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On Representing Strain Gradient Elastic Solutions of Boundary Value Problems by Encompassing the Classical Elastic Solution

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Abstract: The present work aims to primarily provide a general representation of the solution of the simplified elastostatics version of Mindlin's Form II first-strain gradient elastic theory, which converges to the solution of the corresponding classical elastic boundary value problem as the intrinsic gradient parameters become zero. Through functional theory considerations, a solution representation of the one-intrinsic-parameter strain gradient elastostatic equation that comprises the classical elastic solution of the corresponding boundary value problem is rigorously provided for the first time. Next, that solution representation is employed to give an answer to contradictions arising by two well-known first-strain gradient elastic models proposed in the literature to describe the strain gradient elastostatic bending behavior of Bernoulli–Euler beams.

Keywords: strain gradient elastic theory; general solution representation; Bernoulli–Euler beam; material with microstructure

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1. Introduction

It is well known that a structure consisting of a linear, isotropic, classical elastic material, subjected to external time-invariant boundary conditions, behaves according to the Navier–Cauchy equilibrium equation [1–3], which in terms of displacements $\mathbf{u}^{classical}$ and without the presence of body forces reads:

$$\mu\nabla^2\mathbf{u}^{classical} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u}^{classical} = 0, \tag{1}$$

where ∇ is the gradient operator, while λ and μ indicate the Lamé constants.

Despite the discrete nature of real materials, the continuum theory of classical elasticity—described by (1) for the elastostatics case—is deduced by considering that the dimensions of the material microstructure are much smaller than a material representative volume element (RVE), which in turn is much smaller than any dimension of the loaded structure. Additionally, the material properties and the generated elastic fields in the RVE are projected, by averaging, around a point $\mathbf{x}$ lying at the center of the RVE. That projection imposes the local nature of the classical theory of elasticity and requires that displacements, stresses, and strains vary constantly or linearly throughout the material RVE [4,5]. Obviously, the situation becomes problematic when the material inhomogeneity is comparable with the size of the structure and the averaging performed in the RVE requires the consideration of strain gradients in the potential energy density and the introduction of internal length scale parameters, which are able to capture size effect phenomena.

At the beginning of 20th century, the Cosserat brothers [6] proposed the idea of an enhanced elastic theory in which, except strains and stresses, the gradient of rotations and the dual in energy, couple stresses should be considered. Their idea reached maturity almost fifty years later with the general works of Toupin [7], Mindlin and Tiersten [8], Green and Rivlin [9] and Koiter [10]. Meanwhile Mindlin [11], while investigating the influence of couple stresses to stress concentrations mentions that "*Also, it would seem to be desirable to explore the consequences of taking into account the remaining components of the strain gradient and, perhaps, second and higher gradients of the strain*". Indeed, one year later, Mindlin [12] published his general dynamic theory for elastic materials with microstructure, where the microstructure is considered as an additional micro-continuum embedded at every point of the macro-continuum, thus justifying, the presence of higher order strain gradients in the expressions of potential energy density. Ignoring inertia terms, the couple stress theory of Toupin [7], Mindlin's elastic theory with microstructure [12] and later the virtual power theory of Germain [13] lead, for a material with microstructural effects, to the same equilibrium equation and boundary conditions. However, the elastostatic Form II version of Mindlin's theory has the very attractive characteristic of retaining the symmetry of the considered total stress tensor as in the classical elasticity case, and concludes in a simple equilibrium equation of the following form:

$$(\lambda+2\mu)(1-l_1^2\nabla^2)\nabla\nabla\cdot\mathbf{u}^{gradient}-\mu(1-l_2^2\nabla^2)\nabla\times\nabla\times\mathbf{u}^{gradient}=0, \tag{2}$$

where l_1^2, l_2^2 are internal length scale parameters that can facilitate microstructural effects for dilatational and shear deformations, respectively.

Considering uniform microstructural effects for all types of deformation, i.e., $l_1^2=l_2^2=g$, Equation (2) can be further simplified obtaining the form:

$$\left(1-g^2\nabla^2\right)\left[\mu\nabla^2\mathbf{u}^{gradient}+(\lambda+\mu)\nabla\nabla\cdot\mathbf{u}^{gradient}\right]=0 \tag{3}$$

Equation (3) and the corresponding boundary conditions of Mindlin's Form II theory consist of an attractive enhanced elastic theory with a microstructure, known as strain gradient elasticity (SGE) or dipolar gradient elasticity (DGE), since it employs only one internal length scale parameter in addition to the two classical Lamé constants, and the most important strains and stresses, appearing in its constitutive equations, are symmetric as in the classical elastic case. During the last thirty years, many authors exploited the simplicity of SGE to solve analytically and/or numerically elastostatic problems with microstructural effects in many fields of linear elastic continuum mechanics, such as fracture and dislocations mechanics [14–20] and structural and material response [21–30], while interesting remarks on SGE can be found in [5,31–34].

Mindlin [11] first proposed a solution representation of Equation (2) based on Papkovich–Neuber type vector and scalar potentials $\mathbf{B}$, B_0 [3,35], which, in the case of Equation (3), is simplified to [30]:

$$\begin{aligned}
&\mathbf{u}^{gradient}=\mathbf{B}-\tfrac{\lambda+\mu}{2(\lambda+2\mu)}\left[\mathbf{r}\cdot\left(1-g^2\nabla^2\right)\mathbf{B}+B_0\right]\\
&\left(1-g^2\nabla^2\right)\nabla^2\mathbf{B}=0\\
&\left(1-g^2\nabla^2\right)\nabla^2 B_0=0.
\end{aligned} \tag{4}$$

Instead of (4), Charalambopoulos and Polyzos [26] utilized the following representation of the solution of (3):

$$\begin{aligned}
&\mathbf{u}^{gradient}=\mathbf{u}^e+\mathbf{u}^g\\
&\mu\nabla^2\mathbf{u}^e+(\lambda+\mu)\nabla\nabla\cdot\mathbf{u}^e=0\\
&\left(1-g^2\nabla^2\right)\mathbf{u}^g=0.
\end{aligned} \tag{5}$$

without providing any correlation of (5) with the solution representation (4). The proof of the decomposition (5) is provided in this paper through the proof of theorem 1 in Section 2 and via Papkovich–Neuber type potentials in Appendix A. It should be mentioned at this point that the Papkovich–Neuber type gradient elastic solution (4) agrees with the corresponding one provided by Solyaev et al. [36,37], while the solution decomposition (5) is in agreement with the decomposition proposed by Lazar [38] if one considers that for a classical elastic solution $\mathbf{u}^e$ satisfying Equation (1), the vector function $\left(1 - g^2\nabla^2\right)\mathbf{u}^e$ remains a classical elastic solution. In [34], Gourgiotis et al. solve a sharp notch problem in microstructured solids utilizing a Knein–Williams technique, and mentioned that their asymptotic solution shows significant departure from those of classical elasticity. This statement is basically the motivation for the present work, which, among others, proposes a solution representation of (3) by comprising the solution of the corresponding classical elastic boundary value problem, which is absent in both representations (4) and (5).

The first attempt at incorporating the classical elastic solution in the solution of (3) is that of Ru and Aifantis [39]. More specifically, considering the elastic displacement field $\mathbf{u}^{classical}$ that satisfies (1) and the corresponding classical elastic boundary conditions in a domain Ω confined by a surface $\partial\Omega$, they proposed as strain gradient elastic solution of (3) the vector $\mathbf{u}^{gradient}$ that satisfies the non-homogeneous partial differential equation:

$$(1 - g^2\nabla^2)\mathbf{u}^{gradient} = \mathbf{u}^{classical}, \tag{6}$$

and the extra boundary condition

$$\frac{\partial^2}{\partial n^2}\mathbf{u}^{gradient} = 0, \tag{7}$$

where $\partial/\partial n$ denotes differentiation with respect to the unit normal vector of $\partial\Omega$.

Comparing (3) with (6), the obvious advantage of this representation is the reduction in the order of the partial differential equation of the problem by two. However, as it is mentioned in Charalambopoulos and Polyzos [26], the representation (6) and (7) is questionable for the following reasons: (i) the solution of the partial differential equation of second order—as Equation (6)—satisfies a boundary condition of second order as the condition (7) is in contradiction with the mathematically accepted condition that the maximal degree of boundary conditions never exceeds the crucial number $n - 1$, where n stands for the degree of the differential equation. (ii) The boundary condition (7) is arbitrary and not the outcome of a variational process. (iii) The classical solution $\mathbf{u}^{classical}$ satisfies the equilibrium Equation (3), but not Equation (6).

Charalambopoulos et al. [30] utilized a representation of the solution that satisfies Equation (3) and the corresponding Form II boundary conditions and comprises the corresponding classical elastic solution, without, however, providing any systematic proof of that representation. This is, among others, the goal of the present work. The possible application of the presented here methodology to Equation (1) and to equations describing the behavior of linear pantographic sheets [40] or obeying to the generalized Hook's law for isotropic second gradient materials [41] will be the subject of future work. The structure of the present work is the following: The next section illustrates the Form II SGE theory of Mindlin with only one internal length scale parameter. Section 3 is entirely devoted to the mathematical establishment of a solution representation of (3), which encompasses the corresponding classical solution and its convergence behavior as the gradient parameter tends to zero. The same solution representation is exploited in Section 4 to show that the bending stiffness of a Form II SGE Bernoulli–Euler beam depends on the material rigidity EI and the internal length scale parameter g and not on EI, g plus the area of the cross-section of the beam.

2. Strain Gradient Elastostatics with One Internal Length Scale Parameter

The present section reports a boundary value problem in terms of the simplest possible strain gradient elastostatic theory with one intrinsic parameter. Mindlin [42] in the second version of his theory considered that the first gradient elastic potential energy density for an elastic body with microstructure is a quadratic form of the strains ε_{ij} and the gradient of strains κ_{ijk}. For isotropic materials, this theory provides a potential energy density containing the two Lamé material constants and five constants that normalize the terms of strain gradients, i.e.,

$$\begin{aligned}
&U = \tfrac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + \hat{\alpha}_1\kappa_{iik}\kappa_{kjj} + \hat{\alpha}_2\kappa_{ijj}\kappa_{ikk} + \hat{\alpha}_3\kappa_{iik}\kappa_{jjk} \\
&+\hat{\alpha}_4\kappa_{ijk}\kappa_{ijk} + \hat{\alpha}_5\kappa_{ijk}\kappa_{kji} \\
&\varepsilon_{ij} = \tfrac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) = \varepsilon_{ji} \\
&\kappa_{ijk} = \partial_i\varepsilon_{jk} = \tfrac{1}{2}\left(\partial_i\partial_j u_k + \partial_i\partial_k u_j\right) = \kappa_{ikj},
\end{aligned} \tag{8}$$

where ∂_i denotes spatial differentiation, u_i are the displacement components,λ, μ are the well-known Lamé constants having units N/m^2 and $\hat{\alpha}_1 \div \hat{\alpha}_5$ are five constants having units of force, all explicitly provided in [12].

For the case of $\hat{\alpha}_1 = \hat{\alpha}_3 = \hat{\alpha}_5 = 0$ and $\hat{\alpha}_2 = \lambda g^2$, $\hat{\alpha}_4 = \mu g^2$, the potential energy density U obtains the form

$$\begin{aligned}
U = &\tfrac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + \tfrac{1}{2}\lambda g^2\kappa_{ijj}\kappa_{ikk} \\
&+ \mu g^2\kappa_{ijk}\kappa_{ijk},
\end{aligned} \tag{9}$$

where g is the only internal length scale parameter that correlates the microstructure with macrostructure, having units of length (m).

Strains and gradient of strains are dual in energy with the Cauchy-like stresses and double stresses, respectively, defined as:

$$\tau_{ij} = \frac{\partial W}{\partial\varepsilon_{ij}} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}, \tag{10}$$

and

$$\mu_{ijk} = \frac{\partial W}{\partial\kappa_{ijk}} = g^2\partial_i\tau_{jk} \tag{11}$$

If the Young modulus E and Poisson ratio v are used instead of the Lame constants λ, μ, then the replacements $\lambda = \frac{Ev}{(1+v)(1-2v)}$, $\mu = \frac{E}{2(1+v)}$ should be made.

Considering a material with a microstructure of volume V and external boundary S, the variation of the total potential energy (9) provides, after some algebra [12,43,44], the following relation:

$$\begin{aligned}
&\int_V \delta U dV = -\int_V \left[\partial_j\left(\tau_{jk} - \partial_i\mu_{ijk}\right)\right]\delta u_k dV + \int_S R_k D\delta u_k dS \\
&+ \int_{S_1\cup S_2} p_k\delta u_k dS + \langle E_k\delta u_k\rangle,
\end{aligned} \tag{12}$$

where the vectors p_k, R_k represent the traction and double traction vectors, respectively, defined on the boundary S and written as

$$p_k = n_j\left(\tau_{jk} - \partial_i\mu_{ijk}\right) - D_j\left(n_i\mu_{ijk}\right) + (D_m n_m)n_i n_j\mu_{ijk}, \tag{13}$$

and

$$R_k = n_i n_j\mu_{ijk}, \tag{14}$$

where D_j and D represent the tangential and normal gradient operators on S, respectively, and have the form

$$\begin{aligned}
&D_j = (\delta_{jm} - n_j n_m)\partial_m \\
&D = n_m\partial_m.
\end{aligned} \tag{15}$$

The vector E_k in Equation (12) concerns non-smooth boundaries with at least one corner c in two dimensions or at least one closed edge line ℓ in three dimensions, admitting the form:

$$\langle E_k \delta u_k \rangle = \begin{cases} \|n_i m_j \mu_{ijk}\|_{corner\,c} \delta u_k & for\, 2D \\ \oint\limits_{\ell} \|n_i m_j \mu_{ijk}\|_{edge\,\ell} \delta u_k d\ell & for\, 3D \end{cases}, \quad (16)$$

where $\|\bullet\|$ denotes the difference of $\bullet$ at both sides of the corner c or the edge ℓ, while m_j stands for the tangential vector in both sides of a corner or edge.

Equilibrating (12) with the variation of the work performed by external body force F_k, boundary tractions $\overline{p}_k$, double traction $\overline{R}_k$ and jump traction $\overline{E}_k$, we arrive at the following equilibrium equation:

$$\partial_j(\tau_{jk} - \partial_i \mu_{ijk}) + F_k = 0, \quad (17)$$

accompanied by the classical essential and natural boundary conditions where the displacement vector u_k and/or the traction vector p_k must be defined on the global boundary $S \equiv S_1 \cup S_2$, i.e.,

$$\begin{array}{l} u_k(\mathbf{x}) = \overline{u}_k(\mathbf{x}), \quad \mathbf{x} \in S_1 \\ p_k(\mathbf{x}) = \overline{p}_k(\mathbf{x}), \quad \mathbf{x} \in S_2, \end{array} \quad (18)$$

and the non-classical essential and natural boundary conditions where the normal displacement vector $q_k = Du_k$, the double traction vector R_k or jump traction E_k are prescribed on $S \equiv S_3 \cup S_4$, i.e.,

$$\begin{array}{l} q_k(\mathbf{x}) = \overline{q}_k(\mathbf{x}), \quad \mathbf{x} \in S_3 \\ R_k(\mathbf{x}) = \overline{R}_k(\mathbf{x}), \quad \mathbf{x} \in S_4 \\ E_k(\mathbf{x}) = \overline{E}_k(\mathbf{x}), \quad \mathbf{x} \in \text{corner or edge}. \end{array} \quad (19)$$

In terms of the displacement vector $\mathbf{u}(\mathbf{x})$ and free of body forces, Equation (17) obtains the form

$$\left(1 - g^2\nabla^2\right)\left[\mu\nabla^2\mathbf{u}(\mathbf{x}) + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u}(\mathbf{x})\right] = 0. \quad (20)$$

Theorem 1. *The solution of Equation (20) can be written as*

$$\mathbf{u} \equiv \mathbf{u}^{gradient} = \mathbf{u}^{classical} + \mathbf{u}^{g}, \quad (21)$$

with $\mathbf{u}^{classical} \in \ker\left(\mu\nabla^2 + (\lambda+\mu)\nabla\nabla\cdot\right)$ *and* $\mathbf{u}^{g} \in \ker\left(1 - g^2\nabla^2\right)$.

Proof. We denote as Δ^* the classical elastic elliptic differential operator $\mu\Delta + (\lambda+\mu)\nabla(\nabla\cdot)$ Given that $\left(1 - g^2\Delta\right)\Delta^*\mathbf{u}^{gradient} = \Delta^*\left(1 - g^2\Delta\right)\mathbf{u}^{gradient} = 0$, we infer that

$$\Delta^*\mathbf{u}^{gradient} = \mathbf{w}^{g} \in \ker\left(1 - g^2\Delta\right), \quad (22)$$

and

$$\left(1 - g^2\Delta\right)\mathbf{u}^{gradient} = \mathbf{w}^{e} \in \ker(\Delta^*). \quad (23)$$

Consequently,

$$\left(1 - g^2\Delta\right)\mathbf{u}^{gradient} = \mathbf{w}^{e} \Rightarrow \mu\Delta\mathbf{u}^{gradient} = \frac{\mu}{g^2}\mathbf{u}^{gradient} - \frac{\mu}{g^2}\mathbf{w}^{e}. \quad (24)$$

Equations (22) and (24) imply that

$$\begin{array}{l} \frac{\mu}{g^2}\mathbf{u}^{gradient} - \frac{\mu}{g^2}\mathbf{w}^{e} + (\lambda+\mu)\nabla\left(\nabla\cdot\mathbf{u}^{gradient}\right) = \mathbf{w}^{g} \Rightarrow \\ \mathbf{u}^{gradient} = -\frac{\lambda+\mu}{\mu} g^2 \nabla\left(\nabla\cdot\mathbf{u}^{gradient}\right) + \mathbf{w}^{e} + \frac{g^2}{\mu}\mathbf{w}^{g}. \end{array} \quad (25)$$

We consider the Helmholtz decomposition of the field $\mathbf{w}^g : \mathbf{w}^g = \nabla H + \nabla \times \mathbf{K}$. Then, Equation (22) leads to

$$\begin{aligned}
&\nabla \cdot \left[\mu \Delta \mathbf{u}^{gradient} + (\lambda+\mu)\nabla\left(\nabla \cdot \mathbf{u}^{gradient}\right)\right] = \nabla \cdot (\nabla H + \nabla \times \mathbf{K}) \Rightarrow \\
&\Delta\left[(\lambda+2\mu)\nabla \cdot \mathbf{u}^{gradient} - H\right] = 0 \Rightarrow \\
&(\lambda+2\mu)\nabla \cdot \mathbf{u}^{gradient} = H + B_0,
\end{aligned} \tag{26}$$

where B_0, is a harmonic function. Then, Equations (25) and (26) give

$$\mathbf{u}^{gradient} = -\frac{\lambda+\mu}{\mu(\lambda+2\mu)} g^2 \nabla(H+B_0) + \mathbf{w}^e + \frac{g^2}{\mu}\mathbf{w}^g. \tag{27}$$

Additionally, it holds that $\Delta H = \nabla \cdot \mathbf{w}^g \in \ker\left(1-g^2\Delta\right)$ and therefore $\Delta\left[\left(1-g^2\Delta\right)H\right] = 0 \Rightarrow \left(1-g^2\Delta\right)H = B_1$, where B_1 is another harmonic function. One partial solution of this equation is clearly exactly the function B_1 and so the general solution is $H = u^g + B_1$, with $u^g \in \ker\left(1-g^2\Delta\right)$. Then, (27) is rewritten as

$$\begin{aligned}
\mathbf{u}^{gradient} &= -\tfrac{\lambda+\mu}{\mu(\lambda+2\mu)} g^2 \nabla(B_0+B_1) + \mathbf{w}^e \\
&+ \tfrac{g^2}{\mu}\mathbf{w}^g - \tfrac{\lambda+\mu}{\mu(\lambda+2\mu)} g^2 \nabla u^g.
\end{aligned} \tag{28}$$

The first two terms of the decomposition (28) form the classical part $\mathbf{u}^e \in \ker\Delta^*$ while the remaining terms form the component $\mathbf{u}^g$ obeying to the homogeneous modified Helmholtz equation. □

3. On Representing Strain Gradient Elastic Solutions via the Solution of the Corresponding Classical Elastic Boundary Value Problem

All the representations of the solution of a strain gradient elastostatic problem appearing in the literature, do not include as a constituent the respective classical elastic solution, while the convergence behavior of these solutions when the gradient parameters fade away is not studied. As in [45,46], in the following, we present a general result considering all the above concerns.

Consider a bounded open region $\Omega \subset R^d$, $d = 2, 3$, with its boundary $\partial\Omega$ being a Lipschitz surface. Assuming that the body forces are absent, we consider the solution of the boundary value problem consisting of the fourth order partial differential Equation (20) and a set of classical and non-classical boundary conditions as those illustrated in Section 2. More precisely, we partition the surface $\partial\Omega$ twice, first in two subdomains $\partial\Omega_D$ and $\partial\Omega_N$ (with $meas(\partial\Omega_D) > 0$), where classical conditions are imposed, and secondly, in the subdomains $\partial\Omega_Q$ and $\partial\Omega_R$, whose common boundary Γ is of dimension $(d-2)$ and represents the corners c or the edges l of the surface $\partial\Omega$. Then, the general set of mixed-type classical conditions can be formulated as

$$\begin{aligned}
\mathbf{u}(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \mathbf{x} \in \partial\Omega_D \ (a) \\
\mathbf{P}(\mathbf{x}) = \mathbf{g}(\mathbf{x}), \mathbf{x} \in \partial\Omega_N \ (b),
\end{aligned} \tag{29}$$

along with the set of non-classical conditions

$$\begin{aligned}
\tfrac{\partial \mathbf{u}(\mathbf{x})}{\partial n} = \mathbf{h}(\mathbf{x}; g),\ \mathbf{x} \in \partial\Omega_Q \ (a) \\
\mathbf{R}(\mathbf{x}) = \mathbf{r}(\mathbf{x}; g),\ \mathbf{x} \in \partial\Omega_R \ (b) \\
\mathbf{E}(\mathbf{x}) = s(\mathbf{x}; g),\ \mathbf{x} \in \Gamma \ (c),
\end{aligned} \tag{30}$$

where $\mathbf{P} = P_i\hat{\mathbf{x}}_i$, $\mathbf{R} = R_i\hat{\mathbf{x}}_i$ and $\mathbf{E} = E_i\hat{\mathbf{x}}_i$.

The solution $\mathbf{u}(\mathbf{x})$ and the fields $\mathbf{P}(\mathbf{x})$, $\mathbf{R}(\mathbf{x})$, $\mathbf{E}(\mathbf{x})$ depend of course on the parameter g, but this is omitted for simplicity and will be notified only when it is necessary.

It is assumed that the given functions $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$, $\mathbf{r}$ and $\mathbf{s}$ share all the required regularity for the well-posedness of the traces of the solution of Equation (20)—and its derivatives—on $\partial\Omega$. To clarify rigorously the last remark, we could additionally invoke the functional theoretic framework of variational problems settled in Sobolev spaces. This approach is described extensively in Wloka [47] for the general case of elliptic boundary value problems of a higher order and is very profitable since it provides results with crucial influence in the solvability of the problem under consideration with existence, uniqueness, and stability. It is out of the scope of the present work to give an extensive investigation via the aforementioned alternative framework, but it would be very helpful to give some brief concepts, facilitating the comprehension of the inner structure of the boundary value problems under investigation. Then, we recall the Sobolev spaces $H^s(\Omega)$ and $H^s(\partial\Omega)$, which are complete Hilbert spaces built by functions (or distributions) with specific integration behavior over Ω and $\partial\Omega$. Among the more recognizable Sobolev spaces, we encounter the space of square integrable measurable functions $H^0(\Omega) = L^2(\Omega)$ and its subspace $H^1(\Omega)$, whose elements possess distributional derivatives with the same square integrable behavior. Every Hilbert space $H^m(\Omega)$, with $m \in \mathrm{N}$, has a usual inner product and an induced norm consisting of the L^2 norms of the derivatives up to degree m. Therefore, the inclusion $H^{m_1}(\Omega) \subset H^{m_2}(\Omega)$ when $m_1 > m_2$ is an obvious relation. For real and positive order s, the norm is defined in a more complicated manner, but generalizes naturally what happens for an integer order. The linear spaces inclusion above still holds. When $s \geq 0$, the elements of the space $H^s(\Omega)$ belong to $L^2(\Omega)$, but this is not the case when $s < 0$ and constitute then pure distributions without functional representative.

Although not necessarily classical functions, the elements of $H^s(\Omega)$ have trace on the boundary $\partial\Omega$, which are generalized functions belonging to $H^{s-\frac{1}{2}}(\partial\Omega)$. The distributional surface normal derivative (of order k) of an element in $H^s(\Omega)$ belongs to $H^{s-k-\frac{1}{2}}(\partial\Omega)$ in the case that the smoothness of the surface allows the induced differentiability. The space $H_0^s(\Omega)$ is a subspace of $H^s(\Omega)$, with elements that, along with all their normal derivatives of order less than s, have zero traces on $\partial\Omega$.

Notice at this point that, for the case of the closed surface $\partial\Omega$, the space $H^{-s}(\partial\Omega)$ is the dual space of $H^s(\partial\Omega)$ and so surface terms of the form $\langle h, f\rangle|_{\partial\Omega}$ naturally arise, where $h \in H^{-s}(\partial\Omega)$ and $f \in H^s(\partial\Omega)$. This term expresses the action of h on f and it is very reminiscent of the virtual work of a force over a displacement. Only when $s = 0$, we have the reduction in the dual pairing $\langle h, f\rangle|_{\partial\Omega}$ to the usual inner product $\int_{\partial\Omega} h(\mathbf{x}) f(\mathbf{x}) dS_{\mathbf{x}}$.

After this brief discussion, we notice that in the framework of a boundary value problem whose differential equation is of order $2m = 4$, the boundary Equations (29) and (30) involve $m(= 2)$ boundary operators (of possibly mixed type) (when $\Gamma = \varnothing$). These operators are characterized by their own orders, which obey to the rule: $0 \leq m_j \leq 2m - 1$. If, for example, we had exactly the boundary conditions (29) and (30) valid on the whole surface $\partial\Omega$ (with $\Gamma = \varnothing$), the involved boundary operators B_j, $j = 1, 2$ would be $B_1 = I$ (with order $m_1 = 0$) and $B_2 = \partial/\partial n$ (with order $m_2 = 1$). This example corresponds to the so called Dirichlet boundary value problem of fourth order. This settlement lies totally in the general framework of the abstract theory of elliptic boundary value problems: The main outcome is the existence and uniqueness of the solution of the problem (20), (29) and (30) with the following regularity general result [47]

$$\begin{aligned}\|\mathbf{u}\|_{H^s(\Omega)} \leq C_g \Bigg(&\|\mathbf{f}\|_{H^{s-m_1-\frac{1}{2}}(\partial\Omega_D)} + \|\mathbf{g}\|_{H^{s-m_2-\frac{1}{2}}(\partial\Omega_N)} + \|\mathbf{h}\|_{H^{s-m_3-\frac{1}{2}}(\partial\Omega_Q)} \\ &+ \|\mathbf{r}\|_{H^{s-m_4-\frac{1}{2}}(\partial\Omega_R)} + \|\mathbf{s}\|_{H^{(s-m_4-\frac{1}{2})-\frac{1}{2}}(\Gamma)} \Bigg),\end{aligned}$$

and so, assigning to boundary operators their exact orders, we obtain

$$\|\mathbf{u}\|_{H^s(\Omega)} \leq C_g\bigg(\|\mathbf{f}\|_{H^{s-\frac{1}{2}}(\partial\Omega_D)} + \|\mathbf{g}\|_{H^{s-\frac{7}{2}}(\partial\Omega_N)} + \|\mathbf{h}\|_{H^{s-\frac{3}{2}}(\partial\Omega_Q)} + \|\mathbf{r}\|_{H^{s-\frac{5}{2}}(\partial\Omega_R)} + \|\mathbf{s}\|_{H^{s-3}(\Gamma)}\bigg). \tag{31}$$

We would like to note the double consecutive dimension reduction $\Omega \to \partial\Omega \to \Gamma$, which takes place in the treatment of the fifth term of the r.h.s of the last equation. The regularity of the solutions depends on the regularity of the data in a specific manner. It is not explicitly apparent, but when the parameter s increases, the validity of Equation (31) passes through further assumptions for additional smoothness of the boundary $\partial\Omega$. The most encountered evocation of the representation (31) is with the selection $s = 2m = 4$ (the order of the differential equation). Therefore, for the solution $\mathbf{u}$ to have square integrable (in Ω) derivatives up to fourth order, all the data must belong to "smooth" Sobolev spaces of positive order. Then, the differential equation is satisfied in L^2-sense. However, this is accompanied with the hypothesis of a smooth $C^{1,1}$ boundary. To permit a boundary with corners or edges and to obtain the broader class of admissible solutions, it is preferable to work with $m = 2$. Then,

$$\|\mathbf{u}\|_{H^2(\Omega)} \leq C_g\bigg(\|\mathbf{f}\|_{H^{\frac{3}{2}}(\partial\Omega_D)} + \|\mathbf{g}\|_{H^{-\frac{3}{2}}(\partial\Omega_N)} + \|\mathbf{h}\|_{H^{\frac{1}{2}}(\partial\Omega_Q)} + \|\mathbf{r}\|_{H^{-\frac{1}{2}}(\partial\Omega_R)} + \|\mathbf{s}\|_{H^{-1}(\Gamma)}\bigg). \tag{32}$$

In this case, the data $\mathbf{f}$, $\mathbf{h}$ loose regularity, but still belong to the realm of square integrable functions. However, the data $\mathbf{g}$, $\mathbf{r}$, and $\mathbf{s}$ pertaining to stresses and jumps of double stresses over corners (edges) might become not square integrable distributions, ready to act—via the mentioned above dual pairings—on their reciprocal fields. In this point, we would like to say that our analysis has been facilitated from the fact that we are in absence of body forces. The differential equation is not any more valid classically, but in the distributional sense. It is noticeable that due to the special coercivity behavior of the bilinear form corresponding to the gradient elasticity operator, the generic constant C_g appearing in relation (32) cannot present worse behavior than the asymptotic convergence $O(g^{-2})$ for $g \to 0$. In addition, (32) implies a fortiori the boundedness

$$\|\mathbf{u}\|_{H^1(\Omega)} \leq C\bigg(\|\mathbf{f}\|_{H^{\frac{3}{2}}(\partial\Omega_D)} + \|\mathbf{g}\|_{H^{-\frac{3}{2}}(\partial\Omega_N)} + \|\mathbf{h}\|_{H^{\frac{1}{2}}(\partial\Omega_Q)} + \|\mathbf{r}\|_{H^{-\frac{1}{2}}(\partial\Omega_R)} + \|\mathbf{s}\|_{H^{-1}(\Gamma)}\bigg). \tag{33}$$

It is noticeable here that the constant C is independent of g since suppressing downwards the energy bilinear form and keeping only norms of first derivatives involve exclusively the Lamé constants of classical elasticity.

After the brief introduction of the functional theoretic setting, we are in position to present the two main accomplishments of the current work. First, we are going to present the construction of a very useful decomposition of the unique solution of the problem under consideration. In the sequel, we will state the necessary assumptions on the data so that this representation obtains a stable (with respect to g) behavior, incorporating appropriately the classical solution.

Consider the auxiliary second order classical boundary value problem (titled Problem I), satisfied by the solution $\mathbf{u}^{classical}(\mathbf{x})$:

$$\begin{aligned} &\Delta^*\mathbf{u}^{classical}(\mathbf{x}) = 0, \mathbf{x} \in \Omega \\ &\Delta^* \equiv \mu\nabla^2 + (\lambda+\mu)\nabla\nabla, \end{aligned} \tag{34}$$

$$\mathbf{u}^{classical}(\mathbf{x}) = \mathbf{f}(\mathbf{x}),\ \mathbf{x} \in \ \partial\Omega_D, \tag{35}$$

$$\mathbf{t}^{classical}(\mathbf{x}) = \mathbf{g}(\mathbf{x}),\ \mathbf{x} \in \ \partial\Omega_N, \tag{36}$$

where $\mathbf{t}^{classical} = \tau_{ij}\hat{\mathbf{x}}_i\hat{\mathbf{x}}_j \cdot \hat{\mathbf{n}}$, is the classical surface traction field.

The above defined problem involving Equations (34)–(36) used the classical data from the gradient problem and "ignores" the non-classical ones. We emphasize that in Problem I, the gradient boundary term $\mathbf{P}(\mathbf{x})$ offers its place to its classical counterpart $\mathbf{t}^{classical}(\mathbf{x})$. Applying the classical well-known framework of the preceding analysis concerning this time the traditional second order elliptic boundary value problems, we deduce easily that the unique classical solution satisfies the stability relation

$$\|\mathbf{u}^{classical}\|_{H^1(\Omega)} \leq C\left(\|\mathbf{f}\|_{H^{\frac{1}{2}}(\partial\Omega_D)} + \|\mathbf{g}\|_{H^{-\frac{1}{2}}(\partial\Omega_N)}\right).$$

It is worthwhile to mention here that the last equation defines exactly the needed regularity of the data $\mathbf{f}, \mathbf{g}$ for the stated stability to be guaranteed. However, the gradient problem has already assigned specific regularity assumptions on the data. Compromising the two groups of requirements, we deduce that $\mathbf{f} \in H^{\frac{3}{2}}(\partial\Omega_D) \subset H^{\frac{1}{2}}(\partial\Omega_D)$ as well as $\mathbf{g} \in H^{-\frac{1}{2}}(\partial\Omega_N) \subset H^{-\frac{3}{2}}(\partial\Omega_N)$.

We are in position to state the main representation theorem of this work. We set first, as induced by the discussion above, the broader possible space in which the data are permitted to belong:

$$(\mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{r}, \mathbf{s}) \in B = H^{\frac{3}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N) \times H^{\frac{1}{2}}\left(\partial\Omega_Q\right) \times H^{-\frac{1}{2}}(\partial\Omega_R) \times H^{-1}(\Gamma).$$

The following theorem holds:

Theorem 2. *Let the boundary value problem consist of Equations (20), (29) and (30) where the data* $(\mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{r}, \mathbf{s}) \in B$*. This problem is a well-posed fourth order elliptic boundary value problem with a unique solution. This solution can be represented as follows*

$$\begin{aligned} \mathbf{u}(\mathbf{x}; g) &= \mathbf{u}^{classical}(\mathbf{x}) + \mathbf{w}(\mathbf{x}; g),\ \mathbf{x} \in \Omega \\ \mathbf{w}(\mathbf{x}; g) &= \mathbf{u}^N(\mathbf{x}; g) + g^2\mathbf{u}^G(\mathbf{x}; g), \end{aligned} \tag{37}$$

where $\mathbf{u}^{classical}$*satisfies Problem I,* $\mathbf{u}^N$*satisfies the classical elastostatic equation and* $\mathbf{u}^G$*obeys to the modified Helmholtz equation:*

$$\mu\nabla^2\mathbf{u}^N(\mathbf{x}; g) + (\lambda + \mu)\nabla\nabla \cdot \mathbf{u}^N(\mathbf{x}; g) = 0, \quad \mathbf{x} \in \Omega, \tag{38}$$

$$(1 - g^2\nabla^2)\mathbf{u}^g(\mathbf{x}; g) = 0, \quad \mathbf{x} \in \Omega. \tag{39}$$

Proof. As explained above, the problem (20), (29) and (30) disposes a unique solution $\mathbf{u}(\mathbf{x}, g)$, which is stable with respect to the data as Equations (32) and (33) guarantee. We consider the decomposition

$$\mathbf{u}(\mathbf{x}; g) = \mathbf{u}^{classical}(\mathbf{x}) + \mathbf{w}(\mathbf{x}; g).$$

The function $\mathbf{w}(\mathbf{x}; g)$ satisfies Equation (20) given that both $\mathbf{u}^{classical}(\mathbf{x})$ and the field $\mathbf{u}(\mathbf{x}; g)$ obey to this equation too. Based on the Papkovich-type representation [26,30], the field $\mathbf{w}(\mathbf{x}; g)$ can be written as

$$\begin{aligned} \mathbf{w}(\mathbf{x}; g) = \mathbf{B} - \tfrac{\lambda+\mu}{2(\lambda+2\mu)}\{&(1 - g^2\nabla^2)\mathbf{B} + \\ &\left[(1 - g^2\nabla^2)\nabla\mathbf{B}\right] \cdot \mathbf{r} + \nabla B_0\}, \end{aligned} \tag{40}$$

where

$$\left(1 - g^2\nabla^2\right)\nabla^2\mathbf{B}(\mathbf{x};g) = 0, \tag{41}$$

$$\left(1 - g^2\nabla^2\right)\nabla^2 B_0(\mathbf{x};g) = 0. \tag{42}$$

Working with Equation (41), we have that necessarily $(1 - g^2\nabla^2)\mathbf{B}(\mathbf{x};g)$ is a harmonic function $\mathbf{B}^L(\mathbf{x};g)$. Then,

$$\mathbf{B}(\mathbf{x};g) = (1 - g^2\nabla^2)\mathbf{B}(\mathbf{x};g) + g^2\nabla^2\mathbf{B}(\mathbf{x};g) = \mathbf{B}^L(\mathbf{x};g) + g^2\mathbf{B}^G(\mathbf{x};g),$$

where $\mathbf{B}^G(\mathbf{x};g)$ satisfies the modified Helmholtz equation $(1 - g^2\nabla^2)\mathbf{B}^G(\mathbf{x};g) = 0$.

A similar treatment applies to Equation (42) and the result is that the fields of Equations (41) and (42) have the decomposed form

$$\begin{aligned} \mathbf{B}(\mathbf{x};g) &= \mathbf{B}^L(\mathbf{x};g) + g^2\mathbf{B}^G(\mathbf{x};g) \\ B_0(\mathbf{x};g) &= B_0^L(\mathbf{x};g) + g^2 B_0^G(\mathbf{x};g), \end{aligned} \tag{43}$$

$$\begin{aligned} \nabla^2\mathbf{B}^L(\mathbf{x};g) = 0, &\quad (1 - g^2\nabla^2)\mathbf{B}^G(\mathbf{x};g) = 0 \\ \nabla^2 B_0^L(\mathbf{x};g) = 0, &\quad (1 - g^2\nabla^2) B_0^G(\mathbf{x};g) = 0. \end{aligned} \tag{44}$$

Inserting these representations in Equation (40) leads to the decomposition

$$\mathbf{w}(\mathbf{x};g) = \mathbf{u}^N(\mathbf{x};g) + g^2\mathbf{u}^G(\mathbf{x};g), \tag{45}$$

where

$$\mathbf{u}^N = \frac{\lambda + 3\mu}{2(\lambda + 2\mu)}\mathbf{B}^L - \frac{\lambda + \mu}{2(\lambda + 2\mu)}\nabla\mathbf{B}^L \cdot \mathbf{r} - \frac{\lambda + \mu}{2(\lambda + 2\mu)}\nabla B_0^L, \tag{46}$$

and

$$\mathbf{u}^G = \mathbf{B}^G - \frac{\lambda + \mu}{2(\lambda + 2\mu)}\nabla B_0^G. \tag{47}$$

It is evident that $\mathbf{u}^G \in \ker(1 - g^2\nabla^2)$. In addition, it comes out easily that $\mathbf{u}^N$ belongs to the kernel of the operator ∇^2. Consequently, we deduce that

$$\begin{aligned} 0 &= (1 - g^2\nabla^2)\left[\mu\nabla^2\mathbf{w} + (\lambda + \mu)\nabla\nabla \cdot \mathbf{w}\right] \\ &= \mu\nabla^2\mathbf{u}^N + (\lambda + \mu)\nabla\nabla \cdot \mathbf{u}^N + (1 - g^2\nabla^2)\left[\mu\nabla^2\mathbf{u}^G + (\lambda + \mu)\nabla\nabla \cdot \mathbf{u}^G\right] \\ &= \mu\nabla^2\mathbf{u}^N + (\lambda + \mu)\nabla\nabla \cdot \mathbf{u}^N, \end{aligned}$$

from where we infer that $\mathbf{u}^N \in \ker\left[\mu\nabla^2 + (\lambda + \mu)\nabla\nabla\cdot\right]$. $\square$

As a conclusion, the decomposition (37) with the differential properties (38) and (39) was derived from the point of view of the underlying differential equations and always can be applied to the unique solution of the fourth order boundary value problem under discussion. The implication of boundary conditions is of course the next step. The representation (37) has the advantage that it disposes additional degrees of freedom since the involved "free" functions satisfy the second order differential equations in which the gradient elasticity law decomposes. In addition, the representation (37) involves the classical solution of the problem in the absence of the microstructure $g = 0$ as a cornerstone constituent. It is very interesting to examine whether taking the limit of the expression (37) as $g \to 0$ leads to the classical solution. What matters of course is primary the construction of the solution of the boundary value problem independently of its convergence behavior. Nevertheless, it is essential to construct sufficient conditions on the data assuring this desirable convergence property.

The next theorem verifies that under specific assumptions on the data, convergence is established. The first requirement is in accordance with the underlying constitutive equations concerning double stresses. Indeed, it is natural that the magnitude of the double stresses and the relevant jump fields imposed on the structure obey to specific order analysis with respect to the microstructure parameter. More precisely, the boundary tensors $\mathbf{R}(\mathbf{x})$ and $\mathbf{E}(\mathbf{x})$ are selected as follows:

$$\mathbf{r}(\mathbf{x};s) = g^2\widetilde{\mathbf{r}}(\mathbf{x}),\ \mathbf{x} \in\ \partial\Omega_R, \tag{48}$$

$$\mathbf{s}(\mathbf{x};g) = g^2\widetilde{\mathbf{s}}(\mathbf{x}),\ \mathbf{x} \in\ \Gamma, \tag{49}$$

and as stated, this choice has a physical origin. The additional requirements are related to the necessary regularity of the rest of the data. It is necessary, in the convergence setting for the boundary data $\mathbf{g}$, to be a genuine square integrable function, in fact an element of $H^{\frac{1}{2}}(\partial\Omega_N)$ and that $\|\mathbf{h}(\cdot,g)\|_{H^{\frac{1}{2}}(\partial\Omega_Q)}$ remains bounded as g varies. In practice, the term $\mathbf{h}(\mathbf{x},g)$ is usually independent of g.

As far as the stated above restriction of $\mathbf{g}$ is concerned, the concept stems from the treatment of the classical solution. As we will see, under this assumption, the function $\frac{\partial}{\partial n}\mathbf{u}^{classical}(\mathbf{x})$ becomes a square integrable function with a crucial role in the convergence analysis.

The following theorem holds.

Theorem 3. *Let the boundary value problem consist of Equations (20), (29) and (30). The data* $(\mathbf{f},\mathbf{g},\mathbf{h},\mathbf{r},\mathbf{s})$ *are elements of B with the restriction that* $\mathbf{r}$, $\mathbf{s}$ *obey to Equations (48) and (49),* $\mathbf{g} \in H^{\frac{1}{2}}(\partial\Omega_N) \subset H^{-\frac{1}{2}}(\partial\Omega_N)$ *and* $\|\mathbf{h}(\cdot,g)\|_{H^{\frac{1}{2}}(\partial\Omega_Q)}$ *is a bounded function of g. Two mutually exclusive alternatives arise:*

(A) $\mathbf{h}(\mathbf{x};g) = \mathbf{h}(\mathbf{x}) = \frac{\partial}{\partial n}\mathbf{u}^{classical}(\mathbf{x}),\ \mathbf{x} \in \partial\Omega_Q$ a.e.
(B) $\mathbf{h}(\mathbf{x};g) - \frac{\partial}{\partial n}\mathbf{u}^{classical}(\mathbf{x}) \neq 0,\ \mathbf{x} \in \partial\Omega_Q$

In case (B), we impose further regularity on the data by demanding $\mathbf{f} \in H^{\frac{5}{2}}(\partial\Omega_D)$ *and* $\mathbf{g} \in H^{\frac{3}{2}}(\partial\Omega_N)$.

Let $\mathbf{u}(\mathbf{x};g)$ be the unique solution of the gradient elasticity boundary value problem constructed in Theorem 2. Then, the following asymptotic analysis holds:

$$\mathbf{u}^G = \mathbf{B}^G - \frac{\lambda+\mu}{2(\lambda+2\mu)}\nabla B_0^G, \tag{50}$$

rendering Problem I, the asymptotic classical limit of the gradient problem, when the impact of the microstructure fades away.

Proof. The classical elastic displacement field satisfies the following stability condition:

$$\|\mathbf{u}^{classical}\|_{H^1(\Omega)} \leq C\left(\|\mathbf{f}\|_{H^{\frac{1}{2}}(\partial\Omega_D)} + \|\mathbf{g}\|_{H^{-\frac{1}{2}}(\partial\Omega_N)}\right), \tag{51}$$

as stated before. However, due to the additional regularity $\mathbf{f} \in H^{\frac{3}{2}}(\partial\Omega_D) \subset H^{\frac{1}{2}}(\partial\Omega_D)$ (valid throughout the work) and the extra requirement $\mathbf{g} \in H^{\frac{1}{2}}(\partial\Omega_N) \subset H^{-\frac{1}{2}}(\partial\Omega_N)$ introduced in the assumptions of the current theorem, we are in position to invoke the well-known regularity theory for second order elliptic boundary value problems guaranteeing that

$$\|\mathbf{u}^{classical}\|_{H^2} \leq C\left(\|\mathbf{f}\|_{H^{\frac{3}{2}}(\partial\Omega_D)} + \|\mathbf{g}\|_{H^{\frac{1}{2}}(\partial\Omega_N)}\right). \tag{52}$$

Consequently, the classical solution has square integrable derivatives of the second order and the theory of traces on the boundary implies that all the terms $\partial_i\partial_j\mathbf{u}^{classical}\Big|_{\partial\Omega}$ belong to $H^{-\frac{1}{2}}(\partial\Omega)$ as well as $\partial_i\partial_j\partial_k\mathbf{u}^{classical}\Big|_{\partial\Omega} \in H^{-\frac{3}{2}}(\partial\Omega)$.

On the basis of Equation (7) and via tensor symbolism—just for condensing the form of the expressions—the following Betti's form type result, referring to the elastic field $\mathbf{w}$, can be constructed:

$$\begin{aligned} 0 = \int_{\Omega} (\nabla \cdot (\widetilde{\boldsymbol{\tau}}_w - \nabla \cdot \widetilde{\boldsymbol{\mu}}_w)) \cdot \mathbf{w} dx = \langle \mathbf{R}_w, D\mathbf{w} \rangle|_{\partial\Omega} + \langle \mathbf{P}_w, \mathbf{w} \rangle|_{\partial\Omega} \\ + \langle \mathbf{E}_w, \mathbf{w} \rangle|_{\Gamma} \ \ - \int_{\Omega} \left(\widetilde{\boldsymbol{\tau}}_w : \widetilde{\boldsymbol{\varepsilon}}_w + \widetilde{\boldsymbol{\mu}}_w \vdots \widetilde{\boldsymbol{\varepsilon}}_w \nabla \right) dx. \end{aligned} \tag{53}$$

Here, we encounter the dual pairings $\langle \mathbf{R}_w, D\mathbf{w} \rangle|_{\partial\Omega} = \langle \mathbf{R}_w, D\mathbf{w} \rangle|_{H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)}$, $\langle \mathbf{P}_w, \mathbf{w} \rangle|_{\partial\Omega} = \langle \mathbf{P}_w, \mathbf{w} \rangle|_{H^{-\frac{3}{2}}(\partial\Omega) \times H^{\frac{3}{2}}(\partial\Omega)}$ and $\langle \mathbf{E}_w, \mathbf{w} \rangle|_{\Gamma} = \langle \mathbf{E}_w, \mathbf{w} \rangle|_{H^{-1}(\Gamma) \times H^{1}(\Gamma)}$ between dual spaces that represent surface and curve virtual actions. Only when all the involving fields are regular enough (square integrable functions) these terms give place to the well-known surface L^2 inner products.

On the basis of decomposition (45), the boundary conditions satisfied by the two partners of this decomposition and the splitting imposed by Equation (19), we remark that

$$\mathbf{w}(\mathbf{x}; g) = 0, \ \mathbf{x} \in \partial\Omega_D, \tag{54}$$

$$\begin{aligned} &\mathbf{P}_w(\mathbf{x}; g) = \mathbf{g}(\mathbf{x}) - \mathbf{P}_{\mathbf{u}^{classical}}(\mathbf{x}, g) = \\ &-((\mathbf{D} \cdot \hat{\mathbf{n}}(\mathbf{x}))\hat{\mathbf{n}}(\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}) - \mathbf{D}\hat{\mathbf{n}}(\mathbf{x})) : \widetilde{\boldsymbol{\mu}}_{\mathbf{u}^{classical}}(\mathbf{x}; g) \\ &+\hat{\mathbf{n}}(\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}) : D\widetilde{\boldsymbol{\mu}}_{\mathbf{u}^{classical}}(\mathbf{x}; g) + \\ &\hat{\mathbf{n}}(\mathbf{x}) \cdot (\mathbf{D} \cdot \widetilde{\boldsymbol{\mu}}_{\mathbf{u}^{classical}}(\mathbf{x}; g)) + \hat{\mathbf{n}}(\mathbf{x}) \left(\mathbf{D} \cdot \widetilde{\boldsymbol{\mu}}^{213}_{\mathbf{u}^{classical}}(\mathbf{x}; g) \right), \ \mathbf{x} \in \partial\Omega_N \Rightarrow \\ &\mathbf{P}_w(\mathbf{x}; g) = g^2(\mathbf{D}\hat{\mathbf{n}}(\mathbf{x}) - (\mathbf{D} \cdot \hat{\mathbf{n}}(\mathbf{x}))\hat{\mathbf{n}}(\mathbf{x})\hat{\mathbf{n}}(\mathbf{x})) : \nabla\widetilde{\boldsymbol{\tau}}_{\mathbf{u}^{classical}} \\ &\qquad + g^2\hat{\mathbf{n}}(\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}) : D\nabla\widetilde{\boldsymbol{\tau}}_{\mathbf{u}^{classical}} \\ &\qquad + g^2\hat{\mathbf{n}}(\mathbf{x}) \cdot (\mathbf{D} \cdot \nabla\widetilde{\boldsymbol{\tau}}_{\mathbf{u}^{classical}}) + g^2\hat{\mathbf{n}}(\mathbf{x}) \cdot \left(\mathbf{D} \cdot \nabla\widetilde{\boldsymbol{\tau}}^{213}_{\mathbf{u}^{classical}} \right), \end{aligned} \tag{55}$$

where $\mathbf{D} = \hat{\mathbf{x}}_i D_i$ (see Equation (13)).

Handling the non-classical boundary conditions of $\mathbf{w}(\mathbf{x}; g)$ leads to

$$\begin{gathered} D\mathbf{w}(\mathbf{x}; g) = \frac{\partial}{\partial n}\mathbf{u}(\mathbf{x}; g) - \frac{\partial}{\partial n}\mathbf{u}^{classical}(\mathbf{x}) = \\ \mathbf{h}(\mathbf{x}; g) - \frac{\partial}{\partial n}\mathbf{u}^{classical}(\mathbf{x}), \ \mathbf{x} \in \partial\Omega_Q, \end{gathered}$$

and

$$\begin{gathered} \mathbf{R}_{\mathbf{w}}(\mathbf{x}; g) = g^2\widetilde{\mathbf{r}}(\mathbf{x}) - \mathbf{R}_{\mathbf{u}^{classical}}(\mathbf{x}; g) = \\ g^2(\widetilde{\mathbf{r}}(\mathbf{x}) - \hat{\mathbf{n}}(\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}) : \nabla\widetilde{\boldsymbol{\tau}}_{\mathbf{u}^{classical}}), \ \mathbf{x} \in \partial\Omega_R. \end{gathered}$$

Then, Equation (53) obtains the form:

$$\begin{aligned} &\int_{\Omega} \left(\widetilde{\boldsymbol{\tau}}_w : \widetilde{\boldsymbol{\varepsilon}}_w + \widetilde{\boldsymbol{\mu}}_w \vdots \widetilde{\boldsymbol{\varepsilon}}_w \nabla \right) dx = g^2 \langle \mathbf{k}(\cdot), \mathbf{w}(\cdot; g) \rangle|_{\partial\Omega_N} + \\ &+ \left\langle \mathbf{R}_w(\mathbf{x}; g), \mathbf{h}(\mathbf{x}; g) - \frac{\partial}{\partial n}\mathbf{u}^{classical}(\mathbf{x}) \right\rangle \Big|_{\partial\Omega_Q} + \\ &g^2 \langle (\widetilde{\mathbf{r}} - \hat{\mathbf{n}}(\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}) : \nabla\widetilde{\boldsymbol{\tau}}_{\mathbf{u}^{classical}}), D\mathbf{w} \rangle|_{\partial\Omega_R} + g^2 \langle \widetilde{\mathbf{s}}(\mathbf{x}), \mathbf{w} \rangle|_{\Gamma} - \\ &g^2 \langle \mathbf{s}_{\mathbf{u}^{classical}}(\mathbf{x}), \mathbf{w} \rangle|_{\Gamma}, \end{aligned} \tag{56}$$

where

$$\begin{aligned} &\mathbf{k}(\mathbf{x}) = \{ (\mathbf{D}\hat{\mathbf{n}}(\mathbf{x}) - (\mathbf{D} \cdot \hat{\mathbf{n}}(\mathbf{x}))\hat{\mathbf{n}}(\mathbf{x})\hat{\mathbf{n}}(\mathbf{x})) : \nabla\widetilde{\boldsymbol{\tau}}_{\mathbf{u}^{classical}} \\ &+\hat{\mathbf{n}}(\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}) : D\nabla\widetilde{\boldsymbol{\tau}}_{\mathbf{u}^{classical}} + \hat{\mathbf{n}}(\mathbf{x}) \cdot (\mathbf{D} \cdot \nabla\widetilde{\boldsymbol{\tau}}_{\mathbf{u}^{classical}}) + \hat{\mathbf{n}}(\mathbf{x}) \cdot \left(\mathbf{D} \cdot \nabla\widetilde{\boldsymbol{\tau}}^{213}_{\mathbf{u}^{classical}} \right) \}. \end{aligned}$$

Thanks to the introductory discussion of this theorem, pertaining to the regularity of surface terms generated by the classical field, it is clear that all the terms participating in $\mathbf{k}(\mathbf{x})$ are well defined and belong to $H^{-\frac{3}{2}}(\partial\Omega_N)$. In addition, given that in our elliptic boundary value problem, every surface norm of traces of the classical field and its derivatives is controlled by the volume norm $\|\mathbf{u}^{classical}\|_{H^2(\Omega)}$, which is bounded by the data as Equation (52) implies, the surface field $g^2\mathbf{k}(\mathbf{x})$ is uniformly bounded in $H^{-\frac{3}{2}}(\partial\Omega_N)$ with magnitude of order g^2.

The fields $\mathbf{w}$ and $D\mathbf{w}$, on surfaces and (or) curves have appropriate norms, which are bounded by the norm of the solution $\|\mathbf{w}\|_{H^1(\Omega)}$ (due to the continuity of the traces with respect to the solution of the boundary value problem), which is, by its turn, bounded uniformly with respect to the gradient parameter g, via an estimate of the form (33) (applied to $\mathbf{w}$).

If the case (A) holds, the second dual pairing in r.h.s. of Equation (56) disappears and no need to handle the functional $\mathbf{R}_w(\mathbf{x};g)$ on $\partial\Omega_D$ arises. When we have case (B), we impose further regularity on the displacement and stresses leading to higher regularity on the classical solution. This regularity is optimally selected to guarantee that the crucial field $\frac{\partial}{\partial n}\mathbf{u}^{classical}(\mathbf{x})$ belongs to $H^{\frac{3}{2}}(\partial\Omega)$ (since under this choice, the norm $\|\mathbf{u}^{classical}\|_{H^3(\Omega)}$ is kept bounded).

Finally, the field $\mathbf{R}_w(\mathbf{x};g)$ on $\partial\Omega_Q$ equals with the field $g^2\hat{\mathbf{n}}(\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}) : \nabla\boldsymbol{\tau}_w$, which incorporates second order derivatives of $\mathbf{w}$. Then, the adequate surface $H^{-\frac{3}{2}}$—norm of $R_w(\mathbf{x};g)$ is again uniformly bounded (due to the continuity of the traces with respect to the solution of the boundary value problem) by $g^2\|\mathbf{w}\|_{H^1(\Omega)}$, which via (33) is bounded by the norms of the data multiplied by g^2C. This coefficient goes to 0 as $g \to 0$, since, as mentioned before, the generic constant does not depend on g.

In any case, taking the limit as $g \to 0$, in expression (56), all the terms of the right-hand side converge uniformly to 0, therefore

$$\int\left(\widetilde{\boldsymbol{\tau}}_{w_{g=0}} : \hat{\boldsymbol{\varepsilon}}_{w_{g=0}} + \widetilde{\boldsymbol{\mu}}_{w_{g=0}} \vdots \hat{\boldsymbol{\varepsilon}}_{w_{g=0}}\nabla\right)dx = 0. \tag{57}$$

The positivity of the bilinear elastic form guarantees that $\widetilde{\boldsymbol{\varepsilon}}_{w_{g=0}} = \widetilde{0}$, which implies that $\mathbf{w}(\mathbf{x},0)$ is a constant. Given that the trace $\mathbf{w}(\mathbf{x},0)|_{\partial\Omega_D} = 0$, we infer that

$$\lim_{g\to 0}\mathbf{w}(\mathbf{x},g) = 0,\ x \in \Omega, \tag{58}$$

from where we obtain immediately the asymptotic behavior (50). □

Much effort was places in finding the necessary regularity of the data assuring the desired convergence when microstructure behavior disappears. It is interesting that this is necessarily valid when the data are smooth analytic functions of their arguments. Indeed, we have the following corollary, which would be the main outcome of this work if the data were considered analytic, but in our opinion, this could not be proven without the herein adopted generalization to abstract functional spaces.

Corollary 1. *Consider that the boundary value problem consists of Equations (20), (29) and (30). The data* $(\mathbf{f},\mathbf{g},\mathbf{h},\mathbf{r},\mathbf{s})$ *are analytic functions of their arguments with the restriction that* $\mathbf{r}$, $\mathbf{s}$ *obey to Equations (48) and (49). Let* $\mathbf{u}(\mathbf{x};g)$ *be the unique solution of the gradient elasticity boundary value problem, expressed on the basis of the decomposition constructed in Theorem 2. Then, the following asymptotic result is obtained:*

$$\mathbf{u}(\mathbf{x};g) \to \mathbf{u}^{classical}(\mathbf{x}) \text{ as } g \to 0,\ \mathbf{x} \in \Omega.$$

Proof. The assumed analytic smoothness of the data confirms that the assumptions of the alternative (B) of Theorem 3 are always valid and then the convergence outcome holds in any case. □

Remark 1. *As far as the field $\mathbf{R}_w(\mathbf{x};g)$ on $\partial\Omega_Q$ is concerned, we could easily find that the surface $H^{-\frac{1}{2}}$—norm of $\mathbf{R}_w(\mathbf{x};g)$ is again uniformly bounded by $g^2\|\mathbf{w}\|_{H^2(\Omega)}$. On the basis of (24a), this term is bounded by the norms of the data multiplied by g^2C_g. Therefore, if no extra assumption on data was made, the limit $\lim\limits_{g\to 0} g^2C_g$ would be ambiguous.*

Remark 2. *We would like to mention once again here that, despite the convergence, which needs some special settlement of the nature of the imposed data, the decomposition (37) is always valid, and when the conditions are arbitrary, what is exactly stated for the participants of the decomposition is that they satisfy the corresponding differential equations, while their superposition satisfies the boundary value problem.*

A number of simple one-dimensional problems can reveal the essence of the previous theorems. Let us consider, for example, the elastostatic response of a finite length string. The considered differential equation is given by $u_{xx} - g^2u_{xxxx} = 0,\ x \in (0,1)$ accompanied by the classical boundary conditions $u(0) = 1$, $u_x(1) - g^2u_{xxx}(1) = 2$ and the non-classical ones $g^2u_{xxx}(0) = 3g^2$ and $u_x(1) = 0$.

The solution is

$$u(x) = 2x + 1 - eg^2 + \left[\left(3g^2 + 2ge^{-\frac{1}{g}}\right)e^{-\frac{x}{g}} + g\left(3ge^{-\frac{1}{g}} - 2\right)e^{\frac{x-1}{g}}\right]\left(1 + e^{-\frac{2}{g}}\right)^{-1},$$

while the classical elastic solution that satisfies the conditions $u(0) = 1$ and $u_x(1) = 2$ is $u^{cl}(x) = 2x + 1$.

The same differential equation with the boundary conditions $u(0) = 0$, $u(1)= 1$, $u_x(0) = 2$ and $u_{xxx}(1) = 0$ has the following solution:

$$u(x) = x + g\left[\left(1 - e^{-\frac{2}{g}}\right)(x-1) + e^{-\frac{x}{g}} - e^{\frac{x-2}{g}}\right]\left(g - 1 - (g+1)e^{-\frac{2}{g}}\right)^{-1}$$

with corresponding classical solution, the function $u^{cl.}(x) = x$.

Remark 3. *The second example of Remark 2 implies that sometimes it is useful to write Equation (37) in the following form*

$$\mathbf{u}(\mathbf{x},g) = (1+\delta(g))\mathbf{u}^{classical}(\mathbf{x}) + \mathbf{u}^N(\mathbf{x},g) + g^2\mathbf{u}^G(\mathbf{x},g) \tag{59}$$

with $\delta(g) \to 0$ as $g \to 0$, where $\mathbf{u}^N(\mathbf{x},\xi)$ obeys to the same differential regime. This approach consists of a repartition of the partners of the decomposition, which seems to be more flexible in applications. This reordering is realizable since both fields $\mathbf{u}^{classical}(\mathbf{x})$ and $\mathbf{u}^N(\mathbf{x})$ belong to the kernel of the classical elasticity operator.

Remark 4. *The result stated by Theorem 3 could be violated if the boundary conditions are set arbitrarily. As an example, consider the same differential equation as in Remark 2, accompanied with the boundary conditions $u(0) = 0$, $u(1) = 1$, $u_x(0) - g^2u_{xxx}(0) = 2$ and $u_{xx}(1) = 0$. This problem has the unique solution $u(x) = 2x - 1 + \left[e^{-\frac{x}{g}} - e^{-\frac{1}{g}}e^{\frac{x-1}{g}}\right]\left(1 - e^{-\frac{2}{g}}\right)^{-1}$. However, the classical solution $2x - 1$ satisfies the boundary conditions $u(1) = 1$, $u_x(0) = 2$, but does not satisfy the classical boundary condition $u(0) = 0$. Additionally, the part of the solution containing the displacement $\mathbf{u}^G$ does not converge to zero as $g \to 0$ at the endpoint $x = 0$. The critical factor here, is that two classical boundary conditions are prescribed simultaneously at the same part of the boundary.*

Remark 5. *In the statement of the theorem, we demanded that meas$(\partial\Omega_D) > 0$). In mathematical terms, this is needed for the uniqueness of the solution. Physically, it reflects the necessity of some anchoring of the structure. However, we would like to say that this condition is not of major* *importance for the validity of Theorem 3. The modifications needed could be stated mathematically—they include some compatibility conditions for the data—but the essence is very simple: The results are exactly the same modulo motions of a rigid body. Consequently, imposing a-posteriori a condition not allowing rigid motions is sufficient to construct the unique solution of the problem.*

4. Revisiting Bending Theories of Strain Gradient Elastic Beams

In this section, the solution representation addressed in Section 3 is employed to present an answer to contradictions arising by two well-known first-strain gradient elastic models proposed in the literature to describe the strain gradient elastostatic bending behavior of Bernoulli–Euler beams.

During the last two decades, a plethora of papers dealing with the static and dynamic response of Bernoulli–Euler strain gradient elastic beams have appeared in the literature. Most of them are based on variational approaches and, for the elastostatic case, the equilibrium equation they conclude has either the following form [48–52]

$$EIu'''' - EIg^2u'''''' + q(x) = 0 \tag{60}$$

where x coincides with the neutral axis of the beam, E stands for the material Young modulus, I is the moment of inertia of the beam's cross-section A, $u(x)$ is the transverse beam deflection, $q(x)$ is the transverse external load, and g^2 is the intrinsic strain gradient elastic parameter, or the form [53–61]

$$\left(EI + g^2A\right)u'''' - EIg^2u'''''' + q(x) = 0. \tag{61}$$

The essential difference between these two equations is that the first is derived by considering the strain e_{xx} and the gradient of strain e_{xxx} in the expression of the potential energy density of the beam, while the second one considers the strain e_{xx} and the strain gradients e_{xxx}, e_{yxx} with e_{yxx} being the differentiation of e_{xx} with respect to axis y directed along the thickness of the beam. The result is that the bending stiffness in Equation (60) is the same as that of classical elasticity, while the bending stiffness in (61) depends on the internal length scale parameter g^2 and the cross-section of beam A. The interesting point here is that the same categorization is valid for other works dealing with the bending of strain gradient elastic Timoshenko beams and plates, as well as for experimental and numerical validations on the bending response of strain gradient elastic beams. Here, one can mention the works of Papargyri-Beskou and Beskos [62], Papargyri-Beskou et al. [63], Triantafyllou and Giannakopoulos [64], and Gortsas et al. [29] for the strain gradient model of Equation (60), and the works of Lazopoulos and Lazopoulos [65], Khakalo and Niiranen [57,58,60,66], and Korshunova et al. [67] for the model of Equation (61).

Since there is a principal difference between the two above-mentioned bending models, the question here is which of them is the correct one. Lurie et al. [68] and Lurie and Solyaev [69,70] proposed elegant answers to that question by proving that Equation (60) is the only correct one for the bending response of a strain gradient elastic Bernoulli–Euler beam. Among others, they mentioned that " … *a formal variational procedure for obtaining the governing equilibrium equations in the beam theories, ignoring boundary conditions on the top and bottom surfaces of the beam leads to an erroneous result of abnormal increasing of the beam normalized bending stiffness with decreasing its thickness*. … ". Polizzotto [31] upholds this argument because the normal derivative of displacements identically vanishes at the beam lateral surface, and thus cannot play any role as a boundary layer. In the present section, we reach the same conclusion under the light of the theorems proved in the previous section.

The starting point of our analysis is the pure bending of a classical elastic, isotropic two-dimensional orthogonal rectangle subjected to pure bending under plane strain conditions, as depicted in Figure 1. The boundary conditions of the problem are $p_k(x_1, \pm a) = 0$ and

$p_2(0, x_2) = p_2(L, x_2) = 0$, $p_1(0, x_2) = p_1(L, x_2) = P_1 x_2/a$ and the corresponding solution is provided by Selvadurai [3] in the following form:

$$\begin{aligned} u_1^{classical} &= \tfrac{P_1(1-\nu^2)}{Ea} x_1 x_2 \\ u_2^{classical} &= -\tfrac{P_1}{2Ea}\left[\nu(1+\nu)x_2^2 + (1-\nu^2)x_1^2\right]. \end{aligned} \tag{62}$$

Figure 1. Pure bending of an elastic beam.

Consider a rectangular 3D plate of length L and cross-section $2a \times b$. The torque at both boundaries of the plate is defined as

$$M = \int_{-a}^{a} x_2 \left(P_1 \frac{x_2}{a}\right) dS = \int_{-a}^{a} x_2 \left(P_1 \frac{x_2}{a}\right) b dx_2 = \frac{P_1 I}{a} \tag{63}$$

with $I = 2ba^3/3$ being the moment of inertia of the cross-section $2a \times b$. Equation (62), valid for the midplane of the plate, obtain the form

$$\begin{aligned} u_1^{classical} &= \tfrac{M(1-\nu^2)}{EI} x_1 x_2 \\ u_2^{classical} &= -\tfrac{M}{2EI}\left[\nu(1+\nu)x_2^2 + (1-\nu^2)x_1^2\right] \end{aligned} \tag{64}$$

while the deflection of the neutral axis is given by $u_2^{classical}$ for $x_2 = 0$, i.e.,

$$u_2^{classical} = -\frac{M}{2EI}(1-\nu^2)x_1^2 \tag{65}$$

which, for $\nu = 0$, is identical to the deflection of a classical elastic Bernoulli–Euler beam subjected to pure bending [3,69], i.e.,

$$u_{Bernoulli-Euler}^{classical} = -\frac{M}{2EI}x_1^2. \tag{66}$$

Next, we consider the same pure bending problem presented in Figure 1, however, for a strain gradient elastic material. The classical boundary conditions remain the same as in the classical elastic problem, i.e.,

$$\begin{aligned} &p_k(x_1, \pm a) = 0 \\ &p_2(0, x_2) = p_2(L, x_2) = 0 \\ &p_1(0, x_2) = p_1(L, x_2) = \tfrac{M}{I} x_2 \end{aligned} \tag{67}$$

while in non-classical boundary conditions, we consider the zeroing of double tractions at all the external boundaries, for example:

$$
\begin{aligned}
&R_k(x_1, \pm a) = 0 \\
&R_k(0, x_2) = R_k(0, x_2) = 0.
\end{aligned}
\tag{68}
$$

According to the theorem presented in Section 3, the strain gradient elastic solution of this problem is written as

$$
\begin{aligned}
u_1(x_1, x_2; g) &= u_1^{classical}(x_1, x_2) + u_1^N(x_1, x_2; g) + g^2 u_1^G(x_1, x_2; g) \\
u_2(x_1, x_2; g) &= u_2^{classical}(x_1, x_2) + u_2^N(x_1, x_2; g) + g^2 u_2^G(x_1, x_2; g).
\end{aligned}
\tag{69}
$$

Evidently, the solution (69) imposes the following forms for tractions and double tractions, respectively

$$
\begin{aligned}
p_1(\hat{\mathbf{n}}, x_1, x_2; g) &= p_1^{classical}(\hat{\mathbf{n}}, x_1, x_2; g) + p_1^N(\hat{\mathbf{n}}, x_1, x_2; g) \\
&\quad + g^2 p_1^G(\hat{\mathbf{n}}, x_1, x_2; g) \\
p_2(\hat{\mathbf{n}}, x_1, x_2; g) &= p_2^{classical}(\hat{\mathbf{n}}, x_1, x_2; g) + p_2^N(\hat{\mathbf{n}}, x_1, x_2; g) \\
&\quad + g^2 p_2^G(\hat{\mathbf{n}}, x_1, x_2; g)
\end{aligned}
\tag{70}
$$

$$
\begin{aligned}
R_1(\hat{\mathbf{n}}, x_1, x_2; g) &= R_1^{classical}(\hat{\mathbf{n}}, x_1, x_2; g) + R_1^N(\hat{\mathbf{n}}, x_1, x_2; g) \\
&\quad + g^2 R_1^G(\hat{\mathbf{n}}, x_1, x_2; g) \\
R_2(\hat{\mathbf{n}}, x_1, x_2; g) &= R_2^{classical}(\hat{\mathbf{n}}, x_1, x_2; g) + R_2^N(\hat{\mathbf{n}}, x_1, x_2; g) \\
&\quad + g^2 R_2^G(\hat{\mathbf{n}}, x_1, x_2; g)
\end{aligned}
\tag{71}
$$

with $\hat{\mathbf{n}}$ being the unit normal vector of the surface, to which both tractions and double tractions are referred.

Concentrating our attention on the classical part of the solution (69), it is easy to observe that [30]

$$
\begin{aligned}
\tau_{11}^{classical} &= (\lambda + 2\mu)\partial_1 u_1^{classical} + \lambda \partial_2 u_2^{classical} &&= \frac{M}{I} x_2 \\
\tau_{22}^{classical} &= \lambda \partial_1 u_1^{classical} + (\lambda + 2\mu)\partial_2 u_2^{classical} &&= 0 \\
\tau_{12}^{classical} &= \tau_{21}^{classical} = \mu(\partial_2 u_1^{classical} + \partial_1 u_2^{classical}) = 0
\end{aligned}
\tag{72}
$$

and

$$
\begin{aligned}
\mu_{111}^{classical} &= (\lambda + 2\mu) g^2 \partial_1^2 u_1^{classical} + \lambda g^2 \partial_1 \partial_2 u_2^{classical} &&= 0 \\
\mu_{222}^{classical} &= \lambda g^2 \partial_1 \partial_2 u_1^{classical} + (\lambda + 2\mu) g^2 \partial_2^2 u_2^{classical} &&= 0 \\
\mu_{112}^{classical} &= \mu_{121}^{classical} = \mu g^2 (\partial_1 \partial_2 u_1^{classical} + \partial_1^2 u_2^{classical}) &&= 0 \\
\mu_{122}^{classical} &= \lambda g^2 \partial_1^2 u_1^{classical} + (\lambda + 2\mu) g^2 \partial_1 \partial_2 u_2^{classical} &&= 0 \\
\mu_{211}^{classical} &= (\lambda + 2\mu) g^2 \partial_1 \partial_2 u_1^{classical} + \lambda g^2 \partial_2^2 u_2^{classical} &&= g^2 \frac{M}{I} \\
\mu_{212}^{classical} &= \mu_{221}^{classical} = \mu g^2 (\partial_2^2 u_1^{classical} + \partial_1 \partial_2 u_2^{classical}) &&= 0.
\end{aligned}
\tag{73}
$$

As it is explained by Charalambopoulos et al. [30], the two components of the gradient elastic traction and double traction vectors, defined on a surface with unit normal vector $\hat{\mathbf{n}}(n_1, n_2)$, have the form, respectively,

$$
\begin{aligned}
&p_1 = n_1 \tau_{11} + n_2 \tau_{21} + n_1(n_1^2 - 2)\partial_1 \mu_{111} + n_2(n_2^2 - 2)\partial_2 \mu_{221} \\
&+ n_2(n_1^2 - 1)(\partial_1 \mu_{121} + \partial_1 \mu_{211}) + n_1(n_2^2 - 1)(\partial_2 \mu_{121} + \partial_2 \mu_{211}) \\
&+ n_2 n_1^2 \partial_2 \mu_{111} + n_1 n_2^2 \partial_1 \mu_{221} \\
&p_2 = n_2 \tau_{22} + n_1 \tau_{12} + n_1(n_1^2 - 2)\partial_1 \mu_{112} + n_2(n_2^2 - 2)\partial_2 \mu_{222} \\
&+ n_2(n_1^2 - 1)(\partial_1 \mu_{212} + \partial_1 \mu_{122}) + n_1(n_2^2 - 1)(\partial_2 \mu_{122} + \partial_2 \mu_{212}) \\
&+ n_2 n_1^2 \partial_2 \mu_{112} + n_1 n_2^2 \partial_1 \mu_{222}
\end{aligned}
\tag{74}
$$

and

$$\begin{aligned} R_1 &= n_1^2\mu_{111} + n_1 n_2 \mu_{121} + n_1 n_2 \mu_{211} + n_2^2 \mu_{221} \\ R_2 &= n_2^2\mu_{222} + n_1^2 \mu_{112} + n_1 n_2 \mu_{122} + n_1 n_2 \mu_{212} \end{aligned} \tag{75}$$

In view of (72)–(75), the classical part of the gradient elastic traction and double traction vectors defined on a surface with unit normal vector $\hat{\mathbf{n}}(n_1, n_2)$ exhibit the following forms, respectively

$$\begin{aligned} R_1 &= n_1^2\mu_{111} + n_1 n_2 \mu_{121} + n_1 n_2 \mu_{211} + n_2^2 \mu_{221} \\ R_2 &= n_2^2\mu_{222} + n_1^2 \mu_{112} + n_1 n_2 \mu_{122} + n_1 n_2 \mu_{212} \end{aligned} \tag{76}$$

and

$$\begin{aligned} &R_1^{classical}(\hat{\mathbf{n}}, x_1, x_2; g) = n_1 n_2 \mu_{211}^{classical} = n_1 n_2 \frac{M}{I} \\ &R_2^{classical}(\hat{\mathbf{n}}, x_1, x_2; g) = 0. \end{aligned} \tag{77}$$

Inserting Equations (76) and (77) into (70) and (71), respectively, and satisfying the boundary conditions (67) and (68) one obtains:

$$\begin{aligned} &p_k^N(n_2, x_1, \pm a; g) + g^2 p_k^G(n_2, x_1, \pm a; g) = 0,\ k = 1,2 \\ &p_k^N(n_1, 0, x_2; g) + g^2 p_k^G(n_1, 0, x_2; g) = 0,\ k = 1,2 \\ &p_k^N(n_1, L, x_2; g) + g^2 p_k^G(n_1, L, x_2; g) = 0,\ k = 1,2 \end{aligned} \tag{78}$$

and

$$\begin{aligned} &R_k^N(n_2, x_1, \pm a; g) + g^2 R_k^G(n_2, x_1, \pm a; g) = 0,\ k = 1,2 \\ &R_k^N(n_1, 0, x_2; g) + g^2 R_k^G(n_1, 0, x_2; g) = 0,\ k = 1,2 \\ &R_k^N(n_1, L, x_2; g) + g^2 R_k^G(n_1, L, x_2; g) = 0,\ k = 1,2. \end{aligned} \tag{79}$$

Equations (78) and (79) indicate that the parts $u_k^N(x_1, x_2; g)$, $u_k^G(x_1, x_2; g)$ satisfy a homogeneous system of algebraic equations for arbitrary material properties. Apparently, this leads to the conclusion that

$$u_k^N(x_1, x_2; g) = u_k^G(x_1, x_2; g) = 0 \tag{80}$$

which means that the pure bending of the strain gradient elastic plate presented in Figure 1 has absolutely the same response as that of the classical elastic one.

Extending this result to the behavior of the neutral axis of the plate for $\nu = 0$, we obtain

$$u_{Bernoulli-Euler}^{classical} \equiv u_{Bernoulli-Euler}^{gradient} = -\frac{M}{2EI} x_1^2. \tag{81}$$

This result is possible only when Equation (60) is valid. Equation (61) is misleading since the double stress μ_{yxx} does not contribute to the solution of the problem.

5. Conclusions

A material with microstructural effects obeys the simplified elastostatic version of Mindlin's Form II first-strain gradient elastic theory, and its displacement field $\mathbf{u}(\mathbf{x})$ satisfies the fourth-order partial differential Equation (3) and the relevant classical and non-classical boundary conditions. In the present work, it has been rigorously proved that the solution of Equation (3) admits the following representation:

$$\mathbf{u}(\mathbf{x}, g) = (1 + \delta(g))\mathbf{u}^{classical}(\mathbf{x}) + \mathbf{u}^N(\mathbf{x}, g) + g^2 \mathbf{u}^G(\mathbf{x}, g), \tag{82}$$

which has the following convenient advantages:

1. Incorporates the solution $\mathbf{u}^{classical}(\mathbf{x})$ of the respective classical elastic boundary value problem, that satisfies the same classical boundary conditions with the strain gradient elastic problem.
2. Converges to the classical elastic solution as $g \to 0$.

3. Comprises two displacements fields $\mathbf{u}^N(\mathbf{x}, g)$, $\mathbf{u}^G(\mathbf{x}, g)$, which satisfy the simpler equations $\mu\nabla^2\mathbf{u}^N + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u}^N = 0$ and $(1-g^2\nabla^2)\mathbf{u}^G = 0$, respectively.

The representation of the solution presented above was employed to prove that a strain gradient elastic Bernoulli–Euler beam subjected to pure bending does not present microstructural effects and its behavior is identical to that of a classical elastic Bernoulli–Euler beam. This result is in full agreement with the corresponding conclusions provided by Lurie and Solyaev [69] on the same subject.

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Appendix A

In this Appendix, the proof of decomposition (5) through the solution representation (4) is provided.

It is well known that the solution of Equation (1), via Papkovich–Neuber potentials, has the form [3,64]

$$\mathbf{u}^{classical} = \mathbf{B}^{classical} - \frac{1}{4(1-v)}\nabla\left(\mathbf{r}\cdot\mathbf{B}^{classical} + B_0^{classical}\right), \tag{A1}$$

with

$$\begin{aligned}\nabla^2\mathbf{B}^{classical} &= 0\\ \nabla^2 B_0^{classical} &= 0.\end{aligned} \tag{A2}$$

On the other hand, the solution representation (4) reads

$$\mathbf{u}^{gradient} = \mathbf{B}^{gradient} - \frac{1}{4(1-v)}\left[\mathbf{r}\cdot\left(1-g^2\nabla^2\right)\mathbf{B}^{gradient} + B_0^{gradient}\right], \tag{A3}$$

with

$$\begin{aligned}\left(1-g^2\nabla^2\right)\nabla^2\mathbf{B}^{gradient} &= 0\\ \left(1-g^2\nabla^2\right)\nabla^2 B_0^{gradient} &= 0.\end{aligned} \tag{A4}$$

Comparing (A2) with (A4), it is apparent that $\left(1-g^2\nabla^2\right)\mathbf{B}^{gradient} \equiv \mathbf{B}^{classical}$, and subsequently (A3) can be written as

$$\mathbf{u}^{gradient} = \mathbf{B}^{gradient} - \frac{1}{4(1-v)}\left[\mathbf{r}\cdot\mathbf{B}^{classical} + B_0^{gradient}\right]. \tag{A5}$$

By adding and subtracting $\mathbf{B}^{classical}$ and $B_0^{classical}$ in (A5), we obtain

$$\begin{aligned}&\mathbf{u}^{gradient} = \mathbf{B}^{gradient} + \mathbf{B}^{classical} - \mathbf{B}^{classical}\\ &-\tfrac{1}{4(1-v)}\left[\mathbf{r}\cdot\mathbf{B}^{classical} + B_0^{gradient} + B_0^{classical} - B_0^{classical}\right] \Rightarrow\\ &\mathbf{u}^{gradient} = \mathbf{B}^{classical} - \tfrac{1}{4(1-v)}\nabla\left[\mathbf{r}\cdot\mathbf{B}^{classical} + B_0^{classical}\right]\\ &+\mathbf{B}^{gradient} - \mathbf{B}^{classical} - \tfrac{1}{4(1-v)}\nabla\left[B_0^{gradient} - B_0^{classical}\right],\end{aligned}$$

and because of (A1), it is apparent that

$$\mathbf{u}^{gradient} = \mathbf{u}^{classical} + \mathbf{B}^{gradient} - \mathbf{B}^{classical} - \frac{1}{4(1-v)}\nabla\left[B_0^{gradient} - B_0^{classical}\right]. \tag{A6}$$

However,

$$\left.\begin{array}{l}\left(1-g^2\nabla^2\right)\left(\mathbf{B}^{gradient} - \mathbf{B}^{classical}\right) \\ \left(1-g^2\nabla^2\right)\left(B_0^{gradient} - B_0^{classical}\right)\end{array}\right\} \overset{(A.4)}{=} \left.\begin{array}{l}\mathbf{B}^{classical} - \mathbf{B}^{classical} \\ B_0^{classical} - B_0^{classical}\end{array}\right\} = 0. \tag{A7}$$

Equations (A6) and (A7) easily imply that

$$\begin{array}{l}\mathbf{u}^{gradient} = \mathbf{u}^{classical} + \mathbf{u}^{g} \\ \mathbf{u}^{classical} \in \ker\left(\mu\nabla^2(\lambda+\mu)\nabla\nabla\cdot\right) \\ \mathbf{u}^{g} \in \ker\left(1-g^2\nabla^2\right)\end{array} \tag{A8}$$

which confirms (5).

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