





## Article

# Hamiltonian Cycles in Cayley Graphs of Gyrogroups

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**Abstract:** In this study, we investigate Hamiltonian cycles in the right-Cayley graphs of gyrogroups. More specifically, we give a gyrogroup version of the factor group lemma and show that some right-Cayley graphs of certain gyrogroups are Hamiltonian.

**Keywords:** gyrogroup; Cayley graph; Hamiltonian cycle

**MSC:** 20C99; 20N05; 05C25



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## 1. Introduction

The gyrogroup structure is a group-like structure discovered by Ungar during his study of Einstein's relativistic velocity addition law, see [1,2] for more details. A gyrogroup can be considered as a generalization of a group, where the associative property is replaced by the left gyroassociative property and the left loop property. The past decade has seen a rise in research interest regarding algebraic properties of gyrogroups and topological properties of topological gyrogroups.

As for combinatorial properties, Cayley graphs of gyrogroups were first studied by Bussaban, Kawekhao, and Suantai in [3]. After that, some of us have studied some relationships between algebraic properties of gyrogroups and combinatorial properties of the Cayley graphs, see [4,5]. In these two studies, definitions of left and right Cayley graphs of gyrogroups were given, and some properties such as transitivity, connectedness, and preservation of edge coloring have been explored.

In this research, we continue our investigation on the right-Cayley graphs of gyrogroups. In particular, we study Hamiltonian cycles in the graphs. We state and prove a gyrogroup version of the factor group lemma, and we also show that the right-Cayley graphs  $R\text{-Cay}(G(n), \{1, m\})$  of the gyrogroups  $G(n)$  constructed in [6] are Hamiltonian.

Outline of the paper: In Section 2, preliminary knowledge, including the definition of a gyrogroup, the definition of its right-Cayley graph, and necessary results of these two structures, are provided. In Section 3, we give the statement and the proof of the factor group lemma, then the statement and the proof of its gyrogroup version, and some examples regarding the gyrogroup version. Later in this section, we show that the Cayley graphs of the gyrogroups constructed in [6], with the generating sets also given there, are Hamiltonian. Lastly, in Section 4, we discuss the results and leave some questions.

## 2. Background

This section contains basic knowledge of gyrogroups and their Cayley graphs. The section consists of two parts. In the first part, the definition of a gyrogroup, some important algebraic identities, and related properties are included. In the second part, the definitions of right Cayley graphs and some properties that were studied in [3–5] are provided.

For more detailed knowledge of gyrogroups, we recommend readers to see [1,2,7]. For basic knowledge of graph theory, we refer readers to [8].

### 2.1. Gyrogroups

Let  $(G, \oplus)$  be a groupoid. We will simply call it  $G$  if there is no ambiguity in the operation. An automorphism  $f$  of  $G$  is a bijection from  $G$  to itself with the property that  $f(g_1 \oplus g_2) = f(g_1) \oplus f(g_2)$  for all  $g_1, g_2 \in G$ . The set of all automorphisms on  $G$  is denoted by  $\text{Aut}(G, \oplus)$ .

**Definition 1** (Definition 2.7 of [1]). Let  $(G, \oplus)$  be a non-empty groupoid. We say that  $G$  is a gyrogroup if the following properties hold:

1. There is a unique identity element  $e \in G$  such that

$$e \oplus x = x = x \oplus e \quad \text{for all } x \in G.$$

2. For each  $x \in G$ , there exists a unique inverse element  $\ominus x \in G$  such that

$$\ominus x \oplus x = e = x \oplus (\ominus x).$$

3. For all  $x, y \in G$ , there exists an automorphism  $\text{gyr}[x, y] \in \text{Aut}(G, \oplus)$  such that

$$x \oplus (y \oplus z) = (x \oplus y) \oplus \text{gyr}[x, y]z \quad (\text{left gyroassociative law})$$

for all  $z \in G$ .

4. For all  $x, y \in G$ ,  $\text{gyr}[x \oplus y, y] = \text{gyr}[x, y]$ . (left loop property)

For all elements  $a, b, c$  in a gyrogroup  $G$ , the gyroautomorphism  $\text{gyr}[a, b]$  is given by the following identity:

$$\text{gyr}[a, b]c = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c)). \quad (\text{gyrator identity})$$

Algebraic properties of gyrogroup parallel to those of groups were rigorously studied by Suksumran and his colleagues. Among their work, the following definitions and theorems are necessary to our work. Readers are recommended to see [7] for more details.

**Definition 2.** A non-empty subset  $H$  of a gyrogroup  $(G, \oplus)$  is a subgyrogroup of  $G$  if  $H$  is a gyrogroup under the operation inherited from  $G$  and  $\text{gyr}[a, b](H) = H$  for all  $a, b \in H$ . It is called an  $L$ -subgyrogroup of  $G$  if  $\text{gyr}[a, h](H) = H$  for all  $a \in G$  and  $h \in H$ .

**Theorem 1.** If  $H$  is an  $L$ -subgyrogroup of a gyrogroup  $G$ , then the set  $\{g \oplus H \mid g \in G\}$  forms a partition of  $G$ .

Analogous to groups, when  $H$  is an  $L$ -subgyrogroup of a gyrogroup  $G$ , we will call each  $g \oplus H$ , a left coset.

**Theorem 2** (Theorem 21 of [9], Lagrange's Theorem for  $L$ -Subgyrogroups). If  $H$  is an  $L$ -subgyrogroup of a finite gyrogroup  $G$ , then  $|H|$  divides  $|G|$ .

Writing  $[G : H]$  as the number of left cosets of  $H$  in  $G$ , we have the following corollary as a consequence of Theorem 2.

**Corollary 1.** If  $H$  is an  $L$ -subgyrogroup of a finite gyrogroup  $G$ , then  $|G| = [G : H]|H|$ .

In this study, we focus on finite gyrogroups, and the following class of gyrogroups which are constructed in [6] will be used throughout.

**Example 1.** In [6], Mahdavi, Ashrafi, Salahshour, and Ungar constructed a class of gyrogroups whose every proper subgyrogroup is either a cyclic or a dihedral group. They call the gyrogroups in this class dihedral gyrogroups because the (normal) subgyrogroup lattice of each gyrogroup in this class is isomorphic to the (normal) subgroup lattice of the dihedral group with the same order. Later in this paper, we will see a similarity between the Cayley graphs of the dihedral gyrogroups and those of the dihedral groups. We show the construction of the dihedral gyrogroups in this example. For an integer  $n \geq 3$ , let  $P(n) = \{0, 1, \dots, 2^{n-1} - 1\}$ ,  $H(n) = \{2^{n-1}, 2^{n-1} + 1, \dots, 2^n - 1\}$ , and  $G(n) = P(n) \cup H(n)$ . Let  $m = 2^{n-1}$ . The binary operation of the gyrogroup  $(G(n), \oplus)$  is defined as follows:

$$i \oplus j = \begin{cases} t & (i, j) \in P(n) \times P(n) \\ t + m & (i, j) \in P(n) \times H(n) \\ s + m & (i, j) \in H(n) \times P(n) \\ k & (i, j) \in H(n) \times H(n), \end{cases}$$

where  $t, s, k \in P(n)$  are the following non-negative integers:

$$\begin{cases} t \equiv i + j & \text{mod } m \\ s \equiv i + (\frac{m}{2} - 1)j & \text{mod } m \\ k \equiv (\frac{m}{2} + 1)i + (\frac{m}{2} - 1)j & \text{mod } m. \end{cases}$$

In [6], the gyroaddition tables and the gyration tables of  $G(3)$  and  $G(4)$  are provided, and we include them here as Tables 1–3 for reference.

**Table 1.** The gyroaddition table (left) and the gyration table (right) for the gyrogroup  $G(3)$ . The gyroautomorphism  $A$  is given in cycle decomposition by  $A = (4\ 6)(5\ 7)$ .

$\oplus$	0	1	2	3	4	5	6	7	gyr	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7	0	I	I	I	I	I	I	I	I
1	1	2	3	0	5	6	7	4	1	I	I	I	I	A	A	A	A
2	2	3	0	1	6	7	4	5	2	I	I	I	I	I	I	I	I
3	3	0	1	2	7	4	5	6	3	I	I	I	I	A	A	A	A
4	4	5	6	7	0	1	2	3	4	I	A	I	A	I	A	I	A
5	5	6	7	4	3	0	1	2	5	I	A	I	A	A	I	A	I
6	6	7	4	5	2	3	0	1	6	I	A	I	A	I	A	I	A
7	7	4	5	6	1	2	3	0	7	I	A	I	A	A	I	A	I

**Table 2.** The gyroaddition table for the gyrogroup  $G(4)$ .

$\oplus$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	6	7	0	9	10	11	12	13	14	15	8
2	2	3	4	5	6	7	0	1	10	11	12	13	14	15	8	9
3	3	4	5	6	7	0	1	2	11	12	13	14	15	8	9	10
4	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	5	6	7	0	1	2	3	4	13	14	15	8	9	10	11	12
6	6	7	0	1	2	3	4	5	14	15	8	9	10	11	12	13
7	7	0	1	2	3	4	5	6	15	8	9	10	11	12	13	14
8	8	11	14	9	12	15	10	13	0	3	6	1	4	7	2	5
9	9	12	15	10	13	8	11	14	5	0	3	6	1	4	7	2
10	10	13	8	11	14	9	12	15	2	5	0	3	6	1	4	7
11	11	14	9	12	15	10	13	8	7	2	5	0	3	6	1	4
12	12	15	10	13	8	11	14	9	4	7	2	5	0	3	6	1
13	13	8	11	14	9	12	15	10	1	4	7	2	5	0	3	6
14	14	9	12	15	10	13	8	11	6	1	4	7	2	5	0	3
15	15	10	13	8	11	14	9	12	3	6	1	4	7	2	5	0

**Table 3.** The gyration table for the gyrogroup  $G(4)$ . The gyroautomorphism  $A$  is given in cycle decomposition by  $A = (8\ 12)(9\ 13)(10\ 14)(11\ 15)$ .

gyr	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$
1	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$A$	$A$	$A$	$A$	$A$	$A$	$A$	$A$
2	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$
3	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$A$	$A$	$A$	$A$	$A$	$A$	$A$	$A$
4	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$
5	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$A$	$A$	$A$	$A$	$A$	$A$	$A$	$A$
6	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$
7	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$I$	$A$	$A$	$A$	$A$	$A$	$A$	$A$	$A$
8	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$
9	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$
10	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$
11	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$
12	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$
13	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$
14	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$
15	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$	$I$	$A$

Let us turn to some algebras of gyrogroups. To solve the equation  $x \oplus a = b$  for  $x$ , Ungar introduced a second binary operation in  $G$  called the gyrogroup coaddition or coaddition  $\boxplus$ , defined by

$$a \boxplus b = a \oplus \text{gyr}[a, \ominus b]b$$

for all  $a, b \in G$ . We write  $a \boxminus b$  for  $a \boxplus \ominus b$ . The solution to the equation  $x \oplus a = b$  is  $x = b \boxminus a$ .

Many identities regarding the gyrogroup addition and coaddition have been studied in [1], and we list some of them here.

**Theorem 3 ([1]).** Let  $G$  be a gyrogroup. For all  $a, b, c \in G$ , the following properties hold:

1. if  $a \oplus b = a \oplus c$ , then  $b = c$ ; (general left cancellation law)
2.  $\ominus a \oplus (a \oplus b) = b$ ; (left cancellation law)
3.  $(a \ominus b) \boxplus b = a$ ; (right cancellation law I)
4.  $(a \boxminus b) \oplus b = a$ ; (right cancellation law II)
5.  $(a \oplus b) \oplus c = a \oplus (b \oplus \text{gyr}[b, a]c)$ ; (right gyroassociative law)
6.  $\text{gyr}[a, b](\ominus c) = \ominus \text{gyr}[a, b]c$ .

## 2.2. Cayley Graphs of Gyrogroups

In this subsection, we give the definition of a right-Cayley graph of a gyrogroup and collect some properties that will be used to prove our results. For more details about the study of Cayley graphs of gyrogroups so far, we encourage readers to see [3–5].

A directed graph is an ordered pair  $\vec{\mathcal{D}} = (V, \vec{E})$ , where  $V$  is a set of vertices and  $\vec{E} \subseteq \{(u, v) | u, v \in V \text{ and } u \neq v\}$  is a set of edges. In this paper, we will often write  $u \rightarrow v$  instead of  $(u, v)$  to emphasize the direction. A directed graph is said to be undirected if, for any vertices  $u$  and  $v$ ,  $u \rightarrow v$  implies  $v \rightarrow u$ . Given a directed graph  $\vec{\mathcal{D}}$ , we may also consider  $\mathcal{D} = (V, E)$ , where  $E = \{\{u, v\} | (u, v) \in \vec{E}\}$  as the underlying undirected graph of  $\vec{\mathcal{D}}$ . A directed graph  $\vec{\mathcal{D}}$  is connected if  $\mathcal{D}$  is connected. A graph automorphism of  $\mathcal{D}$  is a bijection  $\sigma$  from the vertex set of  $\mathcal{D}$  to itself such that for all vertices  $u, v$ ,  $\{u, v\}$  is an edge in  $\mathcal{D}$  if and only if  $\{\sigma(u), \sigma(v)\}$  is an edge in  $\mathcal{D}$  (respectively,  $(u, v)$  is an edge in  $\vec{\mathcal{D}}$  if and only if  $(\sigma(u), \sigma(v))$  is an edge in  $\vec{\mathcal{D}}$ ). An undirected graph  $\mathcal{D}$  (respectively, a directed graph  $\vec{\mathcal{D}}$ ) is vertex-transitive if, for all vertices  $u, v$ , there is an automorphism  $\sigma$  of  $\mathcal{D}$  (respectively,  $\vec{\mathcal{D}}$ ) such that  $\sigma(u) = v$ .

**Definition 3** (Right-Cayley graph). Let  $G$  be a gyrogroup and let  $S$  be a subset of  $G$  not containing the identity. The (color) right-Cayley graph or (color) R-Cayley graph of  $G$  generated by  $S$ , denoted by  $\text{R-Cay}(G, S)$ , is a directed graph whose vertices are the gyrogroup elements, and for any two vertices  $u$  and  $v$ , there is an edge  $u \rightarrow v$  with color  $s$  if  $v = u \oplus s$  for some  $s \in S$ . We will conflate the gyrogroup elements and the vertices of graph whenever there are no confusions.

The left-Cayley graphs can be defined in the same way as in Definition 3 by adding  $s$  to the left-hand side instead. We give our attention to the right-Cayley graphs because of their connection to the L-subgyrogroups. For example, Theorem 4 shows a relationship between the cosets of L-subgyrogroups and the connected components of R-Cayley graphs.

**Theorem 4** (Theorem 4.8 in [4]). Let  $G$  be a finite gyrogroup and let  $S$  be a symmetric subset of  $G$  such that it right-generates an L-subgyrogroup  $H$  and  $\text{gyr}[g, h](S) = S$  for all  $g \in G$  and for all  $h \in H$ . Then, two vertices  $u$  and  $v$  are in the same connected component of  $\text{R-Cay}(G, S)$  if and only if  $u$  and  $v$  are in the same left-coset of  $H$ .

Unlike groups, transitivity of the Cayley graphs of gyrogroups does not always hold, due to the lack of associativity. In the case of groups, the essence is that a left (right)-multiplication by an element is always an automorphism on the right (left)-Cayley graph. For gyrogroups, some of us showed in [5] that, under a certain condition, any left-addition by an element is an automorphism on the right-Cayley graph.

**Theorem 5** (Theorem 4.5 in [4]). Let  $G$  be a finite gyrogroup and let  $S$  be a symmetric subset of  $G$ . Then, every left addition induces an automorphism of  $\text{R-Cay}(G, S)$ , in particular,  $\text{R-Cay}(G, S)$  is  $L_S(G)$ -transitive if and only if  $\text{gyr}[g, g'](S) = S$  for all  $g, g' \in G$ .

For the last part of this subsection, we talk about normal subgyrogroups and give the definition of a Cayley graph of a quotient gyrogroup. A gyrogroup homomorphism is a map between two gyrogroups that preserves the gyrogroup operations. Let  $\varphi : G \rightarrow K$  be a gyrogroup homomorphism. The kernel of  $\varphi$  is the set  $\{a \in G \mid \varphi(a) = e_K\}$ , where  $e_K$  is the identity of  $K$ . A normal subgyrogroup  $H$  of a gyrogroup  $G$  is defined to be the kernel of a gyrogroup homomorphism with domain  $G$ , denoted by  $H \trianglelefteq G$ . In this case, the quotient space of the left cosets  $G/H$  is a gyrogroup (Theorem 29 in [7]). The gyroaddition in the quotient space is defined by

$$(a \oplus H) \oplus (b \oplus H) = (a \oplus b) \oplus H,$$

and the gyration is defined by

$$\text{gyr}[a \oplus H, b \oplus H](c \oplus H) = \text{gyr}[a, b]c \oplus H.$$

Let  $S$  be a subset of  $G$ . We can define the right-Cayley graph of  $G/H$  induced by  $S$  and call it the quotient right-Cayley graph, denoted by  $\text{R-Cay}(G/H, S^*)$ , where  $S^* = S/H - \{H\}$ . The coset  $H$  is excluded from  $S/H$  to avoid self loops in the graph.

We delay examples of Cayley graphs of gyrogroups and Cayley graphs of quotient gyrogroups to the next section, where our results are given.

### 3. Hamiltonian Cycles in Right-Cayley Graphs of Gyrogroups

In this section, we give our results and examples. We begin with some notations in graph theory. We may refer to a walk in a graph by specifying its vertices as  $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_n$ , or by specifying its edge labelling as  $[s_1, s_2, \dots, s_n]$ . However, in the case of edge labelling, a walk is not unique unless the initial vertex is given. In this case, we will write  $v_1[s_1, s_2, \dots, s_n]$  to indicate that the initial vertex is  $v_1$ . We will write  $[s_1, s_2, \dots, s_n]^i$  for a walk that repeats edge labelling  $[s_1, s_2, \dots, s_n]$   $i$  times.

We are ready to talk about our first result in this work. For groups, the factor group lemma gives a sufficient condition for a Cayley graph of a group to be Hamiltonian. We state the lemma and provide a proof that can be adapted for the gyrogroup version.

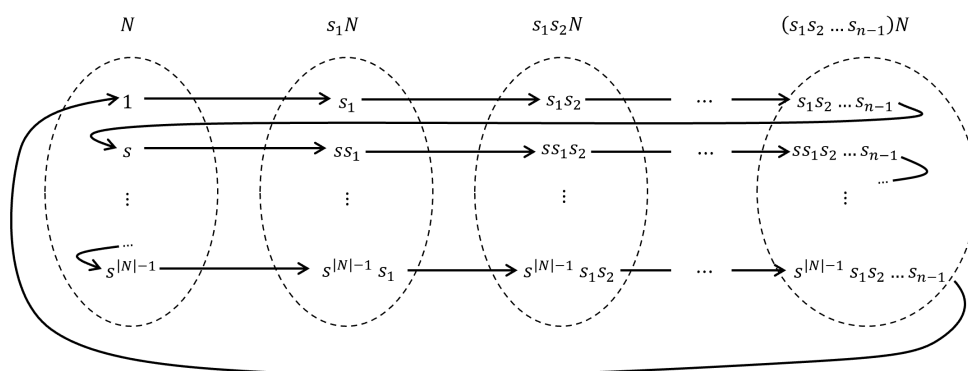
**Lemma 1** (Factor Group Lemma, Section 2.2 in [10]). *Let  $G$  be a finite group and let  $S$  be a generating set of  $G$ . Suppose that there exist elements  $s_1, s_2, \dots, s_n$  in  $S$  satisfying the following two conditions:*

1. *the product  $s_1 s_2 \cdots s_n := s$  generates a cyclic normal subgroup  $N$  of  $G$  with index  $[G : N] = n$ , and*
2. *the quotient Cayley graph  $\text{Cay}(G/N, S^*)$  is Hamiltonian with the cycle  $[s_1 N, s_2 N, \dots, s_n N]$ .*

*Then,  $\text{Cay}(G, S)$  is Hamiltonian with the cycle  $[s_1, s_2, \dots, s_n]^{|N|}$ .*

**Proof.** Left-multiplying each vertex in the right-Cayley graph by an element defines an automorphism on the graph. Thus, we will show that, starting with the identity element 1, the walk  $1[s_1, s_2, \dots, s_n]^{|N|}$  is a cycle containing all elements in  $G$ , e.g., a Hamiltonian cycle. Then, for any  $a \in G$ ,  $a[s_1, s_2, \dots, s_n]^{|N|}$  is a Hamiltonian cycle.

By the first condition of the lemma, the last vertex of the walk  $1[s_1, s_2, \dots, s_n]^{|N|}$  is  $1(s_1 s_2 \cdots s_n)^{|N|} = 1s^{|N|} = 1$  and the length of the walk is  $n|N| = [G : N]|N| = |G|$ . Each non-identity vertex  $u$  in the walk is of form  $s^m x$ , where  $0 \leq m \leq |N| - 1$  and  $x = s_1, s_1 s_2, \dots$ , or  $s_1 s_2 \cdots s_{n-1}$ , see Figure 1 for reference. The second condition tells us that the  $n$   $N$ -cosets are  $N, s_1 N, s_1 s_2 N, \dots$ , and  $(s_1 s_2 \cdots s_{n-1})N$ . Thus, during the walk, the  $N$ -coset of the vertex that we are visiting changes to the corresponding coset. We claim that we visit different vertices until we are back to 1. If not, suppose that vertices  $u$  and  $v$  are in distinct positions in the walk and  $u = v$ . Both of them must be in the same  $N$ -coset, say  $u = s^i x$  and  $v = s^j x$ , where  $i \neq j$ ,  $0 \leq i, j \leq |N| - 1$ , and  $x = s_1, s_1 s_2, \dots$ , or  $s_1 s_2 \cdots s_{n-1}$ . Then, we have  $s^{i-j} = 1$ , which is contrary to the order of  $s$ . We conclude that the walk  $1[s_1, s_2, \dots, s_n]^{|N|}$  is a Hamiltonian cycle.  $\square$



**Figure 1.** The Hamiltonian cycle explained in the proof of the Factor Group Lemma.

Notice that associativity plays an important role in the proof of the lemma. It is so powerful that we can omit parentheses entirely. However, the property is absent in gyrogroups. In particular,  $((x \oplus y) \oplus x) \oplus y$  does not always equal  $(x \oplus y) \oplus (x \oplus y)$ . For example, in  $G(3)$ ,  $((4 \oplus 5) \oplus 4) \oplus 5 = 0$ , whereas  $(4 \oplus 5) \oplus (4 \oplus 5) = 2$ . Hence,  $(\cdots ((s \oplus s_1) \oplus s_2) \oplus \cdots s_{n-1}) \oplus s_n$  is not necessarily equal to  $s \oplus s$ . Moreover, in the lemma, the cycles are given by edge labelling without specifying the initial vertex. This is due to the fact that the left multiplications are automorphisms on the Cayley graphs of groups. However, this is not the case for gyrogroups, see some examples in [4]. The following theorem is a version of this lemma for gyrogroups.

**Theorem 6** (Factor Gyrogroup Lemma). *Let  $G$  be a finite gyrogroup and let  $S$  be a subset of  $G$ . Suppose that there exist elements (possibly repeated)  $s_1, s_2, \dots, s_n$  in  $S$  satisfying the following conditions:*



1. the gyrosum  $s = (\cdots (s_1 \oplus s_2) \oplus \cdots) \oplus s_n$  generates a cyclic normal subgyrogroup  $H$  of  $G$  with index  $[G : H] = n$ ,
2. for  $i = 1, 2, \dots, |H| - 1$ ,  $(\cdots ((is \oplus s_1) \oplus s_2) \oplus \cdots) \oplus s_n = is \oplus s = (i + 1)s$ , and
3. the quotient right-Cayley graph  $\text{R-Cay}(G/H, S^*)$  is Hamiltonian with the cycle

$$H \rightarrow s_1 \oplus H \rightarrow (s_1 \oplus s_2) \oplus H \rightarrow \cdots \rightarrow ((\cdots (s_1 \oplus s_2) \oplus \cdots) \oplus s_n) \oplus H$$

or, equivalently, the cycle

$$H[s_1 \oplus H, s_2 \oplus H, \dots, s_n \oplus H].$$

Then,  $\text{R-Cay}(G, S)$  is Hamiltonian with the cycle

$$\begin{aligned} 0 &\rightarrow s_1 \rightarrow s_1 \oplus s_2 \rightarrow \cdots \rightarrow (\cdots (s_1 \oplus s_2) \oplus \cdots) \oplus s_{n-1} \rightarrow s \rightarrow s \oplus s_1 \rightarrow \cdots \\ &\rightarrow (|H| - 1)s \oplus s_1 \rightarrow \cdots \\ &\rightarrow (\cdots ((|H| - 1)s \oplus s_1) \oplus \cdots \oplus s_{n-2}) \oplus s_{n-1} = (|H| - 1)s \oplus ((\cdots (s_1 \oplus s_2) \oplus \cdots) \\ &\oplus s_{n-1}) \\ &\rightarrow 0, \end{aligned}$$

or, equivalently, the cycle  $0[s_1, s_2, \dots, s_n]^{[H]}$ . In addition to the three conditions above, if

4.  $\text{gyr}[g, g'](S) = S$  for all  $g, g' \in G$ ,

Then, starting with a vertex  $x$ ,

$$x[s_1, \text{gyr}[x, s_1]s_2, \text{gyr}[x, s_1 \oplus s_2]s_3, \text{gyr}[x, (s_1 \oplus s_2) \oplus s_3]s_4, \dots, \text{gyr}[x, s]s_1, \text{gyr}[x, s \oplus s_1]s_2, \dots, \text{gyr}[x, |H|s]s_n]$$

is a Hamiltonian cycle. Notice that the  $i^{\text{th}}$  edge in the labelling is acted by  $\text{gyr}[x, v_i]$ , where  $v_i$  is the  $i^{\text{th}}$  vertex of the Hamiltonian cycle starting at the identity obtained from the first part.

**Proof.** For the first part, the second condition in this theorem together with the right cancellation law I enable us to follow the same proof as in Lemma 1. For the second part, suppose  $\text{gyr}[g, g'](S) = S$ , for all  $g, g' \in G$ . By Theorem 5, the left addition by  $x$  is an automorphism on  $\text{R-Cay}(G, S)$ . Moreover, the edge connecting  $x \oplus ms$  and  $(x \oplus ms) \oplus s_1$  is  $\text{gyr}[x, ms]s_1$ , and the edge connecting  $x \oplus (\cdots ((ms \oplus s_1) \oplus s_2) \cdots \oplus s_q)$  and  $x \oplus ((\cdots ((ms \oplus s_1) \oplus s_2) \cdots \oplus s_q) \oplus s_{q+1})$  is  $\text{gyr}[x, ((ms \oplus s_1) \oplus s_2) \cdots \oplus s_q]s_{q+1}$  obtained by moving the outermost parenthesis to the left-hand side, where  $m = 0, 1, \dots, |H| - 1$  and  $q \leq n - 1$ .  $\square$

**Example 2.** Consider the gyrogroup  $G(3)$  with the generating set  $S_1 = \{1, 6\}$ . The gyrosum  $((1 \oplus 6) \oplus 1) \oplus 6 = 2$  is the generator of the normal cyclic subgyrogroup  $H = \{0, 2\}$ , which satisfies the first condition of Theorem 6. From the gyration table,  $\text{gyr}[g, 2] = \text{id}$  for all  $g \in G(3)$ , which yields the second condition. The quotient Cayley graph  $\text{R-Cay}(G(3)/H, S_1^*)$  is a square with a Hamiltonian cycle  $H[1 \oplus H, 6 \oplus H, 1 \oplus H, 6 \oplus H]$ , satisfying the third condition. Hence, Theorem 6 implies that

$$0[1, 6, 1, 6]^2 = 0[1, 6, 1, 6, 1, 6, 1, 6] \quad \text{or equivalently} \quad 0 \rightarrow 1 \rightarrow 7 \rightarrow 4 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 6 \rightarrow 0$$

is a Hamiltonian cycle in  $\text{R-Cay}[G(3), S_1]$ . However, the cycle depends on the initial vertex, for example, the cycle  $1[1, 6, 1, 6]^2$  is not Hamiltonian. Both  $\text{R-Cay}(G(3)/H, S_1^*)$  and  $\text{R-Cay}(G(3), S_1)$  are depicted together as in Figure 2.

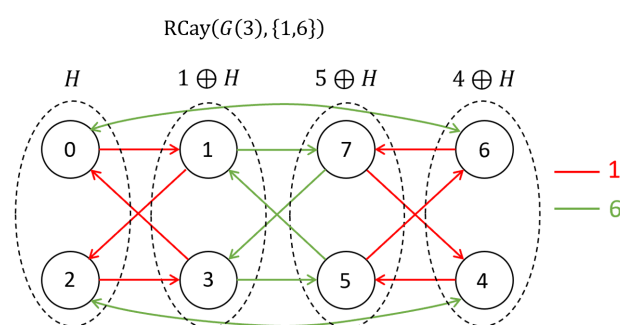
Adding 4 to the generating set, we have a new generating set  $S_2 = \{1, 4, 6\}$  that satisfies the condition  $\text{gyr}[g, g'](S_2) = S_2$  for all  $g, g' \in G(3)$ . Theorem 6 implies that, for any  $x \in G(3)$ , the cycle starting at  $x$ ,

$$x[1, \text{gyr}[x, 1]6, \text{gyr}[x, 7]1, \text{gyr}[x, 4]6, \text{gyr}[x, 2]1, \text{gyr}[x, 3]6, \text{gyr}[x, 5]1, \text{gyr}[x, 6]6],$$

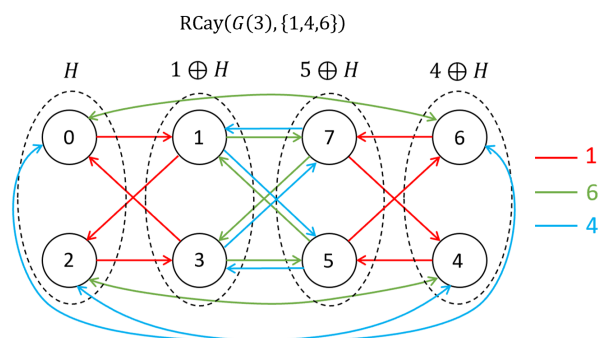
is Hamiltonian in  $\text{R-Cay}(G(3), S_2)$ . Notice that the  $i^{\text{th}}$  edge in the labelling is acted by  $\text{gyr}[x, v_i]$ , where  $v_i$  is the  $i^{\text{th}}$  vertex of the Hamiltonian cycle obtained earlier in this example. For example, starting at 1, the cycle

$$1[1, \text{gyr}[1, 1]6, \text{gyr}[1, 7]1, \text{gyr}[1, 4]6, \text{gyr}[1, 2]1, \text{gyr}[1, 3]6, \text{gyr}[1, 5]1, \text{gyr}[1, 6]6],$$

which is  $1[1, 6, 1, 4, 1, 6, 1, 4]$ , is Hamiltonian. The right Cayley graph  $\text{R-Cay}(G(3), S_2)$  is depicted in Figure 3.



**Figure 2.** The right Cayley graph  $\text{R-Cay}(G(3), S_1)$ , where  $S_1 = \{1, 6\}$ . The elements in  $G(3)$  are grouped by  $H$ -cosets, where  $H = \{0, 2\}$ . The quotient right-Cayley graph  $\text{R-Cay}(G(3)/H, S_1^*)$  can also be seen in this drawing when we consider each coset as a vertex.



**Figure 3.** The right-Cayley graph  $\text{R-Cay}(G(3), \{1, 4, 6\})$ .

**Theorem 7.** In  $\text{R-Cay}(G(n), \{1, m\})$ , the walk  $0[[m][1]^{m-1}]^2$  is a Hamiltonian cycle.

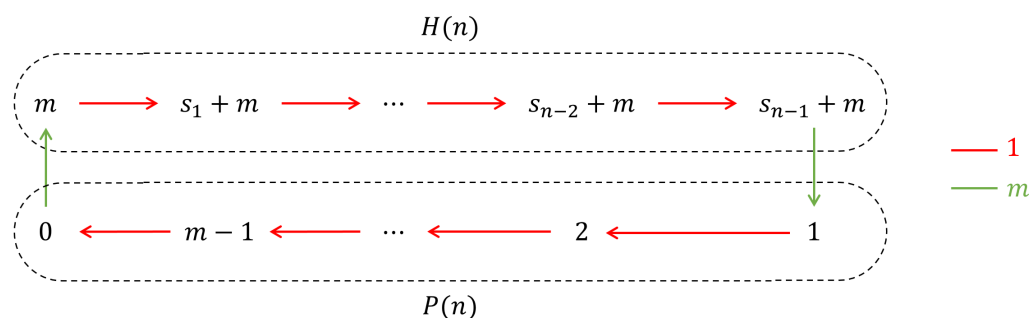
**Proof.** We will show that the walk exhausts all vertices in  $H(n)$  then all vertices in  $P(n)$ , see Figure 4 for reference. The walk starts at the vertex 0 and, after adding to the right by  $m = 2^{n-1}$ , we are at the vertex  $m \in H(n)$ . All vertices in the walk starting at  $m, m[1]^{m-1}$ , are in  $H(n)$  since  $h \oplus 1 \in H(n)$  whenever  $h \in H(n)$  by the definition of addition in  $G(n)$ . We observe that  $h \oplus 1 = s + m$ , where  $s \equiv h + 2^{n-2} - 1 \pmod{m}$  and  $0 \leq s \leq m - 1$ , and also observe that  $\gcd(m, 2^{n-2} - 1) = \gcd(2^{n-1}, 2^{n-2} - 1) = 1$ . Hence, all vertices in the walk  $m[1]^{m-1}$  are all distinct and all of  $H(n)$ . The terminal vertex of the path  $m[1]^{m-1}$  is



$s + m$ , where  $s \equiv (2^{n-2} - 1)(2^{n-1} - 1) \pmod{m}$ . Adding  $m$  to the right of the terminal vertices of the path  $m[1]^{m-1}$ , we obtain  $(s + m) \oplus m = 1 \in P(n)$  because

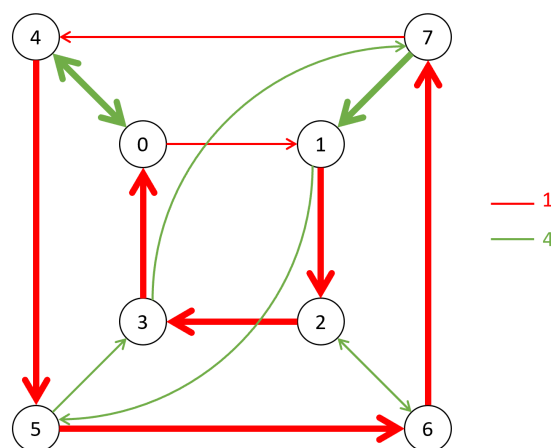
$$\begin{aligned} \left(\frac{m}{2} + 1\right)(s + m) + \left(\frac{m}{2} - 1\right)m &\equiv (2^{n-2} + 1)((2^{n-2} - 1)(2^{n-1} - 1) + 2^{n-1}) + (2^{n-2} - 1)2^{n-1} \\ &\equiv 1 \pmod{2^{n-1}}. \end{aligned}$$

It is easy to see that  $1[1]^{m-1}$  is the path  $1 \rightarrow 2 \rightarrow 3 \rightarrow \dots \rightarrow m - 1 \rightarrow 0$  consisting of all elements in  $P(n)$ .  $\square$



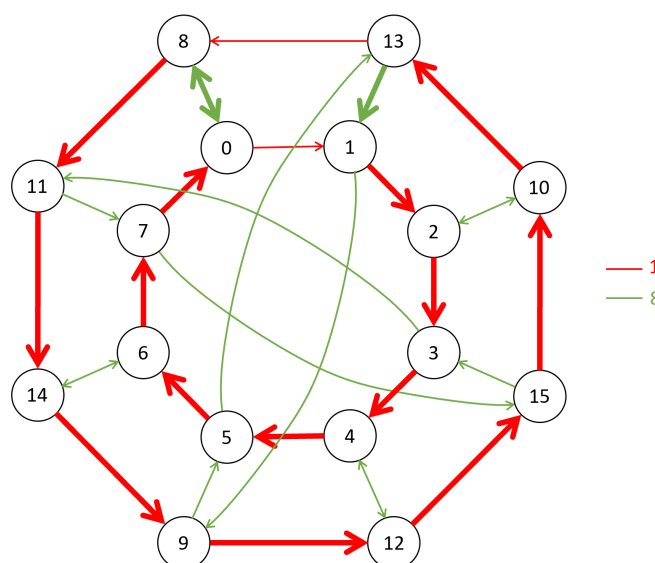
**Figure 4.** The Hamiltonian cycle in  $\text{R-Cay}(G(n), \{1, m\})$  in the proof of Theorem 7.

**Example 3.** Consider  $\text{R-Cay}(G(3), \{1, 4\})$ . It follows from Theorem 7 that  $0[4[1]^3]^2$  is a Hamiltonian cycle, see Figure 5. Notice the similarity between  $\text{R-Cay}(G(3), \{1, 4\})$  and the Cayley graph of the dihedral group of order 4 generated by a rotation and a reflection. In addition, notice the similarity between the Hamiltonian cycle  $0[4[1]^3]^2$  and the Hamiltonian cycle in the dihedral group constructed from the rotation and the reflection.



**Figure 5.** The right-Cayley graph  $\text{R-Cay}(G(3), \{1, 4\})$  with the Hamiltonian cycle  $0[4[1]^3]^2$  shown in thick edges.

**Example 4.** Consider  $\text{R-Cay}(G(4), \{1, 8\})$ . It follows from Theorem 7 that  $0[8[1]^7]^2$  is a Hamiltonian cycle, see Figure 6. Notice the similarity of the figures and Hamiltonian cycles between this Cayley graph and the Cayley graph of the dihedral group of order 8 generated by a rotation and a reflection.



**Figure 6.** The right-Cayley graph  $R\text{-Cay}(G(4), \{1, 8\})$  with the Hamiltonian cycle  $0[8[1]^7]^2$  shown in thick edges.

#### 4. Discussion

With the absence of associativity, many properties of Cayley graphs of groups do not hold true for gyrogroups. However, similar to other algebraic and topological properties of gyrogroups that have been studied, when imposed with the so-called strongly generated property ( $S$  is a generating set of a gyrogroup  $G$  and  $\text{gyr}[g, g'](S) = S$  for all  $g, g' \in G$ ), many group-like properties of gyrogroups are valid to some degree. In our version of the factor gyrogroup lemma, when the strongly generated property holds, the Hamiltonian cycle is independent of the choice of the starting vertex with a specific change in the edge labelling. While many properties of Cayley graphs of groups have been studied, their gyrogroup counterparts are yet to be explored. We would like to end our discussion with some questions similar to the classical ones in group theory and a question of our interest.

**Question 1.** *Is every connected right-Cayley graph of a gyrogroup Hamiltonian?*

In group theory, Question 1 is still an open problem. The progress of the study of Hamiltonicity in Cayley graphs of groups can be found in [11–14], for instance. Therefore, we also ask a more specific question.

**Question 2.** *Is every connected right-Cayley graph of a dihedral gyrogroup Hamiltonian?*

We remark that, even in the case of groups, Question 2 is not completely solved. In [15], the authors prove that the right-Cayley graph of a dihedral group  $D_{2n}$  is Hamiltonian for all even integers  $n$  (see Corollary 4.1 of [15]). Seeing the similarity between the Cayley graphs of dihedral gyrogroups and the Cayley graphs of dihedral groups, we are interested in the following question:

**Question 3.** *Is there a geometric representation of the dihedral gyrogroups?*

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