|^q+|F'(\rho)|^q}{32}\right)^{\frac{1}{q}}\right.
\end{aligned}
$$

$$+\left(\frac{|F'(\sigma)|^q+7|F'(\rho)|^q}{32}\right)^{\frac{1}{q}}\Bigg\}\Bigg].$$

Thus, the proof is completed. □

Corollary 2. *For $\alpha = 1$ in Theorem 2, we have the following Milne's rule type inequality:*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2F\left(\frac{\rho+3\sigma}{4}\right)-F\left(\frac{\rho+\sigma}{2}\right)+2F\left(\frac{3\rho+\sigma}{4}\right)\right]-\frac{1}{\sigma-\rho}\int_\rho^\sigma F(x)dx\right|\\
\leq\ &(\sigma-\rho)\left[\left(\frac{1}{4^{p+1}(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{|F'(\sigma)|^q+7|F'(\rho)|^q}{32}\right)^{\frac{1}{q}}\right.\right.\\
&\left.+\left(\frac{7|F'(\sigma)|^q+|F'(\rho)|^q}{32}\right)^{\frac{1}{q}}\right\}\\
&+\left(\frac{5^{p+1}}{12^{p+1}(p+1)}-\frac{1}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{3|F'(\sigma)|^q+5|F'(\rho)|^q}{32}\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\frac{5|F'(\sigma)|^q+3|F'(\rho)|^q}{32}\right)^{\frac{1}{q}}\right\}\right].
\end{aligned}$$

This inequality helps us find the error bound of Milne's rule.

3. Examples

In this section, we give some mathematical examples to show the validity of the first-time-developed Milne's rule-type inequalities in fractional calculus.

Example 1. *We consider a convex function $F(x) = x^2$ over the interval $[\rho,\sigma] = [1,2]$, and from Theorem 1 for $\alpha = \frac{1}{2}$, we have*

$$\begin{aligned}
L.H.S.\ &=\ \left|\frac{1}{3}\left[2\left(\frac{1+6}{4}\right)^2-\left(\frac{1+2}{2}\right)^2+2\left(\frac{3+2}{4}\right)^2\right]\right.\\
&\quad\left.-\frac{\Gamma\left(\frac{1}{2}+1\right)}{2(2-1)^{\frac{1}{2}}}\left[J_{1+}^{\frac{1}{2}}F(2)+J_{2-}^{\frac{1}{2}}F(1)\right]\right|\\
&=\ \frac{1}{30}
\end{aligned}$$

and

$$\begin{aligned}
R.H.S.\ &=\ \frac{(\sigma-\rho)}{2}\left[\frac{1}{4^{\frac{1}{2}+1}\left(\frac{1}{2}+1\right)}[2(2)+2(1)]+A_1\left(\frac{1}{2}\right)[2(2)+2(1)]\right.\\
&\quad\left.+A_2\left(\frac{1}{2}\right)[2(2)+2(1)]+\frac{1}{4^{\frac{1}{2}+1}\left(\frac{1}{2}+1\right)}[2(2)+2(1)]\right]\\
&=\ \frac{1}{2}\left(\frac{1}{12}(4+1)+0.0165(4+1)+0.1139(4+1)+\frac{1}{12}(4+1)\right)\\
&\approx\ 0.7426.
\end{aligned}$$

Thus,

$$\frac{1}{30} < 0.7426$$

and

$$L.H.S. < R.H.S.$$

which shows that the newly proved inequality (8) is valid.

Example 2. *We consider a convex function $F(x) = x^2$ over the interval $[\rho, \sigma] = [1, 2]$ and from Theorem 2 for $\alpha = \frac{1}{2}$ and $p = q = 2$, we have*

$$\begin{aligned} L.H.S. &= \Bigg| \frac{1}{3}\left[2\left(\frac{1+6}{4}\right)^2 - \left(\frac{1+2}{2}\right)^2 + 2\left(\frac{3+2}{4}\right)^2\right] \\ &\quad - \frac{\Gamma\left(\frac{1}{2}+1\right)}{2(2-1)^{\frac{1}{2}}}\left[J_{1+}^{\frac{1}{2}} F(2) + J_{2-}^{\frac{1}{2}} F(1)\right] \Bigg| \\ &= \frac{1}{30} \end{aligned}$$

and

$$\begin{aligned} R.H.S. = {} & \frac{\sigma-\rho}{2}\Bigg[2\left(\frac{1}{4^{\alpha p+1}(\alpha p+1)}\right)^{\frac{1}{p}}\Bigg\{\left(\frac{|F'(\sigma)|^q + 7|F'(\rho)|^q}{32}\right)^{\frac{1}{q}} \\ & + \left(\frac{7|F'(\sigma)|^q + |F'(\rho)|^q}{32}\right)^{\frac{1}{q}}\Bigg\} \\ & + \left(\int_{\frac{1}{4}}^{\frac{1}{2}} \left|\kappa^\alpha - \frac{2}{3}\right|^p d\kappa\right)^{\frac{1}{p}}\Bigg\{\left(\frac{3|F'(\sigma)|^q + 5|F'(\rho)|^q}{32}\right)^{\frac{1}{q}} \\ & + \left(\frac{5|F'(\sigma)|^q + 3|F'(\rho)|^q}{32}\right)^{\frac{1}{q}}\Bigg\} \\ & + \left(\int_{\frac{1}{2}}^{\frac{3}{4}} \left|\kappa^\alpha - \frac{1}{3}\right|^p d\kappa\right)^{\frac{1}{p}}\Bigg\{\left(\frac{5|F'(\sigma)|^q + 3|F'(\rho)|^q}{32}\right)^{\frac{1}{q}} \\ & + \left(\frac{3|F'(\sigma)|^q + 5|F'(\rho)|^q}{32}\right)^{\frac{1}{q}}\Bigg\}\Bigg] \\ = {} & \frac{1}{2}\Bigg[2\left(\frac{1}{4^2(2)}\right)^{\frac{1}{2}}\Bigg\{\left(\frac{16+7(4)}{32}\right)^{\frac{1}{2}} + \left(\frac{7(16)+4}{32}\right)^{\frac{1}{2}}\Bigg\} \\ & + (0.0017)^{\frac{1}{2}}\Bigg\{\left(\frac{3(16)+5(4)}{32}\right)^{\frac{1}{2}} + \left(\frac{5(16)+3(4)}{32}\right)^{\frac{1}{2}}\Bigg\} \\ & + (0.0524)^{\frac{1}{2}}\Bigg\{\left(\frac{5(16)+3(4)}{32}\right)^{\frac{1}{2}} + \left(\frac{3(16)+5(4)}{32}\right)^{\frac{1}{2}}\Bigg\}\Bigg] \\ \approx {} & 0.9697. \end{aligned}$$

Thus,

$$\frac{1}{30} < 0.9697$$

and

$$L.H.S. < R.H.S.$$

which shows that the newly proved inequality (9) is valid.

4. Conclusions

In this work, we have proven an identity involving Riemann–Liouville fractional integrals and differentiable functions. Then, we have proven some error bounds for Milne's formula by using the newly established identity in the setting of fractional and classical calculus. Finally, we have added some examples to show the validity of the newly proven bounds. The results presented in this paper are more useful than the classical calculus because already established bounds are the special case of our results for $\alpha = 1$ and our results hold for the whole interval given for α. It is an interesting and new problem for upcoming researchers that they can obtain similar inequalities for other fractional integrals and coordinates.

Author Contributions: Conceptualization, M.A.A. and Z.Z.; methodology, M.A.A.; software, M.F.; validation, M.A.A., Z.Z. and M.F.; formal analysis, M.F.; investigation, Z.Z.; resources, Z.Z. and M.F.; data curation, M.A.A.; writing—original draft preparation, M.A.A.; writing—review and editing, Z.Z.; visualization, M.F.; supervision, Z.Z.; project administration, Z.Z. and M.F.; funding acquisition, M.F. All authors have read and agreed to the published version of the manuscript.

Funding: This work is partially supported by the National Natural Science Foundation of China with grant no. 11971241, and by the Slovak Research and Development Agency under the contract No. APVV-18-0308 and by the Slovak Grant Agency VEGA No. 1/0358/20 and No. 2/0127/20.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We thank the worthy referees and editor for their valuable comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Dragomir, S.S.; Agarwal, R. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Appl. Math. Lett.* **1998**, *11*, 91–95. [CrossRef]
2. Kirmaci, U. Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula. *Appl. Math. Comput.* **2004**, *147*, 137–146. [CrossRef]
3. Chen, J.; Huang, X. Some new inequalities of Simpson's type for s-convex functions via fractional integrals. *Filomat* **2017**, *31* , 4989–4997. [CrossRef]
4. Dragomir, S.S.; Agarwal, R.P.; Cerone, P. On Simpson's inequality and applications. *J. Inequal. Appl.* **2000**, *5*, 533–579. [CrossRef]
5. Dragomir, S.S. On Simpson's quadrature formula for mappings of bounded variation and applications. *Tamkang J. Math.* **1999**, *30* , 53–58. [CrossRef]
6. Du, T.; Li, Y.; Yang, Z. A generalization of Simpson's inequality via differentiable mapping using extended (s,m)-convex functions. *Appl. Math. Comput.* **2017**, *293*, 358–369. [CrossRef]
7. Luo, C.; Du, T. Generalized Simpson type inequalities involving Riemann-Liouville fractional integrals and their applications. *Filomat* **2020**, *34*, 751–760. [CrossRef]
8. Set, E.; Akdemir, A.O.; Ozdemir, M.E. Simpson type integral inequalities for convex functions via Riemann-Liouville integrals. *Filomat* **2017**, *31* , 4415–4420. [CrossRef]
9. Budak, H.; Erden, S.; Ali, M.A. Simpson and Newton type inequalities for convex functions via newly defined quantum integrals. *Math. Methods Appl. Sci.* **2021**, *44*, 378–390. [CrossRef]
10. Ali, M.A.; M.Abbas; Budak, H.; Agarwal, P.; Murtaza, G.; Y.-M. New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions. *Adv. Difference Equ.* **2021**, *2021*, 64. [CrossRef]
11. Erden, S.; Iftikhar, S.; Delavar, M.R.; Kumam, P.; Thounthong, P.; Kumam, W. On generalizations of some inequalities for convex functions via quantum integrals. *Rev. R. Acad. Cienc. Exactas Fís. Nat.* **2020**, *114*, 110. [CrossRef]
12. Iftikhar, S.; Kumam, P.; Erden, S. Newton's-type integral inequalities via local fractional integrals. *Fractals* **2020**, *28*, 2050037. [CrossRef]
13. Sitthiwirattham, T.; Nonlaopon, K.; Ali, M.A.; Budak, H. Riemann–Liouville Fractional Newton's Type Inequalities for Differentiable Convex Functions. *Fractal Fract* **2022**. *6*, 175. [CrossRef]
14. Soontharanon, J.; Ali, M.A.; Budak, H.; Kosem, P.; Nonlaopon, K.; Sitthiwirattham, T., Some New Generalized Fractional Newton's Type Inequalities for Convex Functions. *J. Funct. Spaces* **2022**, *2022*, 6261970. [CrossRef]
15. Y. M.Li; Rashid, S.; Hammouch, Z.; Baleanu, D.; Chu, Y.-M. New Newton's type estimates pertaining to local fractional integral via generalized p-convexity with applications. *Fractals* **2021**, *29*, 2140018. [CrossRef]

16. Booth, A.D. *Numerical Methods*, 3rd ed.; Butterworths Scientific Publications: London, UK, 1966.
17. Gorenflo, R.; Mainardi, F. *Fractional Calculus: Integral and Differential Equations of Fractional Order*; Springer: Wien, Austria, 1997.
18. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
19. Rahman, R.U.; Al-Maaitah, A.F.; Qousini, M.; Az-Zo, E.A.; Eldin, S.M.; Abuzar, M. New soliton solutions and modulation instability analysis of fractional Huxley equation. *Results Phys.* **2022**, *44*, 106163. [CrossRef]
20. Green, C.W.H.; Yan, Y. Detailed Error Analysis for a Fractional Adams Method on Caputo-Hadamard Fractional Differential Equations. *Foundations* **2022**, *2*, 839–861. [CrossRef]
21. Ortigueira, M.D.; Bengochea, G. Bilateral tempered fractional derivatives. *Symmetry* **2021**, *13*, 823. [CrossRef]
22. Batiha, I.M.; Obeidat, A.; Alshorm, S.; Alotaibi, A.; Alsubaie, H.; Momani, S.; Albdareen, M.; Zouidi, F.; Eldin, S.; Jahanshahi, H. A Numerical Confirmation of a Fractional-Order COVID-19 Model's Efficiency. *Symmetry* **2022**, *14*, 2583. [CrossRef]
23. Az-Zo, E.A.; Al-Khaled, K.; Darweesh, A. Numeric-analytic solutions for nonlinear oscillators via the modified multi-stage decomposition method. *Mathematics* **2019**, *7*, 550. [CrossRef]
24. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Başak, N. Hermite–Hadamard's inequalities for fractional integrals and relatedfractional inequalities. *Math. Comput. Model.* **2013**, *57*, 2403–2407. [CrossRef]
25. Set, E. New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. *Comput. Math. Appl.* **2012**, *63*, 1147–1154. [CrossRef]
26. İ. İşcan and S. Wu, Hermite–Hadamard type inequalities for harmonically convex functions via fractional integrals. *Appl. Math. Comput.* **2014**, *238*, 237–244. [CrossRef]
27. Awan, M.U.; Talib, S.; Chu, Y.M.; Noor, M.A.; Noor, K.I. Some new refinements of Hermite–Hadamard-type inequalities involving-Riemann—Liouville fractional integrals and applications. *Math. Probl. Eng.* **2020**, *2020*, 3051920. [CrossRef]
28. Peng, C.; Zhou, C.; Du, T.S. Riemann-Liouville fractional Simpson's inequalities through generalized (m, h_1, h_2) -preinvexity. *Ital. J. Pure Appl. Math.* **2017**, *38*, 345–367.
29. Sarikaya, M.Z.; Yildrim, H. On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals. *Miskolc Math. Notes* **2016**, *17*, 1049–1059. [CrossRef]
30. Tunc, M. On new inequalities for h-convex functions via Riemann-Liouville fractional integration. *Filomat* **2013**, *27*, 559–565. [CrossRef]