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Self-Adaptive Method and Inertial Modification for Solving the Split Feasibility Problem and Fixed-Point Problem of Quasi-Nonexpansive Mapping

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Abstract: The split feasibility problem (SFP) has many practical applications, which has attracted the attention of many authors. In this paper, we propose a different method to solve the SFP and the fixed-point problem involving quasi-nonexpansive mappings. We relax the conditions of the operator as well as consider the inertial iteration and the adaptive step size. For example, the convergence generated by our new method is better than that of other algorithms, and the convergence rate of our algorithm greatly improves that of previous algorithms.

Keywords: iterative method; inertial method; quasi-nonexpansive mapping; fixed point; split feasibility problem

MSC: 47H09; 47H10; 47H04

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1. Introduction

Since Censor et al. [1] introduced the SFP, more and more people have paid attention to this problem due to its various applications in resolving practical issues.

Throughout this paper, we suppose that H_1, H_2 are real Hilbert spaces, and C, Q are nonempty convex closed subsets of H_1, H_2, respectively. We consider $A : H_1 \to H_2$ a bounded linear operator, and $A \neq 0$. The SFP can be stated in the following form [2–9]:

Find a point $q \in H_1$, such that

$$q \in C, \quad Aq \in Q. \tag{1}$$

The solution for (1) is denoted by $\mathrm{SFP}(C, Q)$:

$$\mathrm{SFP}(C, Q) := \{q \in C : Aq \in Q\}. \tag{2}$$

We note that the CQ algorithm of Byrne [2] is a very successful approach to (1), where $\{q_n\}$ is generated by the following process:

For any initial estimation as $q_1 \in H_1$,

$$q_{n+1} = P_C(q_n - \tau_n A^*(I - P_Q)Aq_n), \quad \forall n \geq 1. \tag{3}$$

The metric projections of C, Q are P_C, P_Q, and the adjoint operator of A is A^*. We select the step size τ_n with $\tau_n \in \left(0, \frac{2}{\|A\|^2}\right)$. The selection of τ_n is dependent on the operator norm, but the calculation of the operator norm is not easy.

The use of Formula (3) to solve (1) can be further optimized. We introduce the following function:

$$f(q) := \frac{1}{2}\|(I - P_Q)Aq\|^2. \tag{4}$$

According to the above function, we can get the following equation:

$$\nabla f(q) = A^*(I - P_Q)Aq. \tag{5}$$

Therefore, (3) is also included as a particular case of a gradient projection algorithm. In order to conquer the difficulty of numerical calculation, many authors have come up with the variable step size, which does not need to calculate norms $\|A\|$. Later on, based on predecessors, López et al. [4] thought deeply and finally put forward a new variable step size sequence τ_n, expressed in the following form:

$$\tau_n := \frac{\rho_n f(q_n)}{\|\nabla f(q_n)\|^2}, \quad \forall n \geq 1, \tag{6}$$

where ρ_n satisfies these conditions: the upper bound is 4, the lower bound is 0, and ρ_n is a sequence of positive real numbers. If we select the step size (6), we do not need to know any other conditions of the norm $\|A\|$, Q, and A.

In 2019, Qin et al. [5] introduced and studied a fixed point method to solve the SFP (1). Given that $q_1 \in C$, calculate the following iteration as:

$$\begin{cases} y_n = P_C((1-\delta_n)(q_n - \tau_n A^*(I - P_Q)Aq_n) + \delta_n S q_n), \\ q_{n+1} = \alpha_n g(q_n) + \beta_n q_n + \gamma_n y_n, \quad n \geq 1, \end{cases} \tag{7}$$

where $g : C \to C$ is a $k-$contraction, $S : C \to C$ is a nonexpansive mapping, $\mathrm{Fix}(S)$ denotes the set of fixed points of S. $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, and $\{\tau_n\}$ are real sequences and belong to $(0,1)$, satisfying the following:

(C_1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(C_2) $\lim_{n\to\infty} |\tau_n - \tau_{n+1}| = 0$, $0 < \liminf_{n\to\infty} \tau_n \leq \limsup_{n\to\infty} \tau_n < \frac{2}{\|A\|^2}$;
(C_3) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C_4) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$, $\lim_{n\to\infty} |\delta_n - \delta_{n+1}| = 0$;
(C_5) $\alpha_n + \beta_n + \gamma_n = 1$.

Then, $\{q_n\}$ converges strongly to $x^* \in \mathrm{Fix}(S) \cap \mathrm{SFP}(C,Q)$, and x^* is the unique solution of the following variational inequality:

$$\langle q - x^*, g(x^*) - x^* \rangle \leq 0, \quad \forall q \in \mathrm{Fix}(S) \cap \mathrm{SFP}(C,Q).$$

In 2020, Kraikaew et al. [6] further weakened the conditions and simplified the process of proof. They showed that the sequence $\{q_n\}$ produced by (7) converges strongly to $q^* \in \mathrm{Fix}(S) \cap \mathrm{SFP}(C,Q)$ when the following conditions are satisfied:

(C_1) $\limsup_{n\to\infty} \beta_n < 1$;
(C_2) $0 < \liminf_{n\to\infty} \tau_n \leq \limsup_{n\to\infty} \tau_n < \frac{2}{\|A\|^2}$;
(C_3) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C_4) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$;
(C_5) $\alpha_n + \beta_n + \gamma_n = 1$.

Based on previous works, in this paper, we further weaken the conditions and add the inertia method so that the choice of step size does not need to calculate the operator norm.

2. Preliminaries

Throughout this paper, we suppose that H is a real Hilbert space, and D is a nonempty convex closed subset of H. For sequence $\{q_n\}$, and with q in H, we use $q_n \to q$ to represent a strong convergence and $q_n \rightharpoonup q$ to represent a weak convergence. $\mathrm{Fix}(T)$ denotes the fixed points of $T : H \to H$.

The mapping $T : H \to H$ is called:

(i) A nonexpansive mapping if $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in H$;

(ii) A quasi-nonexpansive mapping if $\mathrm{Fix}(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for every $x \in H, y \in \mathrm{Fix}(T)$;
(iii) A firmly nonexpansive mapping if $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$ for any $x, y \in H$;
(iv) A $\iota-$Lipschitz continuous mapping if there is $\iota > 0$ such that $\|Tx - Ty\| \leq \iota\|x - y\|$ for any $x, y \in H$;
(v) A contraction mapping if there exists $\kappa \in [0, 1)$ such that $\|T(x) - T(y)\| \leq \kappa\|x - y\|$, for any $x, y \in H$.

Lemma 1 ([10,11])**.** *For any $x, y \in H$, then*
(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \forall x, y \in X$;
(2) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \forall t \in [0, 1]$.

Recall that P_D is the metric projection operator, that is:

$$P_D y := \arg\min_{x \in D} \|x - y\|^2, \quad y \in H.$$

Lemma 2 ([12–14])**.** *Given $x \in D$ and $y \in H$,*
(1) *$x = P_D y$ is equivalent to $\langle x - y, y - z\rangle \geq 0, \quad \forall z \in D$;*
(2) $\|x - P_D y\|^2 \leq \|x - y\|^2 - \|y - P_D y\|^2$.

From Lemma 2, we can easily prove that $I - P_D$ is firmly nonexpansive.

Lemma 3 ([15])**.** *Let $\{q_n\}$ be a non-negative number sequence, which satisfies:*

$$q_{n+1} \leq (1 - \Gamma_n)q_n + \Gamma_n\Lambda_n, \quad n \geq 1,$$
$$q_{n+1} \leq q_n - \Psi_n + \Phi_n, \quad n \geq 1,$$

where $\{\Gamma_n\}$ is a sequence in the open interval $(0, 1)$, $\{\Psi_n\}$ is a non-negative real sequence, $\{\Lambda_n\}$, $\{\Phi_n\}$ are two sequences on $\mathbb{R}$, satisfying the following:
(1) $\sum_{n=0}^{\infty} \Gamma_n = \infty$;
(2) $\lim_{n\to\infty} \Phi_n = 0$;
(3) $\lim_{k\to\infty} \Psi_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \Lambda_{n_k} \leq 0$, *where $\{n_k\}$ is a subsequence of $\{n\}$.*
Then, $\lim_{n\to\infty} q_n = 0$.

Lemma 4 ([16])**.** *Let $f(q) = \frac{1}{2}\|(I - P_Q)Aq\|^2$. Then ∇f is $\|A\|^2-$ Lipschitz continuous.*

Definition 1 ([17])**.** *Let $T : H \to H$ be a nonlinear operator with* $\mathrm{Fix}(T) \neq \emptyset$*, I be the identity operator. If the following implication holds for $\{q_n\} \in H$:*

$$q_n \rightharpoonup q \text{ and } (I - T)q_n \to 0 \Rightarrow q \in \mathrm{Fix}(T),$$

then we say that $I - T$ is demiclosed at zero.

It is easy to see that this implication holds for Lipschitz continuous quasi-nonexpansive mappings (see [18]).

3. Main Results

Theorem 1. *Let $S : H_1 \to H_1$ be a quasi-nonexpansive mapping. Suppose that $I - S$ is demiclosed at zero, and $g : H_1 \to H_1$ is a κ-contraction. In addition, let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ be sequences in $[0, 1]$, satisfying the following:*
(C_1) $\limsup_{n\to\infty} \beta_n < 1$;
(C_2) $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$;

(C_3) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C_4) $0 < \liminf_{n\to\infty} \delta_n \le \limsup_{n\to\infty} \delta_n < 1$;
(C_5) $\alpha_n + \beta_n + \gamma_n = 1$.

For each $n \ge 1$, we can define the following constant:

$$f(w_n) := \frac{1}{2}\|(I - P_Q)Aw_n\|^2, \tag{8}$$

so that

$$\nabla f(w_n) = A^*(I - P_Q)Aw_n.$$

If $\{q_n\}$ is defined by: $q_0, q_1 \in H_1$ are arbitrarily chosen, and we have the following equation:

$$\begin{cases} w_n = q_n + \mu_n(q_n - q_{n-1}), \\ y_n = P_C((1-\delta_n)(w_n - \tau_n A^*(I - P_Q)Aw_n) + \delta_n S w_n), \\ q_{n+1} = \alpha_n g(q_n) + \beta_n w_n + \gamma_n y_n, \quad n \ge 1, \end{cases} \tag{9}$$

$$\mu_n = \begin{cases} \min\Big\{\mu, \frac{\epsilon_n}{\|q_n - q_{n-1}\|}\Big\}, & \text{if } q_n \ne q_{n-1}, \\ \mu, & \text{otherwise,} \end{cases}$$

where $\mu \ge 0$, $\tau_n = \frac{\rho_n f(w_n)}{\|\nabla f(w_n)\|^2}$, $\rho_n \in (0,4)$, and ρ_n is a sequence of positive real numbers. If $\nabla f(w_n) = 0$, then stop; otherwise, let $n := n+1$ and go to compute the next iteration. Assuming that $\mathrm{Fix}(S) \cap \mathrm{SFP}(C,Q) \ne \emptyset$, then $\{q_n\}$ converges strongly to $x^* \in \mathrm{Fix}(S) \cap \mathrm{SFP}(C,Q)$, and x^* is the unique solution of the following variational inequality:

$$\langle z' - x^*, g(x^*) - x^* \rangle \le 0, \quad \forall z' \in \mathrm{Fix}(S) \cap \mathrm{SFP}(C,Q).$$

Proof. From Lemma 2, we know that x^* is a solution of the following variational inequality:

$$\langle z' - x^*, g(x^*) - x^* \rangle \le 0, \quad \forall z' \in \mathrm{Fix}(S) \cap \mathrm{SFP}(C,Q),$$

if and only if $x^* = P_{\mathrm{Fix}(S)\cap\mathrm{SFP}(C,Q)} g(x^*)$. Since g is contractive and $P_{\mathrm{Fix}(S)\cap\mathrm{SFP}(C,Q)}$ is nonexpansive, we know that $P_{\mathrm{Fix}(S)\cap\mathrm{SFP}(C,Q)} g$ is contractive. Hence, such x^* exists and is unique.

First, let $p \in \mathrm{SFP}(C,Q) \cap \mathrm{Fix}(S)$. Because $p \in C$, according to Lemma 1 and Lemma 2, we find that:

$$\begin{aligned}
& \|y_n - p\|^2 \\
= {} & \|P_C((1-\delta_n)(w_n - \tau_n A^*(I-P_Q)Aw_n) + \delta_n S w_n) - p\|^2 \\
\le {} & \|((1-\delta_n)(w_n - \tau_n A^*(I-P_Q)Aw_n) + \delta_n S w_n) - p\|^2 \\
& -\|(I-P_C)((1-\delta_n)(w_n - \tau_n A^*(I-P_Q)Aw_n) + \delta_n S w_n)\|^2 \\
= {} & \|\delta_n(Sw_n - p) + (1-\delta_n)(w_n - \tau_n A^*(I-P_Q)Aw_n - p)\|^2 \\
& -\|(I-P_C)((1-\delta_n)(w_n - \tau_n A^*(I-P_Q)Aw_n) + \delta_n S w_n)\|^2 \\
= {} & \delta_n\|Sw_n - p\|^2 + (1-\delta_n)\|w_n - \tau_n \nabla f(w_n) - p\|^2 \\
& -\delta_n(1-\delta_n)\|Sw_n - w_n + \tau_n A^*(I-P_Q)Aw_n\|^2 \\
& -\|(I-P_C)((1-\delta_n)(w_n - \tau_n A^*(I-P_Q)Aw_n) + \delta_n S w_n)\|^2 \\
\le {} & \delta_n\|w_n - p\|^2 + (1-\delta_n)\|w_n - \tau_n \nabla f(w_n) - p\|^2 \\
& -\delta_n(1-\delta_n)\|Sw_n - w_n + \tau_n A^*(I-P_Q)Aw_n\|^2 \\
& -\|(I-P_C)((1-\delta_n)(w_n - \tau_n A^*(I-P_Q)Aw_n) + \delta_n S w_n)\|^2 \\
\le {} & \delta_n\|w_n - p\|^2 + (1-\delta_n)(\|w_n - p\|^2 + \tau_n^2\|\nabla f(w_n)\|^2 \\
& -2\tau_n\langle \nabla f(w_n), w_n - p\rangle) - \delta_n(1-\delta_n)\|Sw_n - w_n + \tau_n A^*(I-P_Q)Aw_n\|^2 \\
& -\|(I-P_C)((1-\delta_n)(w_n - \tau_n A^*(I-P_Q)Aw_n) + \delta_n S w_n)\|^2,
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
& \langle \nabla f(w_n), w_n - p \rangle \\
= \ & \langle (I - P_Q)Aw_n - (I - P_Q)Ap, Aw_n - Ap \rangle \\
\geq \ & \|(I - P_Q)Aw_n\|^2 \\
= \ & 2f(w_n). \qquad (11)
\end{aligned}$$

Therefore, by combining (10) and (11), we derive the following:

$$\begin{aligned}
& \|y_n - p\|^2 \\
\leq \ & \|w_n - p\|^2 - 4(1-\delta_n)\tau_n f(w_n) + \tau_n^2(1-\delta_n)\|\nabla f(w_n)\|^2 \\
& -\delta_n(1-\delta_n)\|Sw_n - w_n + \tau_n A^*(I - P_Q)Aw_n\|^2 \\
& -\|(I - P_C)((1-\delta_n)(w_n - \tau_n A^*(I - P_Q)Aw_n) + \delta_n Sw_n)\|^2 \\
= \ & \|w_n - p\|^2 - \rho_n(4-\rho_n)(1-\delta_n)\frac{f^2(w_n)}{\|\nabla f(w_n)\|^2} \\
& -\delta_n(1-\delta_n)\|Sw_n - w_n + \tau_n A^*(I - P_Q)Aw_n\|^2 \\
& -\|(I - P_C)((1-\delta_n)(w_n - \tau_n A^*(I - P_Q)Aw_n) + \delta_n Sw_n)\|^2. \qquad (12)
\end{aligned}$$

Note that $\rho_n \in (0,4)$, $\{\delta_n\}$ is a sequence in $(0,1)$. We thus derive the following equation:

$$\|y_n - p\| \leq \|w_n - p\|. \qquad (13)$$

Putting $z_n = \frac{\beta_n}{1-\alpha_n}w_n + \frac{\gamma_n}{1-\alpha_n}y_n$, by Lemma 1, we can derive that:

$$\begin{aligned}
& \|z_n - p\|^2 \\
= \ & \left\| \frac{\beta_n}{1-\alpha_n}w_n + \frac{\gamma_n}{1-\alpha_n}y_n - p \right\|^2 \\
= \ & \left\| \frac{\beta_n}{1-\alpha_n}(w_n - p) + \frac{\gamma_n}{1-\alpha_n}(y_n - p) \right\|^2 \\
= \ & \frac{\beta_n}{1-\alpha_n}\|w_n - p\|^2 + \frac{\gamma_n}{1-\alpha_n}\|y_n - p\|^2 - \frac{\beta_n}{1-\alpha_n}\frac{\gamma_n}{1-\alpha_n}\|w_n - y_n\|^2.
\end{aligned}$$

From the conditions imposed on $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and (12), we have the following:

$$\begin{aligned}
& \|z_n - p\|^2 \\
\leq \ & \frac{\beta_n}{1-\alpha_n}\|w_n - p\|^2 + \frac{\gamma_n}{1-\alpha_n}\|y_n - p\|^2 \\
\leq \ & \frac{\beta_n}{1-\alpha_n}\|w_n - p\|^2 + \frac{\gamma_n}{1-\alpha_n}\|w_n - p\|^2 \\
& -(1-\delta_n)\frac{\gamma_n}{1-\alpha_n}\rho_n(4-\rho_n)\frac{f^2(w_n)}{\|\nabla f(w_n)\|^2} \\
& -\frac{\gamma_n}{1-\alpha_n}\delta_n(1-\delta_n)\|Sw_n - w_n + \tau_n A^*(I - P_Q)Aw_n\|^2 \\
& -\frac{\gamma_n}{1-\alpha_n}\|(I - P_C)((1-\delta_n)(w_n - \tau_n A^*(I - P_Q)Aw_n) + \delta_n Sw_n)\|^2 \\
= \ & \|w_n - p\|^2 - (1-\delta_n)\frac{\gamma_n}{1-\alpha_n}\rho_n(4-\rho_n)\frac{f^2(w_n)}{\|\nabla f(w_n)\|^2} \\
& -\frac{\gamma_n}{1-\alpha_n}\delta_n(1-\delta_n)\|Sw_n - w_n + \tau_n A^*(I - P_Q)Aw_n\|^2 \\
& -\frac{\gamma_n}{1-\alpha_n}\|(I - P_C)((1-\delta_n)(w_n - \tau_n A^*(I - P_Q)Aw_n) + \delta_n Sw_n)\|^2. \qquad (14)
\end{aligned}$$

From the conditions imposed on $\{\alpha_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$, we have the following equation:

$$\|z_n - p\| \leq \|w_n - p\|. \tag{15}$$

Since $z_n = \frac{\beta_n}{1-\alpha_n} w_n + \frac{\gamma_n}{1-\alpha_n} y_n$, we can get the equations below:

$$\begin{aligned} q_{n+1} &= \alpha_n g(q_n) + \beta_n w_n + \gamma_n y_n \\ &= \alpha_n g(q_n) + (1-\alpha_n) z_n. \end{aligned}$$

Since g is a κ-contraction and by using (15), we can get the following:

$$\begin{aligned} & \|q_{n+1} - p\| \\ = \ & \|\alpha_n g(q_n) + (1-\alpha_n) z_n - p\| \\ = \ & \|\alpha_n (g(q_n) - p) + (1-\alpha_n)(z_n - p)\| \\ \leq \ & \alpha_n \|g(q_n) - p\| + (1-\alpha_n)\|z_n - p\| \\ \leq \ & \alpha_n \|g(q_n) - p\| + (1-\alpha_n)\|w_n - p\| \\ \leq \ & \alpha_n \|g(q_n) - g(p)\| + \alpha_n \|g(p) - p\| + (1-\alpha_n)\|q_n - p + \mu_n(q_n - q_{n-1})\| \\ \leq \ & \alpha_n \kappa \|q_n - p\| + \alpha_n \|g(p) - p\| + (1-\alpha_n)\|q_n - p\| \\ & +(1-\alpha_n)\|\mu_n(q_n - q_{n-1})\| \\ \leq \ & \alpha_n \kappa \|q_n - p\| + \alpha_n \|g(p) - p\| + (1-\alpha_n)\|q_n - p\| + \mu_n \|q_n - q_{n-1}\| \\ \leq \ & \alpha_n \kappa \|q_n - p\| + \alpha_n \|g(p) - p\| + (1-\alpha_n)\|q_n - p\| + \epsilon_n \\ = \ & (1 - \alpha_n(1-\kappa))\|q_n - p\| + \alpha_n(1-\kappa)\left(\frac{\|g(p) - p\|}{1-\kappa} + \frac{\epsilon_n}{\alpha_n(1-\kappa)}\right). \end{aligned}$$

Since $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$, we therefore have $\frac{\epsilon_n}{\alpha_n} < M$, where M is a suitable positive constant. Hence, we have the following:

$$\begin{aligned} & \|q_{n+1} - p\| \\ \leq \ & (1 - \alpha_n(1-\kappa))\|q_n - p\| + \alpha_n(1-\kappa)\left(\frac{\|g(p) - p\| + M}{1-\kappa}\right) \\ \leq \ & \max\left\{\|q_n - p\|, \frac{\|g(p) - p\| + M}{1-\kappa}\right\} \end{aligned}$$

We can thus deduce that:

$$\|q_{n+1} - p\| \leq \max\left\{\|q_1 - p\|, \frac{\|g(p) - p\| + M}{1-\kappa}\right\}. \tag{16}$$

Therefore, the sequence $\{\|q_n - p\|\}$ is bounded.
From Lemma 1, we can get the following:

$$\begin{aligned} \|w_n - p\|^2 &= \|q_n + \mu_n(q_n - q_{n-1}) - p\|^2 \\ &\leq \|q_n - p\|^2 + 2\mu_n \langle q_n - q_{n-1}, w_n - p\rangle \\ &\leq \|q_n - p\|^2 + 2\mu_n \|q_n - q_{n-1}\| \|w_n - p\| \\ &\leq \|q_n - p\|^2 + 2\epsilon_n \|w_n - p\|. \end{aligned}$$

We derive that:

$$\|w_n - p\|^2 \leq \|q_n - p\|^2 + 2\epsilon_n \|w_n - p\|. \tag{17}$$

As p is chosen arbitrarily and g is a κ-contraction, we have the following equations:

$$\begin{aligned}
& \|q_{n+1}-x^*\|^2 \\
= & \|\alpha_n g(q_n)+(1-\alpha_n)z_n-x^*\|^2 \\
= & \|\alpha_n(g(q_n)-x^*)+(1-\alpha_n)(z_n-x^*)\|^2 \\
\leq & \alpha_n^2\|g(q_n)-x^*\|^2+(1-\alpha_n)^2\|z_n-x^*\|^2 \\
& +2\alpha_n\langle g(q_n)-x^*,z_n-x^*\rangle-2\alpha_n^2\langle g(q_n)-x^*,z_n-x^*\rangle \\
\leq & \alpha_n^2\|g(q_n)-x^*\|^2+(1-\alpha_n)^2\|z_n-x^*\|^2 \\
& +2\alpha_n\langle g(q_n)-x^*,z_n-x^*\rangle+2\alpha_n^2\|g(q_n)-x^*\|\|z_n-x^*\| \\
= & \alpha_n^2\|g(q_n)-x^*\|^2+(1-\alpha_n)^2\|z_n-x^*\|^2+2\alpha_n\langle g(q_n)-g(x^*),z_n-x^*\rangle \\
& +2\alpha_n\langle g(x^*)-x^*,z_n-x^*\rangle+2\alpha_n^2\|g(q_n)-x^*\|\|z_n-x^*\| \\
\leq & \alpha_n^2\|g(q_n)-x^*\|^2+(1-\alpha_n)^2\|z_n-x^*\|^2+2\alpha_n\kappa\|q_n-x^*\|\|z_n-x^*\| \\
& +2\alpha_n\langle g(x^*)-x^*,z_n-x^*\rangle+2\alpha_n^2\|g(q_n)-x^*\|\|z_n-x^*\| \\
\leq & \alpha_n^2\|g(q_n)-x^*\|^2+2\alpha_n^2\|g(q_n)-x^*\|\|z_n-x^*\|+(1-\alpha_n)^2\|z_n-x^*\|^2 \\
& +\alpha_n\kappa(\|q_n-x^*\|^2+\|z_n-x^*\|^2)+2\alpha_n\langle g(x^*)-x^*,z_n-x^*\rangle.
\end{aligned} \tag{18}$$

From (15) and (17), we can derive that:

$$\|z_n-x^*\|^2\leq\|q_n-x^*\|^2+2\epsilon_n\|w_n-x^*\|. \tag{19}$$

It thus follows from (18) and (19) that:

$$\begin{aligned}
& \|q_{n+1}-x^*\|^2 \\
\leq & \alpha_n^2\|g(q_n)-x^*\|^2+2\alpha_n^2\|g(q_n)-x^*\|\|z_n-x^*\| \\
& +\alpha_n\kappa(\|q_n-x^*\|^2+\|q_n-x^*\|^2+2\epsilon_n\|w_n-x^*\|) \\
& +2\alpha_n\langle g(x^*)-x^*,z_n-x^*\rangle+(1-\alpha_n)^2(\|q_n-x^*\|^2+2\epsilon_n\|w_n-x^*\|) \\
= & \alpha_n^2\|g(q_n)-x^*\|^2+2\alpha_n^2\|g(q_n)-x^*\|\|z_n-x^*\| \\
& +(\alpha_n^2+(1-2\alpha_n(1-\kappa)))\|q_n-x^*\|^2 \\
& +(2\epsilon_n(1-\alpha_n)^2+2\alpha_n\kappa\epsilon_n)\|w_n-x^*\|+2\alpha_n\langle g(x^*)-x^*,z_n-x^*\rangle \\
\leq & \alpha_n^2\|g(q_n)-x^*\|^2+2\alpha_n^2\|g(q_n)-x^*\|\|z_n-x^*\|+\alpha_n^2\|q_n-x^*\|^2 \\
& +(1-2\alpha_n(1-\kappa))\|q_n-x^*\|^2+4\epsilon_n\|w_n-x^*\|+2\alpha_n\langle g(x^*)-x^*,z_n-x^*\rangle \\
= & (1-2\alpha_n(1-\kappa))\|q_n-x^*\|^2+\alpha_n\Big(\alpha_n\|g(q_n)-x^*\|^2 \\
& +2\alpha_n\|g(q_n)-x^*\|\|z_n-x^*\|+\alpha_n\|q_n-x^*\|^2+\frac{4\epsilon_n}{\alpha_n}\|w_n-x^*\| \\
& +2\langle g(x^*)-x^*,z_n-x^*\rangle\Big) \\
= & (1-2\alpha_n(1-\kappa))\|q_n-x^*\|^2+2\alpha_n(1-\kappa)\frac{1}{2(1-\kappa)}\Big(\alpha_n\|g(q_n)-x^*\|^2 \\
& +2\alpha_n\|g(q_n)-x^*\|\|z_n-x^*\|+\alpha_n\|q_n-x^*\|^2 \\
& +\frac{4\epsilon_n}{\alpha_n}\|w_n-x^*\|+2\langle g(x^*)-x^*,z_n-x^*\rangle\Big).
\end{aligned} \tag{20}$$

On the other hand, by Lemma 1, we can derive that:

$$\begin{aligned}
& \|q_{n+1}-x^*\|^2 \\
= & \|\alpha_n(g(q_n)-z_n)+z_n-x^*\|^2 \\
\leq & \|z_n-x^*\|^2+2\alpha_n\langle g(x_n)-z_n,x_{n+1}-x^*\rangle.
\end{aligned} \tag{21}$$

From (14), (17), (21), we find the following:

$$\begin{aligned}
&\|q_{n+1}-x^*\|^2 \\
\leq\ & \|w_n-x^*\|^2-(1-\delta_n)\frac{\gamma_n}{1-\alpha_n}\rho_n(4-\rho_n)\frac{f^2(w_n)}{\|\nabla f(w_n)\|^2} \\
&-\frac{\gamma_n}{1-\alpha_n}\delta_n(1-\delta_n)\|Sw_n-w_n+\tau_nA^*(I-P_Q)Aw_n\|^2 \\
&-\frac{\gamma_n}{1-\alpha_n}\|(I-P_C)((1-\delta_n)(w_n-\tau_nA^*(I-P_Q)Aw_n) \\
&+\delta_nSw_n)\|^2+2\alpha_n\langle g(q_n)-z_n,q_{n+1}-x^*\rangle \\
\leq\ & \|q_n-x^*\|^2+2\epsilon_n\|w_n-x^*\|-(1-\delta_n)\frac{\gamma_n}{1-\alpha_n}\rho_n(4-\rho_n)\frac{f^2(w_n)}{\|\nabla f(w_n)\|^2} \\
&-\frac{\gamma_n}{1-\alpha_n}\delta_n(1-\delta_n)\|Sw_n-w_n+\tau_nA^*(I-P_Q)Aw_n\|^2 \\
&-\frac{\gamma_n}{1-\alpha_n}\|(I-P_C)((1-\delta_n)(w_n-\tau_nA^*(I-P_Q)Aw_n) \\
&+\delta_nSw_n)\|^2+2\alpha_n\langle g(x_n)-z_n,q_{n+1}-x^*\rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|q_{n+1}-x^*\|^2 \\
\leq\ & \|q_n-x^*\|^2+2\epsilon_n\|w_n-x^*\|-(1-\delta_n)\frac{\gamma_n}{1-\alpha_n}\rho_n(4-\rho_n)\frac{f^2(w_n)}{\|\nabla f(w_n)\|^2} \\
&-\frac{\gamma_n}{1-\alpha_n}\delta_n(1-\delta_n)\|Sw_n-w_n+\tau_nA^*(I-P_Q)Aw_n\|^2 \\
&-\frac{\gamma_n}{1-\alpha_n}\|(I-P_C)((1-\delta_n)(w_n-\tau_nA^*(I-P_Q)Aw_n) \\
&+\delta_nSw_n)\|^2+2\alpha_n\langle g(q_n)-z_n,q_{n+1}-x^*\rangle.
\end{aligned}\tag{22}$$

Set the following:

$$\begin{aligned}
\Gamma_n &= 2\alpha_n(1-\kappa), \\
\Lambda_n &= \frac{1}{2(1-\kappa)}\Big(\alpha_n\|g(q_n)-x^*\|^2 \\
&\quad +2\alpha_n\|g(q_n)-x^*\|\|z_n-x^*\|+\alpha_n\|q_n-x^*\|^2 \\
&\quad +\frac{2\epsilon_n}{\alpha_n}\|w_n-x^*\|+2\langle g(x^*)-x^*,z_n-x^*\rangle\Big), \\
\Psi_n &= (1-\delta_n)\frac{\gamma_n}{1-\alpha_n}\rho_n(4-\rho_n)\frac{f^2(w_n)}{\|\nabla f(w_n)\|^2} \\
&\quad +\frac{\gamma_n}{1-\alpha_n}\delta_n(1-\delta_n)\|Sw_n-w_n+\tau_nA^*(I-P_Q)Aw_n\|^2 \\
&\quad +\frac{\gamma_n}{1-\alpha_n}\|(I-P_C)((1-\delta_n)(w_n-\tau_nA^*(I-P_Q)Aw_n)+\delta_nSw_n)\|^2, \\
\Phi_n &= 2\alpha_n\langle g(q_n)-z_n,q_{n+1}-x^*\rangle.
\end{aligned}$$

Then, (20) and (22) can be rewritten as follows:

$$\begin{aligned}
\|q_{n+1}-x^*\|^2 &\leq (1-\Gamma_n)\|q_n-x^*\|^2+\Gamma_n\Lambda_n, \\
\|q_{n+1}-x^*\|^2 &\leq \|q_n-x^*\|^2-\Psi_n+\Phi_n.
\end{aligned}$$

It is easy to see that $\lim_{n\to\infty}\Gamma_n=0$, $\sum_{n=0}^{\infty}\Gamma_n=\infty$, and $\lim_{n\to\infty}\Phi_n=0$. Therefore, by Lemma 3, we prove that $\lim_{n\to\infty}\|q_n-x^*\|=0$ if we show that $\limsup_{k\to\infty}\Lambda_{n_k}\leq 0$ whenever $\lim_{k\to\infty}\Psi_{n_k}=0$ for any subsequence $\{n_k\}\subset\{n\}$.

Suppose that

$$\lim_{k\to\infty} \Psi_{n_k} = 0. \tag{23}$$

By the conditions of $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, and $\{\gamma_n\}$, we have the following equations:

$$\lim_{k\to\infty} \rho_{n_k}(4-\rho_{n_k})\frac{f^2(w_{n_k})}{\|\nabla f(w_{n_k})\|^2} = 0, \tag{24}$$

$$\lim_{k\to\infty} \|Sw_{n_k} - w_{n_k} + \tau_{n_k}A^*(I-P_Q)Aw_{n_k}\|^2 = 0, \tag{25}$$

$$\lim_{k\to\infty} \|(I-P_C)((1-\delta_{n_k})(w_{n_k} - \tau_{n_k}\nabla f(w_{n_k})) + \delta_{n_k}Sw_{n_k})\| = 0. \tag{26}$$

Equation (24) implies that:

$$\frac{f^2(w_{n_k})}{\|\nabla f(w_{n_k})\|^2} \to 0. \tag{27}$$

From Lemma 4, since $\{\|\nabla f(w_{n_k})\|\}$ is bounded, we derive that $f(w_{n_k}) \to 0$ as $k \to \infty$, so $\lim_{k\to\infty} \|(I-P_Q)Aw_{n_k}\| = 0$. By using (27) and the conditions on $\{\rho_n\}$, we get the following:

$$\tau_{n_k}\|\nabla f(w_{n_k})\| = \frac{\rho_{n_k} f(w_{n_k})}{\|\nabla f(w_{n_k})\|} \to 0. \tag{28}$$

Moreover, according to (25), we can get the equation below:

$$\|Sw_{n_k} - w_{n_k}\| \to 0. \tag{29}$$

From (26), by expanding the formula, since $y_{n_k} = P_C((1-\delta_{n_k})(w_{n_k} - \tau_{n_k}\nabla f(w_{n_k})) + \delta_{n_k}Sw_{n_k})$, we can get:

$$\|(1-\delta_{n_k})(w_{n_k} - \tau_{n_k}\nabla f(w_{n_k})) + \delta_{n_k}Sw_{n_k} - y_{n_k}\| \to 0. \tag{30}$$

By expanding (30), we can get the following equation:

$$\|(1-\delta_{n_k})w_{n_k} - (1-\delta_{n_k})\tau_{n_k}\nabla f(w_{n_k}) + \delta_{n_k}Sw_{n_k} - y_{n_k}\| \to 0. \tag{31}$$

With (31) and (28), we can derive the equation below:

$$\|(1-\delta_{n_k})w_{n_k} + \delta_{n_k}Sw_{n_k} - y_{n_k}\| \to 0, \tag{32}$$

i.e.,

$$\|w_{n_k} - y_{n_k} + \delta_{n_k}(Sw_{n_k} - w_{n_k})\| \to 0. \tag{33}$$

Hence, we arrive at the following:

$$\begin{aligned} & \|w_{n_k} - y_{n_k}\| \\ = \ & \|w_{n_k} - y_{n_k} + \delta_{n_k}(Sw_{n_k} - w_{n_k}) - \delta_{n_k}(Sw_{n_k} - w_{n_k})\| \\ \leq \ & \|w_{n_k} - y_{n_k} + \delta_{n_k}(Sw_{n_k} - w_{n_k})\| + \|\delta_{n_k}(Sw_{n_k} - w_{n_k})\|. \end{aligned}$$

Then, from (29) and (33), we can derive that:

$$\|w_{n_k} - y_{n_k}\| \to 0. \tag{34}$$

From the definition of z_n, we can see the following:

$$\begin{aligned} \|z_{n_k} - w_{n_k}\| &= \left\| \frac{\beta_{n_k}}{1-\alpha_{n_k}} w_{n_k} + \frac{\gamma_{n_k}}{1-\alpha_{n_k}} y_{n_k} - w_{n_k} \right\| \\ &= \left\| -\frac{\gamma_{n_k}}{1-\alpha_{n_k}} w_{n_k} + \frac{\gamma_{n_k}}{1-\alpha_{n_k}} y_{n_k} \right\| \\ &= \frac{\gamma_{n_k}}{1-\alpha_{n_k}} \|y_{n_k} - w_{n_k}\|. \end{aligned}$$

By using (34), we can get the following:

$$\|z_{n_k} - w_{n_k}\| \to 0. \tag{35}$$

Combining (29) and the fact that $I - S$ is demiclosed at zero, we know $\omega_w(w_{n_k}) \subset \mathrm{Fix}(S)$. We select a subsequence $\{w_{n_{k_j}}\}$ of $\{w_{n_k}\}$ to satisfy the following equation:

$$\limsup_{k\to\infty} \langle g(x^*) - x^*, w_{n_k} - x^* \rangle = \lim_{j\to\infty} \langle g(x^*) - x^*, w_{n_{k_j}} - x^* \rangle.$$

Without loss of generality, we can assume that $w_{n_{k_j}} \rightharpoonup z'$. According to $f(w_{n_k}) \to 0$, we can derive that $0 \le f(z') \le \liminf_{j\to\infty} f(w_{n_{k_j}}) = 0$, so $f(z') = 0$, $Az' \in Q$. This means that $z' \in \mathrm{SFP}(C, Q)$ by combining with (34). Therefore, $z' \in \mathrm{Fix}(S) \cap \mathrm{SFP}(C, Q)$. By using (35), we have the following:

$$\begin{aligned} & \limsup_{k\to\infty} \langle g(x^*) - x^*, z_{n_k} - x^* \rangle \\ = \; & \limsup_{k\to\infty} \langle g(x^*) - x^*, w_{n_k} - x^* \rangle \\ = \; & \lim_{j\to\infty} \langle g(x^*) - x^*, w_{n_{k_j}} - x^* \rangle \\ = \; & \langle g(x^*) - x^*, z' - x^* \rangle \\ \le \; & 0. \end{aligned}$$

This means that:

$$\lim_{k\to\infty} \Lambda_{n_k} \le 0.$$

The proof is finished. □

4. Numerical Experiments

Now, we give two numerical experiments. We wrote these programs on Matlab 9.0, performed them on a PC Desktop Intel(R) Core(TM) i5-1035G1 CPU @ 1.00 GHz 1.19 GHz, RAM 16.0 GB.

Example 1. *Solving the system of linear equations $Ax = b$. We assume that $H_1 = H_2 = \mathbb{R}^5$. In the following, we take:*

$$S = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and $g = 0$. Consider $Ax = b$, where

$$A = \begin{pmatrix} 1 & 1 & 2 & 2 & 1 \\ 0 & 2 & 1 & 5 & -1 \\ 1 & 1 & 0 & 4 & -1 \\ 2 & 0 & 3 & 1 & 5 \\ 2 & 2 & 3 & 6 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} \frac{43}{16} \\ 2 \\ \frac{19}{16} \\ \frac{51}{8} \\ \frac{41}{8} \end{pmatrix}.$$

We give the parameters and initial values as follows: For (7) and (9), we choose $\alpha_n = \frac{1}{10n}$, $\beta_n = 0.5$, $\gamma_n = 0.5 - \frac{1}{10n}$, $\delta_n = 0.5$, $q_1 = (1,1,1,1,1)^{\mathrm{T}}$*; for (7), we choose* $\tau_n = \frac{1}{\|A\|^2}$*; for (9), we choose* $\epsilon_n = \frac{1}{n^2}$, $\mu = 1$, $\rho_n = 3 + \frac{1}{n+1}$, $q_0 = (1,1,1,1,1)^{\mathrm{T}}$*. Denote x^* by the solution of $Ax = b$. Then we have* $x^* = (\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1)^{\mathrm{T}}$*. We can see that* $x^* \in \mathrm{Fix}(S)$*. We can see the numerical results of the main algorithms in Table 1 and Figure 1.*

Table 1. Numerical results of scheme (9) as regards Example 1.

$n-1$	$q_n^{(1)}$	$q_n^{(2)}$	$q_n^{(3)}$	$q_n^{(4)}$	$q_n^{(5)}$	E_n
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.5675×10^{0}
10	0.2146	0.1868	0.2938	0.4159	0.8249	2.5806×10^{-1}
50	0.0704	0.1281	0.2543	0.4935	0.9827	2.0782×10^{-2}
100	0.0658	0.1263	0.2519	0.4971	0.9921	9.3812×10^{-3}
500	0.0632	0.1253	0.2504	0.4994	0.9984	1.8944×10^{-3}
1000	0.0628	0.1251	0.2502	0.4997	0.9992	9.4829×10^{-4}
5000	0.0626	0.1250	0.2500	0.4999	0.9998	1.8983×10^{-4}
10000	0.0625	0.1250	0.2500	0.5000	0.9999	9.4925×10^{-5}

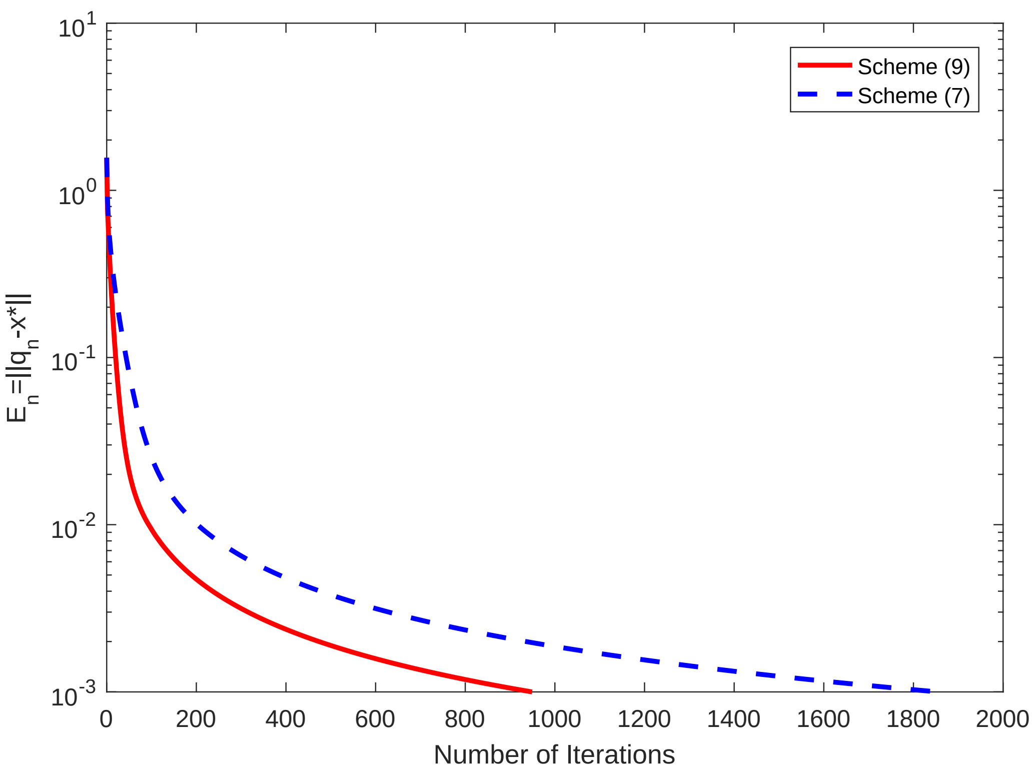

Figure 1. Comparison of scheme (9) and scheme (7) in Example 1.

From Table 1, we can see that with the addition of iterative steps, $\{q_n\}$ is closer to the exact solution x^*. We can also see that these errors are closer to zero. Hence, we can conclude that our *algorithm is reliable. From Figure 1, we can see that our method has fewer iterations than (7), therefore our method has more advantages.*

Example 2. *Seeking the solution to the following problem:*

$$\min\left\{\frac{1}{2}\|Ax-b\|_2^2 : x \in \mathbb{R}^s, \|x\|_1 \le \tau\right\},$$

where $A : \mathbb{R}^s \to \mathbb{R}^m$, $m < s$ is the bounded linear operator, $b \in \mathbb{R}^m$ and $\tau > 0$. A is a sparse matrix, and A is generated by a standard normal distribution. The uniform distribution on the interval (-2,2) generates a real sparse signal x^*. The position of random p is not equal to zero, and the rest remains at zero. We can then obtain the sample data $b = Ax^*$.

The key is to seek the sparse solution of the linear system so that we can use method (9) to solve the problem.

We define $C = \{x : \|x\|_1 \le \tau\}$, $Q = \{b\}$. Because the projection on C has no closed formal solution, we consider the subgradient projection to solve it. Assume that the convex function $c(x)$ and the level set C_n are defined by the following equation:

$$c(x) = \|x\|_1 - \tau, \quad C_n = \{x : c(q_n + \langle \varsigma_n, x - q_n\rangle) \le 0\},$$

then $\varsigma_n \in \partial c(q_n)$. Next, we can calculate the orthogonal projection on C_n according to the following formula:

$$P_{C_n}(x) = \begin{cases} x, & \text{if } c(q_n) + \langle \varsigma_n, x - q_n\rangle \le 0, \\ x - \frac{c(q_n)+\langle \varsigma_n, x - x_n\rangle}{\|\varsigma_n\|^2}, & \text{otherwise.} \end{cases}$$

Note that the subdifferential ∂c on q_n is the following:

$$\partial c(q_n) = \begin{cases} 1, & \text{if } q_n > 0, \\ [-1,1], & \text{if } q_n = 0, \\ -1, & \text{if } q_n < 0. \end{cases}$$

Let $S = I$, $g = 0.4I$. Take $\frac{1}{2}\|Aq_n - b\|_2^2 \le 10^{-3}$ as the stopping criterion. We give the parameters and initial values as follows: For (7) and (9), we choose $\alpha_n = \frac{1}{10n}$, $\beta_n = 0.5$, $\gamma_n = 0.5 - \frac{1}{10n}$, $\delta_n = 0.2$, $q_1 = (1,1,\cdots,1)^{\mathrm{T}}$; for (7), we choose $\tau_n = \frac{1}{\|A\|^2}$; for (9), we choose $\epsilon_n = \frac{1}{n^2}$, $\mu = 1$, $\rho_n = 2$, $q_0 = (1,1,\cdots,1)^{\mathrm{T}}$. We can see the numerical results of the main algorithms in Table 2. Figure 2 shows that when $(m,s,p) = (240,1024,30)$, we can obtain the relationship between the target function and the iterations.

Table 2. Numerical results of scheme (9) and scheme (7) as regards Example 2.

m	*s*	*p*	Scheme (9)		Scheme (7)	
			Iter.	Time (s)	Iter.	Time (s)
240	1024	30	40	0.0584	181	1.7113
480	2048	60	98	0.0933	337	13.8633
720	3072	90	142	0.1578	455	50.5543
960	4096	120	117	0.2073	544	138.2107
1200	5120	150	246	0.3534	795	706.2521
1440	6144	180	291	0.5483	883	1029.8199

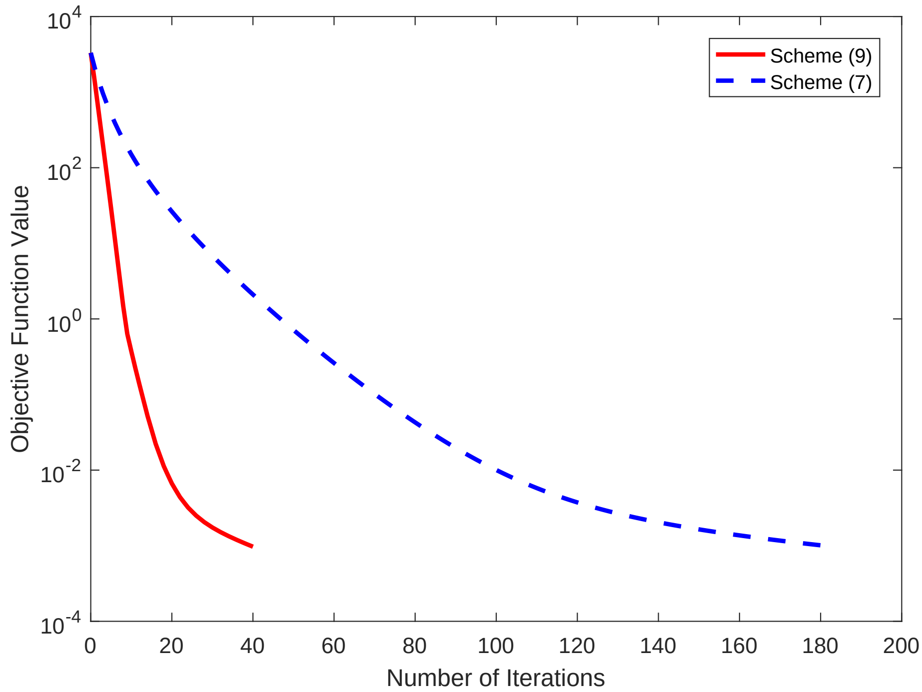

Figure 2. Comparison of scheme (9) and scheme (7) in Example 2, with $(m, s, p) = (240, 1024, 30)$.

From Table 2 and Figure 2, we can see that our iterative method has advantages in both reaction time and the number of iterations.

Example 3. *Let $H_1 = H_2 = L_2[0,1]$, with the inner product given by the following:*

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$

Let $C = \{x \in L_2[0,1] : \|x\| \leq 1\}$, $Q = \left\{x \in L_2[0,1] : \left\langle x, \frac{t}{2} \right\rangle = 0\right\}$ and $(Ax)(t) = \frac{x(t)}{2}$. Let $S = I$, $g = 0.5I$. Take $\|q_n - P_C q_n\|^2 + \|Aq_n - P_Q Aq_n\|^2 \leq 10^{-6}$ as the stopping criterion.

We then give the parameters and initial values as follows: For (7) and (9), we choose $\alpha_n = 0.5n^{-0.7}$, $\beta_n = 0.5$, $\gamma_n = 0.5 - 0.5n^{-0.7}$, $\delta_n = 0.5$; For (7), we choose $\tau_n = \frac{1}{2\|A\|^2}$; For (9), we choose $\epsilon_n = 0.25n^{-1.4}$, $\mu = 0.5$, $\rho_n = 1$, $q_0 = q_1$. The numerical results for each choice of q_1 are shown in Table 3. Figure 3 shows that the error plotting for $q_1 = 4t^2 + t + 3$.

Table 3. Numerical results of scheme (9) and scheme (7) as regards Example 3.

q_1	Scheme (9)		Scheme (7)	
	Iter.	Time (s)	Iter.	Time (s)
$4t^2 + t + 3$	43	0.0413	57	0.0261
$e^t + 2t$	37	0.0406	55	0.0249
$2^t/16$	10	0.0338	25	0.0146
$t^3 + \sin t$	21	0.0363	46	0.0208

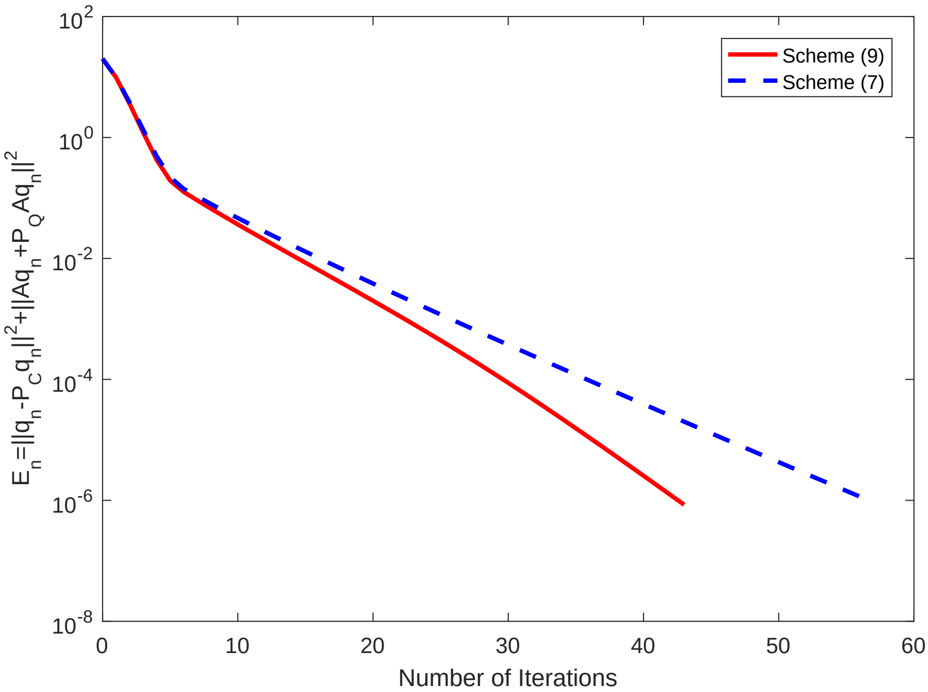

Figure 3. Comparison of scheme (9) and scheme (7) in Example 3, with $q_1 = 4t^2 + t + 3$.

5. Conclusions

In this paper, we proposed a new method to solve the SFP and the fixed-point problem involving quasi-nonexpansive mappings. Compared with the work of (7), the conditions were relaxed, and the nonexpansive mapping was extended to quasi-nonexpansive mapping. The inertia was also added to accelerate the convergence rate further. In addition, the selection of step size no longer depended on the operator norm.

By solving some examples, we have illustrated the effectiveness and practicability of the method. We compared all numerical implementations of this method with (7). As shown in Figures 1 and 2, we can find that (9) is superior. For these reasons, we can see that (9) is more effective than (7).

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