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Hermite–Hadamard-Type Inequalities and Two-Point Quadrature Formula

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Abstract: As convexity plays an important role in many aspects of mathematical programming, e.g., for obtaining sufficient optimality conditions and in duality theorems, and one of the most important inequalities for convex functions is the Hermite–Hadamard inequality, the importance of this paper lies in providing some new improvements for convex functions and new directions in studying new variants of the Hermite–Hadamard inequality. The first part of the article includes some known concepts regarding convex functions and related inequalities. In the second part of the study, a derivation of the Hermite–Hadamard inequality for convex functions of higher order is given, emphasizing the purpose and importance of some quadrature formulas. In the third section, the applications of the main results are presented by obtaining Hermite–Hadamard-type estimates for various classical quadrature formulas such as the Gauss–Legendre two-point quadrature formula and the Gauss–Chebyshev two-point quadrature formulas of the first and second kind.

Keywords: Hermite–Hadamard inequalities; weighted two-point formula; higher-order convex functions; w-harmonic sequences of functions

MSC: 25D15; 65D30; 65D32

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1. Introduction

The well-known Jensen inequality [1] states that if $f : X \to \mathbb{R}$ is a convex mapping defined on the linear space X and $x_i \in X$, $p_i \geq 0$, $i = 1, \ldots, n$, $P = \sum_{i=1}^{n} p_i > 0$, then

$$f\left(\frac{1}{P}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P}\sum_{i=1}^{n} p_i f(x_i).$$

The Hermite–Hadamard inequality gives us an estimate of the (integral) mean value of a continuous convex function as follows.

If $f : [a, b] \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{1}{2}f(a) + \frac{1}{2}f(b). \tag{1}$$

If f is concave, then above inequalities are reversed.

Over the last decades, these inequalities have been investigated in many papers and monographs, since they are very useful in approximation theory, optimization theory, information theory and numerical analysis (see [2] and the references cited therein).

Combining a special case of the integral Jensen inequality and a special case of the integral Lah–Ribarič inequality, the following weighted Hermite–Hadamard inequality is established (see [1], p. 145).

Theorem 1. *Let $p : [a, b] \to \mathbb{R}$ be a non-negative function. If f is a convex function given on an interval I, then we have*

$$f(\lambda) \leq \frac{1}{P(b)} \int_a^b p(x) f(x)\, dx \leq \frac{b-\lambda}{b-a} f(a) + \frac{\lambda - a}{b-a} f(b)$$

or

$$P(b) f(\lambda) \leq \int_a^b p(x) f(x)\, dx \leq P(b) \left[\frac{b-\lambda}{b-a} f(a) + \frac{\lambda - a}{b-a} f(b) \right],$$

where

$$P(t) = \int_a^t p(x)\, dx \quad \text{and} \quad \lambda = \frac{1}{P(b)} \int_a^b p(x) x\, dx.$$

In [3,4], the authors proved some weighted versions of the general integral identities using harmonic sequences of polynomials and w-harmonic sequences of functions. In order to introduce one of these identities, we consider the subdivision $\sigma = \{a = x_0 < x_1 < \ldots < x_m = b\}$ of the segment $[a, b]$, $m \in \mathbb{N}$. If $w : [a, b] \to \mathbb{R}$ is an arbitrary integrable function, then for each segment $[x_{k-1}, x_k]$, $k = 1, \ldots, m$, we define w-harmonic sequences of functions $\{w_{kj}\}_{j=1,\ldots,n}$ by

$$\begin{aligned} w'_{k1}(t) &= w(t), \quad t \in [x_{k-1}, x_k], \\ w'_{kj}(t) &= w_{k,j-1}(t), \quad t \in [x_{k-1}, x_k],\ j = 2, 3, \ldots, n \end{aligned} \tag{2}$$

and the function $W_{n,w}$ by

$$W_{n,w}(t, \sigma) = \begin{cases} w_{1n}(t), & t \in [a, x_1], \\ w_{2n}(t), & t \in (x_1, x_2], \\ \ldots \\ w_{mn}(t), & t \in (x_{m-1}, b]. \end{cases} \tag{3}$$

An approximation of an integral $\int_a^b w(t) g(t)\, dt$ via w-harmonic sequences of functions is given in the general integral identity proved in the following theorem (see [3]).

Theorem 2. *If $g : [a, b] \to \mathbb{R}$ is such that $g^{(n)}$ is piecewise continuous on $[a, b]$, then the following identity holds:*

$$\begin{aligned} \int_a^b w(t) g(t)\, dt &= \sum_{j=1}^{n} (-1)^{j-1} \Big[w_{mj}(b) g^{(j-1)}(b) \\ &\quad + \sum_{k=1}^{m-1} \big[w_{kj}(x_k) - w_{k+1,j}(x_k) \big] g^{(j-1)}(x_k) - w_{1j}(a) g^{(j-1)}(a) \Big] \\ &\quad + (-1)^n \int_a^b W_{n,w}(t, \sigma) g^{(n)}(t)\, dt. \end{aligned} \tag{4}$$

The following Hermite–Hadamard-type inequality is obtained in [5] using identity (4).

Theorem 3. *Suppose $w : [a, b] \to \mathbb{R}$ is an arbitrary integrable function, and w-harmonic sequences of functions $\{w_{kj}\}_{j=1,\ldots,n}$ are defined by (2). Let the function $W_{n,w}$, defined by (3), be non-negative. Then,*

(a) *if $g : [a,b] \to \mathbb{R}$ is an $(n+2)$-convex function, the following inequalities hold*

$$\begin{aligned}
&(-1)^n \cdot P(b) \cdot g^{(n)}(\lambda) \\
&\leq \int_a^b w(t) g(t)\, dt - \sum_{j=1}^{n} (-1)^{j-1} \Big[w_{mj}(b) g^{(j-1)}(b) \\
&+ \sum_{k=1}^{m-1} \Big[w_{kj}(x_k) - w_{k+1,j}(x_k) \Big] g^{(j-1)}(x_k) - w_{1j}(a) g^{(j-1)}(a) \Big] \\
&\leq (-1)^n \cdot P(b) \cdot \left[\frac{b-\lambda}{b-a} g^{(n)}(a) + \frac{\lambda - a}{b-a} g^{(n)}(b) \right],
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
P(b) = (-1)^n &\left[\frac{1}{n!} \int_a^b w(t) \cdot t^n \, dt - \sum_{j=1}^{n} \frac{(-1)^{j-1}}{(n-j+1)!} \right. \\
&\left. \cdot \left(w_{mj}(b) b^{n-j+1} + \sum_{k=1}^{m-1} \Big(w_{kj}(x_k) - w_{k+1,j}(x_k) \Big) x_k^{n-j+1} - w_{1j}(a) a^{n-j+1} \right) \right]
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
\lambda = (-1)^n &\left[\frac{1}{(n+1)! P(b)} \int_a^b w(t) \cdot t^{n+1} \, dt - \frac{1}{P(b)} \sum_{j=1}^{n} \frac{(-1)^{j-1}}{(n-j+2)!} \right. \\
&\left. \cdot \left(w_{mj}(b) b^{n-j+2} + \sum_{k=1}^{m-1} \Big(w_{kj}(x_k) - w_{k+1,j}(x_k) \Big) x_k^{n-j+2} - w_{1j}(a) a^{n-j+2} \right) \right],
\end{aligned} \tag{7}$$

(b) *if g is an $(n+2)$-concave function, then (5) holds with the sign of inequalities reversed.*

If w-harmonic sequences of functions $\{w_{kj}\}_{j=1,\dots,n}$ are expanded by $w_{k,n+1}$, such that $w'_{k,n+1}(t) = w_{k,n}(t)$ for $t \in [x_{k-1}, x_k]$, the function $W_{n+1,w}$ becomes

$$W_{n+1,w}(t,\sigma) = \begin{cases} w_{1,n+1}(t), & t \in [a, x_1], \\ w_{2,n+1}(t), & t \in (x_1, x_2], \\ . \\ . \\ . \\ w_{m,n+1}(t), & t \in (x_{m-1}, b] \end{cases} \tag{8}$$

and the following result is obtained ([5]).

Theorem 4. *Assume $g : [a,b] \to \mathbb{R}$ is an $(n+2)$-convex function. Suppose $w : [a,b] \to \mathbb{R}$ is an arbitrary integrable function and $\{w_{kj}\}_{j=1,\dots,n+1}$ are w-harmonic sequences of functions. Let the function $W_{n+1,w}$, defined by (8), be non-negative. Then, inequality (5) is valid for*

$$P(b) = w_{m,n+1}(b) + \sum_{k=1}^{m-1} [w_{k,n+1}(x_k) - w_{k+1,n+1}(x_k)] - w_{1,n+1}(a)$$

and

$$\lambda = \frac{1}{P(b)}\left[bw_{m,n+1}(b) - aw_{1,n+1}(a) + \sum_{k=1}^{m-1}(x_k w_{k,n+1}(x_k) - x_k \cdot w_{k+1,n+1}(x_k)) - w_{m,n+2}(b) - \sum_{k=1}^{m-1}(w_{k,n+2}(x_k) - w_{k+1,n+2}(x_k)) + w_{1,n+2}(a)\right].$$

If $W_{n,w}(t,\sigma) \leq 0$ or g is an $(n+2)$-concave function, then (5) holds with the sign of inequalities reversed.

2. Two-Point Formula

Now, we use the weighted version of the integral identity given in Theorem 2 and the inequalities from Theorems 3 and 4 to establish Hermite–Hadamard-type inequalities for the weighted two-point formula.

We observe the function $g : [a,b] \to \mathbb{R}$, the integrable function $w : [a,b] \to \mathbb{R}$ and the w-harmonic sequences of functions $\{w_{kj}\}_{j=0,1,\ldots,n}$ on $[x_{k-1}, x_k]$, where $k = 1,2,3$. We consider the subdivision $\sigma = \{a = x_0 < x_1 = x < x_2 = a+b-x < x_3 = b\}$ of the segment $[a,b]$, and we assume $w_{1j}(a) = 0$ and $w_{3j}(b) = 0$, for $j = 1,\ldots,n$. In [4,6] the authors proved the following theorem.

Theorem 5. *Let $w : [a,b] \to \mathbb{R}$ be an integrable function and $x \in [a, \frac{a+b}{2}]$, and let $\{Q_{j,x}\}_{j\in\mathbb{N}}$ be a sequence of polynomials such that deg $Q_{j,x} \leq j-1$, $Q'_{j,x}(t) = Q_{j-1,x}(t)$, $j \in \mathbb{N}$ and $Q_{0,x} \equiv 0$. Suppose $\{w_{kj}\}_{j=1,\ldots,n}$ are w-harmonic sequences of functions on $[x_{k-1}, x_k]$, for $k = 1,2,3$ and some $n \in \mathbb{N}$, defined by the following relations:*

$$w_{1j}(t) = \frac{1}{(j-1)!}\int_a^t (t-s)^{j-1}w(s)\,ds, \quad t \in [a,x],$$

$$w_{2j}(t) = \frac{1}{(j-1)!}\int_x^t (t-s)^{j-1}w(s)\,ds + Q_{j,x}(t), \quad t \in (x, a+b-x],$$

$$w_{3j}(t) = -\frac{1}{(j-1)!}\int_t^b (t-s)^{j-1}w(s)\,ds, \quad t \in (a+b-x, b],$$

for $j = 1,\ldots,n$. If $g : [a,b] \to \mathbb{R}$ is such that $g^{(n)}$ is piecewise continuous on $[a,b]$, then we have

$$\int_a^b w(t)g(t)\,dt = \sum_{j=1}^{n}\left[A_j(x)g^{(j-1)}(x) + B_j(x)g^{(j-1)}(a+b-x)\right] + (-1)^n \int_a^b W_{n,w}(t,x)g^{(n)}(t)\,dt, \tag{9}$$

where for $j = 1,\ldots,n$

$$A_j(x) = (-1)^{j-1}\left[\frac{1}{(j-1)!}\int_a^x (x-s)^{j-1}w(s)\,ds - Q_{j,x}(x)\right] \tag{10}$$

and

$$B_j(x) = (-1)^{j-1}\left[\frac{1}{(j-1)!}\int_x^b (a+b-x-s)^{j-1}w(s)\,ds + Q_{j,x}(a+b-x)\right], \tag{11}$$

such that

$$W_{n,w}(t,x) = \begin{cases} w_{1n}(t), & t \in [a,x] \\ w_{2n}(t), & t \in (x, a+b-x] \\ w_{3n}(t), & t \in (a+b-x, b]. \end{cases} \tag{12}$$

Remark 1. *The polynomials $Q_{j,x}$ satisfy*

$$Q_{j,x}(t) = \sum_{k=0}^{j-1} Q_{j-k,x}(x) \frac{(t-x)^k}{k!},$$

and hence the polynomial $Q_{j,x}$ is uniquely determined by the values $Q_{k,x}(x)$, for $k = 0, 1, \ldots, j$.

From Theorems 1 and 3, the properties of n-convex functions and the properties of w-harmonic sequences of functions, we now obtain new Hermite–Hadamard-type inequalities for the weighted two-point quadrature Formula (9).

Theorem 6. *Let $w : [a,b] \to \mathbb{R}$ be an integrable function and $x \in [a, \frac{a+b}{2}]$ be fixed. Suppose $\{w_{kj}\}_{j=1,\ldots,n}$ are w-harmonic sequences of functions on $[x_{k-1}, x_k]$, for $k = 1,2,3$ and $n \in \mathbb{N}$, as defined in Theorem 5. Let the function $W_{n,w}$, defined by (12), be non-negative. If $g : [a,b] \to \mathbb{R}$ is an $(n+2)$-convex function, then*

$$\begin{aligned} &(-1)^n \cdot P(b) \cdot g^{(n)}(\lambda) \\ &\leq \int_a^b w(t)g(t)\,dt - \sum_{j=1}^{n} \Big[A_j(x) g^{(j-1)}(x) + B_j(x) g^{(j-1)}(a+b-x) \Big] \\ &\leq (-1)^n \cdot P(b) \cdot \left[\frac{b-\lambda}{b-a} g^{(n)}(a) + \frac{\lambda - a}{b-a} g^{(n)}(b) \right], \end{aligned} \tag{13}$$

where

$$\begin{aligned} P(b) &= (-1)^n \Bigg[\frac{1}{n!} \int_a^b w(t) \cdot t^n \,dt \\ &\quad - \sum_{j=1}^{n} \left[\frac{x^{n-j+1}}{(n-j+1)!} \cdot A_j(x) + \frac{(a+b-x)^{n-j+1}}{(n-j+1)!} \cdot B_j(x) \right] \Bigg], \end{aligned}$$

$$\begin{aligned} \lambda &= \frac{(-1)^n}{P(b)} \Bigg[\frac{1}{(n+1)!} \int_a^b w(t) \cdot t^{n+1} \,dt \\ &\quad - \sum_{j=1}^{n} \left[\frac{x^{n-j+2}}{(n-j+2)!} \cdot A_j(x) + \frac{(a+b-x)^{n-j+2}}{(n-j+2)!} \cdot B_j(x) \right] \Bigg] \end{aligned}$$

and A_j and B_j are defined as in Theorem 5. If $W_{n,w}(t,\sigma) \leq 0$ or g is $(n+2)$-concave, then (13) holds with the sign of inequalities reversed.

Proof. As g is an $(n+2)-$convex function, then $g^{(n)}$ is convex and inequalities (13) follow directly from Theorem 1, replacing the non-negative function p with the non-negative function $W_{n,w}$ and the convex function f with the convex function $g^{(n)}$, and then applying the identity (9) on $(-1)^n \int_a^b W_{n,w}(t,x) g^{(n)}(t)\,dt$. Further, using identity (6) from Theorem 3 for $m = 2$, $x_1 = x$ and $x_1 = a+b-x$, we obtain

$$
\begin{aligned}
P(b) &= (-1)^n \left[\frac{1}{n!} \int_a^b w(t) t^n \, dt - \sum_{j=1}^{n} \frac{(-1)^{j-1}}{(n-j+1)!} \right. \\
&\quad \cdot \Big(w_{3j}(b) b^{n-j+1} + w_{1j}(x) \cdot x^{n-j+1} - w_{2j}(x) \cdot x^{n-j+1} \\
&\quad + w_{2j}(a+b-x) \cdot (a+b-x)^{n-j+1} - w_{3j}(a+b-x) \cdot (a+b-x)^{n-j+1} \\
&\quad \left. - w_{1j}(a) a^{n-j+1} \Big) \right].
\end{aligned}
$$

Since, $w_{1j}(a) = 0$ and $w_{3j}(b) = 0$, for $j = 1, \ldots, n$, we obtain

$$
\begin{aligned}
P(b) &= (-1)^n \left[\frac{1}{n!} \int_a^b w(t) t^n \, dt - \sum_{j=1}^{n} \left(\frac{(-1)^{j-1}}{(n-j+1)!} \cdot \big(w_{1j}(x) - w_{2j}(x) \big) \cdot x^{n-j+1} \right. \right. \\
&\quad \left. \left. + \frac{(-1)^{j-1}}{(n-j+1)!} \cdot \big(w_{2j}(a+b-x) - w_{3j}(a+b-x) \big) \cdot (a+b-x)^{n-j+1} \right) \right].
\end{aligned}
$$

Applying the definitions of $\{w_{kj}\}$ from Theorem 5, we derive

$$
w_{1j}(x) - w_{2j}(x) = \frac{1}{(j-1)!} \int_a^x (x-s)^{j-1} w(s) \, ds - Q_{j,x}(x),
$$

and

$$
w_{2j}(a+b-x) - w_{3j}(a+b-x) = \frac{1}{(j-1)!} \int_x^b (a+b-x-s)^{j-1} w(s) \, ds + Q_{j,x}(a+b-x).
$$

Now, according to the definitions of A_j and B_j given by (10) and (11), respectively, we obtain

$$
\begin{aligned}
P(b) &= (-1)^n \left[\frac{1}{n!} \int_a^b w(t) \cdot t^n \, dt \right. \\
&\quad \left. - \sum_{j=1}^{n} \left[\frac{x^{n-j+1}}{(n-j+1)!} \cdot A_j(x) + \frac{(a+b-x)^{n-j+1}}{(n-j+1)!} \cdot B_j(x) \right] \right].
\end{aligned}
$$

Similarly, using identity (7) from Theorem 3 for $m = 2$, $x_1 = x$ and $x_1 = a + b - x$, the definitions of $\{w_{kj}\}$ from Theorem 5 and the definitions of A_j and B_j given by (10) and (11), we can calculate λ.

$$
\begin{aligned}
\lambda &= \frac{(-1)^n}{P(b)} \left[\frac{1}{(n+1)!} \int_a^b w(t) \cdot t^{n+1} \, dt - \sum_{j=1}^{n} \frac{(-1)^{j-1}}{(n-j+2)!} \right. \\
&\quad \cdot \Big(w_{3j}(b) b^{n-j+2} + w_{1j}(x) \cdot x^{n-j+2} - w_{2j}(x) \cdot x^{n-j+2} \\
&\quad + w_{2j}(a+b-x) \cdot (a+b-x)^{n-j+2} - w_{3j}(a+b-x) \cdot (a+b-x)^{n-j+2} \\
&\quad \left. - w_{1j}(a) a^{n-j+2} \Big) \right] \\
&= \frac{(-1)^n}{P(b)} \left[\frac{1}{(n+1)!} \int_a^b w(t) \cdot t^{n+1} \, dt \right. \\
&\quad \left. - \sum_{j=1}^{n} \left(\frac{x^{n-j+2}}{(n-j+2)!} \cdot A_j(x) + \frac{(a+b-x)^{n-j+2}}{(n-j+2)!} \cdot B_j(x) \right) \right].
\end{aligned}
$$

□

We continue now by expanding the w-harmonic sequences of functions $\{w_{kj}\}_{j=1,\dots,n}$ with $w_{k,n+1}$, such that $w'_{k,n+1}(t) = w_{k,n}(t)$ for $t \in [x_{k-1}, x_k]$, so that function $W_{n+1,w}$ is equal to

$$W_{n+1,w}(t,x) = \begin{cases} w_{1,n+1}(t), & t \in [a,x], \\ w_{2,n+1}(t), & t \in (x, a+b-x], \\ w_{3,n+1}(t), & t \in (a+b-x, b]. \end{cases} \tag{14}$$

For the new subdivision $\sigma = \{a = x_0 < x_1 = x < x_2 = a+b-x < x_3 = b\}$ of the segment $[a,b]$ and the values $w_{1j}(a) = 0$ and $w_{3j}(b) = 0$, for $j = 1, \dots, n+2$, we obtain the following results.

Theorem 7. *Suppose $w : [a,b] \to \mathbb{R}$ is an integrable function and $x \in [a, \frac{a+b}{2}]$ is fixed. Suppose $\{w_{kj}\}_{j=1,\dots,n+1}$ are w-harmonic sequences of functions on $[x_{k-1}, x_k]$, $k = 1,2,3$ and $n \in \mathbb{N}$. Let the function $W_{n+1,w}$, defined by (14), be non-negative. If $g : [a,b] \to \mathbb{R}$ is an $(n+2)$-convex function, then inequalities (13) are valid for*

$$\begin{aligned} P(b) &= w_{1,n+1}(x) - w_{2,n+1}(x) + w_{2,n+1}(a+b-x) - w_{3,n+1}(a+b-x) \\ &= (-1)^n A_{n+1}(x) + (-1)^n B_{n+1}(x) \end{aligned}$$

and

$$\begin{aligned} \lambda &= \frac{1}{P(b)}[x(w_{1,n+1}(x) - w_{2,n+1}(x)) \\ &\quad + (a+b-x)(w_{2,n+1}(a+b-x) - w_{3,n+1}(a+b-x)) \\ &\quad - w_{1,n+2}(x) + w_{2,n+2}(x) - w_{2,n+2}(a+b-x) + w_{3,n+2}(a+b-x)] \\ &= \frac{1}{P(b)}\Big[(-1)^n(xA_{n+1}(x) + (a+b-x)B_{n+1}(x)) \\ &\quad + (-1)^{n+1}(A_{n+2}(x) + B_{n+2}(x))\Big]. \end{aligned}$$

If $W_{n,w}(t,\sigma) \le 0$ or g is $(n+2)$-concave, then (13) holds with the sign of inequalities reversed.

Proof. Applying Theorem 4 for $m = 3$, $x_1 = x$, $x_2 = a+b-x$, $w_{1j}(a) = 0$ and $w_{3j}(b) = 0$, for $j = 1, \dots, n+2$, and the definitions of $\{w_{kj}\}$ from Theorem 5, we obtain values of $P(b)$ and λ. □

Using the integral mean value theorem for $\int_a^b W_{2n,w}(t,x) g^{(2n)}(t)\,dt$, where $g : [a,b] \to \mathbb{R}$ is such that $g^{(2n)}$ is a continuous function, the authors in [3] proved that there exists an $\eta \in (a,b)$ such that

$$\begin{aligned} &\int_a^b w(t)g(t)\,dt - \sum_{j=1}^{2n} \Big(A_j(x) g^{(j-1)}(x) + B_j(x) g^{(j-1)}(a+b-x) \Big) \\ &= (A_{2n+1}(x) + B_{2n+1}(x)) g^{(2n)}(\eta). \end{aligned} \tag{15}$$

Applying this integral identity to our result in inequalities (13), we obtain the following theorem.

Theorem 8. *Assume $\{w_{kj}\}$ satisfies the conditions of Theorem 7 for $j = 1, \dots, 2n+1$. Let A_j and B_j be defined as in (10) and (11). Let $w : [a,b] \to [0,\infty)$ be a continuous function on (a,b), and let*

$$Q_{2n,x}(t) \ge -\frac{1}{(2n-1)!} \int_x^t (t-s)^{2n-1} \cdot w(s)\,ds, \quad \forall t \in [x, a+b-x]$$

for some $n \in \mathbb{N}$. If $g : [a,b] \to \mathbb{R}$ is such that $g^{(2n)}$ is a continuous function, then there exists an $\eta \in (a,b)$ such that

$$\begin{aligned} P(b) \cdot g^{(2n)}(\lambda) &\leq \frac{g^{(2n)}(\eta)}{(2n)!} \left[\int_a^x (x-s)^{2n} \cdot w(s)\, ds - Q_{2n,x}(x) \right. \\ &\quad \left. + \int_x^b (a+b-x-s)^{2n} \cdot w(s)\, ds + Q_{2n,x}(a+b-x) \right] \\ &\leq P(b) \cdot \left[\frac{b-\lambda}{b-a} g^{(2n)}(a) + \frac{\lambda-a}{b-a} g^{(2n)}(b) \right], \end{aligned} \tag{16}$$

where

$$\begin{aligned} P(b) &= \frac{1}{(2n)!} \int_a^b w(t) \cdot t^{2n}\, dt \\ &\quad - \sum_{j=1}^{2n} \left(\frac{x^{2n-j+1}}{(2n-j+1)!} \cdot A_j(x) + \frac{(a+b-x)^{2n-j+1}}{(2n-j+1)!} \cdot B_j(x) \right) \end{aligned}$$

and

$$\begin{aligned} \lambda &= \frac{1}{P(b)} \left[\frac{1}{(2n+1)!} \int_a^b w(t) \cdot t^{2n+1}\, dt \right. \\ &\quad \left. - \sum_{j=1}^{2n} \left(\frac{x^{2n-j+2}}{(2n-j+2)!} \cdot A_j(x) + \frac{(a+b-x)^{2n-j+2}}{(2n-j+2)!} \cdot B_j(x) \right) \right]. \end{aligned}$$

Proof. Inequality (16) follows directly from (13), replacing its middle term by

$$(A_{2n+1}(x) + B_{2n+1}(x)) \cdot g^{(2n)}(\eta),$$

according to the integral identity (15), and then applying (10) and (11) to A_{2n+1} and B_{2n+1}, respectively. □

The coefficients $A_j(x)$ and $B_j(x)$ defined with (10) and (11) are not symmetric. If we assume $w(s) = w(a+b-s)$, for $s \in [a,b]$, and

$$(-1)^j Q_{j,x}(x) - Q_{j,x}(a+b-x) = \frac{1}{(j-1)!} \int_x^{a+b-x} (s-x)^{j-1} \cdot w(s)\, ds, \tag{17}$$

then we obtain $A_j(x) = (-1)^{j-1} B_j(x)$.

To obtain the maximum degree of exactness of the quadrature formula in Equation (9) for fixed $x \in \left[a, \frac{a+b}{2}\right]$, we choose the sequence of polynomials $\{Q_{j,x}\}_{j=0,1,\ldots,n}$ which is, according to Remark 1, uniquely determined by the formula

$$\begin{aligned} Q_{1,x}(x) &= \frac{1}{2x-a-b} \left(\int_a^x (x-s) w(s)\, ds + \int_x^b (a+b-x-s) w(s)\, ds \right), \\ Q_{j,x}(x) &= \frac{1}{(j-1)!} \int_a^x (x-s)^{j-1} w(s)\, ds, \; j = 2,3,4, \\ Q_{j,x}(x) &= 0, \; for\ j \geq 5. \end{aligned} \tag{18}$$

Hence, we have $A_1(x) = B_1(x) = \frac{1}{2} \int_a^b w(s)\, ds$ and $A_j(x) = B_j(x) = 0$, for $j = 2,3,4$.

Finally, from identity (9) for $x \in \left[a, \frac{a+b}{2}\right]$, we obtain the following two-point weighted integral formula:

$$\begin{aligned} \int_a^b w(t)g(t)\,dt &= A_1(x)[g(x)+g(a+b-x)] + T_{n,w}(x) \\ &\quad +(-1)^n \int_a^b W_{n,w}(t,x)g^{(n)}(t)\,dt, \end{aligned} \tag{19}$$

where

$$T_{n,w}(x) = \sum_{j=5}^{n} \left[A_j(x)g^{(j-1)}(x) + B_j(x)g^{(j-1)}(a+b-x)\right].$$

Now, applying the results from Theorems 6 and 7 to identity (19), we obtain the following corollaries.

Corollary 1. *Let $w : [a,b] \to \mathbb{R}$ be an integrable function such that $w(t) = w(a+b-t)$ for each $t \in [a,b]$, and let equality (17) hold. Suppose $\{w_{kj}\}_{j=1,\ldots,n}$ are w-harmonic sequences of functions on $[x_{k-1}, x_k]$, for $k = 1,2,3$ and $n \in \mathbb{N}$, as defined in Theorem 5, and let $Q_{j,x}(t)$ be defined by (18). Let the function $W_{n,w}$, defined by (12), be non-negative and let $x \in [a, \frac{a+b}{2}]$. If $g : [a,b] \to \mathbb{R}$ is an $(n+2)$-convex function, then*

$$\begin{aligned} &(-1)^n \cdot P(b) \cdot g^{(n)}(\lambda) \\ &\le \int_a^b w(t)g(t)\,dt - A_1(x)[g(x)+g(a+b-x)] - T_{n,w}(x) \\ &\le (-1)^n \cdot P(b) \cdot \left[\frac{b-\lambda}{b-a} g^{(n)}(a) + \frac{\lambda-a}{b-a} g^{(n)}(b)\right], \end{aligned}$$

where

$$\begin{aligned} P(b) &= (-1)^n \Bigg[\frac{1}{n!}\int_a^b w(t)\cdot t^n\,dt - A_1(x)\left(\frac{x^n + (a+b-x)^n}{n!}\right) \\ &\quad - \sum_{j=5}^{n} A_j(x) \cdot \frac{x^{n-j+1} + (-1)^{j-1}(a+b-x)^{n-j+1}}{(n-j+1)!}\Bigg], \end{aligned}$$

and

$$\begin{aligned} \lambda &= \frac{(-1)^n}{P(b)} \Bigg[\frac{1}{(n+1)!}\int_a^b w(t)\cdot t^{n+1}\,dt - A_1(x)\left(\frac{x^{n+1} + (a+b-x)^{n+1}}{(n+1)!}\right) \\ &\quad - \sum_{j=5}^{n} A_j(x) \cdot \frac{x^{n-j+2} + (-1)^{j-1}(a+b-x)^{n-j+2}}{(n-j+2)!}\Bigg]. \end{aligned}$$

and A_j is defined as in Theorem 5. If $W_{n,w}(t,\sigma) \le 0$ or g is $(n+2)$-concave, then (13) holds with the sign of inequalities reversed.

Proof. The proof follows from Theorem 6 for the special choice of the polynomials $Q_{j,x}$. □

Corollary 2. *Let $w : [a,b] \to \mathbb{R}$ be an integrable function such that $w(t) = w(a+b-t)$ for each $t \in [a,b]$, and let equality (17) hold. Suppose $\{w_{kj}\}_{j=1,\ldots,2n}$ are w-harmonic sequences of functions on $[x_{k-1}, x_k]$, for $k = 1,2,3$ and $n \ge 2$, as defined in Theorem 5, and let $Q_{j,x}(t)$ be defined by (18). Let the function $W_{2n+1,w}$, defined by (14), be non-negative and let $x \in [a, \frac{a+b}{2}]$. If $g : [a,b] \to \mathbb{R}$ is a $(2n+2)$-convex function, then*

$$
\begin{aligned}
&P(b) \cdot g^{(n)}\left(\frac{a+b}{2}\right) \\
&\leq \int_a^b w(t)g(t)\,dt - A_1(x)[g(x) + g(a+b-x)] - T_{2n,w}(x) \\
&\leq P(b) \cdot \left[\frac{1}{2}g^{(2n)}(a) + \frac{1}{2}g^{(2n)}(b)\right],
\end{aligned} \tag{20}
$$

where

$$
P(b) = \frac{2}{(2n)!}\int_a^x (x-s)^{2n} w(s)\,ds.
$$

If g is a $(2n+2)$-concave function, then (20) holds with the sign of inequalities reversed.

Proof. The proof follows from Theorem 7 for the special choice of the polynomials $Q_{j,x}$. □

3. Applications

Considering some special cases of the function w, we here obtain new bounds for the Gauss–Legendre two-point quadrature formula and for the Gauss–Chebyshev two-point quadrature formulas of the first and second kind.

3.1. Gauss–Legendre Two-Point Quadrature Formula

Suppose that $w(t) = 1$, $t \in [a,b]$ and $x \in \left[a, \frac{a+b}{2}\right]$. Now, from Theorem 5, we calculate

$$
W_{n,w}^{LG}(t,x) = \begin{cases} w_{1n}(t) = \frac{(t-a)^n}{n!}, & t \in [a,x] \\ w_{2n}(t) = \frac{(t-x)^n}{n!} + Q_{n,x}(t), & t \in (x, a+b-x] \\ w_{3n}(t) = \frac{(t-b)^n}{n!}, & t \in (a+b-x, b], \end{cases} \tag{21}
$$

and for $j \geq 1$

$$
A_j^{LG}(x) = (-1)^{j-1}\left[\frac{(x-a)^j}{j!} - Q_{j,x}(x)\right]
$$

and

$$
B_j^{LG}(x) = (-1)^{j-1}\left[\frac{(a+b-2x)^j}{j!} - \frac{(a-x)^j}{j!} + Q_{j,x}(a+b-x)\right].
$$

In order to provide the non-negativity of $W_{n,w}^{LG}$, we will replace n, in the definition of $W_{n,w}^{LG}$, by $2n$.

Corollary 3. *Let*

$$
Q_{2n,x}(t) \geq -\frac{(t-x)^{2n}}{(2n)!}, \quad \forall t \in (x, a+b-x],
$$

for $n \in \mathbb{N}$. If $g : [a,b] \to \mathbb{R}$ is a $(2n+2)$-convex function, then

$$
\begin{aligned}
&P(b) \cdot g^{(2n)}(\lambda) \\
&\leq \int_a^b g(t)\,dt - \sum_{j=1}^{2n}\left[A_j^{LG}(x) \cdot g^{(j-1)}(x) + B_j^{LG}(x) \cdot g^{(j-1)}(a+b-x)\right] \\
&\leq P(b) \cdot \left[\frac{b-\lambda}{b-a}g^{(2n)}(a) + \frac{\lambda-a}{b-a}g^{(2n)}(b)\right],
\end{aligned} \tag{22}
$$

where

$$P(b) = \left[\frac{b^{2n+1}-a^{2n+1}}{(2n+1)!} - \sum_{j=1}^{2n}\left(\frac{x^{2n-j+1}}{(2n-j+1)!}\cdot A_j^{LG}(x) + \frac{(a+b-x)^{2n-j+1}}{(2n-j+1)!}\cdot B_j^{LG}(x)\right)\right]$$

and

$$\lambda = \frac{1}{P(b)}\left[\frac{b^{2n+2}-a^{2n+2}}{(2n+2)!} - \sum_{j=1}^{2n}\left(\frac{x^{2n-j+2}}{(2n-j+2)!}\cdot A_j^{LG}(x) + \frac{(a+b-x)^{2n-j+2}}{(2n-j+2)!}\cdot B_j^{LG}(x)\right)\right].$$

If g is $(2n+2)$-concave, then (22) holds with the sign of inequalities reversed.

Proof. Inequality (22) follows from Theorem 6 if $w(t)=1$ and $W_{2n,w}^{LG}$ is the non-negative function given in (21). □

Corollary 4. *Let*

$$Q_{2n,x}(t) \geq -\frac{(t-x)^{2n}}{(2n)!}, \quad \forall t \in (x, a+b-x],$$

for $n \in \mathbb{N}$. If $g : [a,b] \to \mathbb{R}$ is a $(2n+2)$-convex function, then inequalities (22) hold for

$$P(b) = A_{2n+1}^{LG}(x) + B_{2n+1}^{LG}(x)$$

and

$$\lambda = \frac{1}{P(b)}\left[xA_{2n+1}^{LG}(x) + (a+b-x)B_{2n+1}^{LG}(x) - A_{2n+2}^{LG}(x) - B_{2n+2}^{LG}(x)\right].$$

If g is $(2n+2)$-concave, then (22) holds with the sign of inequalities reversed.

Proof. The obtained results follow from Theorem 7 if $w(t)=1$ and $W_{2n,w}^{LG}$ is the non-negative function defined in (21). □

If the polynomials $Q_{j,x}(t)$ are as follows:

$$\begin{aligned} Q_{1,x}(x) &= x - a - \frac{b-a}{2}, \\ Q_{j,x}(x) &= \frac{(x-a)^j}{(j)!}, \text{ for } j = 2,3,4, \\ Q_{j,x}(x) &= 0, \text{ for } j \geq 5, \end{aligned}$$

we have $A_1^{LG}(x) = B_1^{LG}(x) = \frac{b-a}{2}$ and $A_j^{LG}(x) = B_j^{LG}(x) = 0$, for $j = 2,3,4$, and hence we obtain the non-weighted two-point quadrature formulas with a maximum degree of exactness

$$\int_a^b g(t)\,dt = \frac{b-a}{2}[g(x) + g(a+b-x)] + T_{n,w}^{LG}(x) + (-1)^n \int_a^b W_{n,w}^{LG}(t,x) g^{(n)}(t)\,dt,$$

where

$$T_{n,w}^{LG}(x) = \sum_{j=5}^{n}\left[A_j^{LG}(x) g^{(j-1)}(x) + B_j^{LG}(x) g^{(j-1)}(a+b-x)\right].$$

Specifically, for $x = \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}$, the generalization of the Legendre–Gauss two-point formula follows. Now, we derive Hermite–Hadamard-type estimates for this generalization of the Legendre–Gauss two-point formula.

If the assumptions of Corollary (1) hold, for $w(t) = 1$ and $t \in [a,b]$ and if $g : [a,b] \to \mathbb{R}$ is a $(2n+2)$-convex function, we derive

$$\begin{aligned}
& P_n^{LG}\left(\frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right) \cdot g^{(2n)}\left(\lambda^{LG}\left(\frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right)\right) \\
\leq \; & \int_a^b g(t)\,dt - \frac{b-a}{2}\left[g\left(\frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right) + g\left(\frac{a+b}{2} + \frac{b-a}{2\sqrt{3}}\right)\right] \\
& -T_{2n,w}^{LG}\left(\frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right) \\
\leq \; & P_n^{LG}\left(\frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right) \cdot \left[\frac{b - \lambda^{LG}\left(\frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right)}{b-a} g^{(2n)}(a)\right. \\
& \left. + \frac{\lambda^{LG}\left(\frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right) - a}{b-a} g^{(2n)}(b)\right],
\end{aligned}$$

where

$$\begin{aligned}
P_n^{LG}(x) \; &= \; \frac{b^{2n+1} - a^{2n+1}}{(2n+1)!} - \frac{b-a}{2}\left(\frac{x^{2n} + (a+b-x)^{2n}}{(2n)!}\right) \\
& \quad - \sum_{j=5}^{2n} \frac{(x-a)^j}{j!}\left[\frac{(-1)^{j-1}x^{2n-j+1} + (a+b-x)^{2n-j+1}}{(2n-j+1)!}\right],
\end{aligned}$$

$$\begin{aligned}
\lambda^{LG}(x) \; &= \; \frac{1}{P_n^{LG}(x)}\left[\frac{b^{2n+2} - a^{2n+2}}{(2n+2)!} - \frac{b-a}{2}\left(\frac{x^{2n+1} + (a+b-x)^{2n+1}}{(2n+1)!}\right)\right. \\
& \quad \left. - \sum_{j=5}^{2n} \frac{(x-a)^j}{j!}\left[\frac{(-1)^{j-1}x^{2n-j+2} + (a+b-x)^{2n-j+2}}{(2n-j+2)!}\right]\right].
\end{aligned}$$

In the special case of $n = 2$, we obtain

$$\begin{aligned}
& \frac{(b-a)^5}{4320} \cdot g^{(4)}\left(\frac{a+b}{2}\right) \\
\leq \; & \int_a^b g(t)\,dt - \frac{b-a}{2}\left[g\left(\frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right) + g\left(\frac{a+b}{2} + \frac{b-a}{2\sqrt{3}}\right)\right] \\
\leq \; & \frac{(b-a)^5}{4320} \cdot \left[\frac{1}{2}g^{(4}(a) + \frac{1}{2}g^{(4)}(b)\right].
\end{aligned}$$

If the assumptions of Corollary (2) hold, for $w(t) = 1$ and $t \in [a,b]$ and if g is a $(2n+2)$-convex function for $n \geq 2$, we obtain

$$
\begin{aligned}
& \frac{6^{-2n}(3-\sqrt{3})(b-a)^{2n+1}}{(6n+3)(2n)!} \cdot g^{(2n)}\left(\frac{a+b}{2}\right) \\
\leq \; & \int_a^b g(t)\,dt - \frac{b-a}{2}\left[g\left(\frac{a+b}{2}-\frac{b-a}{2\sqrt{3}}\right)+g\left(\frac{a+b}{2}+\frac{b-a}{2\sqrt{3}}\right)\right] \\
& -T_{n,w}^{LG}\left(\frac{a+b}{2}-\frac{b-a}{2\sqrt{3}}\right) \\
\leq \; & \frac{6^{-2n}(3-\sqrt{3})(b-a)^{2n+1}}{(6n+3)(2n)!} \cdot \left[\frac{1}{2}g^{(2n)}(a)+\frac{1}{2}g^{(2n)}(b)\right].
\end{aligned}
$$

In the special case of $n = 2$, we obtain

$$
\begin{aligned}
& \frac{(3-\sqrt{3})^5(b-a)^5}{466560} \cdot g^{(4)}\left(\frac{a+b}{2}\right) \\
\leq & \int_a^b g(t)\,dt - \frac{b-a}{2}\left[g\left(\frac{a+b}{2}-\frac{b-a}{2\sqrt{3}}\right)+g\left(\frac{a+b}{2}+\frac{b-a}{2\sqrt{3}}\right)\right] \\
\leq & \frac{(3-\sqrt{3})^5(b-a)^5}{466560} \cdot \left[\frac{1}{2}g^{(4}(a)+\frac{1}{2}g^{(4)}(b)\right].
\end{aligned}
$$

3.2. Gauss–Chebyshev Two-Point Quadrature Formula of the First Kind

Suppose that $w(t) = \frac{1}{\sqrt{1-t^2}}$, $t \in [-1,1]$ and $x \in [-1,0]$. Now, from Theorem 5, we calculate

$$
W_{n,w}^{GC1}(t,x) = \begin{cases} w_{1n}(t) = \frac{1}{(n-1)!}\int_{-1}^{t} \frac{(t-s)^{n-1}}{\sqrt{1-s^2}}\,ds, & t \in [-1,x], \\ w_{2n}(t) = \frac{1}{(n-1)!}\int_{x}^{t} \frac{(t-s)^{n-1}}{\sqrt{1-s^2}}\,ds + Q_{n,x}(t), & t \in (x,-x], \\ w_{3n}(t) = -\frac{1}{(n-1)!}\int_{t}^{1} \frac{(t-s)^{n-1}}{\sqrt{1-s^2}}\,ds, & t \in (-x,1], \end{cases} \tag{23}
$$

$$
A_j^{GC1}(x) = (-1)^{j-1}\left[\frac{2^{j-1/2}(x+1)^{j-1/2}}{(2j-1)!!}F\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}+j,\frac{x+1}{2}\right) - Q_{j,x}(x)\right]
$$

and

$$
B_j^{GC1}(x) = (-1)^{j-1}\left[\frac{1}{(j-1)!}\int_x^1 \frac{(-x-s)^{j-1}}{\sqrt{1-s^2}}\,ds + Q_{j,x}(-x)\right].
$$

In what follows, B denotes the beta function, defined by

$$
B(u,v) = \int_0^1 s^{u-1}(1-s)^{v-1}\,ds
$$

and

$$
F(\alpha,\beta;\gamma;z) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\,dt
$$

is the hypergeometric function with $\gamma > \beta > 0$, $z < 1$.

Corollary 5. *Let $w_{2,2n}(t) \geq 0$, for all $t \in [x, -x]$ and for $n \in \mathbb{N}$. If $g : [-1,1] \to \mathbb{R}$ is a $(2n+2)$-convex function, then*

$$\begin{aligned}
&P(b) \cdot g^{(2n)}(\lambda) \\
&\leq \int_{-1}^{1} \frac{g(t)}{\sqrt{1-t^2}}\, dt - \sum_{j=1}^{2n} \Big[A_j^{GC1}(x) \cdot g^{(j-1)}(x) + B_j^{GC1}(x) \cdot g^{(j-1)}(-x) \Big] \\
&\leq P(b) \cdot \left[\frac{1-\lambda}{2} g^{(2n)}(-1) + \frac{\lambda+1}{2} g^{(2n)}(1) \right],
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
P(b) \quad = \quad & \left[\frac{1}{(2n)!} B\left(\frac{1}{2}, \frac{1}{2} + n\right) - \sum_{j=1}^{2n} \left(\frac{x^{2n-j+1}}{(2n-j+1)!} \cdot A_j^{GC1}(x) \right.\right. \\
& \left.\left. + \frac{(-x)^{2n-j+1}}{(2n-j+1)!} \cdot B_j^{GC1}(x) \right) \right]
\end{aligned}$$

and

$$\lambda = \frac{1}{P(b)} \sum_{j=1}^{2n} \left(-\frac{x^{2n-j+2}}{(2n-j+2)!} \cdot A_j^{GC1}(x) - \frac{(-x)^{2n-j+2}}{(2n-j+2)!} \cdot B_j^{GC1}(x) \right).$$

If g is a $(2n+2)$-concave function, then (24) holds with the sign of inequalities reversed.

Proof. The obtained results follow from Theorem 6 for $w(t) = \frac{1}{\sqrt{1-t^2}}$, $t \in [-1,1]$ and the non-negative function $W_{2n,w}^{GC1}$, defined by (23). □

Corollary 6. *Let $w_{2,2n}(t) \geq 0$, for all $t \in [x, -x]$ and for $n \in \mathbb{N}$. If $g : [-1,1] \to \mathbb{R}$ is a $(2n+2)$-convex function, then (24) holds for*

$$P(b) = A_{2n+1}^{GC1}(x) + B_{2n+1}^{GC1}(x)$$

and

$$\lambda = \frac{1}{P(b)} \Big[x(A_{2n+1}^{GC1}(x) - B_{2n+1}^{GC1}(x)) - A_{2n+2}^{GC1}(x) - B_{2n+2}^{GC1}(x) \Big].$$

If g is a $(2n+2)$-concave function, then (24) holds with the sign of inequalities reversed.

Proof. These results are a special case of Theorem 7 for $w(t) = \frac{1}{\sqrt{1-t^2}}$, $t \in [-1,1]$ and the non-negative function $W_{2n,w}^{GC1}$, defined by (23). □

If we assume that the polynomials $Q_{j,x}(t)$ are such that

$$\begin{aligned}
Q_{j,x}(x) \quad &= \quad \frac{1}{(j-1)!} \int_{-1}^{x} \frac{(x-s)^{j-1}}{\sqrt{1-s^2}}\, ds, \text{ for } j = 2,3,4, \\
Q_{j,x}(x) \quad &= \quad 0, \text{ for } j \geq 5,
\end{aligned}$$

we have $A_1^{GC1}(x) = B_1^{GC1}(x) = \frac{\pi}{2}$ and $A_j^{GC1}(x) = B_j^{GC1}(x) = 0$, for $j = 2,3,4$, and hence we obtain

$$\int_{-1}^{1} \frac{g(t)}{\sqrt{1-t^2}}\, dt = \frac{\pi}{2}[g(x)+g(-x)] + T_{n,w}^{GC1}(x) + (-1)^n \int_a^b W_{n,w}^{GC1}(t,x) g^{(n)}(t)\, dt,$$

where

$$T_{n,w}^{GC1}(x) = \sum_{j=5}^{n} \left[A_j^{GC1}(x) g^{(j-1)}(x) + B_j^{GC1}(x) g^{(j-1)}(-x) \right].$$

Specifically, for $x = -\frac{\sqrt{2}}{2}$, we obtain the generalization of the Gauss–Chebyshev two-point quadrature formula of the first kind. Now, we obtain Hermite–Hadamard-type estimates for the Gauss–Chebyshev two-point quadrature formula of the first kind.

Applying Corollary (1) for $w(t) = \frac{1}{\sqrt{1-t^2}}$, $t \in [-1,1]$, $x = -\frac{\sqrt{2}}{2}$ and a 6-convex function g, we obtain

$$\begin{aligned} &\frac{\pi}{192} \cdot g^{(4)}(0) \\ &\leq \int_{-1}^{1} \frac{g(t)}{\sqrt{1-t^2}}\, dt - \frac{\pi}{2}\left[g\left(\frac{-\sqrt{2}}{2} \right) + g\left(\frac{\sqrt{2}}{2} \right) \right] \\ &\leq \frac{\pi}{192} \cdot \left[\frac{1}{2} g^{(4)}(-1) + \frac{1}{2} g^{(4)}(1) \right]. \end{aligned}$$

Further, if the assumptions of Corollary (2) hold, for $w(t) = \frac{1}{\sqrt{1-t^2}}$, $t \in [-1,1]$ and a $(2n+2)$-convex function g we obtain

$$\begin{aligned} &P(b) \cdot g^{(2n)}(0) \\ &\leq \int_{-1}^{1} \frac{g(t)}{\sqrt{1-t^2}}\, dt - \frac{\pi}{2}\left[g\left(\frac{-\sqrt{2}}{2} \right) + g\left(\frac{\sqrt{2}}{2} \right) \right] - T_{2n,w}^{GC1}\left(-\frac{\sqrt{2}}{2} \right) \\ &\leq P(b) \cdot \left[\frac{1}{2} g^{(2n)}(-1) + \frac{1}{2} g^{(2n)}(1) \right], \end{aligned}$$

where

$$P(b) = \frac{2(2-\sqrt{2})^{2n+1/2}}{(4n+1)!!} F\left(\frac{1}{2}, \frac{1}{2} : \frac{3}{2} + 2n; \frac{2-\sqrt{2}}{4} \right).$$

In the special case of $n = 2$, we obtain

$$\begin{aligned} &P(b) \cdot g^{(4)}(0) \\ &\leq \int_{-1}^{1} \frac{g(t)}{\sqrt{1-t^2}}\, dt - \frac{\pi}{2}\left[g\left(\frac{-\sqrt{2}}{2} \right) + g\left(\frac{\sqrt{2}}{2} \right) \right] \\ &\leq P(b) \cdot \left[\frac{1}{2} g^{(4)}(-1) + \frac{1}{2} g^{(4)}(1) \right]. \end{aligned}$$

where

$$P(b) = \frac{(-2+\sqrt{2})^4 (51\pi - 160) \sqrt{577 + 480\sqrt{2}}}{4608} \approx 0.00019203.$$

3.3. Gauss–Chebyshev Two-Point Quadrature Formula of the Second Kind

Let us assume that $w(t) = \sqrt{1-t^2}$, $t \in [-1,1]$ and $x \in [-1,0]$. Now, from Theorem 5, we calculate

$$W_{n,w}^{GC2}(t,x) = \begin{cases} w_{1n}(t) = \frac{1}{(n-1)!} \int\limits_{-1}^{t} (t-s)^{n-1}\sqrt{1-s^2}\, ds, & t \in [-1,x], \\ w_{2n}(t) = \frac{1}{(n-1)!} \int\limits_{x}^{t} (t-s)^{n-1}\sqrt{1-s^2}\, ds + Q_{n,x}(t), & t \in (x,-x], \\ w_{3n}(t) = -\frac{1}{(n-1)!} \int\limits_{t}^{1} (t-s)^{n-1}\sqrt{1-s^2}\, ds, & t \in (-x,1], \end{cases} \tag{25}$$

$$A_j^{GC2}(x) = (-1)^{j-1}\left[\frac{2^{j+1/2}(x+1)^{j+1/2}}{(2j+1)!!} F\left(-\frac{1}{2}, \frac{3}{2}; \frac{3}{2}+j; \frac{x+1}{2}\right) - Q_{j,x}(x)\right]$$

and

$$B_j^{GC2}(x) = (-1)^{j-1}\left[\frac{1}{(j-1)!}\int\limits_{x}^{1} (-x-s)^{j-1}\sqrt{1-s^2}\, ds + Q_{j,x}(-x)\right].$$

Corollary 7. *Let $w_{2,2n}(t) \geq 0$, for all $t \in [x,-x]$ and for $n \in \mathbb{N}$. If $g : [-1,1] \to \mathbb{R}$ is a $(2n+2)$-convex function, then*

$$\begin{aligned} & P(b) \cdot g^{(2n)}(\lambda) \\ & \leq \int\limits_{-1}^{1} g(t)\sqrt{1-t^2}\, dt - \sum_{j=1}^{2n}\left[A_j^{GC2}(x) \cdot g^{(j-1)}(x) + B_j^{GC2}(x) \cdot g^{(j-1)}(-x)\right] \\ & \leq P(b) \cdot \left[\frac{1-\lambda}{2} g^{(2n)}(-1) + \frac{\lambda+1}{2} g^{(2n)}(1)\right], \end{aligned} \tag{26}$$

where

$$\begin{aligned} P(b) \quad = \quad & \left[\frac{1}{(2n)!} B\left(\frac{3}{2}, \frac{1}{2}+n\right) - \sum_{j=1}^{2n}\left(\frac{x^{2n-j+1}}{(2n-j+1)!} \cdot A_j^{GC2}(x)\right.\right. \\ & \left.\left. + \frac{(-x)^{2n-j+1}}{(2n-j+1)!} \cdot B_j^{GC2}(x)\right)\right] \end{aligned}$$

and

$$\lambda = \frac{1}{P(b)} \sum_{j=1}^{2n}\left(-\frac{x^{2n-j+2}}{(2n-j+2)!} \cdot A_j^{GC2}(x) - \frac{(-x)^{2n-j+2}}{(2n-j+2)!} \cdot B_j^{GC2}(x)\right).$$

If g is $(2n+2)$-concave, then (26) holds with the sign of inequalities reversed.

Proof. This is a special case of Theorem 6 for $w(t) = \sqrt{1-t^2}$, $t \in [-1,1]$ and the non-negative function $W_{2n,w}^{GC2}$, defined by (25). □

Corollary 8. *Let $w_{2,2n}(t) \geq 0$, for all $t \in [x,-x]$ and for $n \in \mathbb{N}$. If $g : [-1,1] \to \mathbb{R}$ is a $(2n+2)$-convex function, then (26) holds for*

$$P(b) = A_{2n+1}^{GC2}(x) + B_{2n+1}^{GC2}(x)$$

and

$$\lambda = \frac{1}{P(b)}\left[x(A_{2n+1}^{GC2}(x) - B_{2n+1}^{GC2}(x)) - A_{2n+2}^{GC2}(x) - B_{2n+2}^{GC2}(x)\right].$$

If g is $(2n+2)$-concave, then (26) holds with the sign of inequalities reversed.

Proof. This is a special case of Theorem 7 for $w(t) = \sqrt{1-t^2}$, $t \in [-1,1]$ and the non-negative function $W_{2n,w}^{GC2}$, defined by (25). □

If we assume that the polynomials $Q_{j,x}(t)$ are such that

$$\begin{aligned} Q_{j,x}(x) &= \frac{1}{(j-1)!}\int_{-1}^{x}(x-s)^{j-1}\sqrt{1-s^2}\,ds, \text{ for } j=2,3,4, \\ Q_{j,x}(x) &= 0, \text{ for } j \geq 5, \end{aligned}$$

we have $A_1^{GC2}(x) = B_1^{GC2}(x) = \frac{\pi}{4}$ and $A_j^{GC2}(x) = B_j^{GC2}(x) = 0$, for $j = 2,3,4$, and hence we obtain

$$\begin{aligned} \int_{-1}^{1} g(t)\sqrt{1-t^2}\,dt &= \frac{\pi}{4}[g(x)+g(-x)] + T_{n,w}^{GC2}(x) \\ &\quad + (-1)^n \int_a^b W_{n,w}^{GC2}(t,x)g^{(n)}(t)\,dt, \end{aligned}$$

where

$$T_{n,w}^{GC2}(x) = \sum_{j=5}^{n}\Big[A_j^{GC2}(x)g^{(j-1)}(x) + B_j^{GC2}(x)g^{(j-1)}(-x)\Big].$$

Specifically, for $x = -\frac{1}{2}$, the generalization of the Gauss–Chebyshev two-point quadrature formula of the second kind follows. Now, we derive Hermite–Hadamard-type estimates for the Gauss–Chebyshev two-point quadrature formula of the second kind.

If the assumptions of Corollary (2) hold, for $w(t) = \sqrt{1-t^2}$, $t \in [-1,1]$ and the $(2n+2)$-convex function g we obtain

$$\begin{aligned} &P(b)\cdot g^{(2n)}(0) \\ &\leq \int_{-1}^{1} g(t)\sqrt{1-t^2}\,dt - \frac{\pi}{4}\left[g\left(-\frac{1}{2}\right)+g\left(\frac{1}{2}\right)\right] - T_{2n,w}^{GC2}\left(-\frac{1}{2}\right) \\ &\leq P(b)\cdot\left[\frac{1}{2}g^{(2n)}(-1)+\frac{1}{2}g^{(2n)}(1)\right], \end{aligned}$$

where

$$P(b) = \frac{2}{(4n+3)!!}F\left(-\frac{1}{2},\frac{3}{2},\frac{5}{2}+2n,\frac{1}{4}\right).$$

In the special case of $n = 2$, we obtain

$$\begin{aligned} &P(b)\cdot g^{(4)}(0) \\ &\leq \int_{-1}^{1} g(t)\sqrt{1-t^2}\,dt - \frac{\pi}{4}\left[g\left(-\frac{1}{2}\right)+g\left(\frac{1}{2}\right)\right] \\ &\leq P(b)\cdot\left[\frac{1}{2}g^{(4)}(-1)+\frac{1}{2}g^{(4)}(1)\right]. \end{aligned}$$

where

$$P(b) = \frac{5\pi - 9\sqrt{3}}{640} \approx 0.000186728.$$

4. Conclusions

The results presented in this paper are an extension of the investigation started in [5], in which the new method of calculating estimates for some quadrature rules using the weighted Hermite–Hadamard inequality for higher-order convex functions was introduced. The obtained results were applied to a weighted two-point formula for numerical integration to derive new estimates of the definite integral values. The Hermite–Hadamard inequality is one of the most important inequalities, and several variants and improvements have been proposed in the literature. However, this paper offers new research directions that could be useful and could motivate application in different types of convexity ([7,8]). We suggest this as an open problem for future work.

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