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Boundedness of the Vector-Valued Intrinsic Square Functions on Variable Exponents Herz Spaces

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Abstract: In this article, the authors study the boundedness of the vector-valued inequality for the intrinsic square function and the boundedness of the scalar-valued intrinsic square function on variable exponents Herz spaces $\dot{K}^{\alpha,q(\cdot)}_{\rho(\cdot)}(\mathbb{R}^n)$. In addition, the boundedness of commutators generated by the scalar-valued intrinsic square function and BMO class is also studied on $\dot{K}^{\alpha,q(\cdot)}_{\rho(\cdot)}(\mathbb{R}^n)$.

Keywords: intrinsic square function; vector-valued inequality; Herz spaces; BMO function; variable exponent; sublinear operators

MSC: 42B20; 42B25; 42B35

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1. Introduction

Before adequately introducing the subject of this article, it should be noted that our primary goal is to study the boundedness of the vector-valued operators on variable exponent Herz spaces $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$. Our main results apply to the intrinsic square function and sublinear operators. In addition, the boundedness of the scalar-valued intrinsic square function and its commutators are also discussed in the aforementioned Herz spaces.

Wilson was the first to propose the intrinsic square functions in [1,2]. In [2], the author obtained the boundedness of the intrinsic square functions on $L^p(\omega)$ spaces. Since then, numerous researchers have paid a great deal of attention to the study of the boundedness of the vector-valued and scalar-valued intrinsic square functions in various functions spaces. For instance, in [3], the author proved sharp weighted norm inequalities for the intrinsic square function in terms of the A_p (Muckenhoupt weight class) characteristic of ω for all $p \in (1, \infty)$. In a series of papers by Wang [4–7], he has proposed the boundedness of the scalar-valued intrinsic square functions on weighted Hardy spaces and weighted Morrey spaces, respectively. Liang et al. [8] discovered the boundedness of intrinsic Littlewood–Paley functions on Musielak–Orlicz Morrey and Campanato spaces. Moreover, we refer to Wang [9], who studied the boundedness of the vector-valued intrinsic square functions on weighted Morrey-type spaces. More applications of such intrinsic square functions can be found in [10–15].

The classical notion of Herz spaces $\dot{K}^{\alpha,q}_{p}(\mathbb{R}^n)$ was originally introduced by Herz in [16]. These spaces were extended and studied by many authors [17–21]. The topic of variable exponents function spaces is currently a very active research area (see, for example [22–27], and so on), in part due to the breadth of their applications (e.g., in fluid dynamics [28] and in differential equations [29–31]). Herz spaces with variable exponents have been studied by many authors using different approaches. The Herz spaces $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ with one variable exponent $p(\cdot)$ were defined by Izuki [25], who also established some boundedness results for certain sublinear operators on such spaces. Wang et al. [32] investigated the Herz spaces $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ with two variable exponents $p(\cdot)$ and $p(\cdot)$, as well as obtaining the boundedness results for parameterized Littlewood–Paley operators on $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$. Yang et al. [33] proved the boundedness of the θ-type Calderón–Zygmund operators and commutators on the homogeneous Herz space with variable exponents $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$. Cai et al. [34] established the boundedness for the rough singular integral operators and commutators on $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$. Another approach including variable exponents was used in [35] to study Herz spaces.

We also mention that Zhuo et al. [36] established the intrinsic square function characterizations of the variable Hardy spaces. Ho [37] proved the boundedness of the vector-valued intrinsic square function on variable Morrey and Bloke spaces. On variable exponent function spaces, Wang [38] investigated the boundedness of commutator of intrinsic square function. For some recent developments, we refer to [39,40].

Motivated by [32,37,38], we study the boundedness of the vector-valued intrinsic square function and the boundedness of the scalar-valued intrinsic square function on the variable exponent Herz spaces. The boundedness of the commutator generated by the scalar-valued intrinsic square function and BMO function is also discussed on $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$.

In this article, by Q, we denote a cube in $\mathbb{R}^n$. If $E \subseteq \mathbb{R}^n$, $|E|$ denotes its Lebesgue measure set in $\mathbb{R}^n$ and χ_E its characteristic function. We always denote a positive constant by C, which is not necessarily the same at each occurrence. By $\hbar \lesssim g$, we mean that $\hbar \le Cg$. The expression $\hbar \approx g$ means that $\hbar \lesssim g \lesssim \hbar$.

2. Definitions and Preliminaries

We review some fundamental lemmas and definitions about variable exponent functions spaces.

Definition 1 ([22]). *Let $p(\cdot) : \Sigma \to [1, \infty)$ be a measurable function. The variable exponent Lebesgue space is defined as*

$$L^{p(\cdot)}(\Sigma) = \left\{ \hbar \text{ is measurable} : \int_\Sigma \left(\frac{|\hbar(x)|}{\vartheta} \right)^{p(x)} \mathrm{d}x < \infty \text{ for some constant } \vartheta > 0 \right\}.$$

The space $L^{p(\cdot)}_{\mathrm{loc}}(\Sigma)$ is defined as

$$L^{p(\cdot)}_{\mathrm{loc}}(\Sigma) := \left\{ \hbar : \hbar \in L^{p(\cdot)}(E) \text{ for every compact subsets } E \subset \Sigma \right\}.$$

$L^{p(\cdot)}(\Sigma)$ is a Banach space with norm given below [27]:

$$\|\hbar\|_{L^{p(\cdot)}(\Sigma)} = \inf \left\{ \vartheta > 0 : \int_\Sigma \left(\frac{|\hbar(x)|}{\vartheta} \right)^{p(x)} \mathrm{d}x \le 1 \right\}.$$

For brevity, let us set $p_- := p^-_\Sigma := ess \inf\{p(x) : x \in \Sigma\}$, $p_+ = p^+_\Sigma := ess \sup\{p(x) : x \in \Sigma\}$.

The notations $\mathcal{P}^*(\Sigma)$,$\mathcal{P}^*_0(\Sigma)$ and $\mathcal{B}^*(\Sigma)$ are defined, respectively, as follows:

$$\mathcal{P}^*(\Sigma) := \{p(\cdot) \text{ is measurable} : p_- > 1 \text{ and } p_+ < +\infty\},$$
$$\mathcal{P}^*_0(\Sigma) := \{p(\cdot) \text{ is measurable} : p_- > 0 \text{ and } p_+ < +\infty\}$$

and

$$\mathcal{B}^*(\Sigma) := \left\{ p(\cdot) \in \mathcal{P}^*(\Sigma) : M_{HL} \text{ is bounded on } L^{p(\cdot)}(\Sigma) \right\},$$

where M_{HL} is the standard Hardy–Littlewood maximal operator, which can be defined as

$$(M_{HL}\hbar)(x) = \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)} |\hbar(y)| dy, \qquad \hbar \in L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

We say $p(\cdot) \in \mathcal{B}^*(\mathbb{R}^n)$ if $p(\cdot) \in \mathcal{P}^*(\mathbb{R}^n)$ and satisfies the following two inequalities:

$$|p(x) - p(y)| \lesssim -\frac{1}{\log(|x-y|)}, \quad \text{if } |x-y| \leq 1/2; \tag{1}$$

$$|p(x) - p(y)| \lesssim \frac{1}{\log(e+|x|)}, \quad \text{if } |y| \geq |x|. \tag{2}$$

See [23,24]. In the sequel, the conjugate of $p(x)$ is given by the symbol $p'(x)$, which means $p'(x) = p(x)/(p(x)-1)$.

Before recalling the definition of Herz spaces with two variable exponents, let us present the following notations: for all $l \in \mathbb{Z}$, we denote $B_l := \{x \in \mathbb{R}^n : |x| \leq 2^l\}$, $\Re_l := B_l \setminus B_{l-1}$ and $\chi_l := \chi_{R_l}$. Denote $\mathbb{Z}^+$ as the sets of positive integers, $\tilde{\chi}_l := \chi_{R_l}$ if $l \in \mathbb{Z}^+$ and $\tilde{\chi}_0 := \chi_{B_0}$. The notation of the mixed Lebesgue-sequence space $\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)$, which is first introduced by Almeida et al. in [41]. Let $p(\cdot)$ and $q(\cdot) \in \mathcal{P}_0^*(\mathbb{R}^n)$. Given a sequence of functions $\{\hbar_J\}_{J\in\mathbb{Z}}$, define the modular

$$\wp_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\{\hbar_J\}_{J\in\mathbb{Z}}\right) := \sum_{J\in\mathbb{Z}} \inf\left\{\vartheta_J > 0 : \int_{\mathbb{R}^n}\left(\frac{|\hbar_J(x)|}{\vartheta_J^{1/q(x)}}\right)^{p(x)} \mathrm{d}x \leq 1\right\}.$$

The quasi-norm is defined as follows:

$$\left\|\{\hbar_J\}_{J\in\mathbb{Z}}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} := \inf\left\{\vartheta > 0 : \wp_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\frac{\{\hbar_J\}_{J\in\mathbb{Z}}}{\vartheta}\right) \leq 1\right\}. \tag{3}$$

When $p(\cdot) \in \mathcal{P}_0^*(\mathbb{R}^n)$, then the space $L^{p(\cdot)}(\mathbb{R}^n)$ can be defined as follows:

$$L^{p(\cdot)}(\mathbb{R}^n) := \left\{\hbar : |\hbar|^{p_0} \in L^{q(\cdot)}(\mathbb{R}^n) \text{ for some } p_0 \text{ with } p_0 \in (0, p_-) \text{ and } q(x) = p(x)p_0^{-1}\right\}.$$

and its quasi-norm is defined as follows:

$$\|\hbar\|_{L^{p(\cdot)}} := \left\||\hbar|^{p_0}\right\|_{L^{q(\cdot)}}^{1/p_0}.$$

Thus, we may substitute (3) for the simple formula

$$\wp_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\{\hbar_J\}_{J\in\mathbb{Z}}\right) := \sum_{J\in\mathbb{Z}}\left\||\hbar_J|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)}.$$

Definition 2 ([32]). *Let $p(\cdot), q(\cdot) \in \mathcal{P}_0^*(\mathbb{R}^n)$ and let $\alpha \in \mathbb{R}$. The homogeneous Herz space $\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)$ is defined as*

$$\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n) = \left\{\hbar \in L_{\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n \backslash \{0\}) : \|\hbar\|_{\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\begin{aligned}\|\hbar\|_{\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)} &:= \left\|\left\{2^{l\alpha}|\hbar\chi_l|\right\}_{l=-\infty}^{\infty}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ &= \inf\left\{\vartheta > 0 : \sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}|\hbar\chi_l|}{\vartheta}\right)^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq 1\right\}.\end{aligned}$$

The non-homogeneous Herz space $K^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ is defined as

$$K^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) := \left\{\hbar \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\}) : \|\hbar\|_{\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\begin{aligned}\|\hbar\|_{K^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} &:= \left\|\left\{2^{l\alpha}|\hbar\chi_l|\right\}_{l=0}^{\infty}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ &= \inf\left\{\vartheta > 0 : \sum_{l=0}^{\infty}\left\|\left(\frac{2^{l\alpha}|\hbar\chi_l|}{\vartheta}\right)^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq 1\right\}.\end{aligned}$$

Remark 1. *(1) If $q(\cdot)$ is a constant, then $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$. If both $p(\cdot), q(\cdot)$ are constant functions, then $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha,q}_{p}(\mathbb{R}^n)$. Furthermore, if $\alpha = 0$ and $p(\cdot) = q(\cdot)$, then $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = K^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n)$.*

(2) If $q_1(\cdot)$ and $q_2(\cdot) \in \mathcal{P}^*_0(\mathbb{R}^n)$ satisfying $(q_1)_+ \leq (q_2)_-$, then (see [32])

$$\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n) \subset \dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(\mathbb{R}^n).$$

Proposition 1 ([34])**.** *Let $\theta_l \in [0, \infty)$, $1 \leq p_l \leq \sup_{l\in\mathbb{Z}} p_l < \infty$ $(l \in \mathbb{Z})$. Then*

$$\sum_{l\in\mathbb{Z}} \theta_l^{p_l} \leq 2 \cdot \left(\sum_{l\in\mathbb{Z}} \theta_l\right)^{p_*},$$

where

$$p_* = \begin{cases} \inf\limits_{l\in\mathbb{Z}} p_l & \text{if } \sum\limits_{l\in\mathbb{Z}} \theta_l \leq 1, \\ \sup\limits_{l\in\mathbb{Z}} p_l & \text{if } \sum\limits_{l\in\mathbb{Z}} \theta_l > 1. \end{cases}$$

When $p(\cdot) \in \mathcal{P}^*(\mathbb{R}^n)$, Hölder's inequality holds true in the following form:

$$\int_{\mathbb{R}^n} |\hbar(x)f(x)|dx \leq \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right)\|\hbar\|_{L^{p(\cdot)}}\|f\|_{L^{p'(\cdot)}}. \tag{4}$$

See [22].

Lemma 1 ([25])**.** *Let $p(\cdot) \in \mathcal{B}^*(\mathbb{R}^n)$. Then, we have for any ball B in $\mathbb{R}^n$,*

$$|B|^{-1}\|\chi_B\|_{L^{p(\cdot)}}\|\chi_B\|_{L^{p'(\cdot)}} \leq C.$$

Lemma 2 ([25])**.** *Let $p_\aleph(\cdot) \in \mathcal{B}^*(\mathbb{R}^n)$, $\aleph = 1, 2$. Then, there is a positive constant C such that for every ball B in $\mathbb{R}^n$ and any measurable subset $S \subset B$,*

$$\frac{\|\chi_S\|_{L^{p_\aleph(\cdot)}}}{\|\chi_B\|_{L^{p_\aleph(\cdot)}}} \leq C\frac{|S|}{|B|}, \quad \frac{\|\chi_S\|_{L^{p'_\aleph(\cdot)}}}{\|\chi_B\|_{L^{p'_\aleph(\cdot)}}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_{\aleph 2}}, \quad \text{and} \quad \frac{\|\chi_R\|_{L^{p_\aleph(\cdot)}}}{\|\chi_B\|_{L^{p_\aleph(\cdot)}}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_{\aleph 1}},$$

where $\delta_{\aleph 1}$, $\delta_{\aleph 2}$ are constants with $0 < \delta_{\aleph 1}, \delta_{\aleph 2} < 1$.

The following result was studied by Wang et al. in [32].

Lemma 3. *Let $p(\cdot)$ and $q(\cdot) \in \mathcal{P}^*(\mathbb{R}^n), \hbar \in L^{p(\cdot)q(\cdot)}$. Then we have*

$$\min(\|\hbar\|^{q_+}_{L^{p(\cdot)q(\cdot)}}, \|\hbar\|^{q_-}_{L^{p(\cdot)q(\cdot)}}) \leq \||\hbar|^{q(\cdot)}\|_{L^{p(\cdot)}} \leq \max(\|\hbar\|^{q_+}_{L^{p(\cdot)q(\cdot)}}, \|\hbar\|^{q_-}_{L^{p(\cdot)q(\cdot)}}).$$

The following Lemmas are due to [37].

Lemma 4. *Let $0 < \beta \le 1$. If $p(\cdot) \in \mathcal{B}^*(\mathbb{R}^n)$, then we have for all $\hbar \in L^{p(\cdot)}$,*

$$\left\|S_\beta(\hbar)\right\|_{L^{p(\cdot)}} \lesssim \|\hbar\|_{L^{p(\cdot)}}. \tag{5}$$

Lemma 5. *Let $r \in (1,\infty), 0 < \beta \le 1$. If $p(\cdot) \in \mathcal{B}^*(\mathbb{R}^n)$, then for any $\{\hbar_\imath\}_{\imath\in\mathbb{N}}$ satisfying $\left\|\{\hbar_\imath\}_{\imath\in\mathbb{N}}\right\|_{\ell^r} \in L^{p(\cdot)}$, we have*

$$\left\|\left\|\left\{S_\beta(\hbar_\imath)\right\}_\imath\right\|_{\ell^r}\right\|_{L^{p(\cdot)}} \lesssim \left\|\|\{\hbar_\imath\}_\imath\|_{\ell^r}\right\|_{L^{p(\cdot)}}. \tag{6}$$

3. Boundedness of Operators on Herz Spaces

3.1. Vector-Valued Intrinsic Square Function

For $0 < \beta \le 1$, the family $\mathcal{C}_\beta$ consists of those functions $\psi : \mathbb{R}^n \to (0,\infty)$ such that $\operatorname{supp}\psi \subseteq \{x \in \mathbb{R}^n : |x| \le 1\}$, $\int_{\mathbb{R}^n} \psi(x)\mathrm{d}x = 0$ and for x_1, x_2,

$$|\psi(x_1) - \psi(x_2)| \le |x_1 - x_2|^\beta. \tag{7}$$

For any $\hbar \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $(y,t) \in \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$, let us set

$$\mathcal{A}_\beta(y,t) := \sup_{\psi\in\mathcal{C}_\beta} |(\hbar * \psi_t)(y)| := \sup_{\psi\in\mathcal{C}_\beta}\left|\int_{\mathbb{R}^n} \psi_t(y-z)\hbar(z)\mathrm{d}z\right|, \tag{8}$$

where $\psi_t(y) = t^{-n}\psi\left(\frac{y}{t}\right)$ and $\mathcal{A}_\beta$ is Lebesgue measurable.

The intrinsic square function of $\hbar$ of order β is defined as follows:

$$\mathcal{S}_\beta(\hbar)(x) = \left(\iint_{\Gamma(x)} (\mathcal{A}_\beta(y,t))^2 \frac{\mathrm{d}t\,\mathrm{d}y}{t^{1+n}}\right)^{1/2} \quad \forall x \in \mathbb{R}^n, \tag{9}$$

where $\Gamma(x) = \left\{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\right\}$.

Theorem 1. *Suppose that $1 < r < \infty, q_1(\cdot)$ and $q_2(\cdot) \in \mathcal{P}_0^*(\mathbb{R}^n)$ with $(q_1)_+ \le (q_2)_-$ and let $\beta \in (0,1]$, $p(\cdot) \in \mathcal{B}^*(\mathbb{R}^n)$. If $\alpha \in (-n\delta_{11}, n\delta_{12})$, where $\delta_{11}, \delta_{12} \in (0,1)$ are constants stated in Lemma 2, then for any $\{\hbar_\imath\}_{\imath\in\mathbb{N}}$ with $\left\|\|\{\hbar_\imath\}_\imath\|_{\ell^r}\right\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} < \infty$, the following vector-valued inequality holds:*

$$\left\|\left(\sum_\imath \left|S_\beta(\hbar_\imath)\right|^r\right)^{\frac{1}{r}}\right\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}} \le C\left\|\left(\sum_\imath |\hbar_\imath|^r\right)^{\frac{1}{r}}\right\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}}.$$

Proof. For any $\{\hbar_\imath\}_{\imath\in\mathbb{N}}$ satisfies $\left\|\|\{\hbar_\imath\}_\imath\|_{\ell^r}\right\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}} < \infty$, we can write $\hbar_\imath = \sum_{j=-\infty}^{\infty} \hbar_\imath\chi_j = \sum_{j=-\infty}^{\infty} \hbar_\imath^j$. By the definition of $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$, we have

$$\left\|\left\|\left\{S_\beta(\hbar_\imath)\right\}_\imath\right\|_{\ell^r}\chi_l\right\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\vartheta > 0 : \sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\left\|\left\{S_\beta(\hbar_\imath)\right\}_\imath\right\|_{\ell^r}\chi_l}{\vartheta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \le 1\right\}$$

Since

$$\left\|\left(\frac{2^{l\alpha}\left\|\left\{S_\beta(\hbar_\imath)\right\}_\imath\right\|_{\ell^r}\chi_l}{\vartheta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \le \left\|\left(\frac{2^{l\alpha}\left\|\left\{\sum_{j=-\infty}^{\infty} S_\beta(\hbar_\imath^j)\right\}_\imath\right\|_{\ell^r}\chi_l}{\vartheta_{11}+\vartheta_{12}+\vartheta_{13}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}$$

$$\leq C\left\|\left(\frac{2^{l\alpha}\left\|\left\{\sum_{j=-\infty}^{l-3} S_\beta(\hbar_\iota^j)\right\}_\iota\right\|_{\ell^r}\chi_l}{\vartheta_{11}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}$$
$$+C\left\|\left(\frac{2^{l\alpha}\left\|\left\{\sum_{j=l-2}^{l+2} S_\beta(\hbar_\iota^j)\right\}_\iota\right\|_{\ell^r}\chi_l}{\vartheta_{12}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}$$
$$+C\left\|\left(\frac{2^{l\alpha}\left\|\left\{\sum_{j=l+3}^{\infty} S_\beta(\hbar_\iota^j)\right\}_\iota\right\|_{\ell^r}\chi_l}{\vartheta_{13}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}},$$

where

$$\vartheta_{11} = \left\|\left\{2^{l\alpha}\left\|\left\{\sum_{j=-\infty}^{l-3} S_\beta(\hbar_\iota^j)\right\}_\iota\right\|_{\ell^r}\chi_l\right\}_{l=-\infty}^{\infty}\right\|_{\ell^{q_2(\cdot)}(L^{p(\cdot)})},$$
$$\vartheta_{12} = \left\|\left\{2^{l\alpha}\left\|\left\{\sum_{j=l-2}^{l+2} S_\beta(\hbar_\iota^j)\right\}_\iota\right\|_{\ell^r}\chi_l\right\}_{l=-\infty}^{\infty}\right\|_{\ell^{q_2(\cdot)}(L^{p(\cdot)})},$$
$$\vartheta_{13} = \left\|\left\{2^{l\alpha}\left\|\left\{\sum_{j=l+3}^{\infty} S_\beta(\hbar_\iota^j)\right\}_\iota\right\|_{\ell^r}\chi_l\right\}_{l=-\infty}^{\infty}\right\|_{\ell^{q_2(\cdot)}(L^{p(\cdot)})}.$$

If $\vartheta := (\vartheta_{11} + \vartheta_{12} + \vartheta_{13}) := \sum_{\varrho=1}^{3} \vartheta_{1\varrho}$, thus,

$$\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\left\|\{S_\beta(\hbar_\iota)\}_\iota\right\|_{\ell^r}\chi_l}{\vartheta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq C.$$

Then,

$$\left\|\left\|\{S_\beta(\hbar_\iota)\}_\iota\right\|_{\ell^r}\right\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} \leq C\vartheta := C\left(\sum_{\varrho=1}^{3}\vartheta_{1\varrho}\right).$$

Hence, if we prove that

$$\vartheta_{11} \leq C\left\|\|\{\hbar_\iota\}_\iota\|_{\ell^r}\right\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)},\ \vartheta_{12} \leq C\left\|\|\{\hbar_\iota\}_\iota\|_{\ell^r}\right\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)},\ \vartheta_{13} \leq C\left\|\|\{\hbar_\iota\}_\iota\|_{\ell^r}\right\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)},$$

we are done. Let $\vartheta_0 = \|\|\{\hbar_\iota\}_\iota\|_{\ell^r}\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}$.

First, we consider ϑ_{12}. For each $l \in \mathbb{Z}$, we define

$$(q_2^1)_l = \begin{cases} (q_2)_- \text{ if} \left\|\dfrac{2^{l\alpha}\left\|\sum_{j=l-2}^{l+2}\left\{S_\beta(\hbar_\iota^j)\right\}_\iota\right\|_{\ell^r}\chi_l}{\vartheta_0}\right\|_{L^{p(\cdot)}} \leq 1, \\ (q_2)_+ \text{ otherwise.} \end{cases}$$

Using Lemmas 3 and 5, it follows that

$$\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\left\|\left\{\sum_{j=l-2}^{l+2} S_\beta(\hbar_\iota^j)\right\}_\iota\right\|_{\ell^r}\chi_l}{\vartheta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}$$

$$\leq C\sum_{l=-\infty}^{\infty}\left\|\frac{2^{l\alpha}\left\|\left\{\sum_{j=l-2}^{l+2}S_\beta(\hbar_\imath^j)\right\}_\imath\right\|_{\ell^r}\chi_l}{\vartheta_0}\right\|_{L^{p(\cdot)}}^{(q_2^1)_l}$$

$$\leq C\sum_{l=-\infty}^{\infty}\left(\sum_{j=l-2}^{l+2}\left\|\frac{2^{l\alpha}\left\|\left\{S_\beta(\hbar_\imath^j)\right\}_\imath\right\|_{\ell^r}\chi_l}{\vartheta_0}\right\|_{L^{p(\cdot)}}\right)^{(q_2^1)_l}$$

$$\leq C\sum_{l=-\infty}^{\infty}\left(\sum_{j=l-2}^{l+2}\left\|\left\|\left\{\frac{D_1 2^{l\alpha}\hbar_\imath^j}{\vartheta_0}\right\}_\imath\right\|_{\ell^r}\right\|_{L^{p(\cdot)}}\right)^{(q_2^1)_l}$$

$$\leq C\sum_{l=-\infty}^{\infty}\left(\sum_{j=l-2}^{l+2}\left\|\left\|\left\{\frac{2^{l\alpha}\hbar_\imath^j}{\vartheta_0}\right\}_\imath\right\|_{\ell^r}\right\|_{L^{p(\cdot)}}\right)^{(q_2^1)_l},$$

where $C:=\max\left\{D_1^{(q_2)_-},D_1^{(q_2)_+}\right\}$. Since $\|\{\hbar_\imath\}_\imath\|_{\ell^r}\in\dot{K}_{p(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)$, it is not difficult for us to get $\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\|\{\hbar_\imath\}_\imath\|_{\ell^r}\chi_l}{\vartheta_0}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}\leq 1$ and $\left\|\frac{2^{l\alpha}\|\{\hbar_\imath\}_\imath\|_{\ell^r}\chi_l}{\vartheta_0}\right\|_{L^{p(\cdot)}}\leq 1$. Hence, from Lemma 3 and Proposition 1, we deduce

$$\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\left\|\left\{\sum_{j=l-2}^{l+2}S_\beta(\hbar_\imath^j)\right\}_\imath\right\|_{\ell^r}\chi_l}{\vartheta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}$$

$$\leq C\sum_{l=-\infty}^{\infty}\left\|\left(2^{l\alpha}\left\|\left\{\frac{\hbar_\imath}{\vartheta_0}\right\}_\imath\right\|_{\ell^r}\chi_l\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{\frac{(q_2^1)_l}{(q_1)_+}}$$

$$\leq C\left\{\sum_{l=-\infty}^{\infty}\left\|\left(2^{l\alpha}\left\|\left\{\frac{\hbar_\imath}{\vartheta_0}\right\}_\imath\right\|_{\ell^r}\chi_l\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}\right\}^{q_*^1}\leq C,$$

where $(q_2^1)_l\geq(q_2)_-\geq(q_1)_+$ and $q_*^1=\inf_{l\in\mathbb{N}}(q_2^1)_l/(q_1)_+$. Consequently, we get

$$\vartheta_{12}\leq C\vartheta_0:=C\big\|\|\{\hbar_\imath\}_\imath\|_{\ell^r}\big\|_{\dot{K}_{p(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}.$$

Now, let us turn to ϑ_{11}. For any $\beta\in(0,1]$, $(y,t)\in\Gamma(x)$ and $\psi\in\mathcal{C}_\beta$, we have

$$\sup_{\psi\in\mathcal{C}_\beta}\left|(\hbar_\imath^j)*\psi_t\right|=\sup_{\psi\in\mathcal{C}_\beta}\left|\int_{\mathbb{R}^n}\psi_t(y-z)\hbar_\imath^j(z)\right|\mathrm{d}z\lesssim\frac{1}{t^n}\int_{\Re_j\cap\{z:|y-z|\leq t\}}|\hbar_\imath^j(z)|\mathrm{d}z.$$

Taking $\|\cdot\|_{\ell^r}$ of each side of the above inequality, we find that

$$\|\{\sup_{\psi\in\mathcal{C}_\beta}\left|(\hbar_\imath^j)*\psi_t\right|\}_\imath\|_{\ell^r}\lesssim\frac{1}{t^n}\int_{\Re_j\cap\{z:|y-z|\leq t\}}\|\{\hbar_\imath^j(z)\}_\imath\|_{\ell^r}\mathrm{d}z. \tag{10}$$

For any $x\in\Re_l$, $(y,t)\in\Gamma(x)$, $j\leq l-2$ and $z\in\Re_j\cap\{z:|y-z|\leq t\}$, it is easy to get

$$t=(t+t)/2\geq(|x-y|+|y-z|)/2\geq|x-z|/2\geq|x|/4. \tag{11}$$

Thus, by combining (10) and (11), together with Minkowski's inequality, it follows that

$$
\begin{aligned}
\left\|\left\{S_\beta\left(\hbar_\iota^j\right)(x)\right\}_\iota\right\|_{\ell^r} &\lesssim \left(\iint_{\Gamma(x)}\left|\frac{1}{t^n}\int_{\Re_j\cap\{z:|y-z|\le t\}}\|\{\hbar_\iota^j(z)\}_\iota\|_{\ell^r}\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{1+n}}\right)^{\frac{1}{2}}\\
&\lesssim \left(\int_{|x|/4}^{\infty}\int_{|x-y|<t}\left|\frac{1}{t^n}\int_{\Re_j\cap\{z:|y-z|\le t\}}\|\{\hbar_\iota^j(z)\}_\iota\|_{\ell^r}\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{1+n}}\right)^{\frac{1}{2}}\\
&\lesssim \left(\int_{\Re_j}\|\{\hbar_\iota^j(z)\}_\iota\|_{\ell^r}\mathrm{d}z\right)\left(\int_{|x|/4}^{\infty}\frac{\mathrm{d}t}{t^{1+2n}}\right)^{\frac{1}{2}}\\
&\lesssim \left(\int_{\Re_j}\|\{\hbar_\iota^j(z)\}_\iota\|_{\ell^r}\mathrm{d}z\right)|x|^n \lesssim 2^{-ln}\int_{\Re_j}\|\{\hbar_\iota^j(z)\}_\iota\|_{\ell^r}\mathrm{d}z.
\end{aligned}
$$

Then, by virtue of Lemma 3, it follows that

$$
\begin{aligned}
&\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\|\sum_{j=-\infty}^{l-3}\left\{S_\beta(\hbar_\iota^j)\right\}_\iota\|_{\ell^r}\chi_l}{\vartheta_0}\right)^{q_2(\cdot)}\right\|\\
\le C&\sum_{l=-\infty}^{\infty}\left\|\frac{2^{l\alpha}\|\sum_{j=-\infty}^{l-3}\left\{S_\beta(\hbar_\iota^j)\right\}_\iota\|_{\ell^r}\chi_l}{\vartheta_0}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}^{(q_2^2)_l}\\
\le C&\sum_{l=-\infty}^{\infty}\left(\sum_{j=-\infty}^{l-3}\left\|\frac{2^{l\alpha}\|\{S_\beta(\hbar_\iota^j)\}_\iota\|_{\ell^r}\chi_l}{\vartheta_0}\right\|_{L^{p(\cdot)}}\right)^{(q_2^2)_l}\\
\le C&\sum_{l=-\infty}^{\infty}\left(\sum_{j=-\infty}^{l-3}\left\|\frac{2^{l\alpha}2^{-nl}\int_{\Re_j}\left\|\left\{\hbar_\iota^j\right\}_\iota\right\|_{\ell^r}\mathrm{d}z\chi_l}{\vartheta_0}\right\|_{L^{p(\cdot)}}\right)^{(q_2^2)_l}\\
\le C&\sum_{l=-\infty}^{\infty}\left(\sum_{j=-\infty}^{l-3}\left\|\left\|\left\{\frac{\hbar_\iota^j}{\vartheta_0}\right\}_\iota\right\|_{\ell^r}\right\|_{L^1(\mathbb{R}^n)}2^{l\alpha}2^{-nl}\|\chi_l\|_{L^{p(\cdot)}}\right)^{(q_2^2)_l}
\end{aligned}
$$

where

$$
(q_2^2)_l=\begin{cases}(q_2)_- & \text{if } \left\|\frac{2^{l\alpha}\left\|\sum_{j=-\infty}^{l-3}\left\{S_\beta(\hbar_\iota^j)\right\}_\iota\|_{\ell^r}\chi_l\right.}{\vartheta_0}\right\|_{L^{p(\cdot)}}\le 1,\\ (q_2)_+ & \text{otherwise.}\end{cases}
$$

From Hölder's inequality (4), and using Lemmas 1 and 2, we infer that

$$
\begin{aligned}
&\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\|\sum_{j=-\infty}^{l-3}\left\{S_\beta(\hbar_\iota^j)\right\}_\iota\|_{\ell^r}\chi_l}{\vartheta_0}\right)^{q_2(\cdot)}\right\|\\
\le C&\sum_{l=-\infty}^{\infty}\left(\sum_{j=-\infty}^{l-3}\left\|\left\|\left\{\frac{\hbar_\iota^j}{\vartheta_0}\right\}_\iota\right\|_{\ell^r}\right\|_{L^{p(\cdot)}}2^{l\alpha}\frac{\|\chi_j\|_{L^{p'(\cdot)}}}{\|\chi_l\|_{L^{p'(\cdot)}}}\frac{|B_l|}{2^{nl}}\right)^{(q_2^2)_l}\\
\le C&\sum_{l=-\infty}^{\infty}\left\{\sum_{j=-\infty}^{l-3}\left\|\left(\frac{2^{j\alpha}\|\{\hbar_\iota\}_\iota\|_{\ell^r}\chi_j}{\vartheta_0}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{\frac{1}{(q_1)_+}}2^{(l-j)(\alpha-\delta_{12})}\right\}^{(q_2^2)_l}.
\end{aligned}
$$

Now, we can distinguish the following two cases: $(q_1)_+>1$ and $0<(q_1)_+\le 1$.

*Case*1°: If $0 < (q_1)_+ \leq 1$, then from Proposition 1 and Lemma 3, we obtain

$$\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\|\sum_{j=-\infty}^{l-3}\left\{S_\beta(\hbar_\imath^j)\right\}_\imath\|_{\ell^r}\chi_l}{\vartheta_0}\right)^{q_2(\cdot)}\right\|$$

$$\leq C\sum_{l=-\infty}^{\infty}\left\{\sum_{j=-\infty}^{l-3}\left\|\left(\frac{2^{j\alpha}\|\{\hbar_\imath\}_\imath\|_{\ell^r}\chi_j}{\vartheta_0}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}2^{(q_1)_+(l-j)(\alpha-\delta_{12})}\right\}^{\frac{(q_2^2)_l}{(q_1)_+}}$$

$$\leq C\left\{\sum_{j=-\infty}^{\infty}\left\|\left(\frac{2^{j\alpha}\|\{\hbar_\imath\}_\imath\|_{\ell^r}\chi_j}{\vartheta_0}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}\sum_{l=j+3}^{\infty}2^{(q_1)_+(l-j)(\alpha-\delta_{12})}\right\}^{q_*^2}\leq C,$$

where $q_*^2 = \inf_{l\in\mathbb{N}}(q_2^2)_l/(q_1)_+$.

Case 2°: If $(q_1)_+ > 1$, then $1 \leq (q_1)_+ \leq (q_2)_- \leq (q_2^2)_l$. Hence, for $\alpha < n\delta_{12}$, by using Hölder's inequality (4) and Proposition 1, we obtain that

$$\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\|\sum_{j=-\infty}^{l-3}\left\{S_\beta(\hbar_\imath^j)\right\}_\imath\|_{\ell^r}\chi_l}{\vartheta_0}\right)^{q_2(\cdot)}\right\|$$

$$\leq C\sum_{l=-\infty}^{\infty}\left\{\sum_{j=-\infty}^{l-3}\left\|\left(\frac{2^{j\alpha}\|\{\hbar_\imath\}_\imath\|_{\ell^r}\chi_j}{\vartheta_0}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}2^{\frac{1}{2}(l-j)(\alpha-\delta_{12})(q_1)_+}\right\}^{\frac{(q_2^2)_l}{(q_1)_+}}$$

$$\cdot\left\{\sum_{j=-\infty}^{l-3}2^{\frac{1}{2}(l-j)(\alpha-\delta_{12})b}\right\}^{\frac{(q_2^2)_l}{b}}$$

$$\leq C\left\{\sum_{j=-\infty}^{\infty}\left\|\left(\frac{2^{j\alpha}\|\{\hbar_\imath\}_\imath\|_{\ell^r}\chi_j}{\vartheta_0}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}\sum_{l=j+3}^{\infty}2^{\frac{1}{2}(l-j)(\alpha-\delta_{12})(q_1)_+}\right\}^{q_*^2}\leq C,$$

where $b = ((q_1)_+)'$ and $q_*^2 = \inf_{l\in\mathbb{N}}(q_2^2)_l/(q_1)_+$.

Then, based on the computations presented above, it follows that

$$\vartheta_{11} \leq C\vartheta_0 := C\left\|\|\{\hbar_\imath\}_\imath\|_{\ell^r}\right\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}.$$

Finally, we estimate ϑ_{13}. For any $\beta \in (0,1]$, $(y,t) \in \Gamma(x)$ and $\psi \in \mathcal{C}_\beta$, we have

$$\sup_{\psi\in\mathcal{C}_\beta}\left|(\hbar_\imath^j) * \psi_t\right| = \sup_{\psi\in\mathcal{C}_\beta}\left|\int_{\mathbb{R}^n}\psi_t(y-z)\hbar_\imath^j(z)\right|\mathrm{d}z \lesssim \frac{1}{t^n}\int_{\Re_j\cap\{z:|y-z|\leq t\}}|\hbar_\imath^j(z)|\mathrm{d}z.$$

By applying the norm $\|\cdot\|_{\ell^r}$ on each side of the above inequality, we find that

$$\|\{\sup_{\psi\in\mathcal{C}_\beta}\left|(\hbar_\imath^j) * \psi_t\right|\}_\imath\|_{\ell^r} \lesssim \frac{1}{t^n}\int_{\Re_j\cap\{z:|y-z|\leq t\}}\|\{\hbar_\imath^j(z)\}_\imath\|_{\ell^r}\mathrm{d}z. \tag{12}$$

Let $z \in \Re_j \cap \{z : |y-z| \leq t\}$, $(y,t) \in \Gamma(x)$ and $j \geq l+3$, it is easy to get

$$t = (t+t)/2 \geq (|x-y| + |y-z|)/2 \geq |x-z|/2 \geq ||z| - |x||/2 \geq |z|/4. \tag{13}$$

Thus, by combining (12) and (13), we obtain

$$
\begin{aligned}
\left\|\left\{S_\beta\left(\hbar_\iota^j\right)(x)\right\}_\iota\right\|_{\ell^r} &\lesssim \left(\iint_{\Gamma(x)}\left|\frac{1}{t^n}\int_{\Re_j\cap\{z:|y-z|\le t\}}\left\|\left\{\hbar_\iota^j(z)\right\}_\iota\right\|_{\ell^r}\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{1+n}}\right)^{\frac{1}{2}}\\
&\lesssim \left(\int_{|z|/4}^{\infty}\int_{|x-y|<t}\left|\frac{1}{t^n}\int_{\Re_j\cap\{z:|y-z|\le t\}}\left\|\left\{\hbar_\iota^j(z)\right\}_\iota\right\|_{\ell^r}\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{1+n}}\right)^{\frac{1}{2}}\\
&\lesssim \left(\int_{\Re_j}\left\|\left\{\hbar_\iota^j(z)\right\}_\iota\right\|_{\ell^r}\mathrm{d}z\right)\left(\int_{|z|/4}^{\infty}\frac{dt}{t^{1+2n}}\right)^{\frac{1}{2}}\\
&\lesssim \left(\int_{\Re_j}\left\|\left\{\hbar_\iota^j(z)\right\}_\iota\right\|_{\ell^r}\mathrm{d}z\right)|z|^n \lesssim 2^{-nj}\int_{\Re_j}\left\|\left\{\hbar_\iota^j(z)\right\}_\iota\right\|_{\ell^r}\mathrm{d}z.
\end{aligned}
$$

Then, from Lemmas 1–3, it follows that

$$
\begin{aligned}
&\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\,\|\sum_{j=l+3}^{\infty}\left\{S_\beta(\hbar_\iota^j)\,\right\}_\iota\|_{\ell^r}\chi_l}{\vartheta_0}\right)^{q_2(\cdot)}\right\|\\
&\le C\sum_{l=-\infty}^{\infty}\left\|\frac{2^{l\alpha}\,\|\sum_{j=l+3}^{\infty}\left\{S_\beta(\hbar_\iota^j)\,\right\}_\iota\|_{\ell^r}\chi_l}{\vartheta_0}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}^{(q_2^3)_l}\\
&\le C\sum_{l=-\infty}^{\infty}\left\{\sum_{j=l+3}^{\infty}\left\|\frac{2^{l\alpha}2^{-nj}\int_{\Re_j}\left\|\left\{\hbar_\iota^j\right\}_\iota\right\|_{\ell^r}\mathrm{d}z\chi_l}{\vartheta_0}\right\|_{L^{p(\cdot)}}\right\}^{(q_2^3)_l}\\
&\le C\sum_{l=-\infty}^{\infty}\left\{\sum_{j=l+3}^{\infty}\left\|\left\|\left\{\frac{\hbar_\iota^j}{\vartheta_0}\right\}_\iota\right\|_{\ell^r}\right\|_{L^1(\mathbb{R}^n)}2^{l\alpha}2^{-nj}\|\chi_l\|_{L^{p(\cdot)}}\right\}^{(q_2^3)_l}\\
&\le C\sum_{l=-\infty}^{\infty}\left\{\sum_{j=l+3}^{\infty}\left\|\left\|\left\{\frac{\hbar_\iota^j}{\vartheta_0}\right\}_\iota\right\|_{\ell^r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}2^{l\alpha}\frac{\|\chi_l\|_{L^{p(\cdot)}}}{\|\chi_j\|_{L^{p(\cdot)}}}\frac{|B_j|}{2^{nj}}\right\}^{(q_2^3)_l}\\
&\le C\sum_{l=-\infty}^{\infty}\left\{\sum_{j=l+3}^{\infty}\left\|\left(\frac{2^{j\alpha}\|\{\hbar_\iota\}_\iota\|_{\ell^r}\chi_j}{\vartheta_0}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{\frac{1}{(q_1)_+}}2^{(l-j)(\alpha+\delta_{11})}\right\}^{(q_2^3)_l}.
\end{aligned}
$$

where

$$
(q_2^3)_l=\begin{cases}(q_2)_- & \text{if } \left\|\frac{2^{l\alpha}\left\|\sum_{j=l+3}^{\infty}\left\{S_\beta(\hbar_\iota^j)\right\}_\iota\|_{\ell^r}\chi_l\right.}{\vartheta_0}\right\|_{L^{p(\cdot)}}\le 1,\\ (q_2)_+ & \text{otherwise.}\end{cases}
$$

Observing that $(q_1)_+\le(q_2)_-$ and the fact $\alpha>-n\delta_{11}$, by using a similar argument to that used for ϑ_{11}, we deduce that

$$
\vartheta_{13}\le C\vartheta_0:=C\left\|\|\{\hbar_\iota\}_\iota\|_{\ell^r}\right\|_{\dot{K}_{p(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}.
$$

This completes the proof of Theorem 1. □

3.2. Intrinsic Square Function

Theorem 2. *Suppose that $q_1(\cdot)$ and $q_2(\cdot) \in \mathcal{P}_0^*(\mathbb{R}^n)$ with $(q_1)_+ \leq (q_2)_-$ and let $\beta \in (0,1]$, $p(\cdot) \in \mathcal{B}^*(\mathbb{R}^n)$. If $\alpha \in (-n\delta_{11}, n\delta_{12})$, where $\delta_{11}, \delta_{12} \in (0,1)$ are constants stated in Lemma 2, then S_β is bounded from $\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$.*

Proof. Let $\hbar \in \dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$. We can write $\hbar(x) = \sum_{j=-\infty}^{\infty} \hbar(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} \hbar_j(x)$. By the definition of $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$, we have

$$\big\|S_\beta(\hbar)\big\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\vartheta > 0 : \sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\big|S_\beta(\hbar)\chi_l\big|}{\vartheta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 1\right\}.$$

Since

$$\begin{aligned}
\left\|\left(\frac{2^{l\alpha}\big|S_\beta(\hbar)\chi_l\big|}{\vartheta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} &\leq \left\|\left(\frac{2^{l\alpha}\Big|\sum_{j=-\infty}^{\infty} S_\beta(\hbar_j)\chi_l\Big|}{\vartheta_{21}+\vartheta_{22}+\vartheta_{23}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
&\leq C\left\|\left(\frac{2^{l\alpha}\Big|\sum_{j=-\infty}^{l-3} S_\beta(\hbar_j)\chi_l\Big|}{\vartheta_{21}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
&\quad + C\left\|\left(\frac{2^{l\alpha}\Big|\sum_{j=l-2}^{l+2} S_\beta(\hbar_j)\chi_l\Big|}{\vartheta_{22}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
&\quad + C\left\|\left(\frac{2^{l\alpha}\Big|\sum_{j=l+2}^{\infty} S_\beta(\hbar_j)\chi_l\Big|}{\vartheta_{23}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}},
\end{aligned}$$

here

$$\begin{aligned}
\vartheta_{21} &= \left\|\left\{2^{l\alpha}\left|\sum_{j=-\infty}^{l-3} S_\beta(\hbar_j)\chi_l\right|\right\}_{l=-\infty}\right\|_{\ell^{q_2(\cdot)}(L^{p(\cdot)})}, \\
\vartheta_{22} &= \left\|\left\{2^{l\alpha}\left|\sum_{j=l-2}^{l+2} S_\beta(\hbar_j)\chi_l\right|\right\}_{l=-\infty}^{\infty}\right\|_{\ell^{q_2(\cdot)}(L^{p(\cdot)})}, \\
\vartheta_{23} &= \left\|\left\{2^{l\alpha}\left|\sum_{j=l+3}^{\infty} S_\beta(\hbar_j)\chi_l\right|\right\}_{l=-\infty}^{\infty}\right\|_{\ell^{q_2(\cdot)}(L^{p(\cdot)})},
\end{aligned}$$

and $\vartheta := (\vartheta_{21}+\vartheta_{22}+\vartheta_{23}) := \sum_{\varrho=1}^{3}\vartheta_{2\varrho}$. That is,

$$\big\|S_\beta(\hbar)\big\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} \leq C\vartheta := C\left(\sum_{\varrho=1}^{3}\vartheta_{1\varrho}\right).$$

Hence, once we establish that

$$\vartheta_{21} \leq C\|\hbar\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}, \vartheta_{22} \leq C\|\hbar\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}, \vartheta_{23} \leq C\|\hbar\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)},$$

we are done. Let $\vartheta_{20} = \|\hbar\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}$.

First, we consider ϑ_{22}. By the boundedness of S_β on $L^{p(\cdot)}$, and using the same technique as for ϑ_{12} in the proof of Theorem 1, we find immediately that

$$\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\left|\sum_{j=l-2}^{l+2} S_\beta(\hbar_j)\chi_l\right|}{\vartheta_{20}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \le C.$$

That is,

$$\vartheta_{22} \le C\vartheta_{20} := C\|\hbar\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}.$$

Now, let us turn to the estimates of ϑ_{21}. For any $\beta \in (0,1], x \in \Re_l, (y,t) \in \Gamma(x)$ and $j \le l-3$, we have

$$\sup_{\psi\in\mathcal{C}_\beta}|(\hbar_j)*\psi_t| = \sup_{\psi\in\mathcal{C}_\beta}\left|\int_{\mathbb{R}^n}\psi_t(y-z)\hbar_j(z)\mathrm{d}z\right| \lesssim \frac{1}{t^n}\int_{\Re_j\cap\{z:|y-z|\le t\}}|\hbar_j(z)\mathrm{d}z|. \tag{14}$$

Let $z \in \Re_j \cap \{z : |y-z| \le t\}$. Since $(y,t) \in \Gamma(x)$, we have

$$t = (t+t)/2 \ge (|x-y| + |y-z|)/2 \ge |x-z|/2 \ge |x|/4. \tag{15}$$

Thus, by combining (14) and (15), we deduce that

$$\begin{aligned}
S_\beta(\hbar_j)(x) &\lesssim \left(\iint_{\Gamma(x)}\left|\frac{1}{t^n}\int_{\Re_j\cap\{z:|y-z|\le t\}}|\hbar_j(z)|\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{1+n}}\right)^{\frac{1}{2}}\\
&\lesssim \left(\int_{|x|/4}^{\infty}\int_{|x-y|<t}\left|\frac{1}{t^n}\int_{\Re_j\cap\{z:|y-z|\le t\}}|\hbar_j(z)|\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{1+n}}\right)^{\frac{1}{2}}\\
&\lesssim \left(\int_{\Re_j}|\hbar_j(z)|\mathrm{d}z\right)\left(\int_{|x|/4}^{\infty}\frac{\mathrm{d}t}{t^{1+2n}}\right)^{\frac{1}{2}}\\
&\lesssim \left(\int_{\Re_j}|\hbar_j(z)|\mathrm{d}z\right)|x|^n \lesssim 2^{-ln}\|\hbar_j\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

Then, by applying Lemma 3 and Hölder's inequality (4), we find that

$$\begin{aligned}
&\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\left|\sum_{j=-\infty}^{l-3} S_\beta(\hbar_j)\chi_l\right|}{\vartheta_{20}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\\
&\le C\sum_{l=-\infty}^{\infty}\left\|\frac{2^{l\alpha}\left|\sum_{j=-\infty}^{l-3} S_\beta(\hbar_j)\chi_l\right|}{\vartheta_{20}}\right\|_{L^{p(\cdot)}}^{(q_2^1)_l}\\
&\le C\sum_{l=-\infty}^{\infty}\left(2^{l\alpha}\sum_{j=-\infty}^{l-3}\left\|\frac{|\hbar_j|}{\vartheta_{20}}\right\|_{L^{p(\cdot)}}\|\chi_l\|_{L^{p(\cdot)}}\|\chi_j\|_{L^{p_1'(\cdot)}}2^{-nl}\right)^{(q_2^1)_l}
\end{aligned}$$

where

$$(q_2^1)_l = \begin{cases}(q_2)_- & \text{if } \left\|\frac{2^{l\alpha}\left|\sum_{j=-\infty}^{l-3} S_\beta(\hbar_j)\right|\chi_l}{\vartheta_{20}}\right\|_{L^{p(\cdot)}} \le 1,\\ (q_2)_+ & \text{otherwise.}\end{cases}$$

From Lemmas 1–3, we deduce that

$$\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\left|\sum_{j=-\infty}^{l-3}S_\beta(\hbar_j)\chi_l\right|}{\vartheta_{20}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}$$
$$\le C\sum_{l=-\infty}^{\infty}\left(2^{l\alpha}\sum_{j=-\infty}^{l-3}\left\|\frac{|\hbar_j|}{\vartheta_{20}}\right\|_{L^{p(\cdot)}}\frac{\|\chi_j\|_{L^{p_1'(\cdot)}}}{\|\chi_l\|_{L^{p_1'(\cdot)}}}\frac{|B_l|}{2^{nl}}\right)^{(q_2^1)_l}$$
$$\le C\sum_{l=-\infty}^{\infty}\left\{\sum_{j=-\infty}^{l-3}\left\|\left(\frac{|2^{j\alpha}\hbar\chi_j|}{\vartheta_{20}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{\frac{1}{(q_1)_+}}2^{(l-j)(\alpha-n\delta_{12})}\right\}^{(q_2^1)_l}$$

Moreover, using the same approach as ϑ_{11} in the proof of Theorem 1, we infer that

$$\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\left|\sum_{j=-\infty}^{l-3}S_\beta(\hbar_j)\chi_l\right|}{\vartheta_{20}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}$$
$$\le C\begin{cases}\left\{\sum\limits_{j=-\infty}^{\infty}\left\|\left(\frac{2^{j\alpha}|\hbar\chi_j|}{\vartheta_{20}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\sum\limits_{l=j+3}^{\infty}2^{\frac{1}{2}(l-j)(\alpha-n\delta_{12})(q_1)_+}\right\}^{q_*^1}, (q_1)_+>1,\\ \left\{\sum\limits_{j=-\infty}^{\infty}\left\|\left(\frac{2^{j\alpha}|\hbar\chi_j|}{\vartheta_{20}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\sum\limits_{l=j+3}^{\infty}2^{(l-j)(\alpha-n\delta_{12})(q_1)_+}\right\}^{q_*^1}, \quad 0<(q_1)_+\le 1,\end{cases}$$

where $q_*^1=\inf_{l\in\mathbb{N}}(q_2^1)_l/(q_1)_+$. This implies that

$$\vartheta_{21}\le C\vartheta_{20}:=C\|\hbar\|_{\dot K_p^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}.$$

Finally, we estimate ϑ_{23}. For any $x\in\Re_j,(y,t)\in\Gamma(x),z\in\Re_j\cap\{z:|y-z|\le t\}$ and $j\ge l+3$, we find that

$$t=(t+t)/2\ge(|x-y|+|y-z|)/2\ge|x-z|/2\ge||z|-|x||/2\ge|z|/4, \tag{16}$$

Thus, by combining (14) and (16), we get

$$\begin{aligned}S_\beta(\hbar_j)(x)&\lesssim\left(\iint_{\Gamma(x)}\left|\frac{1}{t^n}\int_{\Re_j\cap\{z:|y-z|\le t\}}|\hbar_j(z)|\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{1+n}}\right)^{\frac12}\\&\lesssim\left(\int_{|z|/4}^{\infty}\int_{|x-y|<t}\left|\frac{1}{t^n}\int_{\Re_j\cap\{z:|y-z|\le t\}}|\hbar_j(z)|\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{1+n}}\right)^{\frac12}\\&\lesssim\left(\int_{\Re_j}|\hbar_j(z)|\mathrm{d}z\right)\left(\int_{|z|/4}^{\infty}\frac{\mathrm{d}t}{t^{1+2n}}\right)^{\frac12}\\&\lesssim\left(\int_{\Re_j}|\hbar_j(z)|\mathrm{d}z\right)|z|^n\lesssim 2^{-jn}\|\hbar_j\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

Thus, when $\alpha>-n\delta_{11}$, as argued for ϑ_{21}, we deduce that

$$
\begin{aligned}
&\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\left|\sum_{j=l+3}^{\infty}S_\beta(\hbar_j)\chi_l\right|}{\vartheta_{20}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\\
&\leq C\sum_{l=-\infty}^{\infty}\left\|\frac{2^{l\alpha}\left|\sum_{j=l+3}^{\infty}S_\beta(\hbar_j)\chi_l\right|}{\vartheta_{20}}\right\|_{L^{p(\cdot)}}^{(q_2^2)_l}\\
&\leq C\sum_{l=-\infty}^{\infty}\left\{2^{l\alpha}\sum_{j=l+3}^{\infty}\left\|\frac{|\hbar_j|}{\vartheta_{20}}\right\|_{L^{p(\cdot)}}\|\chi_l\|_{L^{p(\cdot)}}\|\chi_j\|_{L^{p_1'(\cdot)}}2^{-nj}\right\}^{(q_2^2)_l}\\
&\leq C\sum_{l=-\infty}^{\infty}\left\{2^{l\alpha}\sum_{j=l+3}^{\infty}2^{-j\alpha}\left\|\frac{2^{j\alpha}|\hbar_j|}{\vartheta_{20}}\right\|_{L^{p(\cdot)}}\frac{\|\chi_l\|_{L^{p(\cdot)}}}{\|\chi_j\|_{L^{p(\cdot)}}}\frac{|B_j|}{2^{nj}}\right\}^{(q_2^2)_l}\\
&\leq C\sum_{l=-\infty}^{\infty}\left\{\sum_{j=l+3}^{\infty}\left\|\left(\frac{2^{j\alpha}|\hbar\chi_j|}{\vartheta_{20}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}^{\frac{1}{(q_1)_+}}2^{(l-j)(\alpha+n\delta_{11})}\right\}^{(q_2^2)_l}\leq C,
\end{aligned}
$$

here

$$
(q_2^2)_l=\begin{cases}(q_2)_- & \text{if } \left\|\frac{2^{l\alpha}\left|\sum_{j=l+3}^{\infty}S_\beta(\hbar_j)\chi_l\right|}{\vartheta_{20}}\right\|_{L^{p(\cdot)}}\leq 1,\\ (q_2)_+ & \text{otherwise.}\end{cases}
$$

Hence,

$$
\vartheta_{23}\leq C\vartheta_{20}:=C\|\hbar\|_{\dot{K}_{p(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}.
$$

This finishes the proof of Theorem 2. □

3.3. Commutator of the Intrinsic Square Function

Let B be any ball with a radius of $r>0$ and a centre at $x\in\mathbb{R}^n$. If a locally integrable function b satisfies the following conditions, it is said to be a BMO function.

$$
\|b\|_*:=\sup_{x\in\mathbb{R}^n,r>0}|B|^{-1}\int_B|b(y)-b_B|\mathrm{d}y<\infty,
$$

where $\|b\|_*$ is the norm in $\mathrm{BMO}(\mathbb{R}^n)$ and $b_B:=|B|^{-1}\int_B b(t)\mathrm{d}t$.

The commutator of the intrinsic square function $S_\beta(\hbar)$ and $b\in\mathrm{BMO}(\mathbb{R}^n)$ is defined as follows:

$$
[b,S_\beta](\hbar)(x)=\left(\iint_{\Gamma(x)}\sup_{\psi\in\mathcal{C}_\beta}\left|\int_{\mathbb{R}^n}(b(x)-b(z))\psi_t(y-z)\hbar(z)\mathrm{d}z\right|^2\frac{dydt}{t^{1+n}}\right)^{\frac{1}{2}}.
$$

Theorem 3. *Suppose that $q_1(\cdot)$ and $q_2(\cdot)\in\mathcal{P}_0^*(\mathbb{R}^n)$ with $(q_1)_+\leq(q_2)_-$ and let $\beta\in(0,1]$, $p(\cdot)\in\mathcal{B}^*(\mathbb{R}^n)$, $b\in\mathrm{BMO}(\mathbb{R}^n)$. If $\alpha\in(-n\delta_{11},n\delta_{12})$, where $\delta_{11},\delta_{12}\in(0,1)$ are constants stated in Lemma 2, then $[b,S_\beta]$ is bounded from $\dot{K}_{p(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}_{p(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)$.*

To give the proof Theorem 3, we use the following lemmas.

Lemma 6 ([25])**.** *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$, v be a positive integer and $p(\cdot) \in \mathcal{B}^*(\mathbb{R}^n)$. Then there is a positive C such that for each $\kappa, \jmath \in \mathbb{Z}$ with $\kappa > \jmath$,*

$$\sup_{B \subset \mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}}} \left\| (b - b_B)^v . \chi_B \right\|_{L^{p(\cdot)}} \approx C \|b\|_*^v,$$
$$\left\| \left(b - b_{B_\jmath} \right)^v \chi_{B_\kappa} \right\|_{L^{p(\cdot)}} \lesssim (\kappa - \jmath)^v \|b\|_*^v \|\chi_{B_\kappa}\|_{L^{p(\cdot)}}.$$

The following Lemma 7 is due to Wang [38].

Lemma 7. *Let $p(\cdot) \in \mathcal{B}^*(\mathbb{R}^n)$. Then for any $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $\hbar \in L^{p(\cdot)}$, the commutator $[b, S_\beta]$ is bounded on $L^{p(\cdot)}$.*

Proof. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $\hbar \in \dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$. We can write $\hbar(x) = \sum_{j=-\infty}^{\infty} \hbar(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} \hbar_j(x)$. By the definition of $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$, we have

$$\left\| [b, S_\beta](\hbar) \right\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} = \inf\left\{ \vartheta > 0 : \sum_{l=-\infty}^{\infty} \left\| \left(\frac{2^{l\alpha} \left| [b, S_\beta](\hbar)\chi_l \right|}{\vartheta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 1 \right\}.$$

Since

$$\begin{aligned}
\left\| \left(\frac{2^{l\alpha} \left| [b, S_\beta](\hbar)\chi_l \right|}{\vartheta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} &\leq \left\| \left(\frac{2^{l\alpha} \left| \sum_{j=-\infty}^{\infty} [b, S_\beta](\hbar_j)\chi_l \right|}{\vartheta_{31} + \vartheta_{32} + \vartheta_{33}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
&\leq C \left\| \left(\frac{2^{l\alpha} \left| \sum_{j=-\infty}^{l-3} [b, S_\beta](\hbar_j)\chi_l \right|}{\vartheta_{31}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
&\quad + C \left\| \left(\frac{2^{l\alpha} \left| \sum_{j=l-2}^{l+2} [b, S_\beta](\hbar_j)\chi_l \right|}{\vartheta_{32}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
&\quad + C \left\| \left(\frac{2^{l\alpha} \left| \sum_{j=l+2}^{\infty} [b, S_\beta](\hbar_j)\chi_l \right|}{\vartheta_{33}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}},
\end{aligned}$$

let

$$\vartheta_{31} = \left\| \left\{ 2^{l\alpha} \left| \sum_{j=-\infty}^{l-3} [b, S_\beta](\hbar_j)\chi_l \right| \right\}_{l=-\infty} \right\|_{\ell^{q_2(\cdot)}(L^{p(\cdot)})},$$
$$\vartheta_{32} = \left\| \left\{ 2^{l\alpha} \left| \sum_{j=l-2}^{l+2} [b, S_\beta](\hbar_j)\chi_l \right| \right\}_{l=-\infty}^{\infty} \right\|_{\ell^{q_2(\cdot)}(L^{p(\cdot)})},$$
$$\vartheta_{33} = \left\| \left\{ 2^{l\alpha} \left| \sum_{j=l+3}^{\infty} [b, S_\beta](\hbar_j)\chi_l \right| \right\}_{l=-\infty}^{\infty} \right\|_{\ell^{q_2(\cdot)}(L^{p(\cdot)})},$$

and $\vartheta := (\vartheta_{31} + \vartheta_{32} + \vartheta_{33}) := \sum_{\varrho=1}^{3} \vartheta_{3\varrho}$. That is,

$$\sum_{l=-\infty}^{\infty} \left\| \left(\frac{2^{l\alpha} \left| [b, S_\beta](\hbar)\chi_l \right|}{\vartheta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq C.$$

Therefore, once we prove that

$$\vartheta_{31} \le C\|b\|_*\|\hbar\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)},\ \vartheta_{32} \le C\|b\|_*\|\hbar\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)},\ \vartheta_{33} \le C\|b\|_*\|\hbar\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)},$$

we are done. Let $\vartheta_{20} = \|\hbar\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}$.

First, we consider ϑ_{32}. By the $L^{p(\cdot)}$-boundedness of $[b, S_\beta]$ (Lemma 7), as discussed about ϑ_{12} in the proof of Theorem 1, we obtain

$$\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\left|\sum_{j=l-2}^{l+2}[b,S_\beta](\hbar_j)\chi_l\right|}{\vartheta_{20}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \le C.$$

That is,

$$\vartheta_{32} \le C\|b\|_*\vartheta_{20} := C\|b\|_*\|\hbar\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}.$$

Now, we estimate ϑ_{31}. Noting that $x \in \Re_j, j \le l-3$. Similar to the estimation of $S_\beta(\hbar_j)(x)$ in the proof of Theorem 2, we can obtain

$$S_\beta(\hbar_j)(x) \lesssim 2^{-nl}\|\hbar_j\|_{L^1(\mathbb{R}^n)}.$$

Thus,

$$[b,S_\beta](\hbar_j)(x) := |S_\beta[(b(x)-b)\hbar_j](x)| \lesssim 2^{-nl}\|(b(\cdot)-b)\hbar_j\|_{L^1(\mathbb{R}^n)}.$$

Hence, by Lemma 3 and Hölder's inequality (4), we get

$$\begin{aligned}
&\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\left|\sum_{j=-\infty}^{l-3}[b,S_\beta](\hbar_j)\chi_l\right|}{\vartheta_{20}\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\\
&\le C\sum_{l=-\infty}^{\infty}\left\|\frac{2^{l\alpha}\left|\sum_{j=-\infty}^{l-3}2^{-nl}\|(b(\cdot)-b)\hbar_j\|_{L^1(\mathbb{R}^n)}\chi_l\right|}{\vartheta_{20}\|b\|_*}\right\|_{L^{p(\cdot)}}^{(q_2^1)_l}\\
&\le C\sum_{l=-\infty}^{\infty}\left(\sum_{j=-\infty}^{l-3}2^{l\alpha}\left\|\frac{|\hbar_j|}{\vartheta_{20}}\right\|_{L^{p(\cdot)}}\left\|(b-b_j)\chi_{B_j}\right\|_{L^{p'(\cdot)}}\frac{2^{-nl}}{\|b\|_*}\|\chi_l\|_{L^{p(\cdot)}}\right)^{(q_2^1)_l}\\
&\quad + C\sum_{l=-\infty}^{\infty}\left(\sum_{j=-\infty}^{l-3}2^{l\alpha}\left\|\frac{|\hbar_j|}{\vartheta_{20}}\right\|_{L^{p(\cdot)}}\|(b-b_j)\chi_{B_l}\|_{L^{p(\cdot)}}\frac{2^{-nl}}{\|b\|_*}\|\chi_j\|_{L^{p'(\cdot)}}\right)^{(q_2^1)_l},
\end{aligned}$$

where

$$(q_2^1)_l = \begin{cases} (q_2)_- & \text{if } \left\|\frac{2^{l\alpha}\left|\sum_{j=-\infty}^{l-3}[b,S_\beta](\hbar_j)\right|\chi_l}{\vartheta_{20}\|b\|_*}\right\|_{L^{p(\cdot)}} \le 1,\\ (q_2)_+ & \text{otherwise.}\end{cases}$$

It follows from Lemmas 1–3 and 6 that

$$\begin{aligned}
&\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\left|\sum_{j=-\infty}^{l-3}[b,S_\beta](\hbar_j)\chi_l\right|}{\vartheta_{20}\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\\
&\le C\sum_{l=-\infty}^{\infty}\left(\sum_{j=-\infty}^{l-3}2^{-j\alpha}\left\|\frac{2^{j\alpha}|\hbar_j|}{\vartheta_{20}}\right\|_{L^{p(\cdot)}}(l-j)2^{l\alpha}\frac{\left\|\chi_{B_j}\right\|_{L^{p'(\cdot)}}}{\|\chi_{B_l}\|_{L^{p_1'(\cdot)}}}\frac{|B_l|}{2^{nl}}\right)^{(q_2^1)_l}
\end{aligned}$$

$$
\leq C\sum_{l=-\infty}^{\infty}\left(\sum_{j=-\infty}^{l-3}\left\|\left(\frac{2^{j\alpha}|\hbar\chi_j|}{\vartheta_{20}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}^{\frac{1}{(q_1)_+}}(l-j)2^{(l-j)(\alpha-n\delta_{11})}\right)^{(q_2^1)_l}
$$

$$
\leq C\begin{cases}\left\{\sum_{l=-\infty}^{\infty}\sum_{j=-\infty}^{l-3}\left\|\left(\frac{2^{j\alpha}|\hbar\chi_j|}{\vartheta_{20}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}(l-j)^{(q_1)_+}2^{\frac{1}{2}(l-j)(\alpha-n\delta_{11})(q_1)_+}\right\}^{q_*^1}, & (q_1)_+>1,\\ \left\{\sum_{l=-\infty}^{\infty}\sum_{j=-\infty}^{l-3}\left\|\left(\frac{2^{j\alpha}|\hbar\chi_j|}{\vartheta_{20}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}(l-j)^{(q_1)_+}2^{(l-j)(\alpha-n\delta_{11})(q_1)_+}\right\}^{q_*^1}, & 0<(q_1)_+\leq 1,\end{cases}
$$

$$
\leq C,
$$

where $q_*^1=\inf_{l\in\mathbb{N}}(q_2^1)_l/(q_1)_+$. This implies that

$$
\vartheta_{31}\leq C\|b\|_*\vartheta_{20}:=C\|b\|_*\|\hbar\|_{\dot K^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}.
$$

Finally, we estimate ϑ_{32}. Let $x\in\Re_j, j\geq l+3$. Similar to the estimation of $S_\beta(\hbar_j)$ in the proof of Theorem 2, we can find that

$$
S_\beta(\hbar_j)(x)\lesssim 2^{-jn}\|\hbar_j\|_{L^1(\mathbb{R}^n)}.
$$

By using the inequality stated above, we can conclude

$$
[b,S_\beta](\hbar_j)(x):=|S_\beta[(b(x)-b)\hbar_j](x)|\lesssim 2^{-nj}\|(b(\cdot)-b)\hbar_j\|_{L^1(\mathbb{R}^n)}.
$$

Therefore, when $\alpha>-n\delta_{12}$, as argued for ϑ_{31}, it follows that

$$
\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\left|\sum_{j=l+3}^{\infty}[b,S_\beta](\hbar_j)\chi_l\right|}{\vartheta_{20}\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}
$$

$$
\leq C\sum_{l=-\infty}^{\infty}\left\|\frac{2^{l\alpha}\left|\sum_{j=l+3}^{\infty}2^{-nj}\|(b(\cdot)-b)\hbar_j\|_{L^1(\mathbb{R}^n)}\chi_l\right|}{\vartheta_{20}\|b\|_*}\right\|_{L^{p(\cdot)}}^{(q_2^2)_l}
$$

$$
\leq C\sum_{l=-\infty}^{\infty}\left(\sum_{j=l+3}^{\infty}2^{l\alpha}\left\|\frac{|\hbar_j|}{\vartheta_{20}}\right\|_{L^{p(\cdot)}}\left\|(b-b_j)\chi_{B_j}\right\|_{L^{p'(\cdot)}}\frac{2^{-nl}}{\|b\|_*}\|\chi_l\|_{L^{p(\cdot)}}\right)^{(q_2^2)_l}
$$

$$
+C\sum_{l=-\infty}^{\infty}\left(\sum_{j=l+3}^{\infty}2^{l\alpha}\left\|\frac{|\hbar_j|}{\vartheta_{20}}\right\|_{L^{p(\cdot)}}\|(b-b_j)\chi_{B_l}\|_{L^{p(\cdot)}}\frac{2^{-nl}}{\|b\|_*}\|\chi_j\|_{L^{p'(\cdot)}}\right)^{(q_2^1)_l}
$$

$$
\leq C\sum_{l=-\infty}^{\infty}\left(\sum_{j=l+3}^{\infty}2^{-j\alpha}\left\|\frac{2^{j\alpha}|\hbar_j|}{\vartheta_{20}}\right\|_{L^{p(\cdot)}}(l-j)2^{l\alpha}\frac{\left\|\chi_{B_j}\right\|_{L^{p'(\cdot)}}}{\left\|\chi_{B_l}\right\|_{L^{p'(\cdot)}}}\frac{|B_l|}{2^{nl}}\right)^{(q_2^2)_l}
$$

$$
\leq C\sum_{l=-\infty}^{\infty}\left(\sum_{j=l+3}^{\infty}\left\|\left(\frac{2^{j\alpha}|\hbar\chi_j|}{\vartheta_{20}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}^{\frac{1}{(q_1)_+}}(l-j)2^{(l-j)(\alpha-n\delta_{12})}\right)^{(q_2^2)_l}
$$

$$
\leq C,
$$

where

$$
(q_2^1)_l=\begin{cases}(q_2)_- & \text{if } \left\|\frac{2^{l\alpha}\left|\sum_{j=l+3}^{\infty}[b,S_\beta](\hbar_j)\right|\chi_l}{\vartheta_{20}\|b\|_*}\right\|_{L^{p(\cdot)}}\leq 1,\\ (q_2)_+ & \text{otherwise.}\end{cases}
$$

$$\vartheta_{32} \le C\vartheta_{20} := C\|\hbar\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}$$

This completes the proof of Theorem 3. □

3.4. Vector-Valued Inequalities for Sublinear Operators

In this section, we shall establish the boundedness of vector-valued sublinear operators on $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$. The boundedness of this operator on the Herz spaces with one variable $p(\cdot)$ exponents is well known [42]. For further research about the sublinear operators on different spaces with variable exponents, we refer readers to [43–46] and so on.

Theorem 4. *Suppose that $1 < r < \infty, q_1(\cdot)$ and $q_2(\cdot) \in \mathcal{P}^*_0(\mathbb{R}^n)$ with $(q_1)_+ \le (q_2)_-$ and let $p(\cdot) \in \mathcal{B}^*(\mathbb{R}^n), \alpha \in (-n\delta_{11}, n\delta_{12})$, where $\delta_{11}, \delta_{12} \in (0,1)$ are constants stated in Lemma 2. If a sublinear operator Λ verifying the size condition*

$$|\Lambda(\hbar)(x)| \lesssim \int_{\mathbb{R}^n} |x-y|^{-n}|\hbar(y)|\mathrm{d}y, \qquad x \notin supp\,\hbar, \tag{17}$$

for any $\hbar \in L^1(\mathbb{R}^n)$ with compact support and the vector-valued inequality on $L^{p(\cdot)}$

$$\left\|\left(\sum_\imath |\Lambda(\hbar_\imath)|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}} \le C\left\|\left(\sum_\imath |\hbar_\imath|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}, \tag{18}$$

then, for any $\{\hbar_\imath\}_{\imath\in\mathbb{N}}$ satisfying $\left\|\|\{\hbar_\imath\}_\imath\|_{\ell^r}\right\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} < \infty$, the following vector-valued inequality holds:

$$\left\|\left(\sum_\imath |\Lambda(\hbar_\imath)|^r\right)^{\frac{1}{r}}\right\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}} \le C\left\|\left(\sum_\imath |\hbar_\imath|^r\right)^{\frac{1}{r}}\right\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}}.$$

Proof. For any $\{\hbar_\imath\}_{\imath\in\mathbb{N}}$ satisfies $\left\|\|\{\hbar_\imath\}_\imath\|_{\ell^r}\right\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}} < \infty$, we can write $\hbar_\imath = \sum_{j=-\infty}^{\infty} \hbar_\imath \chi_j = \sum_{j=-\infty}^{\infty} \hbar_\imath^j$. By the definition of $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$, we have

$$\left\|\|\{\Lambda(\hbar_\imath)\}_\imath\|_{\ell^r}\chi_l\right\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\vartheta > 0 : \sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\|\{\Lambda(\hbar_\imath)\}_\imath\|_{\ell^r}\chi_l}{\vartheta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \le 1\right\}$$

Since

$$\begin{aligned}
\left\|\left(\frac{2^{l\alpha}\|\{\Lambda(\hbar_\imath)\}_\imath\|_{\ell^r}\chi_l}{\vartheta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \le & \left\|\left(\frac{2^{l\alpha}\left\|\left\{\sum_{j=-\infty}^{\infty}\Lambda(\hbar_\imath^j)\right\}_\imath\right\|_{\ell^r}\chi_l}{\vartheta_{41}+\vartheta_{42}+\vartheta_{43}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
\le & C\left\|\left(\frac{2^{l\alpha}\left\|\left\{\sum_{j=-\infty}^{l-3}\Lambda(\hbar_\imath^j)\right\}_\imath\right\|_{\ell^r}\chi_l}{\vartheta_{41}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
& + C\left\|\left(\frac{2^{l\alpha}\left\|\left\{\sum_{j=l-2}^{l+2}\Lambda(\hbar_\imath^j)\right\}_\imath\right\|_{\ell^r}\chi_l}{\vartheta_{42}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}
\end{aligned}$$

$$+C\left\|\left(\frac{2^{l\alpha}\left\|\left\{\sum_{j=l+3}^{\infty}\Lambda(\hbar_\imath^j)\right\}_\imath\right\|_{\ell^r}\chi_l}{\vartheta_{43}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}},$$

here

$$\vartheta_{41}=\left\|\left\{2^{l\alpha}\left\|\left\{\sum_{j=-\infty}^{l-3}\Lambda(\hbar_\imath^j)\right\}_\imath\right\|_{\ell^r}\chi_l\right\}_{l=-\infty}^{\infty}\right\|_{\ell^{q_2(\cdot)}(L^{p(\cdot)})},$$

$$\vartheta_{42}=\left\|\left\{2^{l\alpha}\left\|\left\{\sum_{j=l-2}^{l+2}\Lambda(\hbar_\imath^j)\right\}_\imath\right\|_{\ell^r}\chi_l\right\}_{l=-\infty}^{\infty}\right\|_{\ell^{q_2(\cdot)}(L^{p(\cdot)})},$$

$$\vartheta_{43}=\left\|\left\{2^{l\alpha}\left\|\left\{\sum_{j=l+2}^{\infty}\Lambda(\hbar_\imath^j)\right\}_\imath\right\|_{\ell^r}\chi_l\right\}_{l=-\infty}^{\infty}\right\|_{\ell^{q_2(\cdot)}(L^{p(\cdot)})},$$

and $\vartheta:=(\vartheta_{41}+\vartheta_{42}+\vartheta_{43}):=\sum_{\varrho=1}^{3}\vartheta_{4\varrho}$, thus,

$$\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\|\{\Lambda(\hbar_\imath)\}_\imath\|_{\ell^r}\chi_l}{\vartheta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\leq C.$$

Then,

$$\big\|\|\{\Lambda(\hbar_\imath)\}_\imath\|_{\ell^r}\big\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}\leq C\vartheta:=C\left(\sum_{\varrho=1}^{3}\vartheta_{4\varrho}\right).$$

Hence, if we establish that

$$\vartheta_{41}\leq C\big\|\|\{\hbar_\imath\}_\imath\|_{\ell^r}\big\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)},\ \vartheta_{42}\leq C\big\|\|\{\hbar_\imath\}_\imath\|_{\ell^r}\big\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)},\ \vartheta_{43}\leq C\big\|\|\{\hbar_\imath\}_\imath\|_{\ell^r}\big\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)},$$

we are done. Let $\vartheta_0=\big\|\|\{\hbar_\imath\}_\imath\|_{\ell^r}\big\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}$.

First, we consider ϑ_{42}. Since Λ satisfies (18), then by using the same approach as ϑ_{12} in the proof of Theorem 1, we can easily obtain

$$\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\left\|\left\{\sum_{j=l-2}^{l+2}\Lambda(\hbar_\imath^j)\right\}_\imath\right\|_{\ell^r}\chi_l}{\vartheta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\leq C.$$

That is, $\vartheta_{42}\leq C\vartheta_0:=C\big\|\|\{\hbar_\imath\}_\imath\|_{\ell^r}\big\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}$.

Now, we estimate ϑ_{42}. For all $l\in\mathbb{Z}, j\leq l-3$ and $x\in\Re_l$, since Λ satisfies (17), by applying Minkowski's inequality and the inequality (4), it follows that

$$\begin{aligned}\left\|\left\{\Lambda\left(\hbar_\imath^j\right)(x)\right\}_\imath\right\|_{\ell^r}\cdot\chi_l&\lesssim\left\|\left\{\int_{\Re_j}|\hbar_\imath(y)||x-y|^{-n}\mathrm{d}y\right\}_\imath\right\|_{\ell^r}\cdot\chi_l\\&\lesssim 2^{-nl}\left\|\left\{\int_{\Re_j}|\hbar_\imath(y)|dy\right\}_\imath\right\|_{\ell^r}\cdot\chi_l\\&\lesssim 2^{-nl}\left\|\left\|\left\{\hbar_\imath^j\right\}_\imath\right\|_{\ell^r}\right\|_{L^{p(\cdot)}}\|\chi_j\|_{L^{p'(\cdot)}}\cdot\chi_l.\end{aligned}$$

From Lemmas 1–3, it follows that

$$\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\,\|\sum_{j=-\infty}^{l-3}\left\{\Lambda(\hbar_\imath^j)\right\}_\imath\|_{\ell^r}\chi_l}{\vartheta_0}\right)^{q_2(\cdot)}\right\|$$
$$\leq C\sum_{l=-\infty}^{\infty}\left(\sum_{j=-\infty}^{l-3}\left\|\left\|\left\{\frac{\hbar_\imath^j}{\vartheta_0}\right\}_\imath\right\|_{\ell^r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}2^{l\alpha}\frac{\|\chi_j\|_{L^{p'(\cdot)}}}{\|\chi_l\|_{L^{p'(\cdot)}}}\frac{|B_l|}{2^{nl}}\right)^{(q_2^1)_l}$$
$$\leq C\sum_{l=-\infty}^{\infty}\left\{\sum_{j=-\infty}^{l-3}\left\|\left(\frac{2^{j\alpha}\|\{\hbar_\imath\}_\imath\|_{\ell^r}\chi_j}{\vartheta_0}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{\frac{1}{(q_1)_+}}2^{(l-j)(\alpha-\delta_{12})}\right\}^{(q_2^1)_l},$$

where

$$(q_2^1)_l=\begin{cases}(q_2)_- & \text{if } \left\|\frac{2^{l\alpha}\left|\sum_{j=-\infty}^{l-3}\Lambda(\hbar_j)\right|\chi_l}{\vartheta_{20}}\right\|_{L^{p(\cdot)}}\leq 1,\\ (q_2)_+ & \text{otherwise.}\end{cases}$$

Moreover, using the same approach as ϑ_{11} in the proof of Theorem 1, we find

$$\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\,\|\sum_{j=-\infty}^{l-3}\left\{\Lambda(\hbar_\imath^j)\right\}_\imath\|_{\ell^r}\chi_l}{\vartheta_0}\right)^{q_2(\cdot)}\right\|\leq C.$$

Finally, we estimate ϑ_{43}. For all $l\in\mathbb{Z}$, $j\leq l-3$ and $x\in\Re_l$, since Λ satisfies (17), by applying Minkowski's inequality and the inequality (4), it follows that

$$\begin{aligned}\left\|\left\{\Lambda\left(\hbar_\imath^j\right)(x)\right\}_\imath\right\|_{\ell^r}\cdot\chi_l &\lesssim \left\|\left\{\int_{\Re_j}|\hbar_\imath(y)||x-y|^{-n}\mathrm{d}y\right\}_\imath\right\|_{\ell^r}\cdot\chi_l\\ &\lesssim 2^{-nj}\left\|\left\{\int_{\Re_j}|\hbar_\imath(y)|dy\right\}_\imath\right\|_{\ell^r}\cdot\chi_l\\ &\lesssim 2^{-nj}\left\|\left\|\left\{\hbar_\imath^j\right\}_\imath\right\|_{\ell^r}\right\|_{L^{p(\cdot)}}\|\chi_j\|_{L^{p'(\cdot)}}\cdot\chi_l.\end{aligned}$$

Thus, when $\alpha>-n\delta_{11}$, similar to the approach taken to estimate ϑ_{41} before, we can obtain

$$\sum_{l=-\infty}^{\infty}\left\|\left(\frac{2^{l\alpha}\,\|\sum_{j=l+3}^{\infty}\left\{\Lambda(\hbar_\imath^j)\right\}_\imath\|_{\ell^r}\chi_l}{\vartheta_0}\right)^{q_2(\cdot)}\right\|$$
$$\leq C\sum_{l=-\infty}^{\infty}\left\|\frac{2^{l\alpha}\,\|\sum_{j=l+3}^{\infty}\left\{\Lambda(\hbar_\imath^j)\right\}_\imath\|_{\ell^r}\chi_l}{\vartheta_0}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}^{(q_2^2)_l}$$
$$\leq C\sum_{l=-\infty}^{\infty}\left\{\sum_{j=l+3}^{\infty}\left\|\left\|\left\{\frac{\hbar_\imath^j}{\vartheta_0}\right\}_\imath\right\|_{\ell^r}\right\|_{L^{p(\cdot)}}2^{l\alpha}\frac{\|\chi_l\|_{L^{p(\cdot)}}}{\|\chi_j\|_{L^{p(\cdot)}}}\frac{|B_j|}{2^{nj}}\right\}^{(q_2^2)_l}$$
$$\leq C\sum_{l=-\infty}^{\infty}\left\{\sum_{j=l+3}^{\infty}\left\|\left(\frac{2^{j\alpha}\|\{\hbar_\imath\}_\imath\|_{\ell^r}\chi_j}{\vartheta_0}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{\frac{1}{(q_1)_+}}2^{(l-j)(\alpha+\delta_{11})}\right\}^{(q_2^2)_l}$$
$$\leq C.$$

where

$$(q_2^2)_l = \begin{cases} (q_2)_- & \text{if } \left\| \frac{2^{l\alpha} \left\| \sum_{j=l+3}^{\infty} \left\{ \Lambda(\hbar_\imath^j) \right\}_\imath \right\|_{\ell^r} \chi_l}{\vartheta_0} \right\|_{L^{p(\cdot)}} \le 1, \\ (q_2)_+ & \text{otherwise.} \end{cases}$$

This completes the proof of Theorem 4. □

Remark 2. *(1) It is worth pointing out that even in the particular case that $q(\cdot)$ is constant, our main results, namely Theorems 1, and 2, are also new.*

(2) All of these results can also be proved for non-homogeneous Herz spaces with two variable exponents. Due to the similarity of the arguments, details are omitted.

4. Conclusions

We obtain the boundedness of the vector-valued intrinsic square function and the boundedness of the scalar-valued intrinsic square function on Herz spaces with two variable exponents. The boundedness of the corresponding commutators generated by a BMO function and scalar-valued intrinsic square function is also obtained. As a supplement, the vector-valued inequality for sublinear operators is obtained on $\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)$.

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