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Invariant Finite-Difference Schemes for Plane One-Dimensional MHD Flows That Preserve Conservation Laws

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Abstract: Invariant finite-difference schemes are considered for one-dimensional magnetohydrodynamics (MHD) equations in mass Lagrangian coordinates for the cases of finite and infinite conductivity. The construction of these schemes makes use of results of the group classification of MHD equations previously obtained by the authors. On the basis of the classical Samarskiy–Popov scheme, new schemes are constructed for the case of finite conductivity. These schemes admit all symmetries of the original differential model and have difference analogues of all of its local differential conservation laws. New, previously unknown, conservation laws are found using symmetries and direct calculations. In the case of infinite conductivity, conservative invariant schemes are constructed as well. For isentropic flows of a polytropic gas the proposed schemes possess the conservation law of energy and preserve entropy on two time layers. This is achieved by means of specially selected approximations for the equation of state of a polytropic gas. In addition, invariant difference schemes with additional conservation laws are proposed. A new scheme for the case of finite conductivity is tested numerically for various boundary conditions, which shows accurate preservation of difference conservation laws.

Keywords: classical symmetries; conservation law; numerical scheme

MSC: 65M06; 76W05

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1. Introduction

Magnetic hydrodynamics equations describe the flows of electrically conductive fluids such as plasma, liquid metals, and electrolytes and are widely used in modeling processes in various fields from engineering to geophysics and astrophysics.

In the present publication, we restrict ourselves to considering plane one-dimensional MHD flows under the assumption that the medium is inviscid and thermally non-conducting. A group classification of the MHD equations under the above conditions was carried out recently in [1] (for some particular results see also [2–5]). The group classification splits into four essentially different cases according to whether the conductivity of the medium is finite or infinite, and the longitudinal component of the magnetic field vector is zero or a non-zero constant.

The MHD equations are nonlinear, so that even in the one-dimensional case only their particular solutions are known [6–10]. Therefore, numerical modeling in magnetohydrodynamics is of great practical interest. There are many approaches to numerically modeling MHD equations, including finite-difference, finite element, and finite volume methods (see, e.g., [11–19]). Further we consider finite-difference schemes taking as a starting point the classical Samarsky–Popov schemes [12,13] for the MHD equations for the case of finite conductivity. The main properties of the considered schemes are invariance, i.e., preservation of the symmetries of the original differential equations, and the presence of difference analogues of local differential conservation laws. It is known that there is a connection between the invariance of equations and the presence of conservation laws [20–23]. Recall that local conservation laws in the form of a divergence of some vector make it possible to calculate flows through the boundary of a region using the Gauss–Ostrogradsky theorem [24]. The latter has a clear physical meaning—the vector flux (electromagnetic field, etc.) through the computational domain is preserved if there are no sources inside the domain. The presence of a finite-difference analog of such a local conservation law also makes it possible to sum it over the entire computational domain and preserve the vector flow through the boundary of a domain. Thus, such a difference scheme retains the geometric and physical meaning inherent in the differential model.

Invariant schemes have been studied for a long time [25–28], and over the past decades, significant progress has been made in the development of methods for their construction and integration. For schemes for ordinary differential equations with Lagrangian or Hamiltonian functions, a number of methods [29] have been developed that make it possible to decrease the order or even integrate the schemes. A method based on the Lagrangian identity has also been developed for the case when the equations do not admit a variational formulation [30].

For partial difference schemes, the main methods used are the method of differential invariants [28,31,32] and the difference analogue of the direct method [28]. Using these methods, the authors have constructed invariant schemes for various shallow water models [33–36]. In addition, some previously known schemes have been investigated from a group analysis point of view. In particular, symmetries and conservation laws of the Samarskiy–Popov schemes for the one-dimensional gas dynamics equations of a polytropic gas have been investigated in [37–39]. Based on the results of the group classification [1] and Samarskiy–Popov schemes for the MHD equations, we further construct invariant finite-difference schemes possessing conservation laws. The set and number of conservation laws depend on the conductivity, the form of the magnetic field vector, and the equation of state of the medium.

This paper is organized as follows. In Section 2, the simplest version of the finite-conductivity MHD equations in mass Lagrangian coordinates in the case of one-dimensional plane flows is considered. Electric and magnetic fields are represented by one-component vectors, which greatly simplifies the form of the equations. This was the main case considered in Samarsky and Popov's publications [12,13]. The section also provides basic notation and definitions. Then, symmetries and conservation laws of the Samarsky–Popov scheme for the MHD equations are investigated. In addition to the previously known conservation laws, the center-of-mass conservation law is given, as well as new conservation laws for the specific conductivity function, obtained on the basis of the group classification [1].

Section 3 is devoted to various generalizations of the scheme of Section 2. In Section 3.1, the scheme for arbitrary electric and magnetic fields is considered. Its symmetries are investigated and conservation laws are given. The case of infinite conductivity is considered in Section 3.2. It is shown that in this case the Samarsky–Popov scheme requires some additional modifications in order to possess the conservation law of angular momentum. In the case of a polytropic gas, it turns out to be possible to preserve not only energy, but also entropy along pathlines. This can be done using a specially selected equation of state for a polytropic gas. At the end of the section, an example of an invariant scheme is given that does not possess a conservation law of energy, but preserves entropy and has additional conservation laws in the case of isentropic flows. In Section 4, one of the invariant schemes for the case of finite conductivity is numerically implemented for the example of plasma bunch deceleration by crossed electromagnetic fields. The results are discussed in the Conclusions.

2. Conservative Schemes for MHD Equations with Finite Conductivity

Problems of continuum mechanics and plasma physics are often considered in mass Lagrangian coordinates [13,40] since for them the formulation of boundary conditions is greatly simplified. In particular, conservative Samarsky–Popov schemes for the equations of gas dynamics and magnetohydrodynamics have been constructed in mass Lagrangian coordinates.

In mass Lagrangian coordinates the MHD equations, describing the plane one-dimensional MHD flows, are [1,13]

$$\left(\frac{1}{\rho}\right)_t = u_s, \tag{1a}$$

$$u_t = -\left(p + \frac{(H^y)^2 + (H^z)^2}{2}\right)_s, \tag{1b}$$

$$v_t = H^0 H^y_s, \qquad y_t = v, \tag{1c}$$

$$w_t = H^0 H^z_s, \qquad z_t = w, \tag{1d}$$

$$\left(\frac{H^y}{\rho}\right)_t = (H^0 v + E^z)_s, \tag{1e}$$

$$\left(\frac{H^z}{\rho}\right)_t = (H^0 w - E^y)_s, \tag{1f}$$

$$\sigma E^y = -\rho H^z_s, \qquad \sigma E^z = \rho H^y_s, \tag{1g}$$

$$\varepsilon_t = -p u_s + \frac{1}{\rho}(\mathbf{i}\cdot\mathbf{E}), \tag{1h}$$

$$x_t = u, \qquad x_s = 1/\rho, \tag{1i}$$

where t is time, s is the mass Lagrangian coordinate, x is the Eulerian coordinate, ρ is density, p is pressure, ε is internal energy, $\mathbf{u} = (u, v, w)$ is the velocity of a particle, $\mathbf{E} = (E^x, E^y, E^z)$ is the electric field vector, $\mathbf{H} = (H^x, H^y, H^z)$ is the magnetic field vector, and $\mathbf{i} = (i^x, i^y, i^z)$ is the electric current. The conductivity σ is some function of p and ρ, i.e., $\sigma = \sigma(p, \rho)$.

Following [13] we first consider the simplest case of one-component electric and magnetic fields. Here, we also introduce the notation and some basic concepts. In the next section some generalizations are considered, including the case of infinite conductivity.

For simplicity, the longitudinal component of the magnetic field $\mathbf{H}$ is set to zero, and the coordinate system is chosen in such a way that $\mathbf{H} = (0, H, 0)$. Consequently, the electric current $\mathbf{i}$ and the electric field $\mathbf{E}$ are also one-component vectors, i.e., $\mathbf{i} = (0, 0, i)$, $\mathbf{E} = (0, 0, E)$. Electromagnetic force $\mathbf{f} = (f, 0, 0)$ acts in the x-direction, and the velocity is $\mathbf{u} = (u, 0, 0)$ (see Figure 1).

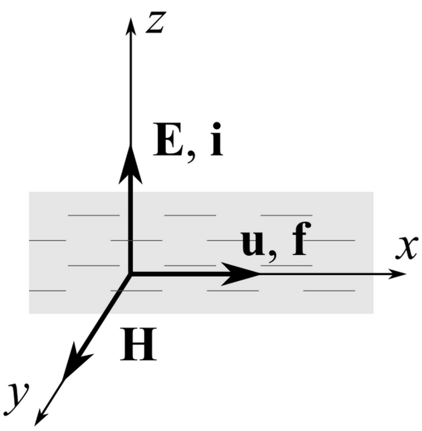

Figure 1. A plane one-dimensional flow for the chosen coordinates.

Given the above, the system of the one-dimensional MHD equations with a finite conductivity $|\sigma| < \infty$ in mass Lagrangian coordinates can be written as [13]

$$\left(\frac{1}{\rho}\right)_t = u_s, \tag{2a}$$

$$u_t = -p_s + f, \qquad f = -iH/\rho, \tag{2b}$$

$$\left(\frac{H}{\rho}\right)_t = E_s, \tag{2c}$$

$$i = \sigma E = \kappa \rho H_s, \tag{2d}$$

$$\varepsilon_t = -p u_s + q, \qquad q = iE/\rho, \tag{2e}$$

$$x_t = u, \qquad x_s = 1/\rho, \tag{2f}$$

where $\kappa = 1/(4\pi)$ and q is Joule heating per unit mass.

In particular, we consider a polytropic gas for which the following relation holds

$$\varepsilon = \frac{1}{\gamma - 1}\frac{p}{\rho}, \qquad \gamma = \text{const} > 1. \tag{3}$$

Equation (2e) for the energy evolution can be rewritten in the semi-divergent form

$$\left(\varepsilon + \frac{u^2}{2}\right)_t = -(pu)_s + fu + q, \tag{4}$$

or in the divergent form

$$\left(\varepsilon + \frac{u^2}{2} + \kappa\frac{H^2}{2\rho}\right)_t = -\left[\left(p + \kappa\frac{H^2}{2}\right)u\right]_s + \kappa(EH)_s. \tag{5}$$

Note that the electromagnetic force $f = -iH/\rho$ can be represented in the divergent form $f = -\kappa(H^2/2)_s$, and Equation (2b) can be rewritten as

$$u_t = -\left(p + \kappa\frac{H^2}{2}\right)_s. \tag{6}$$

Further, we assume $\kappa = 1$ since it can be discarded by means of the scaling transformation

$$\tilde{s} = \kappa s, \quad \tilde{p} = \kappa p, \quad \tilde{\rho} = \kappa\rho, \quad \tilde{\sigma} = \kappa\sigma. \tag{7}$$

2.1. Conservative Samarskiy–Popov Schemes for System (2)

The family of Samarskiy–Popov conservative difference schemes for system (2) is

$$\left(\frac{1}{\rho}\right)_t = u_s^{(0.5)}, \tag{8a}$$

$$u_t = -p_{\bar{s}}^{(\alpha)} + f, \qquad f = -\left(\frac{H\hat{H}}{2}\right)_{\bar{s}} = -\frac{1}{2}[\hat{i}H_*/\hat{\rho}_* + i\hat{H}_*/\rho_*], \tag{8b}$$

$$\left(\frac{H}{\rho}\right)_t = E_s^{(\beta)}, \tag{8c}$$

$$i = \sigma_* E = \rho_* H_{\bar{s}}, \tag{8d}$$

$$\varepsilon_t = -p^{(\alpha)}u_s^{(0.5)} + q, \qquad q = \frac{1}{2}[(i/\rho_*)^{(0.5)}E^{(\beta)} + (i_+/(\rho_+)_*)^{(0.5)}E_+^{(\beta)}], \tag{8e}$$

$$x_t = u^{(0.5)}, \qquad x_s = \frac{1}{\rho}, \tag{8f}$$

where $0 \leqslant (\alpha, \beta) \leqslant 1$ are free parameters. For arbitrary α and β, scheme (8) approximates system (2) up to $O(\tau + h^2)$, and for $\alpha = \beta = 0.5$, the scheme is of order $O(\tau^2 + h^2)$.

Here and further ϕ_t, $\phi_{\check{t}}$ and ϕ_s, $\phi_{\bar{s}}$ denote finite-difference derivatives of some quantity $\phi = \phi(t_n, s_m, u^n_m, ...)$

$$\begin{aligned} \phi_t &= \frac{\underset{+\tau}{S}(\phi) - \phi}{\tau_n}, & \phi_s &= \frac{\underset{+s}{S}(\phi) - \phi}{h_m}, \\ \phi_{\check{t}} &= \frac{\phi - \underset{-\tau}{S}(\phi)}{\tau_{n-1}}, & \phi_{\bar{s}} &= \frac{\phi - \underset{-s}{S}(\phi)}{h_{m-1}}, \end{aligned} \tag{9}$$

which are defined with the help of the finite-difference right and left shifts along the time and space axes correspondingly

$$\begin{aligned} &\underset{\pm\tau}{S}(\phi(t_n, s_m, u^n_m, ...)) = \phi(t_{n\pm1}, s_m, u^{n\pm1}_m, ...), \\ &\underset{\pm s}{S}(\phi(t_n, s_m, u^n_m, ...)) = \phi(t_n, s_{m\pm1}, u^n_{m\pm1}, ...). \end{aligned}$$

The indices n and m are, respectively, changed along time and space axes t and s. The time and space steps are defined as follows

$$\begin{aligned} \tau_n &= \hat{\tau} = t_{n+1} - t_n = \hat{t} - t, & \tau_{n-1} &= \check{\tau} = t_n - t_{n-1} = t - \check{t}, \\ h_m &= h_+ = s_{m+1} - s_m = s_+ - s, & h_{m-1} &= h_- = s_m - s_{m-1} = s - s_-. \end{aligned} \tag{10}$$

Following the Samarskiy–Popov notation throughout the text we denote

$$\underset{+s}{S}(\phi) = \phi_+, \quad \underset{-s}{S}(\phi) = \phi_-, \quad \underset{+\tau}{S}(\phi) = \hat{\phi}, \quad \underset{-\tau}{S}(\phi) = \check{\phi}, \tag{11}$$

$$\phi^{(\alpha)} = \alpha\hat{\phi} + (1 - \alpha)\phi \tag{12}$$

and

$$\phi_* = (\phi_*)^j_i = \frac{h_i \phi^j_{i-1/2} + h_{i-1}\phi^j_{i+1/2}}{h_i + h_{i-1}}. \tag{13}$$

Note that on a uniform lattice $h_i = h = \text{const}$ in its integral nodes (13) becomes

$$\phi_* = \frac{\phi_- + \phi}{2}. \tag{14}$$

Remark 1. *The energy Equation (8e) can be reduced to one of the three following forms [13] using equivalent algebraic transformations:*

$$\varepsilon_t = -p^{(\alpha)} u^{(0.5)}_s + q, \tag{15}$$

$$\left(\varepsilon + \frac{u^2 + u^2_+}{4}\right)_t = -\left(p^{(\alpha)}_* u^{(0.5)}\right)_s + \frac{1}{2}\left[f u^{(0.5)} + f_+ u^{(0.5)}_+\right] + q, \tag{16}$$

$$\left(\varepsilon + \frac{u^2 + u^2_+}{4} + \frac{H^2}{2\rho}\right)_t + \left[\left(p^{(\alpha)}_* + \frac{(H\hat{H})_*}{2}\right) u^{(0.5)} - E^{(\beta)} H^{(0.5)}_*\right]_s = 0. \tag{17}$$

These different forms of equation reflect the balance of certain types of energy, i.e., they express the different physical aspects of energy conservation. To emphasize this property, such schemes are also called completely conservative.

2.2. Invariance of Samarskiy–Popov Schemes

System (2) can be rewritten in the following form that is more convenient for symmetry analysis

$$\left(\frac{1}{\rho}\right)_t = u_s, \tag{18a}$$

$$u_t = -\left(p + \frac{H^2}{2}\right)_s, \tag{18b}$$

$$\left(\frac{H}{\rho}\right)_t = E_s, \tag{18c}$$

$$\sigma E = \rho H_s, \tag{18d}$$

$$p_t = -\gamma \rho p u_s + (\gamma - 1)\sigma E^2, \tag{18e}$$

$$x_t = u, \qquad x_s = 1/\rho. \tag{18f}$$

Remark 2. *Note that for the polytropic gas with the state Equation (3), one can rewrite the energy evolution Equation (8e) of the Samarskiy–Popov scheme as*

$$p_t = -\hat{\rho}(p + (\gamma-1)p^{(\alpha)})u^{(0.5)} + (\gamma-1)\frac{\hat{\rho}}{2}\left[\left(\frac{\sigma_* E}{\rho_*}\right)^{(0.5)} E^{(\beta)} + \left(\frac{(i_+)_* E_+}{(\rho_+)_*}\right)^{(0.5)} E_+^{(\beta)}\right]. \tag{19}$$

In this form the energy evolution equation corresponds to Equation (18e).

Calculations show [1] that the Lie algebra admitted by the system for an arbitrary $\sigma = \sigma(p, \rho)$ is as follows (here and further the notation $\partial_f \equiv \dfrac{\partial}{\partial f}$ is used):

$$X_1 = \partial_t, \qquad X_2 = \partial_s, \qquad X_3 = \partial_x, \qquad X_4 = t\partial_x + \partial_u. \tag{20}$$

The group generator

$$X = \xi^t \partial_t + \xi^s \partial_s + \eta \partial_x \tag{21}$$

is prolonged to the finite-difference space as follows [25,28]

$$\tilde{X} = \sum_{k,l=-\infty}^{\infty} \underset{-\tau}{S^k}\, \underset{-s}{S^l}(X), \tag{22}$$

The scheme of the form

$$\Phi(t, \check{t}, \hat{t}, s, s_+, s_-, u, u_+, u_-, \hat{u}, \check{u}, \hat{u}_+, \check{u}_+, \hat{u}_-, \check{u}_-, \ldots) = 0, \tag{23a}$$

$$h_+ = h_- = h, \qquad \hat{\tau} = \check{\tau} = \tau, \qquad (\vec{\tau}, \vec{h}) = 0 \tag{23b}$$

defined on a uniform orthogonal mesh is *invariant* if the following criterion of invariance holds [28]

$$\begin{aligned} &\tilde{X}\Phi|_{(23)} = 0, \\ &\tilde{X}(\hat{\tau} - \check{\tau})|_{(23)} = 0, \quad \tilde{X}(h_+ - h_-)|_{(23)} = 0. \end{aligned} \tag{24}$$

To preserve uniformness and orthogonality of the mesh it is also required that [25,28]

$$\underset{+s}{D}\,\underset{-s}{D}(\xi^s) = 0, \qquad \underset{+\tau}{D}\,\underset{-\tau}{D}(\xi^t) = 0, \tag{25}$$

$$\underset{\pm s}{D}(\xi^t) = -\underset{\pm\tau}{D}(\xi^s), \tag{26}$$

where $\underset{\pm\tau}{D}$ and $\underset{\pm s}{D}$ are finite-difference differentiation operators

$$\underset{+\tau}{D} = \frac{\underset{+\tau}{S} - 1}{\tau_n}, \quad \underset{-\tau}{D} = \frac{1 - \underset{-\tau}{S}}{\tau_{n-1}}, \quad \underset{+s}{D} = \frac{\underset{+s}{S} - 1}{h_m}, \quad \underset{-s}{D} = \frac{1 - \underset{-s}{S}}{h_{m-1}}.$$

One can verify that scheme (8) is indeed invariant with respect to the generators X_1, ..., X_4 and all the generators (20) satisfy the mesh orthogonality and uniformness conditions (25) and (26). Hence, one can use an orthogonal uniform mesh (23b) that is an invariant one.

2.3. Conservation Laws Possessed by the Samarskiy–Popov Scheme

All the conservation laws of system (18) have their finite-difference counterparts for the Samarskiy–Popov scheme. They are given in Table 1. For convenience, the conservation laws are numbered (see column "#").

An additional conservation law

$$\left(\frac{sH}{\rho}\right)_t + (H - sE)_s = 0. \tag{27}$$

only occurs in case $\sigma = \rho$ [1]. In this case, system (18) admits two more symmetries, namely

$$\begin{gathered} X_5 = s\partial_s - x\partial_x - u\partial_u + 2\rho\partial_\rho - E\partial_E, \\ X_6 = 2t\partial_t + 2s\partial_s - 2u\partial_u + 2p\partial_p + 2\rho\partial_\rho - 3E\partial_E - H\partial_H. \end{gathered} \tag{28}$$

Conservation law (27) has its finite-difference counterpart, which can be found by direct calculations.

Table 1. Differential and difference conservation laws for system (18) and scheme (8).

#	Conservation Laws of the System	Conservation Laws of the Scheme	Physics Interpretation
		$\sigma = \sigma(p, \rho)$	
1	$\left(\frac{1}{\rho}\right)_t - u_s = 0$	$\left(\frac{1}{\rho}\right)_t - \left(u^{(0.5)}\right)_s = 0$	Mass conservation
2	$\left(\frac{H}{\rho}\right)_t - E_s = 0$	$\left(\frac{H}{\rho}\right)_t - (E^{(\beta)})_s = 0$	Magnetic flux conservation
3	$u_t + \left(p + \frac{H^2}{2}\right)_s = 0$	$u_t + \left(p_-^{(\alpha)} + \frac{H_-\hat{H}_-}{2}\right)_s = 0$	Momentum conservation
4	$(tu - x)_t + \left(t\left(p + \frac{H^2}{2}\right)\right)_s = 0$	$\left(\frac{t + \check{t}}{2} u - x\right)_t + \left[t\left(p_-^{(\alpha)} + \frac{H_-\hat{H}_-}{2}\right)\right]_s = 0$	Center of mass law
5	$\left(\varepsilon + \frac{u^2}{2} + \frac{H^2}{2\rho}\right)_t + \left(\left(p + \frac{H^2}{2}\right)u - EH\right)_s = 0$	$\left(\varepsilon + \frac{u^2+u_+^2}{4} + \frac{H^2}{2\rho}\right)_t + \left[\left(p_*^{(\alpha)} + \frac{(H\hat{H})_*}{2}\right)u^{(0.5)} - E^{(\beta)}H_*^{(0.5)}\right]_s = 0$	Energy conservation
		$\sigma = \rho$	
6	$\left(\frac{sH}{\rho}\right)_t + (H - sE)_s = 0$	$\left(\frac{sH}{\rho}\right)_t + (H_-^{(\beta)} - s_-E^{(\beta)})_s = 0$	Unknown

Note that neither conservation law (27) nor the center-of-mass law

$$(tu - x)_t + \left(t\left(p + \frac{H^2}{2}\right)\right)_s = 0 \tag{29}$$

was mentioned in [13]. Perhaps the authors of [13] knew the finite-difference analogue of (29).

3. Generalizations of the Samarskiy–Popov Schemes for MHD Equations

3.1. The Case of Finite Conductivity

We consider a more general case $\mathbf{H} = (H^0, H^y, H^z)$, $\mathbf{E} = (0, E^y, E^z)$, $\mathbf{i} = (0, i^y, i^z)$, $\mathbf{u} = (u, v, w)$, $\mathbf{x} = (x, y, z)$, and $H^0 = \text{const}$. Here we used the fact that the coordinate system can always be chosen in such a way that the first component of the vector field $\mathbf{E}$ is equal to zero.

Further, we consider Equations (1), where, by analogy with (18), the energy evolution Equation (1h) is written as

$$p_t = -\gamma p \rho u_s + (\gamma - 1)\sigma((E^y)^2 + (E^z)^2), \tag{30}$$

A generalization of scheme (8) for $\mathbf{E} = (0, E^y, 0)$ and $\mathbf{H} = (H^0, 0, H^z)$ is given in [13]. Since the MHD equations are almost symmetric in terms of the components E^y, E^z and H^y, H^z, one can extend the scheme proposed in [13] as follows

$$\left(\frac{1}{\rho}\right)_t = u_s^{(0.5)}, \tag{31a}$$

$$u_t = -\left(p^{(\alpha)} + \frac{H^y \hat{H}^y + H^z \hat{H}^z}{2}\right)_{\bar{s}}, \qquad v_t = H^0 (H^y)_{\bar{s}}^{(0.5)}, \qquad w_t = H^0 (H^z)_{\bar{s}}^{(0.5)}, \tag{31b}$$

$$\left(\frac{H^y}{\rho}\right)_t = H^0 v_s^{(0.5)} + (E^z)_s^{(\beta_1)}, \qquad \left(\frac{H^z}{\rho}\right)_t = H^0 w_s^{(0.5)} - (E^y)_s^{(\beta_2)}, \tag{31c}$$

$$i^y = \sigma_* E^y = -\rho_* H^z_{\bar{s}}, \qquad i^z = \sigma_* E^z = \rho_* H^y_{\bar{s}}, \tag{31d}$$

$$\varepsilon_t = -p^{(\alpha)} u_s^{(0.5)} + q^y + q^z, \tag{31e}$$

$$x_t = u^{(0.5)}, \qquad y_t = v^{(0.5)}, \qquad z_t = w^{(0.5)}, \qquad x_s = \frac{1}{\rho}, \tag{31f}$$

where $0 \leqslant (\alpha, \beta_1, \beta_2) \leqslant 1$ and

$$q^y = \frac{1}{2}[(i^y/\rho_*)^{(0.5)} (E^y)^{(\beta_2)} + (i^y_+/(\rho_+)_*)^{(0.5)} (E^y)_+^{(\beta_2)}],$$

$$q^z = \frac{1}{2}[(i^z/\rho_*)^{(0.5)} (E^z)^{(\beta_1)} + (i^z_+/(\rho_+)_*)^{(0.5)} (E^z)_+^{(\beta_1)}].$$

Note that this generalization of the scheme was discussed in [13] but it was not given explicitly.

Remark 3. *One can generalize (19) for scheme (31), (3) as follows*

$$p_t + \hat{\rho} u_s^{(0.5)} (p + (\gamma - 1) p^{(\alpha)}) + (1 - \gamma)\hat{\rho}(q^y + q^z) = 0. \tag{32}$$

According to [1], the symmetries admitted by system (1) are the following.

1. If $H^0 = 0$ and σ is arbitrary, then the admitted Lie algebra is

$$\begin{gathered} X_1 = \partial_t, \qquad X_2 = \partial_s, \qquad X_3 = \partial_x, \qquad X_4 = t\partial_x + \partial_u, \\ X_5 = E^z \partial_{E^y} - E^y \partial_{E^z} + H^z \partial_{H^y} - H^y \partial_{H^z}. \end{gathered} \tag{33}$$

In case $\sigma = \rho$, there are two additional generators that are admitted, namely

$$\begin{gathered} X_{1a} = s\partial_s - x\partial_x - u\partial_u + 2\rho\partial_\rho - E^y \partial_{E^y} - E^z \partial_{E^z}, \\ X_{2a} = 2t\partial_t + 2x\partial_x - 2p\partial_p - 2\rho\partial_\rho - E^y \partial_{E^y} - E^z \partial_{E^z} - H^y \partial_{H^y} - H^z \partial_{H^z}. \end{gathered} \tag{34}$$

There are also two more conservation laws in the latter case (see Table 2). Additional conservation laws do not occur for any other forms of the function σ.

2. If $H^0 \neq 0$ and σ is arbitrary then the admitted Lie algebra is

$$
\begin{gathered}
X_1 = \partial_t, \qquad X_2 = \partial_s, \qquad X_3 = \partial_x, \qquad X_4 = t\partial_x + \partial_u, \\
X_5 = E^z\partial_{E^y} - E^y\partial_{E^z} + H^z\partial_{H^y} - H^y\partial_{H^z} + w\partial_v - v\partial_w + z\partial_y - y\partial_z, \\
X_6 = t\partial_y + \partial_v, \qquad X_7 = t\partial_z + \partial_w, \qquad X_8 = h_1(s)\partial_y, \qquad X_9 = h_2(s)\partial_z,
\end{gathered} \tag{35}
$$

where h_1 and h_2 are arbitrary functions of s.
Additional conservation laws do not occur for any specific σ.

In both the cases above, scheme (31) is *invariant*. The rotation generator X_5 is only admitted for $\beta_1 = \beta_2$. The remaining generators are admitted by the scheme for any set of parameters α, β_1, β_2 and $\gamma > 1$.

The conservation laws possessed by system (31) and their finite-difference counterparts are given in Table 2. Here and further, conservation laws whose fluxes vanish for $H^0 = 0$ are marked with †. In case $H^0 = 0$, their densities preserve along the pathlines.

Table 2. Differential and difference conservation laws for the extended scheme.

#	Conservation Laws of the System	Conservation Laws of the Scheme	Physics Interpretation
		$\sigma = \sigma(p, \rho)$	
1	$\left(\frac{1}{\rho}\right)_t - u_s = 0$	$\left(\frac{1}{\rho}\right)_t - \left(u^{(0.5)}\right)_s = 0$	Mass conservation
2	$\left(\frac{H^y}{\rho}\right)_t - (E^z + H^0 v)_s = 0$	$\left(\frac{H^y}{\rho}\right)_t - \left((E^z)^{(\beta_1)} + H^0 v^{(0.5)}\right)_s = 0$	Magnetic flux conservation
3	$\left(\frac{H^z}{\rho}\right)_t + (E^y - H^0 w)_s = 0$	$\left(\frac{H^z}{\rho}\right)_t + \left((E^y)^{(\beta_2)} - H^0 w^{(0.5)}\right)_s = 0$	Magnetic flux conservation
4	$u_t + \left(p + \frac{(H^y)^2 + (H^z)^2}{2}\right)_s = 0$	$u_t + \left(p_-^{(\alpha)} + \frac{H_-^y \hat{H}_-^y + H_-^z \hat{H}_-^z}{2}\right)_s = 0$	Momentum conservation
5 †	$v_t - (H^0 H^y)_s = 0$	$v_t - \left(H^0 (H_-^y)^{(0.5)}\right)_s = 0$	Momentum conservation
6 †	$w_t - (H^0 H^z)_s = 0$	$w_t - \left(H^0 (H_-^z)^{(0.5)}\right)_s = 0$	Momentum conservation
7	$(tu - x)_t + \left[t\left(p + \frac{(H^y)^2 + (H^z)^2}{2}\right)\right]_s = 0$	$\left(\frac{t + \check{t}}{2} u - x\right)_t + \left[t\left(p_-^{(\alpha)} + \frac{H_-^y \hat{H}_-^y + H_-^z \hat{H}_-^z}{2}\right)\right]_s = 0$	Center of mass law
8 †	$(tv - y)_t - (tH^0 H^y)_s = 0$	$\left(\frac{t + \check{t}}{2} v - y\right)_t - \left(tH^0 (H_-^y)^{(0.5)}\right)_s = 0$	Center of mass law
9 †	$(tw - z)_t - (tH^0 H^z)_s = 0$	$\left(\frac{t + \check{t}}{2} w - z\right)_t - \left(tH^0 (H_-^z)^{(0.5)}\right)_s = 0$	Center of mass law
10	$\left(\varepsilon + \frac{u^2 + v^2 + w^2}{2} + \frac{(H^y)^2 + (H^z)^2}{2\rho}\right)_t + \left[\left(p + \frac{(H^y)^2 + (H^z)^2}{2}\right)u + E^y H^z - E^z H^y - H^0 (vH^y + wH^z)\right]_s = 0$	$\left(\varepsilon + \frac{u^2 + u_+^2 + v^2 + v_+^2 + w^2 + w_+^2}{4} + \frac{(H^y)^2 + (H^z)^2}{2\rho}\right)_t + \left[\left(p_*^{(\alpha)} + \frac{(H^y \hat{H}^y + H^z \hat{H}^z)_*}{2}\right)u^{(0.5)} + (E^y)^{(\beta_2)} (H_*^z)^{(0.5)} - (E^z)^{(\beta_1)} (H_*^y)^{(0.5)} - H^0 \left(v^{(0.5)} (H_*^y)^{(0.5)} + w^{(0.5)} (H_*^z)^{(0.5)}\right)\right]_s = 0$	Energy conservation
		$H^0 = 0$	
11 †	$(zv - yw)_t = 0$	$(zv^{(0.5)} - yw^{(0.5)})_{\check{t}} = 0$	Angular momentum conservation
		$H^0 = 0, \sigma = \rho$	
12	$\left(\frac{sH^y}{\rho}\right)_t + (H^y - sE^z)_s = 0$	$\left(\frac{sH^y}{\rho}\right)_t + ((H_-^y)^{(\beta_1)} - s_-(E^z)^{(\beta_1)})_s = 0$	Unknown
13	$\left(\frac{sH^z}{\rho}\right)_t + (H^z + sE^y)_s = 0$	$\left(\frac{sH^z}{\rho}\right)_t + ((H_-^z)^{(\beta_2)} + s_-(E^y)^{(\beta_2)})_s = 0$	Unknown

3.2. The Case of Infinite Conductivity $\sigma \to \infty$

In this case, system (1) reduces to

$$\left(\frac{1}{\rho}\right)_t = u_s, \tag{36a}$$

$$u_t = -\left(p + \frac{(H^y)^2 + (H^z)^2}{2}\right)_s, \qquad x_t = u, \tag{36b}$$

$$v_t = H^0 H^y_s, \qquad y_t = v, \tag{36c}$$

$$w_t = H^0 H^z_s, \qquad z_t = w, \tag{36d}$$

$$\left(\frac{H^y}{\rho}\right)_t = (H^0 v)_s, \tag{36e}$$

$$\left(\frac{H^z}{\rho}\right)_t = (H^0 w)_s, \tag{36f}$$

$$p_t = -\gamma p \rho u_s, \tag{36g}$$

where the internal energy is given by (3).

In addition to the analogues of conservation laws presented in the previous section, system (36) possesses the conservation law of angular momentum, namely

$$(zv - yw)_t + \Big(H^0(yH^z - zH^y)\Big)_s = 0. \tag{37}$$

As $\sigma \to \infty$, scheme (31) becomes

$$\left(\frac{1}{\rho}\right)_t = u_s^{(0.5)}, \tag{38a}$$

$$u_t = -\left(p^{(\alpha)} + \frac{H^y \hat{H}^y + H^z \hat{H}^z}{2}\right)_{\bar{s}}, \qquad v_t = H^0 (H^y)_{\bar{s}}^{(0.5)}, \qquad w_t = H^0 (H^z)_{\bar{s}}^{(0.5)}, \tag{38b}$$

$$\left(\frac{H^y}{\rho}\right)_t = H^0 v_s^{(0.5)}, \qquad \left(\frac{H^z}{\rho}\right)_t = H^0 w_s^{(0.5)}, \tag{38c}$$

$$\varepsilon_t = -p^{(\alpha)} u_s^{(0.5)}, \tag{38d}$$

$$x_t = u^{(0.5)}, \qquad y_t = v^{(0.5)}, \qquad z_t = w^{(0.5)}, \qquad x_s = \frac{1}{\rho}. \tag{38e}$$

One can verify that scheme (38) is an invariant one. As the symmetries of (36) and the corresponding difference schemes are reviewed in Section 3.2.3, we defer our discussion until then.

3.2.1. Conservation of Angular Momentum and Energy

Apparently, the latter scheme does not preserve angular momentum, i.e., it does not possess a difference analogue of the conservation law (37). One can verify it by algebraic manipulations with the scheme or with the help of the finite-difference analogue of the direct method [33]. We overcome this issue by modifying the latter scheme as follows

$$\left(\frac{1}{\rho}\right)_t = u_s^{(0.5)}, \tag{39a}$$

$$u_t = -\left(p^{(\alpha)} + \frac{H^y \hat{H}^y + H^z \hat{H}^z}{2}\right)_{\bar{s}}, \qquad v_t = H^0 \hat{H}^y_{\bar{s}}, \qquad w_t = H^0 \hat{H}^z_{\bar{s}}, \tag{39b}$$

$$\left(\frac{H^y}{\rho}\right)_t = H^0 v_s, \qquad \left(\frac{H^z}{\rho}\right)_t = H^0 w_s, \tag{39c}$$

$$\varepsilon_t = -p^{(\alpha)} u_s^{(0.5)}, \tag{39d}$$

$$x_t = u^{(0.5)}, \qquad y_t = v, \qquad z_t = w, \qquad x_s = \frac{1}{\rho}. \tag{39e}$$

The latter modification allows one to obtain the whole set of finite-difference analogues of the conservation laws of Equation (36) excluding the conservation of the entropy along the pathlines. The conservation laws are presented in Table 3. Note that the three-layer conservation law of energy given in the table can be rewritten in the following two-layer form by means of (39c)

$$\left[\varepsilon + \frac{u^2 + u_+^2 + v^2 + v_+^2 + w^2 + w_+^2}{4} + \frac{(H^y)^2 + (H^z)^2}{2\rho} + \frac{\tau}{2} H^0 (H^y v_s + H^z w_s)\right]_t$$
$$+ \left[\left(p_*^{(\alpha)} + \frac{(H^y \hat{H}^y + H^z \hat{H}^z)_*}{2}\right) u^{(0.5)} - H^0 (v^{(0.5)} \hat{H}_*^y + w^{(0.5)} \hat{H}_*^z)\right]_s = 0 \tag{40}$$

In addition, in order to verify the conservation law (37), one has to consider the following equations, which can be obtained by integration of (39c)

$$y_s = \frac{H^y}{H^0 \rho}, \qquad z_s = \frac{H^z}{H^0 \rho}. \tag{41}$$

We also note that the modified scheme (39) is still invariant and a completely conservative one.

3.2.2. Conservation of the Entropy along the Pathlines

From the latter system (36) it follows that

$$\left(\frac{p}{\rho^\gamma}\right)_t = S_t = 0. \tag{42}$$

This represents the conservation of the entropy S along pathlines, which is a crucial difference between the finite and infinite conductivity cases.

It is known [37] that the Samarskiy–Popov scheme for polytropic gas does not preserve the entropy S for arbitrary γ. However, the following relation holds on solutions of the system

$$\frac{\hat{p} - p}{p^{(\alpha)}} = \gamma \frac{\hat{\rho} - \rho}{\rho^{(\alpha)}}, \tag{43}$$

which approximates the differential relation

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho}. \tag{44}$$

The latter relation holds along trajectories of the particles up to $O(\tau)$ for $\alpha \neq 0.5$ or up to $O(\tau^2)$ for $\alpha = 0.5$.

In [37], an entropy-preserving invariant scheme for gas dynamics equations in the case of a polytropic gas with $\gamma = 3$ was proposed. This scheme conserves the entropy along the pathlines but has only one conservation law, namely the conservation law of mass. It seems that the conservation of entropy by the difference scheme usually leads to the "loss" of some other conservation laws.

Here, we propose a way of preserving the entropy along the pathlines for polytropic gas with integer values of adiabatic exponent $\gamma \geqslant 2$ for scheme (39). We show that this can be done by choosing appropriate approximations of the state Equation (3).

We note that by means of (3) and (36a), Equation (36g) can be represented as the identity

$$\left(\frac{1}{\gamma-1}\rho^{\gamma-1}\right)_t = \rho^{\gamma-2}\rho_t. \tag{45}$$

In the finite-difference case, the rules of differentiation are different. As a result, not every approximation of the latter identity is a finite-difference identity. For a proper discrete analogue of (45) the right hand side of the identity should also be expressed in the divergent form as well as the left hand side. Choosing the difference approximation for the scheme in the case of a polytropic gas, one has an additional "degree of freedom": the choice of approximation for the state Equation (3). This should be done so that both the left and right hand sides of the resulting approximation for (45) are divergent expressions. Note that this does not affect the conservativeness of the total energy conservation law equation since it does not depend on any specific form of the equation of state.

Further, we consider the shifted version of Equation (38d)

$$\check{\varepsilon}_t = -\check{p}^{(\alpha)}\check{u}_s^{(0.5)} = -\check{p}^{(\alpha)}\left(\frac{1}{\check{\rho}}\right)_t. \tag{46}$$

First, we choose the following approximation of the state Equation (3) for $\gamma = 2$,

$$\varepsilon = \frac{p^{(\alpha)}}{\hat{\rho}}. \tag{47}$$

Substituting (47) into (46), one derives

$$\frac{p^{(\alpha)}}{\hat{\rho}} - \frac{\check{p}^{(\alpha)}}{\rho} = -\tau\check{p}^{(\alpha)}\left(\frac{1}{\check{\rho}}\right)_t. \tag{48}$$

Solving with resect to $\check{p}^{(\alpha)}$, one obtains

$$\check{p}^{(\alpha)} = \frac{p^{(\alpha)}\check{\rho}}{\hat{\rho}}. \tag{49}$$

The latter equation can be rewritten as

$$\frac{p^{(\alpha)}}{\check{p}^{(\alpha)}} = \frac{\rho\hat{\rho}}{\check{\rho}\rho}. \tag{50}$$

Equation (50) can be integrated, i.e.,

$$\left(\frac{\check{p}^{(\alpha)}}{\rho\check{\rho}}\right)_t = S_t = 0. \tag{51}$$

This means conservation of entropy S along pathlines for $\gamma = 2$ on *two* time layers. We have achieved the integrability of the difference analogue of Equation (36g) by choosing a suitable approximation for the state equation.

In a similar way one can arrive at the conservation of entropy for $\gamma = 3$, namely

$$\varepsilon = \frac{\rho p^{(\alpha)}}{\hat{\rho}(\hat{\rho}+\rho)}, \qquad \left(\frac{2\check{p}^{(\alpha)}}{\rho\check{\rho}(\rho+\check{\rho})}\right)_t = 0. \tag{52}$$

Similarly, for $\gamma = 4$

$$\varepsilon = \frac{\rho^2 p^{(\alpha)}}{\hat{\rho}(\hat{\rho}^2+\rho\hat{\rho}+\rho^2)}, \qquad \left(\frac{3\check{p}^{(\alpha)}}{\rho\check{\rho}(\rho^2+\rho\check{\rho}+\check{\rho}^2)}\right)_t = 0, \tag{53}$$

etc.

Thus, by induction, one establishes the following general formula for an arbitrary natural $\gamma \geqslant 2$

$$\varepsilon = \frac{p^{(\alpha)}}{\sum_{k=0}^{\gamma-2} \hat{\rho}^{\gamma-k-1} \rho^{k-\gamma+2}}, \qquad \left(\frac{(\gamma-1)\check{p}^{(\alpha)}}{\sum_{k=0}^{\gamma-2} \rho^{\gamma-k-1} \check{\rho}^{k+1}} \right)_t = 0. \tag{54}$$

Entropy-preservation Formulas (54) are presented in Table 3 among the other conservation laws.

Table 3. Differential and difference conservation laws for the modified scheme (39) (and the corresponding system (36) for an arbitrary entropy $S(s)$ in case $\sigma \to \infty$.

#	Conservation Laws of the System	Conservation Laws of the Scheme	Physics Interpretation
1	$\left(\frac{1}{\rho}\right)_t - u_s = 0$	$\left(\frac{1}{\rho}\right)_t - \left(u^{(0.5)}\right)_s = 0$	Mass conservation
2 †	$\left(\frac{H^y}{\rho}\right)_t - (H^0 v)_s = 0$	$\left(\frac{H^y}{\rho}\right)_t - \left(H^0 v\right)_s = 0$	Magnetic flux conservation
3 †	$\left(\frac{H^z}{\rho}\right)_t - (H^0 w)_s = 0$	$\left(\frac{H^z}{\rho}\right)_t - \left(H^0 w\right)_s = 0$	Magnetic flux conservation
4	$u_t + \left(p + \frac{(H^y)^2 + (H^z)^2}{2}\right)_s = 0$	$u_t + \left(p_-^{(\alpha)} + \frac{H_-^y \hat{H}_-^y + H_-^z \hat{H}_-^z}{2}\right)_s = 0$	Momentum conservation
5 †	$v_t - (H^0 H^y)_s = 0$	$v_t - \left(H^0 H_-^y\right)_s = 0$	Momentum conservation
6 †	$w_t - (H^0 H^z)_s = 0$	$w_t - \left(H^0 H_-^z\right)_s = 0$	Momentum conservation
7	$(tu - x)_t + \left[t\left(p + \frac{(H^y)^2 + (H^z)^2}{2}\right)\right]_s = 0$	$\left(\frac{t + \check{t}}{2} u - x\right)_t +$ $\left[t\left(p_-^{(\alpha)} + \frac{H_-^y \hat{H}_-^y + H_-^z \hat{H}_-^z}{2}\right)\right]_s = 0$	Center of mass law
8 †	$(tv - y)_t - (tH^0 H^y)_s = 0$	$(tv - y)_t - \left(\hat{t} H^0 \hat{H}_-^y\right)_s = 0$	Center of mass law
9†	$(tw - z)_t - (tH^0 H^z)_s = 0$	$(tw - z)_t - \left(\hat{t} H^0 \hat{H}_-^z\right)_s = 0$	Center of mass law
10	$\left(\varepsilon + \frac{u^2+v^2+w^2}{2} + \frac{(H^y)^2 + (H^z)^2}{2\rho}\right)_t +$ $+\left[\left(p + \frac{(H^y)^2 + (H^z)^2}{2}\right)u + E^y H^z - E^z H^y\right.$ $\left.-H^0(vH^y + wH^z)\right]_s = 0$	$\left(\check{\varepsilon} + \frac{\check{u}^2 + \check{u}_+^2 + \check{v}^2 + \check{v}_+^2 + \check{w}^2 + \check{w}_+^2}{4} + \frac{H^y \check{H}^y + H^z \check{H}^z}{2\rho}\right)_t$ $+\left[\left(\check{p}_*^{(\alpha)} + \frac{(H^y \check{H}^y + H^z \check{H}^z)_*}{2}\right)\check{u}^{(0.5)}\right.$ $\left.-H^0(\check{v}^{(0.5)} H_*^y + \check{w}^{(0.5)} H_*^z)\right]_s = 0$	Energy conservation
11 †	$(zv - yw)_t + \left(H^0(yH^z - zH^y)\right)_s = 0.$	$(zv - yw)_t + \left(H^0(\hat{y}\hat{H}_-^z - \hat{z}\hat{H}_-^y)\right)_s = 0$	Angular momentum conservation
	$\varepsilon = \frac{p^{(\alpha)}}{\sum_{k=0}^{\gamma-2} \hat{\rho}^{\gamma-k-1} \rho^{k-\gamma+2}}, \qquad \gamma \in \mathbb{N} \backslash \{1\}$		
12	$\left(\frac{p}{\rho^\gamma}\right)_t = 0$	$\left(\frac{(\gamma-1)\check{p}^{(\alpha)}}{\sum_{k=0}^{\gamma-2} \rho^{\gamma-k-1} \check{\rho}^{k+1}}\right)_t = 0$	Entropy conservation

Remark 4. *From the preservation of entropy $S_t = 0$ in the differential case it follows that (for simplicity, we consider the specific case $\gamma = 2$)*

$$\int_0^T \left(\frac{p}{\rho^2}\right)_t dt = \frac{p(T,s)}{\rho^2(T,s)} - \frac{p(0,s)}{\rho^2(0,s)} = const. \tag{55}$$

Since the constant can be omitted, this means

$$\frac{p(0,s)}{\rho^2(0,s)} = \frac{p(T,s)}{\rho^2(T,s)}.$$

In the finite-difference case, by means of (51), one derives the following analogue of (55)

$$\sum_{k=0}^{N} \left(\frac{\alpha p_m^{n+k} + (1-\alpha) p_m^{n-1+k}}{\rho_m^{n+k} \rho_m^{n-1+k}} \right)_t \tau = \frac{\alpha p_m^{n+N} + (1-\alpha) p_m^{n-1+N}}{\rho_m^{n+N} \rho_m^{n-1+N}} - \frac{\alpha p_m^{n} + (1-\alpha) p_m^{n-1}}{\rho_m^{n} \rho_m^{n-1}} = 0,$$

where $N = \lceil T/\tau \rceil$ *and we recall that* $p^{(\alpha)} = \alpha\hat{p} + (1-\alpha)p$. *Similar to the differential case, the latter gives*

$$\frac{\alpha p_m^n + (1-\alpha)p_m^{n-1}}{\rho_m^n \rho_m^{n-1}} = \frac{\alpha p_m^{n+N} + (1-\alpha)p_m^{n-1+N}}{\rho_m^{n+N}\rho_m^{n-1+N}}$$

which means entropy preservation for a given liquid particle.

Remark 5. *The approach described above can also lead to entropy conservation for* rational *values of* γ. *Without proof of the existence of a general formula we present the result for* $\gamma = 5/3$, *which occurs for one-atomic ideal gas. One can verify that for* $\gamma = 5/3$ *the approximation*

$$\varepsilon = p^{(\alpha)} \frac{\hat{\rho}^{2/3} + (\rho\hat{\rho})^{1/3} + \rho^{2/3}}{\rho^{1/3}\hat{\rho}(\hat{\rho}^{1/3} + \rho^{1/3})} = \frac{3}{2}\frac{p}{\rho} + O(\tau) \tag{56}$$

for the internal energy ε *leads to the following preservation of entropy*

$$\left(\frac{2}{3}\check{p}^{(\alpha)} \frac{\check{\rho}^{2/3} + (\rho\check{\rho})^{1/3} + \rho^{2/3}}{\rho\check{\rho}(\check{\rho}^{1/3} + \rho^{1/3})}\right)_t = 0. \tag{57}$$

Remark 6. *Note that in the case* $H^0 = 0$, *according to Table 3, scheme (39) possesses an infinite set of conservation laws for the following form*

$$\left\{\Phi\left(\frac{(\gamma-1)p^{(\alpha)}}{\sum_{k=0}^{\gamma-2}\hat{\rho}^{\gamma-k-1}\rho^{k+1}}, \frac{H^y}{\rho}, \frac{H^z}{\rho}, v, w, y - tv, z - tw\right)\right\}_t = 0 \tag{58}$$

where $\gamma \in \mathbb{N}\backslash\{1\}$ *and* Φ *is an arbitrary function of its arguments.*

Remark 7. *From (51) it follows that*

$$\frac{\check{p}^{(\alpha)}}{\rho\check{\rho}} - S = 0. \tag{59}$$

The Taylor series expansion of the latter equation is

$$\frac{p}{\rho^2} - S + \left[\frac{p\rho_t}{\rho^3} + (\alpha - 1)\frac{p_t}{\rho^2}\right]\tau + O(\tau^2) = 0. \tag{60}$$

Equation (43) for $\gamma = 2$ *can be represented as*

$$\frac{p^{(\alpha)}}{(\rho^{(\alpha)})^2} - \frac{\rho_t^2}{p_t} = 0. \tag{61}$$

The corresponding expansion is

$$\frac{p}{\rho^2} - \frac{\rho_t^2}{p_t} + \left[\frac{\rho p_t - 2\rho_t p}{\rho^3}\alpha + \frac{(\rho_t p_{tt} - 2p_t\rho_{tt})\rho_t}{2p_t^2}\right]\tau + O(\tau^2) = 0. \tag{62}$$

Equations (61) and (59) approximate the conservation of entropy with the same order $O(\tau)$. *In contrast to (61), approximation (59) can be written in a divergent form. Thus, it represents a conservation law of the scheme, while (61) does not. This gives an advantage in the case of isentropic flows when additional conservation laws include entropy. Then, the expression for the entropy given by Equation (59) can be considered as a constant and included in conserved quantities. Invariant schemes and their conservation laws in the case of isentropic flows are discussed in the next section.*

3.2.3. On Specific Symmetries and Conservation Laws in the Case of Isentropic Flows ($S = \text{const}$)

According to [1], in case $S = p/\rho^\gamma = \text{const}$, Equations (36) admit the following symmetries.

1. If $H^0 \neq 0$, the admitted Lie algebra is

$$\begin{gathered} X_1 = \partial_t, \qquad X_2 = \partial_s, \qquad X_3 = \partial_x, \qquad X_4 = t\partial_x + \partial_u, \\ X_5 = z\partial_y - y\partial_z + w\partial_v - v\partial_w + E^z\partial_{E^y} - E^y\partial_{E^z} + H^z\partial_{H^y} - H^y\partial_{H^z}, \\ X_6 = t\partial_t + 2s\partial_s - u\partial_u - v\partial_v - w\partial_w + 2\rho\partial_\rho, \\ X_7 = -s\partial_s + x\partial_x + y\partial_y + z\partial_z + u\partial_u + v\partial_v + w\partial_w - 2\rho\partial_\rho, \\ X_8 = q_1(s)\partial_y, \qquad X_9 = q_2(s)\partial_z, \qquad X_{10} = t\partial_y + \partial_v, \qquad X_{11} = t\partial_z + \partial_w, \end{gathered} \tag{63}$$

   where q_1, q_2 are arbitrary functions of s.
   Scheme (39) supplemented by the state Equation (54) admits all the generators (63).
2. In case $H^0 = 0$, the admitted Lie algebra is

$$\begin{gathered} X_1 = \partial_t, \quad X_2 = \partial_s, \quad X_3 = \partial_x, \quad X_4 = t\partial_x + \partial_u, \quad X_5 = q_3(s)(H^z\partial_{H^y} - H^y\partial_{H^z}), \\ X_6 = t\partial_t + 2s\partial_s - u\partial_u + 2\rho\partial_\rho, \qquad X_7 = -s\partial_s + x\partial_x + u\partial_u - 2\rho\partial_\rho, \\ X_8 = 2s\partial_s + 2\rho\partial_\rho + p\partial_p + H^y\partial_{H^y} + H^z\partial_{H^z}. \end{gathered} \tag{64}$$

   In case $\gamma = 2$, there are two additional generators, namely

$$X_9 = q_4(s)\rho(\partial_{H^y} - H^y\partial_p), \qquad X_{10} = q_5(s)\rho(\partial_{H^z} - H^z\partial_p). \tag{65}$$

   Here, q_3, q_4, and q_5 are arbitrary functions of s.
   Scheme (39), (54) admits all the generators (64). However, the scheme does not admit the generators X_9 and X_{10}.

There are the following additional conservation laws for system (36).

(a) In case $H^0 \neq 0$, there is an additional conservation law that corresponds to the generator ∂_s

$$\left(\frac{u}{\rho} + \frac{vH^y + wH^z}{H^0\rho}\right)_t + \left(\frac{\gamma S}{\gamma - 1}\rho^{\gamma-1} - \frac{u^2 + v^2 + w^2}{2}\right)_s = 0. \tag{66}$$

(b) Case $H^0 = 0$.

- The conservation law corresponding to the generator ∂_s is

$$\left(\frac{u}{\rho}\right)_t + \left(\frac{\gamma S_0}{\gamma - 1}\rho^{\gamma-1} - \frac{u^2}{2} + \rho B_0\right)_s = 0 \tag{67}$$

  provided

$$S_0 = S = \text{const} \qquad \text{and} \qquad B_0 = \frac{(H^y)^2 + (H^z)^2}{\rho^2} = \text{const}. \tag{68}$$

  The latter follows from system (36). When conductivity of the medium tends to infinity, the phenomenon of frozen-in magnetic field is observed (see, e.g., [41]). In this case, in the absence of the longitudinal component H^0 of the magnetic field, the quantity B_0, which is proportional to the magnetic pressure, turns out to be preserved along the pathlines.

- In case $\gamma = 2$, the admitted generator

$$5X_6 + 3X_7 - 4X_8 \tag{69}$$

corresponds to the conservation law

$$\left(5t\rho A_0 + 5t\frac{u^2}{2} - s\frac{u}{\rho} - 3xu\right)_t + \left(((5tu - 3x)\rho^2 - 2s\rho)A_0 + s\frac{u^2}{2}\right)_s = 0 \tag{70}$$

provided

$$A_0 = \left[\frac{S}{\gamma - 1} + \frac{(H^y)^2 + (H^z)^2}{2\rho^2}\right]_{\gamma=2} = \frac{p}{\rho^2} + \frac{(H^y)^2 + (H^z)^2}{2\rho^2} = \text{const} \tag{71}$$

which follows from system (36).

Remark 8. *Conservation law (70) is a basis one. Its partial derivative with respect to s is equivalent to (67), and its partial derivative with respect to t is*

$$A_0(\rho_t + \rho^2 u_s) + u(u_t + (A_0\rho^2)_s) = 0 \tag{72}$$

which is a combination of (36a) and (36b) provided (71).

By virtue of the content of Remark 6, one can verify that the finite-difference analogues of (68) and (71) hold along the pathlines for scheme (39), namely

$$\left(\frac{(H^y)^2 + (H^z)^2}{\rho^2}\right)_t = 0 \quad \text{or} \quad \left(\frac{H^y\check{H}^y + H^z\check{H}^z}{\rho\check{\rho}}\right)_t = 0 \qquad \text{if} \quad H^0 = 0, \tag{73}$$

and

$$\left(\frac{p^{(\alpha)}}{\check{\rho}\rho} + \frac{(H^y)^2 + (H^z)^2}{2\rho^2}\right)_t = 0 \quad \text{or} \quad \left(\frac{p^{(\alpha)}}{\check{\rho}\rho} + \frac{H^y\check{H}^y + H^z\check{H}^z}{2\rho\check{\rho}}\right)_t = 0 \qquad \text{if} \quad H^0 = 0,\ \gamma = 2. \tag{74}$$

Scheme (39) also admits the generators ∂_s and (69) under the same conditions as for the differential case.

Analyzing scheme (39), one can conclude that for the additional conservation laws (66), (67), and (70) there are no approximations in terms of rational expressions. This means that construction of finite difference analogues of the mentioned conservation laws is extremely hard.

Further, we restrict ourselves to the case $\gamma = 2$ and $S = S_1 = \text{const}$, and consider another invariant scheme on an extended finite-difference stencil.

We introduce the pressure for the polytropic gas as

$$p = S_1\hat{\rho}\hat{\rho}_+. \tag{75}$$

Then, the conservation law of entropy

$$\left(\frac{\check{p}}{\rho\rho_+}\right)_t = (S_1)_t = 0 \tag{76}$$

is defined by the following invariant expression

$$\frac{p}{\hat{\rho}\hat{\rho}_+} = \frac{\check{p}}{\rho\rho_+} = \frac{\check{p}_-}{\rho\rho_-} = \frac{p_-}{\hat{\rho}\hat{\rho}_-} = S_1. \tag{77}$$

The scheme under consideration is based on scheme (39) and it has the following form

$$\left(\frac{1}{\rho}\right)_t = u_s^*, \tag{78a}$$

$$u_t = -\left(p + \frac{\hat{H}_+^y \hat{H}^y + \hat{H}_+^z \hat{H}^z}{2}\right)_{\bar{s}}, \qquad v_t = H^0 \hat{H}_s^y, \qquad w_t = H^0 \hat{H}_s^z, \tag{78b}$$

$$\left(\frac{H^y}{\rho}\right)_t = H^0 v_{\bar{s}}, \qquad \left(\frac{H^z}{\rho}\right)_t = H^0 w_{\bar{s}}, \tag{78c}$$

$$\frac{p}{\hat{\rho}\hat{\rho}_+} = \frac{\check{p}}{\rho\rho_+} = \frac{\check{p}_-}{\rho\rho_-} = \frac{p_-}{\hat{\rho}\hat{\rho}_-} = S_1, \tag{78d}$$

$$x_t = u^*, \qquad y_t^+ = v, \qquad z_t^+ = w, \qquad x_s = \frac{1}{\rho}. \tag{78e}$$

One can verify that the latter scheme is invariant. It admits the same symmetries as scheme (39) and (54). The following quantities hold for (78)

$$\left(\frac{H_+^y H^y + H_+^z H^z}{\rho\rho_+}\right)_t = (B_1)_t = 0 \qquad \text{if} \quad H^0 = 0, \tag{79}$$

$$\left(\frac{\check{p}}{\rho\rho_+} + \frac{H_+^y H^y + H_+^z H^z}{2\rho\rho_+}\right)_t = \frac{1}{2}(2S_1 + B_1)_t = 0 \qquad \text{if} \quad H^0 = 0,\ \gamma = 2. \tag{80}$$

Scheme (78) possesses the difference analogues of (66) and (67), namely

$$\left(\frac{u}{\rho} + \frac{v_* \hat{H}^y + w_* \hat{H}^z}{H^0 \hat{\rho}}\right)_t + \left(2\hat{\rho}_* S_1 - \frac{uu_- + \hat{v}_-^2 + \hat{w}_-^2}{2}\right)_s = 0, \tag{81}$$

$$\left(\frac{u}{\rho}\right)_t + \left(\hat{\rho}_*(2S_1 + B_1) - \frac{uu_-}{2}\right)_s = 0. \tag{82}$$

To construct the latter conservation law one should use the following relation

$$\frac{\hat{H}_+^y \hat{H}^y + \hat{H}_+^z \hat{H}^z}{2} = \frac{\hat{H}_+^y \hat{H}^y + \hat{H}_+^z \hat{H}^z}{2\hat{\rho}\hat{\rho}_+} = \frac{1}{2}\hat{\rho}\hat{\rho}_+ B_1. \tag{83}$$

Remark 9. *The angular momentum and center of mass conservation laws are*

$$(zv_- - yw_-)_t + \left(H^0(\hat{y}\hat{H}_-^z - \hat{z}\hat{H}_-^y)\right)_s = 0, \tag{84}$$

$$(tu^* - x)_t + \left(\hat{t}\left[p_- + \frac{\hat{H}_-^y \hat{H}^y + \hat{H}_-^z \hat{H}^z}{2}\right]_*\right)_s = 0, \tag{85}$$

$$(tv - y_+)_t - \left(\hat{t} H^0 \hat{H}^y\right)_s = 0, \tag{86}$$

$$(tu - z_+)_t - \left(\hat{t} H^0 \hat{H}^z\right)_s = 0. \tag{87}$$

The remaining conservation laws of mass, momentum, magnetic flux, and entropy follow directly from the scheme as it is written in a divergent form.

4. Numerical Experiments

In this section, we consider the problem of deceleration of a plasma bunch in a crossed electromagnetic field under the presence and absence of a longitudinal component of magnetic field. We use scheme (31), and consider how the conservation laws hold on the solutions of this scheme. In addition to the transverse component H^y of the magnetic field, we also consider the case of the presence of a longitudinal magnetic field $H^0 \neq 0$.

A plasma bunch is considered, which moves from left to right in a railgun channel. The channel is filled with a relatively cold weakly conducting gas. With the help of an external electric circuit, a strong transverse magnetic field is generated in the channel, which causes the bunch to decelerate. During its motion, the plasma bunch closes the electric circuit and moves along the background gas; therefore, the magnetic field and pressure at the left boundary of the computational domain are considered equal to zero. The differential boundary conditions are as follows

$$p(0,t) = 0, \qquad H^y(0,t) = 0, \qquad H^z(0,t) = 0, \tag{88a}$$

$$u(S,t) = 0, \qquad E^y(S,t) = 0, \qquad H^y(S,t) = 4\pi J(t), \tag{88b}$$

$$L_0 \frac{dJ}{dt} + R_0 J - V(t) + E^z(S,t) = 0, \tag{88c}$$

$$\frac{dV}{dt} = -J/C_0, \quad V(0) = V_0, \quad J(0) = 0, \tag{88d}$$

where $0 \leqslant s \leqslant S$ and $0 \leqslant t \leqslant t_{\max}$, S is the total mass of the gas, J and V are current and voltage, and C_0, L_0, R_0 are the external circuit parameters. The boundary conditions (88c) and (88d) are approximated in the same way as in [13], namely

$$L_0 J_t + R_0 J^{(0.5)} - V^{(0.5)} + E_M^{z\,(0.5)} = 0, \qquad V_t = -J^{(0.5)}/C_0, \tag{89}$$

where $M = \lfloor S/h \rfloor$.

All calculations were carried out using the dimensionless version of scheme (31) with the value of the coefficient $\kappa = 4\pi$. For the dimensionless form of the scheme, the initial conditions are: $\rho_0 = 1.0$, $p_0 = 0.0056$, $R_0 = 1.17$, $C_0 = 1.64$, $L_0 = 0.0035$, $S = 4.0$, the temperature of the plasma $T_0 = 3.0$, and the initial speed of the plasma bunch $u_0 = 0.75$. The gas is considered polytropic with $\gamma = 5/3$. The uniform mesh steps are $h = 0.067$ and $\tau = 0.003$, and $t_{\max} = 0.7$. The initial voltage V_0 is varied between 1.67 and 2.6 which approximately correspond to the voltage 650 and 1000 V. In experiments where the longitudinal magnetic field H^0 is present, a value close to 1 is taken for H^0. In the calculations, a linear artificial viscosity is used, with a viscosity coefficient $\nu = 2h$.

The problem under consideration is close to the problem described in [42] (see also [43]) in which, however, tabulated real plasma parameters, including electrical conductivity, were used. In our problem, we used the ideal gas equation and an exponential conductivity function.

Scheme (31) is implemented using the iterative methods described in [13]. In this case, the scheme equations are divided into two parts, dynamic and magnetic. The dynamic part is preliminarily linearized using the Newton method, and for the magnetic part a flow version of the sweep method is used [44], which is well suited for the case of finite conductivity, especially when its values are small. The bunch motion is modeled by a shock wave. Conductivity σ of the plasma bunch is proportional to $T^{3/2}$, and the conductivity function $\sigma = \sigma(\rho, T)$ is very sensitive to the density ρ in such a way that in the rarefied background gas region it has values close to zero.

Three essentially different cases are considered:

1. The bunch is decelerated using a transverse magnetic field H^y at a relatively low voltage in the circuit.
2. The bunch is decelerated using a transverse magnetic field at a high voltage in the circuit.
3. A rather strong longitudinal magnetic field H^0 is added to the previous case. (Calculations show that a weak longitudinal magnetic field has little effect on the experimental results.)

In all cases, at the initial moment of time, the gas particles are given a small constant transverse velocity $v > 0$. This is necessary in order to track the influence of the longitudinal magnetic field on the transverse component of the particle velocity, which should be observed only in the third numerical experiment.

Figure 2 shows the evolution of the magnetic field and plasma temperature in the first experiment. The magnetic field is not strong enough to stop the bunch. If the bunch reaches the right boundary of the computational domain, the reflection of the wave can be observed due to the boundary condition $u(S,t) = 0$. Figure 3 shows the second case where the transverse magnetic field is strong enough. The plasma bunch is decelerated by the magnetic field and after a short period of time begins to move backward. Adding a sufficiently strong longitudinal magnetic field H^0 to the previous experiment leads to an intermediate picture: the magnetic field is "smeared" over the computational domain, the plasma deceleration process is not as intense as in the previous case, and is inhomogeneous along the mass coordinate, which leads to a kind of fragmentation of the temperature profile (see Figure 4).

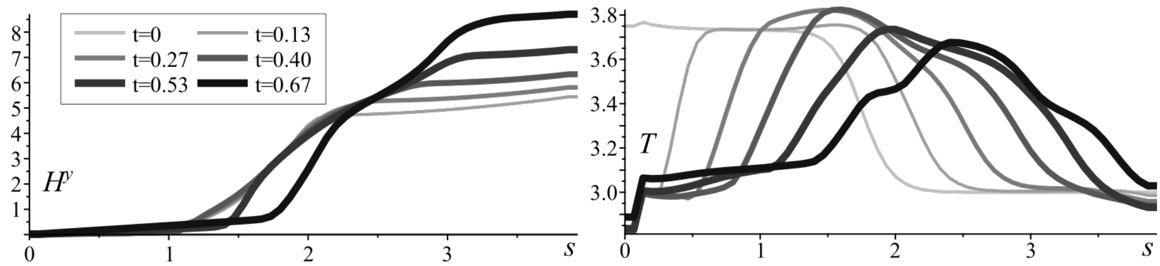

Figure 2. Evolution of the magnetic field H^y and the temperature T for the first experiment.

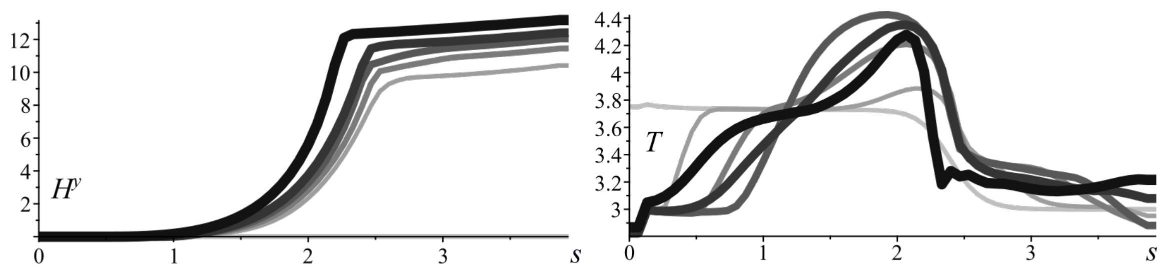

Figure 3. Evolution of the magnetic field H^y and the temperature T for the second experiment.

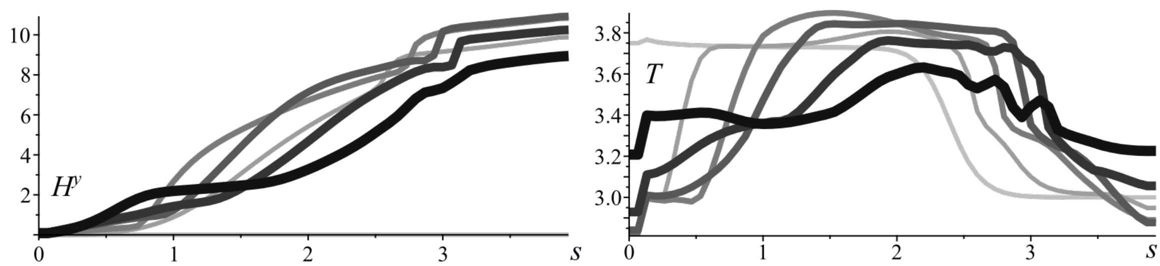

Figure 4. Evolution of the magnetic field H^y and the temperature T for the third experiment.

In Figures 5–7 the evolution of the work $A = -p\,u_s$ performed on the gas and the electromagnetic force f^x in the direction of the x axis are shown. At the beginning of the process at the shock wave front, A is positive, which corresponds to gas heating due to compression. At the left boundary of the computational domain, A is negative, and the rarefaction wave cools the gas. The electromagnetic force $\mathbf{f}$ is mainly localized in the front part of the bunch. With an increase in the total current, the electromagnetic force increases, which leads to the deceleration of the wave and further to its stop and reverse motion.

Note that the component $f^y = H^0 H^y_{\bar{s}}$ appears only in the third experiment ($H^0 \neq 0$). Its evolution is depicted in Figure 8. In addition, recall that, as follows from (31),

$$f^x = -\frac{1}{2}\left(\frac{\hat{\sigma}_*}{\hat{\rho}_*}(\hat{E}^z H^y_* - \hat{E}^y H^z_*) + \frac{\sigma_*}{\rho_*}(E^z \hat{H}^y_* - E^y \hat{H}^z_*)\right). \tag{90}$$

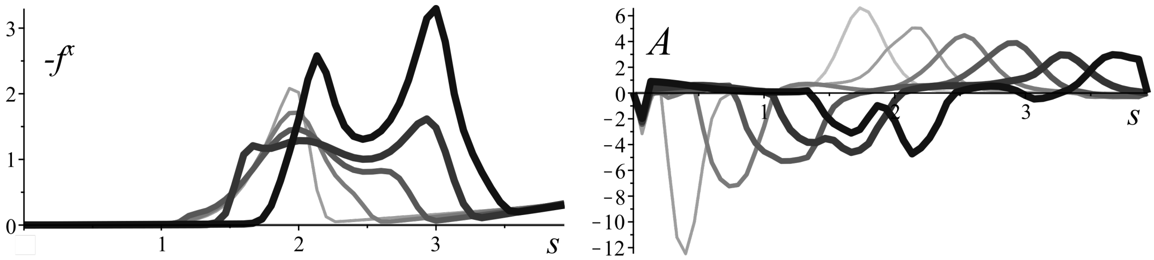

Figure 5. Evolution of the electromagnetic force $-f^x$ and the work on gas A for the first experiment.

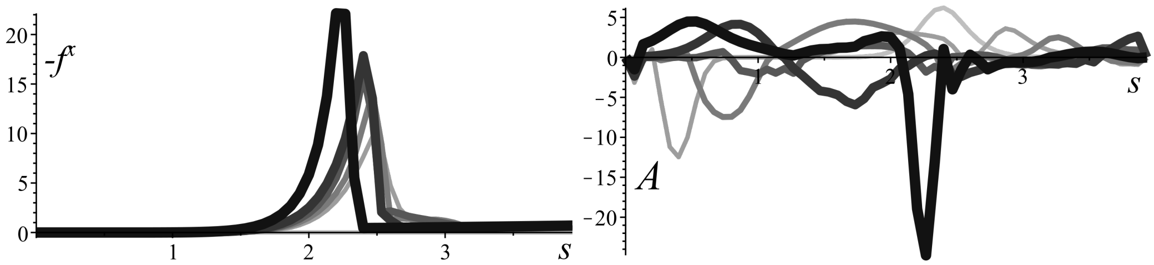

Figure 6. Evolution of the electromagnetic force $-f^x$ and the work on gas A for the second experiment.

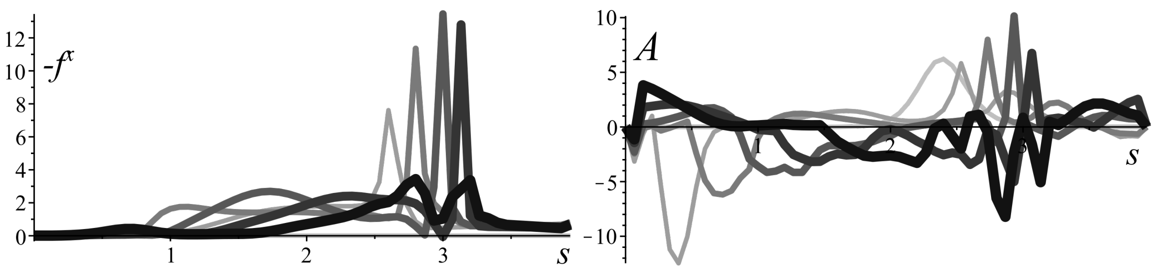

Figure 7. Evolution of the electromagnetic force $-f^x$ and the work on gas A for the third experiment.

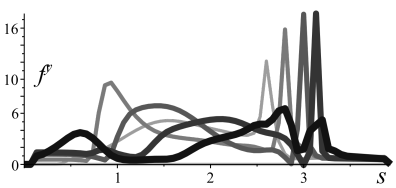

Figure 8. Evolution of the electromagnetic force f^y for the third experiment ($H^0 > 0$).

In Figure 9 the trajectories of particles under the action of magnetic fields are shown. The left part (Figure 9a–c) shows x-trajectories of particles for three experiments. The right side of the figure shows y-trajectories associated with the transverse velocity component v. Figure 9d corresponds to the first and second experiments where v has a constant value and $H^0 = 0$. Figure 9e,f correspond to the third experiment at $H^0 > 0$ and $H^0 < 0$ where under the action of the longitudinal magnetic field the transverse velocity component increases or slows down accordingly. Note that the choice of sign of the value H^0 otherwise does not affect the results of the third experiment.

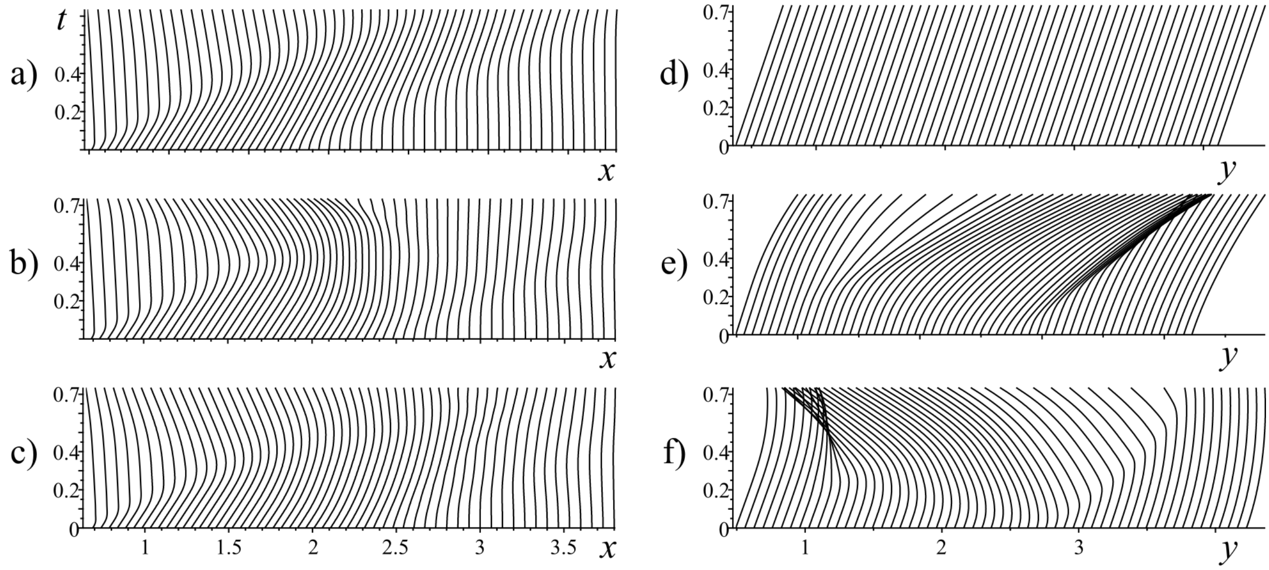

Figure 9. The trajectories of particles under the action of magnetic fields: (**a**–**c**) show x-trajectories for cases 1, 2, and 3; (**d**) y-trajectories for $H^0 = 0$; (**e**) y-trajectories for $H^0 > 0$; (**f**) y-trajectories for $H^0 < 0$.

In Figures 10–12 the finite-difference conservation laws of energy, magnetic flux (along the y axis), momentum, and center-of-mass motion (along the x axis) are given for the selected moment in time, when the interaction of magnetic fields and the plasma bunch is already quite intense. The results are provided only for the third experiment, since in other cases the control of conservation laws gives similar results. The accurate enough preservation of the conservation laws on solutions is due to the conservativeness of scheme (31).

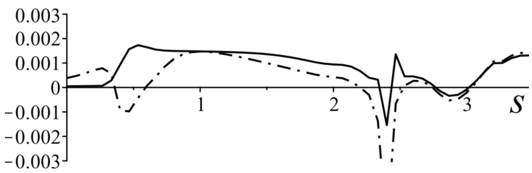

Figure 10. Conservation laws of energy (solid line) and y-flux (dashed line) at $t = 0.64$.

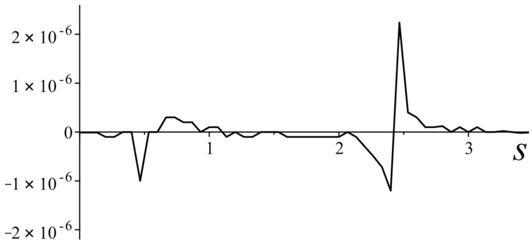

Figure 11. Conservation law of x-momentum at $t = 0.64$.

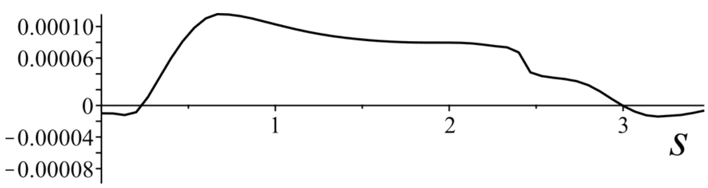

Figure 12. Conservation law of center-of-mass along axis x at $t = 0.64$.

5. Conclusions

Finite-difference schemes for MHD equations in the case of plane one-dimensional flows were considered. The Samarsky–Popov classical scheme for the case of finite conductivity was taken as a starting point. Symmetries and conservation laws of this scheme were investigated. It was shown that the scheme admits the same symmetries as the original differential model. It also has difference analogues of the conservation laws of the original model. In addition to the conservation laws previously known for the scheme, new conservation laws were given, which were obtained on the basis of the group classification recently carried out in [1].

The classical Samarskiy–Popov scheme was generalized to the case of arbitrary vectors of electric and magnetic fields, as well as to the case of infinite conductivity. In the case of finite conductivity the scheme possesses difference analogues of all differential local conservation laws obtained in [1], some of which were not previously known. In the case of infinite conductivity, straightforward generalization of the scheme leads to a scheme that does not preserve angular momentum. The proposed modification makes it possible to obtain an invariant scheme that also possesses the conservation law of angular momentum. In addition, it was shown how to approximate the equation of state for a polytropic gas to preserve the entropy along the pathlines on the extended stencil for two time layers.

A numerical implementation of the generalized Samarsky–Popov scheme for the case of finite conductivity was performed for the problem of deceleration of a plasma bunch by crossed electromagnetic fields. Various cases of the action of fields on a plasma were considered. Calculations showed that the finite-difference conservation laws are preserved on the solutions of the scheme quite accurately.

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