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# Schrödinger Harmonic Functions with Morrey Traces on Dirichlet Metric Measure Spaces

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**Abstract:** Assume that $(X, d, \mu)$ is a metric measure space that satisfies a $Q$-doubling condition with $Q > 1$ and supports an $L^2$-Poincaré inequality. Let $\mathcal{L}$ be a nonnegative operator generalized by a Dirichlet form $\mathscr{E}$ and $V$ be a Muckenhoupt weight belonging to a reverse Hölder class $RH_q(X)$ for some $q \geq (Q+1)/2$. In this paper, we consider the Dirichlet problem for the Schrödinger equation $-\partial_t^2 u + \mathcal{L}u + Vu = 0$ on the upper half-space $X \times \mathbb{R}_+$, which has $f$ as its the boundary value on $X$. We show that a solution $u$ of the Schrödinger equation satisfies the Carleson type condition if and only if there exists a square Morrey function $f$ such that $u$ can be expressed by the Poisson integral of $f$. This extends the results of Song-Tian-Yan [Acta Math. Sin. (Engl. Ser.) 34 (2018), 787-800] from the Euclidean space $\mathbb{R}^Q$ to the metric measure space $X$ and improves the reverse Hölder index from $q \geq Q$ to $q \geq (Q+1)/2$.

**Keywords:** Schrödinger equation; Morrey space; Dirichlet problem; metric measure space

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## 1. Introduction

The Dirichlet problem was originally posed for the Laplace equation. In such a case, the problem can be stated as follows. Assume that $\Omega \subset \mathbb{R}^n$ is a domain and $f$ is a continuous map on $\partial\Omega$. Let us find a continuous function $u$ satisfying

$$\begin{cases} -\Delta u(x) = 0, & x \in \Omega, \\ u(x) = f(x), & x \in \partial\Omega. \end{cases}$$

We call $f$ as the boundary value of $u$. Here, $-\Delta u = 0$ means that

$$\int_\Omega \nabla u \cdot \nabla \phi dx = 0$$

holds for every smooth function $\phi$ on $\mathbb{R}^n$ with compact support in $\Omega$, where $\nabla u$ is the distributional gradient of $u$. For the upper half-space case, the study of the harmonic extension of a function has become one of the elementary tools of harmonic analysis ever since the seminar work of Stein-Weiss [1]. As we know, for any function $f \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$, its Poisson extension $u(x,t) = e^{-t\sqrt{-\Delta}} f(x)$, $(x,t) \in \mathbb{R}^{n+1}_+$, which satisfies

$$\begin{cases} -\partial_t^2 u - \Delta u = 0, & (x,t) \in \mathbb{R}^{n+1}_+, \\ u(x) = f(x), & x \in \mathbb{R}^n. \end{cases}$$

In the study of singular integrals, a natural substitution of the end-point space $L^\infty(\mathbb{R}^n)$ is the space of functions of bounded mean oscillation (BMO). Fefferman-Stein [2] proved that a function $f$ belongs to $\mathrm{BMO}(\mathbb{R}^n)$ if and only if its harmonic extension $u(x,t) = e^{-t\sqrt{-\Delta}}f(x)$ satisfies the following Carleson condition

$$\sup_{x_B, r_B} \int_0^{r_B} \fint_{B(x_B,r_B)} |t\nabla u(x,t)|^2 dx \frac{dt}{t} < \infty, \tag{1}$$

where

$$\fint_{B(x_B,r_B)} := \frac{1}{|B(x_B,r_B)|} \int_{B(x_B,r_B)}.$$

Later, Fabes-Johnson-Neri [3] found that the Carleson condition (1) actually characterizes all harmonic functions $u(x,t)$ on $\mathbb{R}^{n+1}_+$ with BMO traces. Since then, the research on this topic has been widely extended to various settings, including heat equations [4], elliptic equations and systems with complex coefficients [5], degenerate elliptic equations and systems [6], as well as Schrödinger equations [7,8]. We refer the reader to [9–13] and the references therein for more information about this topic.

In this paper, we consider a metric measure space $X$, which satisfies a $Q$-doubling condition with $Q > 1$, and supports an $L^2$-Poincaré inequality. Let $\mathscr{L} = \mathcal{L} + V$ be a Schrödinger operator, where $\mathcal{L}$ is a nonnegative operator generalized by a Dirichlet form $\mathscr{E}$, and the nonnegative potential $V$ is a Muckenhoupt weight belonging to the reverse Hölder class. We study the boundary behavior of Schrödinger harmonic function on $X \times \mathbb{R}_+$. Roughly speaking, we derive that a solution $u$ to the Schrödinger equation

$$-\partial_t^2 u(x,t) + \mathscr{L}u(x,t) = -\partial_t^2 u(x,t) + \mathcal{L}u(x,t) + V(x)u(x,t) = 0$$

satisfies the Carleson type condition analogous to (1) if and only if there exists a square Morrey function $f$ such that $u = e^{-t\sqrt{\mathscr{L}}}f$ holds, where the square Morrey spaces $L^{2,\alpha}(X)$ with $-1/2 < \alpha < 0$ are defined by

$$L^{2,\alpha}(X) = \left\{ f \in L^2_{\mathrm{loc}}(X) : \sup_{B \subset X} \frac{1}{[\mu(B)]^\alpha} \left( \fint_B |f|^2 d\mu \right)^{1/2} < \infty \right\}.$$

We refer the reader to Section 2 for more about the Dirichlet metric measure space, the reverse Hölder classes, the Muckenhoupt weight and the main result. We would like to mention that, when $X = \mathbb{R}^n$, if $V \in RH_q(\mathbb{R}^n)$ for some $q \geq n$, Song-Tian-Yan [8] studied the boundary behavior of Schrödinger harmonic functions. Our result covers more general spaces, such as the Riemannian metric measure space, sub-Riemannian manifold; see [14] (Section 7) for more details.

Regarding their proof, the condition $V \in RH_q(\mathbb{R}^n)$ for some $q \geq n$ is to assure that there exists a pointwise upper bound for the gradient of the Schrödinger Poisson kernel. However, even without the potential $V$, such bounds are not valid in general metric space unless a group structure or strong nonnegative curvature condition is assumed (see [15,16]). Indeed, for uniformly elliptic operators, the pointwise upper bound of the gradient of heat kernel has already failed; see [14,17] for instance.

To overcome this difficulty, we adopt a Caccioppoli inequality for the Schrödinger Poisson semigroup in a tent domain $B(x_B, r_B) \times (0, r_B)$ from [18], and hence the reverse Hölder index can be improved to $q \geq (n+1)/2$ in the case of Euclidean space setting. At this moment, combined with more delicate analysis, we can remove the $C^1$-regularity of the Schrödinger harmonic function. Moreover, based on some new observations, we establish a new Calderón reproducing formula, which plays a crucial role in our proof; see Lemma 6 for more details.

The paper is organized as follows. In Section 2, we begin with a brief overview of our settings, i.e., the metric measure space with a Dirichlet form. Next, we recall the definition of the reverse Hölder class and the Muckenhoupt weight and finally state the main result of this paper. In Section 3, we establish some properties for the Schrödinger harmonic functions, which satisfy Carleson-type conditions. In the last two sections, we prove the main result.

Throughout the paper, we denote by the letter $C$ (or $c$) a positive constant that is independent of the essential parameters but may vary from line to line.

## 2. Main Result

Before stating the main result, we first briefly describe our Dirichlet metric measure space settings; see [19–22] for more details. Suppose that $X$ is a separable, connected, locally compact and metrisable space. Let $\mu$ be a Borel measure that is strictly positive on non-empty open sets and finite on compact sets. We consider a regular and strongly local Dirichlet form $\mathscr{E}$ on $L^2(X,\mu)$ with dense domain $\mathscr{D} \subset L^2(X,\mu)$ (see [20] or [21] for an accurate definition). Suppose that $\mathscr{E}$ admits a "*carré du champ*", which means that, for all $f, g \in \mathscr{D}, \Gamma(f,g)$ is absolutely continuous with respect to the measure $\mu$. Hereafter, for simplicity of notation, let $\langle \nabla_x f, \nabla_x g\rangle$ denote the energy density $\frac{d\Gamma(f,g)}{d\mu}$ and $|\nabla_x f|$ denote the square root of $\frac{d\Gamma(f,f)}{d\mu}$. Assume the space $(X,\mu,\mathscr{E})$ is endowed with the intrinsic (pseudo-)distance on $X$ related to $\mathscr{E}$, which is defined by setting

$$d(x,y) := \sup\{f(x) - f(y) : f \in \mathscr{D}_{\mathrm{loc}} \cap C(X),\ |\nabla_x f| \le 1 \text{ a.e.}\},$$

where $C(X)$ is the space of continuous functions on $X$. Suppose $d$ is indeed a distance and induces a topology equivalent to the original topology on $X$. As a summary of the above situation, we will say that $(X,d,\mu,\mathscr{E})$ is a complete Dirichlet metric measure space.

Let the domain $\mathscr{D}$ be equipped with the norm $(\|f\|_2^2 + \mathscr{E}(f,f))^{1/2}$. We can easily see that it is a Hilbert space and denote it by $W^{1,2}(X)$. Given an open set $U \subset X$, we define the Sobolev spaces $W^{1,p}(U)$ and $W_0^{1,p}(U)$ in the usual sense (see [22–24]). With respect to the Dirichlet form, there exists an operator $\mathcal{L}$ with dense domain $\mathscr{D}(\mathcal{L})$ in $L^2(X,\mu), \mathscr{D}(\mathcal{L}) \subset W^{1,2}(X)$, such that

$$\int_X \mathcal{L}f(x)g(x)d\mu(x) = \mathscr{E}(f,g),$$

for all $f \in \mathscr{D}(\mathcal{L})$ and each $g \in W^{1,2}(X)$.

We denote by $B(x,r)$ the open ball with center $x$ and radius $r$ and set $\lambda B(x,r) := B(x,\lambda r)$. We suppose that $\mu$ is doubling, i.e., there exists a constant $C_d > 0$ such that, for every ball $B(x,r) \subset X$,

$$\mu(B(x,2r)) \le C_d \mu(B(x,r)) < \infty. \tag{2}$$

Note that $\mu$ is doubling implies there exists $Q > 1$ such that, for any $0 < r < R < \infty$ and $x \in X$,

$$\mu(B(x,R)) \le C_d \left(\frac{R}{r}\right)^Q \mu(B(x,r)),$$

and the reverse doubling property holds on a connected space (cf. [25] Remark 8.1.15 or [26] Proposition 5.2), i.e., there exist constants $0 < n \le Q$ and $0 < c < 1$ such that, for any $0 < r < R < \infty$ and $x \in X$,

$$\mu(B(x,r)) \le C\left(\frac{r}{R}\right)^n \mu(B(x,R)). \tag{3}$$

There also exist constants $C > 0$ and $0 \le N \le Q$ such that

$$\mu(B(y,r)) \le C\left(1 + \frac{d(x,y)}{r}\right)^N \mu(B(x,r)) \tag{4}$$

uniformly for all $x, y \in X$ and $r > 0$. Indeed, property (4) with $N = Q$ is a direct consequence of the doubling property (2) and the triangle inequality of the metric $d$. It is worth pointing out that $N$ can be chosen to be zero in the cases of Euclidean space, the Lie group of polynomial growth and metric space with a uniformly distributed measure.

Suppose that $(X, d, \mu, \mathscr{E})$ admits an $L^2$-Poincaré inequality, namely, there exists a constant $C_P > 0$ such that

$$\left(\fint_B |f - f_B|^2 d\mu\right)^{1/2} \leq C_P r_B \left(\fint_B |\nabla_x f|^2 d\mu\right)^{1/2}, \tag{5}$$

for all balls $B = B(x_B, r_B)$ and $W^{1,2}(B)$ functions $f$, where $f_B$ denotes the mean (or average) of $f$ over $B$.

We suppose that $V$ is a non-trivial potential satisfying $0 \leq V \in A_\infty(X) \cap RH_q(X)$, where the Muckenhoupt weight class $A_\infty(X)$ and the reverse Hölder class $RH_q(X)$ are defined as follows (cf. [27,28]).

**Definition 1.**

(i) *We say that a nonnegative function $V$ on $X$ belongs to the Muckenhoupt weight class $A_\infty(X)$, if there exists a constant $C > 0$ such that*

$$\sup_B \fint_B V d\mu \left(\inf_{x \in B} V\right)^{-1} \leq C,$$

*where the infimum is understood as the essential infimum or there exists constant $1 < p < \infty$ and $C > 0$ such that*

$$\sup_B \fint_B V d\mu \left(\fint_B V^{\frac{1}{1-p}} d\mu\right)^{p-1} \leq C.$$

(ii) *For any $1 < q \leq \infty$, we say that a nonnegative function $V$ on $X$ belongs to the reverse Hölder class $RH_q(X)$, if there exists a constant $C > 0$ such that*

$$\left(\fint_B V^q d\mu\right)^{1/q} \leq C \fint_B V d\mu,$$

*for any ball $B \subset X$, with the usual modification when $q = \infty$.*

When $X = \mathbb{R}^n$, it is well known that $A_\infty(\mathbb{R}^n) = \bigcup_{1<q\leq\infty} RH_q(\mathbb{R}^n)$. However, in general metric measure space $X$, this relationship between the reverse Hölder classes and the Muckenhoupt weight may not hold; see [28] (Chapter 1). We point out that, if the measure $\mu$ on $X$ is doubling and the potential $V$ belongs to $A_\infty(X)$, then the induced measure $V d\mu$ is also doubling (cf. [28] Chapter 1).

Let us recall the definition of the critical function $\rho(x)$ associated with the potential $V$ (see [29] Definition 1.3). For all $x \in X$, let

$$\rho(x) := \sup\left\{r > 0 : r^2 \fint_{B(x,r)} V d\mu \leq 1\right\}.$$

Since the potential $V$ is non-trivial, it holds that $0 < \rho(x) < \infty$ for every $x \in X$. Additionally, by the results of Yang-Zhou [30] (Lemma 2.1 & Proposition 2.1), the critical function satisfies the following property. If $V \in A_\infty(X) \cap RH_q(X)$ with $q > \max\{1, Q/2\}$, then there exist constants $k_0 \geq 1$ and $C > 0$ such that, for all $x, y \in X$,

$$C^{-1}\rho(x)\left(1 + \frac{d(x,y)}{\rho(x)}\right)^{-k_0} \leq \rho(y) \leq C\rho(x)\left(1 + \frac{d(x,y)}{\rho(x)}\right)^{k_0/(k_0+1)}. \tag{6}$$

In this paper, we consider the Schrödinger operator

$$\mathscr{L} = \mathcal{L} + V.$$

Throughout this paper, we denote, by $\mathscr{P}_t = e^{-t\sqrt{\mathscr{L}}}$, the Schrödinger Poisson semigroup associated with $\mathscr{L}$ and, by $p_t^v(x,y)$, the kernel of $\mathscr{P}_t = e^{-t\sqrt{\mathscr{L}}}$. Due to the perturbation of $V$, the Schrödinger Poisson kernel and its time derivatives admit the Poisson upper bound with an additional polynomial decay (see [18])—namely, for any $k \in \{0\} \cup \mathbb{N}$ and $K > 0$, there exists a constant $C = C(k, K) > 0$ such that

$$|t^k \partial_t^k p_t^v(x,y)| \leq C \frac{t}{t + d(x,y)} \frac{1}{\mu(B(x, t + d(x,y)))} \left(1 + \frac{t + d(x,y)}{\rho(x)}\right)^{-K}.$$

For more results about the Schrödinger operator and their applications, we refer the reader to [31–44].

Let us recall the definition of $\mathscr{L}_+$-harmonic functions on the upper half-space. A function $u \in W^{1,2}(X \times \mathbb{R}_+)$ is said to be an $\mathscr{L}_+$-harmonic function on $X \times \mathbb{R}_+$, if, for every Lipschitz function $\phi$ with compact support in $X \times \mathbb{R}_+$, it holds that

$$\int_0^\infty \int_X \partial_t u \partial_t \phi d\mu dt + \int_0^\infty \int_X \langle \nabla_x u, \nabla_x \phi \rangle d\mu dt + \int_0^\infty \int_X V u \phi d\mu dt = 0.$$

Suppose $-1/2 < \alpha < 0$. We define $\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}(X \times \mathbb{R}_+)$ as the class of all $\mathscr{L}_+$-harmonic functions $u$ satisfying

$$\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} := \sup_{x_B, r_B} \frac{1}{[\mu(B(x_B, r_B))]^\alpha} \left(\int_0^{r_B} \fint_{B(x_B, r_B)} |t \nabla u(x,t)|^2 d\mu(x) \frac{dt}{t}\right)^{1/2} < \infty.$$

The definition of the Morrey spaces refers to [8,42,45]. For every $-1/2 < \alpha < 0$, the square Morrey space $L^{2,\alpha}(X)$ is defined as

$$L^{2,\alpha}(X) := \left\{ f \in L^2_{\mathrm{loc}}(X) : \sup_{B \subset X} \frac{1}{[\mu(B)]^{2\alpha}} \fint_B |f(x)|^2 d\mu(x) < \infty \right\}.$$

This is a Banach space with respect to the norm

$$\|f\|_{L^{2,\alpha}} := \sup_{B \subset X} \frac{1}{[\mu(B)]^\alpha} \left( \fint_B |f(x)|^2 d\mu(x) \right)^{1/2}.$$

The following theorem is the main result of this paper.

**Theorem 1.** *Assume that $(X, d, \mu, \mathscr{E})$ is a complete Dirichlet metric measure space that satisfies the doubling condition (2) with $Q > 1$, and admits an $L^2$-Poincaré inequality (5). Let $0 \leq V \in A_\infty(X) \cap RH_q(X)$ with $q \geq (Q+1)/2$, and $-1/2 < \alpha < 0$.*

(i) *If $f \in L^{2,\alpha}(X)$, then $u(x,t) = \mathscr{P}_t f(x) \in \mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}(X \times \mathbb{R}_+)$, and there exists a constant $C > 0$, independent of $f$, such that*

$$\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} \leq C \|f\|_{L^{2,\alpha}}.$$

(ii) *Further assume that $\max\{-1/2, -1/2N\} < \alpha < 0$. If $u \in \mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}(X \times \mathbb{R}_+)$, then there exists a function $f \in L^{2,\alpha}(X)$ such that $u(x,t) = \mathscr{P}_t f(x)$. Moreover, there exists a constant $C > 0$, independent of $u$, such that*

$$\|f\|_{L^{2,\alpha}} \leq C \|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}.$$

**Remark 1.**

(i) *In Theorem 1, we assume that the reverse Hölder index $q$ is not less than $(Q+1)/2$. However, the observant readers might notice that, in [29], Shen assumed that the nonnegative potential $V$ belongs to $RH_q(\mathbb{R}^Q)$ for some $q \geq Q/2$. However, we consider the boundary value problem of the Schrödinger equation*

$$-\partial_t^2 u + \mathcal{L}u + Vu = 0$$

*on the upper half-space $X \times \mathbb{R}_+$. In order to make sure the above Schrödinger harmonic function is Hölder continuous on $X \times \mathbb{R}_+$, the critical reverse Hölder index $(Q+1)/2$ seems to be the least condition via the natural extension $V(\cdot,t) := V(\cdot)$ for all $t > 0$. One might wonder if there is any possibility of relaxing the requirement $q \geq (Q+1)/2$ in Theorem 1 to $q > 1$ together with $q \geq Q/2$. From the initial value to the solution, this is ensured by the Caccioppoli inequality for the Schrödinger Poisson semigroup; see Proposition 3 for more details. To the contrary, from the solution to the initial value, this is an interesting problem to be solved.*

(ii) *The range of $\alpha$ in Theorem 1 (ii) is slightly different from that in (i). This assumption $-1/2N < \alpha < 0$ first appears in Lemma 3 below, which is caused by the time regularity of $\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}$-function*

$$|t\partial_t u(x,t)| \leq C[\mu(B(x,t))]^{\alpha}\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}.$$

*Since the pointwise upper bound of the time regularity of $\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}$-function has to do with the measure of some ball to the $\alpha$ power, the condition $2\alpha N + 1 > 0$ ensures the series in Lemma 3 is convergent. In fact, for metric measure space $X$, the nonnegative parameter $N$ arises automatically if we want to calculate the ratio of the volumes of two balls with different centers. However, this would not occur in the cases of Euclidean space, the Lie group of polynomial growth and metric space with a uniformly distributed measure. We remark that $N$ can be chosen to be $0$ under these settings, and hence the assumption $-1/2N < \alpha < 0$ is superfluous.*

## 3. Schrödinger Harmonic Functions Satisfying Carleson

In this section, we will establish some properties of $\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}$-function.

**Lemma 1.** *Assume the Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (2) and (5). Let $V \in A_\infty(X) \cap RH_q(X)$ for some $q > \max\{1, Q/2\}$. If $\mathscr{L}u = \mathcal{L}u + Vu = 0$ holds in a bounded domain $\Omega \subset X$, then there exists a constant $C > 0$ such that, for any ball $B = B(x_B, r_B)$ with $2B \subset \Omega$,*

$$\|u\|_{L^\infty(B)} \leq C \fint_{2B} |u| d\mu.$$

*Furthermore, $u$ is locally Hölder continuous in $\Omega$, and there exists a constant $\theta \in (0, \min\{1, 2 - Q/q\})$ such that, for any $x, y \in \frac{1}{2}B$,*

$$|u(x) - u(y)| \leq C\left(\frac{d(x,y)}{r_B}\right)^{\theta} \|u\|_{L^\infty(B)} \left(1 + r_B^2 \fint_B V d\mu\right).$$

**Proof.** For the proof, we refer to [18] (Proposition 2.12). □

Let us extend the potential $V$ to the upper half-space by defining $V(x,t) := V(x)$ for all $t \in \mathbb{R}$. We can easily find that $V(x,t) \in A_\infty(X \times \mathbb{R}) \cap RH_q(X \times \mathbb{R})$ with $q > (Q+1)/2$, if $0 \leq V(x) \in A_\infty(X) \cap RH_q(X)$ with $q > (Q+1)/2$. Therefore, it follows from Lemma 1 that $\mathscr{L}_+$-harmonic functions are locally Hölder continuous on $X \times \mathbb{R}_+$.

**Lemma 2.** *Suppose the complete Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (2) and (5). Let $0 \leq V \in A_\infty(X) \cap RH_q(X)$ with $q > (Q+1)/2$. If $u \in \mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}(X \times \mathbb{R}_+)$ with $-1/2 < \alpha < 0$, then there exists a constant $C > 0$ such that, for all $x \in X$ and $t > 0$,*

$$|t\partial_t u(x,t)| \leq C[\mu(B(x,t))]^\alpha \|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}.$$

**Proof.** Let $\epsilon > 0$. Given $-\epsilon < h < \epsilon$, for any $x \in X$ and $t > \epsilon$, set

$$u(x,t;h) := \frac{u(x,t+h) - u(x,t)}{h}.$$

It follows that $u(\cdot,\cdot;h)$ is an $\mathscr{L}_+$-harmonic function on $X \times (\epsilon, \infty)$; see the proof of [18] (Lemma 4.1).

Then, by the mean value property in Lemma 1, we conclude that, for any $t > 2\epsilon$,

$$|u(x,t;h)| \leq C\left(\fint_{B(x,t/2)} \fint_{t/2}^{3t/2} |u(y,s;h)|^2 ds d\mu(y)\right)^{1/2}, \tag{7}$$

which, combined with the argument in the proof of Jiang-Li [18] (Lemma 4.1), yields, for each $t > 3\epsilon$, that

$$|tu(x,t;h)| \leq C\left(\fint_{B(x,2t)} \int_0^{2t} |s\partial_s u(y,s)|^2 \frac{ds}{s} d\mu(y)\right)^{1/2}.$$

This implies that, for each $t > 3\epsilon$,

$$|t\partial_t u(x,t)| \leq C[\mu(B(x,t))]^\alpha \|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}.$$

Letting $\epsilon \to 0$ indicates that the above estimate holds for every $t > 0$. □

**Lemma 3.** *Assume the complete Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (2) and (5). Suppose $0 \leq V \in A_\infty(X) \cap RH_q(X)$ with $q > (Q+1)/2$, and $\max\{-1/2, -1/2N\} < \alpha < 0$. If $u \in \mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}(X \times \mathbb{R}_+)$, then there exists a constant $C > 0$ such that, for any $x \in X$ and $t, \epsilon > 0$,*

$$\begin{aligned}
&\int_X \frac{|u(y,\epsilon)|^2}{(t+d(x,y))\mu(B(x,t+d(x,y)))} d\mu(y) \\
&\leq C(1+t^{-1})\|u(\cdot,\epsilon)\|^2_{L^\infty(B(x,2))} + C([\mu(B(x,1))]^{2\alpha} + \epsilon^{2N\alpha}[\mu(B(x,\epsilon))]^{2\alpha})\|u\|^2_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}.
\end{aligned}$$

**Proof.** By Lemma 1, $u(\cdot,\cdot)$ is locally bounded and locally Hölder continuous in $X \times \mathbb{R}_+$. The integral is split into $B(x,1)$ and $X \backslash B(x,1)$. For the local part $B(x,1)$, it holds that

$$\int_{B(x,1)} \frac{|u(y,\epsilon)|^2}{(t+d(x,y))\mu(B(x,t+d(x,y)))} d\mu(y) \leq \frac{C}{t}\|u(\cdot,\epsilon)\|^2_{L^\infty(B(x,1))}.$$

For the global part $X \backslash B(x,1)$, by the annulus argument, we have

$$\begin{aligned}
&\int_{X\backslash B(x,1)} \frac{|u(y,\epsilon)|^2}{(t+d(x,y))\mu(B(x,t+d(x,y)))} d\mu(y) \\
&\leq C\sum_{j=1}^\infty 2^{-j} \fint_{2^{j-1}}^{2^j} \fint_{B(x,2^j)} |u(y,\epsilon)|^2 d\mu(y) d\, s \\
&\leq C\sum_{j=1}^\infty 2^{-j} \fint_{E_j} |u(y,\epsilon) - u(y,s)|^2 d\mu(y) d\, s
\end{aligned}$$

$$
\begin{aligned}
&+C\sum_{j=1}^{\infty}2^{-j}\fint_{E_j}|u(y,s)-u_{E_j}|^2d\mu(y)ds+C\sum_{j=1}^{\infty}2^{-j}|u_{E_j}|^2\\
&=:C(\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3),
\end{aligned}
$$

where we denote the cylinder $B(x,2^j)\times[2^{j-1},2^j)$ by $E_j$ for simplicity.

For the term $\mathrm{I}_1$, it holds by Lemma 2 and $-1/2N<\alpha$ that

$$
\begin{aligned}
\mathrm{I}_1&=\sum_{j=1}^{\infty}2^{-j}\fint_{E_j}\left|\int_{\epsilon}^{s}\partial_r u(y,r)dr\right|^2d\mu(y)d\,s\\
&\le C\|u\|^2_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}\sum_{j=1}^{\infty}2^{-j}\fint_{E_j}\left(\int_{\epsilon}^{s}[\mu(B(y,r))]^{\alpha}\frac{dr}{r}\right)^2d\mu(y)d\,s\\
&\le C\|u\|^2_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}\sum_{j=1}^{\infty}2^{-j}\left\{[\mu(B(x,1))]^{2\alpha}+\left(\frac{2^j}{\epsilon}\right)^{-2N\alpha}[\mu(B(x,\epsilon))]^{2\alpha}\right\}\\
&\le C\Big([\mu(B(x,1))]^{2\alpha}+\epsilon^{2N\alpha}[\mu(B(x,\epsilon))]^{2\alpha}\Big)\|u\|^2_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}.
\end{aligned}
$$

Above, in the second inequality, we used the fact that

$$
\begin{aligned}
&\int_{\epsilon}^{s}[\mu(B(y,r))]^{\alpha}\frac{dr}{r}\\
&\le\int_{\epsilon}^{s}[\mu(B(y,r))]^{\alpha}\frac{dr}{r}\Big(\chi_{(0,2^{j-1})}(\epsilon)+\chi_{(2^{j-1},\infty)}(\epsilon)\Big)\\
&\le\int_{\epsilon}^{\infty}[\mu(B(y,r))]^{\alpha}\frac{dr}{r}\chi_{(0,2^{j-1})}(\epsilon)+\int_{2^{j-1}}^{\infty}[\mu(B(y,r))]^{\alpha}\frac{dr}{r}\\
&\le C\left\{\int_{\epsilon}^{\infty}\left(\frac{r}{\epsilon}\right)^{n\alpha}[\mu(B(y,\epsilon))]^{\alpha}\frac{dr}{r}\chi_{(0,2^{j-1})}(\epsilon)+\int_{2^{j-1}}^{\infty}\left(\frac{r}{2^{j-1}}\right)^{n\alpha}[\mu(B(y,2^{j-1}))]^{\alpha}\frac{dr}{r}\right\}\\
&\le C\left\{\left(1+\frac{d(x,y)}{\epsilon}\right)^{-N\alpha}[\mu(B(x,\epsilon))]^{\alpha}\chi_{(0,2^{j-1})}(\epsilon)+\left(1+\frac{d(x,y)}{2^{j-1}}\right)^{-N\alpha}[\mu(B(x,2^j))]^{\alpha}\right\}\\
&\le C\left\{[\mu(B(x,1))]^{\alpha}+\left(\frac{2^j}{\epsilon}\right)^{-N\alpha}[\mu(B(x,\epsilon))]^{\alpha}\right\}.
\end{aligned}
$$

Now, we put $u_s(\cdot):=u(\cdot,s)$. For the term $\mathrm{I}_2$, we use the Poincaré inequality to deduce that

$$
\begin{aligned}
\mathrm{I}_2&\le 2\sum_{j=1}^{\infty}2^{-j}\left(\fint_{2^{j-1}}^{2^j}\fint_{B(x,2^j)}\left|u(y,s)-(u_s)_{B(x,2^j)}\right|^2d\mu(y)ds+\fint_{2^{j-1}}^{2^j}\left|(u_s)_{B(x,2^j)}-u_{E_j}\right|^2ds\right)\\
&\le C\sum_{j=1}^{\infty}2^{-j}\left(2^{2j}\fint_{2^{j-1}}^{2^j}\fint_{B(x,2^j)}|\nabla_y u(y,s)|^2d\mu(y)ds+\fint_{2^{j-1}}^{2^j}\left|(u_s)_{B(x,2^j)}-u_{E_j}\right|^2ds\right). \qquad (8)
\end{aligned}
$$

By the Hölder inequality and the Poincaré inequality, it holds that

$$
\begin{aligned}
&\fint_{2^{j-1}}^{2^j}\left|(u_s)_{B(x,2^j)}-u_{E_j}\right|^2d\,s\\
&=\fint_{2^{j-1}}^{2^j}\left|\fint_{B(x,2^j)}u(y,s)d\mu(y)-\fint_{2^{j-1}}^{2^j}\fint_{B(x,2^j)}u(y,r)d\mu(y)dr\right|^2d\,s\\
&=\fint_{2^{j-1}}^{2^j}\left|\fint_{2^{j-1}}^{2^j}\fint_{B(x,2^j)}u(y,s)-u(y,r)d\mu(y)dr\right|^2d\,s
\end{aligned}
$$

$$\leq \fint_{B(x,2^j)} \fint_{2^{j-1}}^{2^j} \fint_{2^{j-1}}^{2^j} |u(y,s)-u(y,r)|^2 drdsd\mu(y)$$
$$\leq C2^{2j} \fint_{B(x,2^j)} \fint_{2^{j-1}}^{2^j} |\partial_s u(y,s)|^2 dsd\mu(y).$$

This, together with (8), gives that

$$\begin{aligned}
\mathrm{I}_2 &\leq C\sum_{j=1}^{\infty} 2^{-j}2^{2j} \fint_{2^{j-1}}^{2^j} \fint_{B(x,2^j)} |\nabla u(y,s)|^2 d\mu(y)d\,s \\
&\leq C\sum_{j=1}^{\infty} 2^{-j} \int_0^{2^j} \fint_{B(x,2^j)} |s\nabla u(y,s)|^2 d\mu(y)\frac{ds}{s} \\
&\leq C\sum_{j=1}^{\infty} 2^{-j}[\mu(B(x,2^j))]^{2\alpha} \|u\|^2_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} \\
&\leq C[\mu(B(x,1))]^{2\alpha} \|u\|^2_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}.
\end{aligned}$$

As $E_j = B(x,2^j)\times[2^{j-1},2^j)$, it holds $E_j, E_{j+1} \subset B(x,2^{j+1})\times[2^{j-1},2^{j+1}) =: F_{j+1}$. For the term $\mathrm{I}_3$, one writes

$$\begin{aligned}
\mathrm{I}_3 &\leq \sum_{j=1}^{\infty} 2^{-j}\left(|u_{E_1}| + \sum_{i=2}^{j}|u_{E_i}-u_{E_{i-1}}|\right)^2 \\
&\leq \sum_{j=1}^{\infty} 2^{-j}\left(|(u-u(\cdot,\epsilon))_{E_1}| + \|u(\cdot,\epsilon)\|_{L^\infty(B(x,2))} + \sum_{i=2}^{j}\left(|u_{E_i}-u_{F_i}| + |u_{F_i}-u_{E_{i-1}}|\right)\right)^2.
\end{aligned}$$

It follows from the Poincaré inequality that

$$\begin{aligned}
|u_{E_i}-u_{F_i}| + |u_{F_i}-u_{E_{i-1}}| &\leq C\left(\fint_{2^{i-2}}^{2^i} \fint_{B(x,2^i)} \left|u(y,s)-u_{F_i}\right|^2 d\mu(y)ds\right)^{1/2} \\
&\leq C\left(\fint_{2^{i-2}}^{2^i} \fint_{B(x,2^i)} \left|u(y,s)-(u_s)_{B(x,2^i)}\right|^2 d\mu(y)ds\right)^{1/2} \\
&\quad + C\left(\fint_{2^{i-2}}^{2^i} \left|(u_s)_{B(x,2^i)}-u_{F_i}\right|^2 ds\right)^{1/2} \\
&\leq C2^i\left(\fint_{2^{i-2}}^{2^i} \fint_{B(x,2^i)} |\nabla u(y,s)|^2 d\mu(y)ds\right)^{1/2} \\
&\leq C\left(\int_0^{2^i} \fint_{B(x,2^i)} |s\nabla u(y,s)|^2 d\mu(y)\frac{ds}{s}\right)^{1/2} \\
&\leq C[\mu(B(x,2^i))]^{\alpha}\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} \\
&\leq C[\mu(B(x,1))]^{\alpha}\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}},
\end{aligned} \tag{9}$$

and from Lemma 2 that

$$\begin{aligned}
|(u-u(\cdot,\epsilon))_{E_1}| &\leq \fint_{B(x,2)\times[1,2)} |u(y,s)-u(y,\epsilon)|d\mu(y)ds \\
&\leq \fint_{B(x,2)\times[1,2)} \left|\int_\epsilon^s \partial_r u(y,r)dr\right| d\mu(y)ds
\end{aligned}$$

$$\leq C\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} \fint_{B(x,2)\times[1,2)} \left|\int_\epsilon^s [\mu(B(y,r))]^\alpha \frac{dr}{r}\right| d\mu(y)ds$$
$$\leq C([\mu(B(x,1))]^\alpha + \epsilon^{N\alpha}[\mu(B(x,\epsilon))]^\alpha)\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}. \quad (10)$$

Here, we used the fact that

$$\begin{aligned}\left|\int_\epsilon^s [\mu(B(y,r))]^\alpha \frac{dr}{r}\right| &\leq C\left(\int_\epsilon^\infty [\mu(B(y,r))]^\alpha \frac{dr}{r} + \int_1^\infty [\mu(B(y,r))]^\alpha \frac{dr}{r}\right)\\ &\leq C\left(\int_\epsilon^\infty \left(\frac{r}{\epsilon}\right)^{n\alpha}[\mu(B(y,\epsilon))]^\alpha \frac{dr}{r} + \int_1^\infty r^{n\alpha}[\mu(B(y,1))]^\alpha \frac{dr}{r}\right)\\ &\leq C(\epsilon^{N\alpha}[\mu(B(x,\epsilon))]^\alpha + [\mu(B(x,1))]^\alpha).\end{aligned}$$

The above two estimates (9) and (10) yield that

$$\begin{aligned}\mathrm{I}_3 &\leq C\sum_{j=1}^\infty 2^{-j}\|u(\cdot,\epsilon)\|^2_{L^\infty(B(x,2))}\\ &\quad + C\sum_{j=1}^\infty 2^{-j}\left([\mu(B(x,1))]^\alpha\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} + \epsilon^{N\alpha}[\mu(B(x,\epsilon))]^\alpha\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}\right)^2\\ &\leq C\left(\|u(\cdot,\epsilon)\|^2_{L^\infty(B(x,2))} + [\mu(B(x,1))]^{2\alpha}\|u\|^2_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} + \epsilon^{2N\alpha}[\mu(B(x,\epsilon))]^{2\alpha}\|u\|^2_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}\right).\end{aligned}$$

In combination with the estimates of $\mathrm{I}_1, \mathrm{I}_2$ and $\mathrm{I}_3$, we obtain the required conclusion. □

**Lemma 4.** *Suppose the complete Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (2) with $Q > 1$ and admits (5). Let $0 \leq V \in A_\infty(X) \cap RH_q(X)$ with $q > (Q+1)/2$. Assume that $w$ is a solution to $(-\partial_t^2 + \mathscr{L})w = 0$ on $X \times \mathbb{R}$. If there exists $m > 0$ such that*

$$\int_{\mathbb{R}}\int_X \frac{|w(y,t)|^2}{(1+t+d(x,y))^{m+1}\mu(B(x,1+t+d(x,y)))} d\mu(y)dt < \infty,$$

*then $w \equiv 0$.*

**Proof.** For the proof, we refer to [18] (Corollary 4.5). □

**Proposition 1.** *Suppose the complete Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (2) with $Q > 1$ and admits (5). Let $0 \leq V \in A_\infty(X) \cap RH_q(X)$ with $q > (Q+1)/2$. Assume that $u \in \mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}(X \times \mathbb{R}_+)$ with $\max\{-1/2, -1/2N\} < \alpha < 0$. For any $x \in X$ and $s, t > 0$, it holds that*

$$u(x, t+s) = \mathscr{P}_t(u(\cdot, s))(x).$$

**Proof.** For each $t > 0$, let

$$v(x,t) := u(x, t+s) - \mathscr{P}_t(u(\cdot, s))(x).$$

As $u(\cdot, \cdot + s)$ is Hölder continuous on $X \times (-s, \infty)$ and $u(\cdot, s)$ is Hölder continuous on $X$, we see that

$$v(x, 0) := \lim_{t\to 0^+} v(x,t) = \lim_{t\to 0^+}\{u(x,t+s) - \mathscr{P}_t(u(\cdot,s))(x)\} = 0.$$

We extend $v(x,t)$ to $X \times \mathbb{R}$ as

$$w(x,t) := \begin{cases} v(x,t), & t > 0; \\ 0, & t = 0; \\ -v(x,-t), & t < 0. \end{cases}$$

Then, $w$ is a solution to the Schrödinger equation $(-\partial_t^2 + \mathscr{L})w = 0$ on $X \times \mathbb{R}$. We fix a point $y_0 \in X$. By Lemma 4 and the fact that $w$ is odd with respect to $t$, it is sufficient to show that there exists $m > 0$ such that

$$\int_0^\infty \int_X \frac{|w(x,t)|^2}{(1+t+d(x,y_0))^{m+1}\mu(B(y_0,1+t+d(x,y_0)))} d\mu(x)dt < \infty.$$

By Lemma 3, we have

$$\begin{aligned}
&\int_0^\infty \int_X \frac{|u(x,s+t)|^2}{(1+t+d(x,y_0))^{m+1}\mu(B(y_0,1+t+d(x,y_0)))} d\mu(x)d\,t \\
&\le \int_0^\infty \frac{1}{(1+t)^m} \int_X \frac{|u(x,s+t)|^2}{(1+d(x,y_0))\mu(B(y_0,1+d(x,y_0)))} d\mu(x)d\,t \\
&\le C\int_0^\infty \frac{1}{(1+t)^m} \|u(\cdot,s+t)\|^2_{L^\infty(B(y_0,2))} d\,t \\
&+ C\int_0^\infty \frac{1}{(1+t)^m} \left\{ \left( [\mu(B(y_0,1))]^{2\alpha} + (s+t)^{2N\alpha}[\mu(B(y_0,s+t))]^{2\alpha} \right) \|u\|^2_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} \right\} d\,t \\
&\le C\int_0^\infty \frac{1}{(1+t)^m} \|u(\cdot,s+t)\|^2_{L^\infty(B(y_0,2))} dt \\
&+ C\int_0^\infty \frac{1}{(1+t)^m} \left\{ \left( [\mu(B(y_0,1))]^{2\alpha} + s^{2N\alpha}[\mu(B(y_0,s))]^{2\alpha} \right) \|u\|^2_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} \right\} dt.
\end{aligned}$$

It follows from Lemma 2 that

$$\begin{aligned}
\|u(\cdot,s+t)\|_{L^\infty(B(y_0,2))} &\le \|u(\cdot,s+t) - u(\cdot,s)\|_{L^\infty(B(y_0,2))} + \|u(\cdot,s)\|_{L^\infty(B(y_0,2))} \\
&\le \left\| \int_s^{s+t} |\partial_r u(\cdot,r)| dr \right\|_{L^\infty(B(y_0,2))} + \|u(\cdot,s)\|_{L^\infty(B(y_0,2))} \\
&\le C\left(1+\frac{2}{s}\right)^{-N\alpha} [\mu(B(y_0,s))]^\alpha \|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} + \|u(\cdot,s)\|_{L^\infty(B(y_0,2))} \\
&= C(\alpha, N, y_0, s, \|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}, \|u(\cdot,s)\|_{L^\infty(B(y_0,2))}).
\end{aligned}$$

Above, we used the fact that

$$\begin{aligned}
\sup_{x\in B(y_0,2)} \int_s^{s+t} |\partial_r u(x,r)| dr &\le C \sup_{x\in B(y_0,2)} \int_s^{s+t} [\mu(B(x,r))]^\alpha \|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} \frac{dr}{r} \\
&\le C \sup_{x\in B(y_0,2)} \int_s^\infty \left(\frac{r}{s}\right)^{n\alpha} [\mu(B(x,s))]^\alpha \|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} \frac{dr}{r} \\
&\le C\left(1+\frac{2}{s}\right)^{-N\alpha} [\mu(B(y_0,s))]^\alpha \|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}.
\end{aligned}$$

Therefore, one has

$$\begin{aligned}
&\int_0^\infty \int_X \frac{|u(x,s+t)|^2}{(1+t+d(x,y_0))^{m+1}\mu(B(y_0,1+t+d(x,y_0)))} d\mu(x)dt \\
&\le C(\alpha, N, y_0, s, \|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}, \|u(\cdot,s)\|_{L^\infty(B(y_0,2))}) \int_0^\infty \frac{dt}{(1+t)^m}
\end{aligned}$$

$$\leq C(\alpha, N, y_0, s, \|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}, \|u(\cdot,s)\|_{L^\infty(B(y_0,2))}) < \infty, \tag{11}$$

provided $m > 1$.

For the remaining term, we need to prove that

$$\mathrm{I} := \int_0^\infty \int_X \frac{|\mathscr{P}_t(u(\cdot,s))(x)|^2}{(1+t+d(x,y_0))^{m+1}\mu(B(y_0,1+t+d(x,y_0)))} d\mu(x)dt < \infty.$$

By the Poisson upper bound and the Hölder inequality, it holds that, for all $t > 0$

$$\begin{aligned}|\mathscr{P}_t(u(\cdot,s))(x)|^2 &\leq C|\mathscr{P}_t(1)(x)| \int_X \frac{t|u(y,s)|^2}{(t+d(x,y))\mu(B(x,t+d(x,y)))} d\mu(y)\\ &\leq C\int_X \frac{t|u(y,s)|^2}{(t+d(x,y))\mu(B(x,t+d(x,y)))} d\mu(y).\end{aligned}$$

Hence, we have

$$\begin{aligned}\mathrm{I} &\leq C\int_0^\infty \int_X \int_X \frac{1}{(1+t+d(x,y_0))^{m+1}\mu(B(y_0,1+t+d(x,y_0)))}\\ &\quad\times \frac{t|u(y,s)|^2}{(t+d(x,y))\mu(B(x,t+d(x,y)))} d\mu(x)d\mu(y)dt\\ &\leq C\left\{\int_0^\infty \int_X \int_{B(y_0,d(y,y_0)/2)} + \int_0^\infty \int_X \int_{B(y_0,d(y,y_0)/2)^\complement}\right\} \cdots d\mu(x)d\mu(y)dt\\ &=: \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

For any $x \in B(y_0, d(y,y_0)/2)$, we have $d(x,y) > d(y,y_0) - d(x,y_0) > d(y,y_0)/2$. Hence, by (4) and Lemma 3, we have

$$\begin{aligned}\mathrm{I}_1 &\leq C\int_0^\infty \frac{1}{(1+t)^m} dt \int_X \frac{t|u(y,s)|^2}{(t+d(y,y_0))\mu(B(y_0,t+d(y,y_0)))} d\mu(y)\\ &\quad\times \int_X \frac{d\mu(x)}{(1+d(x,y_0))\mu(B(y_0,1+d(x,y_0)))}\\ &\leq C\int_0^\infty \frac{1}{(1+t)^m} dt \int_X \frac{t|u(y,s)|^2}{(t+d(y,y_0))\mu(B(y_0,t+d(y,y_0)))} d\mu(y)\\ &\leq C\int_0^\infty \frac{(1+t)\|u(\cdot,s)\|^2_{L^\infty(B(y_0,2))}}{(1+t)^m} dt\\ &\quad+ C\int_0^\infty \frac{t[\mu(B(y_0,1))]^{2\alpha}\|u\|^2_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} + ts^{2N\alpha}[\mu(B(y_0,s))]^{2\alpha}\|u\|^2_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}}{(1+t)^m} dt\\ &\leq C(\alpha, N, y_0, s, \|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}, \|u(\cdot,s)\|_{L^\infty(B(y_0,2))}) < \infty,\end{aligned}$$

provided $m > 2$. For any $x \in B(y_0, d(y,y_0)/2)^\complement$, we have $d(x,y_0) > d(y,y_0)/2$. This, together with Lemma 3, yields that

$$\begin{aligned}\mathrm{I}_2 &\leq C\int_0^\infty \frac{dt}{(1+t)^m} \int_X \frac{|u(y,s)|^2 d\mu(y)}{(1+d(y,y_0))\mu(B(y_0,1+d(y,y_0)))}\\ &\quad\times \int_{B(y_0,d(y,y_0)/2)^\complement} \frac{t d\mu(x)}{(t+d(x,y))\mu(B(x,t+d(x,y)))}\\ &\leq C\int_0^\infty \frac{1}{(1+t)^m} dt \int_X \frac{|u(y,s)|^2}{(1+d(y,y_0))\mu(B(y_0,1+d(y,y_0)))} d\mu(y)\end{aligned}$$

$$\leq C\int_0^\infty \frac{\|u(\cdot,s)\|^2_{L^\infty(B(y_0,2))} + \{[\mu(B(y_0,1))]^{2\alpha} + s^{2N\alpha}[\mu(B(y_0,s))]^{2\alpha}\}\|u\|^2_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}}{(1+t)^m}dt$$
$$\leq C(\alpha, N, y_0, s, \|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}, \|u(\cdot,s)\|_{L^\infty(B(y_0,2))}) < \infty,$$

provided $m > 1$. Therefore, it holds that

$$\int_0^\infty \int_X \frac{|\mathscr{P}_t(u(\cdot,s))(x)|^2}{(1+t+d(x,y_0))^{m+1}\mu(B(y_0,1+t+d(x,y_0)))}d\mu(x)dt < \infty,$$

which, together with (11), yields that

$$\int_0^\infty \int_X \frac{|w(x,t)|^2}{(1+t+d(x,y_0))^{m+1}\mu(B(y_0,1+t+d(x,y_0)))}d\mu(x)dt < \infty,$$

provided $m > 2$. The Liouville theorem (Lemma 4) then implies $w(x,t) \equiv 0$, which means $u(x,t+s) \equiv \mathscr{P}_t(u(\cdot,s))(x)$ and thus finishes the proof. □

Next, for every $u \in \mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}(X\times\mathbb{R}_+)$, we will show that $u_s(\cdot) = u(\cdot,s)$ is bounded in $L^{2,\alpha}(X)$ uniformly for all $s > 0$. To this end, we introduce a notation

$$\|\|\mu_{\nabla_t,f}\|\|_\alpha := \sup_{B\subset X}\frac{1}{[\mu(B)]^\alpha}\left(\int_0^{r_B}\fint_B |t\partial_t\mathscr{P}_t f(x)|^2 d\mu(x)\frac{dt}{t}\right)^{1/2},$$

for any

$$f \in \mathcal{M}_2 := \bigcup_{x_0\in X}\bigcup_{0<\beta\leq 1} L^2(X, (1+d(x,x_0))^{-\beta}\mu(B(x_0,1+d(x,x_0)))^{-1}d\mu(x)),$$

and establish Lemmas 5–7 as follows.

**Lemma 5.** *Assume the complete Dirichlet metric measure space $(X,d,\mu,\mathscr{E})$ satisfies (2) with $Q > 1$ and admits (5). Given a ball $B = B(x_B,r_B)$, a function $f \in \mathcal{M}_2$ and an $L^2$-function $g$ supported on $B$, set*

$$F(x,t) := t\partial_t\mathscr{P}_t f(x) \quad \text{and} \quad G(x,t) := t\partial_t\mathscr{P}_t g(x),$$

*for any $(x,t) \in X\times\mathbb{R}_+$. If $\|\|\mu_{\nabla_t,f}\|\|_\alpha < \infty$, then there exists a constant $C > 0$ such that*

$$\int_0^\infty\int_X |F(x,t)G(x,t)|d\mu(x)\frac{dt}{t} \leq C[\mu(B)]^{1/2+\alpha}\|\|\mu_{\nabla_t,f}\|\|_\alpha\|g\|_{L^2(B)}.$$

**Proof.** Let us consider the square function $\mathcal{G}(h)$ given by

$$\mathcal{G}(h)(x) := \left(\int_0^\infty |t\partial_t\mathscr{P}_t h(x)|^2\frac{dt}{t}\right)^{1/2}.$$

By the spectral theory, the function $\mathcal{G}(h)$ is bounded on $L^2(X)$. Let

$$T(B) := \{(x,t)\in X\times\mathbb{R}_+ :\ x\in B,\ 0<t<r_B\} = B\times(0,r_B),$$

and write

$$\int_0^\infty\int_X |F(x,t)G(x,t)|d\mu(x)\frac{dt}{t}$$
$$= \int_{T(2B)} |F(x,t)G(x,t)|d\mu(x)\frac{dt}{t} + \sum_{k=2}^\infty\int_{T(2^kB)\setminus T(2^{k-1}B)} |F(x,t)G(x,t)|d\mu(x)\frac{dt}{t}$$

$$=: \mathrm{A}_1 + \sum_{k=2}^{\infty} \mathrm{A}_k.$$

Using the Hölder inequality and the $L^2$-boundedness of $\mathcal{G}$, we obtain

$$\mathrm{A}_1 \le \left( \int_0^{2r_B} \int_{2B} |t\partial_t \mathscr{P}_t f(x)|^2 d\mu(x) \frac{dt}{t} \right)^{1/2} \|\mathcal{G}(g)\|_{L^2} \le C[\mu(B)]^{1/2+\alpha} |||\mu_{\nabla_t, f}|||_{\alpha} \|g\|_{L^2(B)}.$$

Let us estimate $\mathrm{A}_k$ for $k = 2, 3, \ldots$. Note that, for any $(x,t) \in T(2^k B) \backslash T(2^{k-1}B)$ and $y \in B$, we have $t + d(x,y) \ge 2^{k-2} r_B$. It holds

$$\begin{aligned} |G(x,t)| &= \left| \int_X t\partial_t p_t^{v}(x,y) g(y) d\mu(y) \right| \\ &\le C \int_X \frac{t}{t + d(x,y)} \frac{|g(y)|}{\mu(B(x, t + d(x,y)))} d\mu(y) \\ &\le C \int_X \frac{t}{2^k r_B} \frac{|g(y)|}{\mu(B(x, 2^k r_B))} d\mu(y) \\ &\le C \frac{t}{2^k r_B} \frac{\|g\|_{L^1(B)}}{\mu(2^k B)}, \end{aligned}$$

which, together with the Hölder inequality and (3), implies that

$$\begin{aligned} &\int_{T(2^k B) \backslash T(2^{k-1}B)} |F(x,t) G(x,t)| d\mu(x) \frac{dt}{t} \\ &\le C \left( \int_0^{2^k r_B} ⨍_{2^k B} |t\partial_t \mathscr{P}_t f(x)|^2 d\mu(x) \frac{dt}{t} \right)^{1/2} \|g\|_{L^1(B)} \\ &\le C[\mu(2^k B)]^{\alpha} |||\mu_{\nabla_t, f}|||_{\alpha} \|g\|_{L^1(B)} \\ &\le C 2^{kn\alpha} [\mu(B)]^{1/2+\alpha} |||\mu_{\nabla_t, f}|||_{\alpha} \|g\|_{L^2(B)}. \end{aligned}$$

Summing over $k$ leads to

$$\int_0^{\infty} \int_X |F(x,t) G(x,t)| d\mu(x) \frac{dt}{t} = \sum_{k=1}^{\infty} \mathrm{A}_k \le C[\mu(B)]^{1/2+\alpha} |||\mu_{\nabla_t, f}|||_{\alpha} \|g\|_{L^2(B)}.$$

This completes the proof of Lemma 5. □

**Lemma 6.** *Assume the complete Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (2) with $Q > 1$ and admits (5). Suppose $B, f, g, F, G$ are defined as in Lemma 5. If $|||\mu_{\nabla_t, f}|||_{\alpha} < \infty$, then we have the equality:*

$$\int_X f(x) g(x) d\mu(x) = 4 \int_0^{\infty} \int_X F(x,t) G(x,t) d\mu(x) \frac{dt}{t}.$$

**Proof.** From Lemma 5, we find that

$$\int_0^{\infty} \int_X |F(x,t) G(x,t)| d\mu(x) \frac{dt}{t} < \infty.$$

By the dominated convergence theorem, the following integral converges absolutely and satisfies

$$\int_0^{\infty} \int_X F(x,t) G(x,t) d\mu(x) \frac{dt}{t} = \lim_{\delta \to 0} \int_{\delta}^{1/\delta} \int_X F(x,t) G(x,t) d\mu(x) \frac{dt}{t}.$$

Next, by the commutative property of the semigroup $\{\mathscr{P}_t\}_{t>0}$, we have

$$\int_X F(x,t)G(x,t)d\mu(x) = \int_X f(x)t^2\mathscr{L}\mathscr{P}_{2t}g(x)d\mu(x).$$

This, together with Fubini's theorem, gives

$$\begin{aligned}\int_0^\infty \int_X F(x,t)G(x,t)d\mu(x)\frac{dt}{t} &= \lim_{\delta\to 0}\int_\delta^{1/\delta}\int_X f(x)t^2\mathscr{L}\mathscr{P}_{2t}g(x)d\mu(x)\frac{dt}{t}\\ &= \lim_{\delta\to 0}\int_X f(x)\int_\delta^{1/\delta} t^2\mathscr{L}\mathscr{P}_{2t}g(x)\frac{dt}{t}d\mu(x)\\ &= \lim_{\delta\to 0}\int_X f_1(x)\int_\delta^{1/\delta} t^2\mathscr{L}\mathscr{P}_{2t}g(x)\frac{dt}{t}d\mu(x)\\ &\quad+\lim_{\delta\to 0}\int_X f_2(x)\int_\delta^{1/\delta} t^2\mathscr{L}\mathscr{P}_{2t}g(x)\frac{dt}{t}d\mu(x)\\ &=: \mathrm{I}_1+\mathrm{I}_2,\end{aligned}$$

where $f_1(x) := f\chi_{4B}(x)$ and $f_2(x) := f\chi_{(4B)^\complement}(x)$.

We first consider the term $\mathrm{I}_1$. It follows from the spectral theory that

$$g(x) = 4\lim_{\delta\to 0}\int_\delta^{1/\delta} t^2\mathscr{L}\mathscr{P}_{2t}g(x)\frac{dt}{t}$$

in $L^2(X)$. Hence, it holds

$$\mathrm{I}_1 = \lim_{\delta\to 0}\int_X f_1(x)\int_\delta^{1/\delta} t^2\mathscr{L}\mathscr{P}_{2t}g(x)\frac{dt}{t}d\mu(x) = \frac{1}{4}\int_X f_1(x)g(x)d\mu(x).$$

In order to estimate the term $\mathrm{I}_2$, we need to show that, for any $x\in(4B)^\complement$, there exists a constant $C = C(x_B, r_B) > 0$ such that

$$\sup_{\delta>0}\left|\int_\delta^{1/\delta} t^2\mathscr{L}\mathscr{P}_{2t}g(x)\frac{dt}{t}\right| \le C\frac{\|g\|_{L^2(B)}}{(1+d(x,x_B))\mu(B(x_B,1+d(x,x_B)))}. \tag{12}$$

Recall that $\operatorname{supp} g\subset B$. For any $x\in X\backslash 4B$ and $y\in B$, we have

$$3d(x,x_B)/4 \le d(x,y) \le 5d(x,x_B)/4.$$

Hence, it follows from the Poisson upper bound and (6) that, for any $t>0$,

$$\begin{aligned}&\left|t^2\mathscr{L}\mathscr{P}_{2t}g(x)\right|\\ &\le C\int_B \frac{2t}{(2t+d(x,y))}\frac{1}{\mu(B(x,2t+d(x,y)))}\left(\frac{2t+d(x,y)}{\rho(y)}\right)^{-2}|g(y)|d\mu(y)\\ &\le C\int_B \frac{t}{(t+d(x,x_B))}\frac{1}{\mu(B(x,t+d(x,x_B)))}\left(\frac{\rho(x_B)\left(1+\frac{r_B}{\rho(x_B)}\right)^{k_0/(k_0+1)}}{t+d(x,x_B)}\right)^2|g(y)|d\mu(y)\\ &\le C(x_B,r_B)\frac{t}{(t+d(x,x_B))^3\mu(B(x_B,t+d(x,x_B)))}\|g\|_{L^1(B)}\\ &\le C(x_B,r_B)\frac{\|g\|_{L^2(B)}}{(1+d(x,x_B))\mu(B(x_B,1+d(x,x_B)))}\frac{t}{(t+d(x,x_B))^2}.\end{aligned}$$

The above estimate, together with the fact

$$\int_0^\infty \frac{t}{(t+d(x,x_B))^2}\frac{dt}{t} \le \int_0^\infty \frac{dt}{(t+r_B)^2} \le C(r_B) < \infty$$

yields that

$$\begin{aligned}\left|\int_\delta^{1/\delta} t^2\mathscr{L}\mathscr{P}_{2t}g(x)\frac{dt}{t}\right| &\le \int_0^\infty \left|t^2\mathscr{L}\mathscr{P}_{2t}g(x)\right|\frac{dt}{t}\\ &\le C(x_B,r_B)\frac{\|g\|_{L^2(B)}}{(1+d(x,x_B))\mu(B(x_B,1+d(x,x_B)))}.\end{aligned}$$

Accordingly, (12) follows readily. Now, we estimate the term $\mathrm{I}_2$. Since $f \in \mathcal{M}_2$, the estimate (12) yields that

$$\sup_{\delta>0}\int_X\left|f_2(x)\int_\delta^{1/\delta} t^2\mathscr{L}\mathscr{P}_{2t}g(x)\frac{dt}{t}\right|d\mu(x) \le C(g,x_B,r_B) < \infty.$$

This allows us to pass the limit inside the integral of $\mathrm{I}_2$. Hence, we conclude

$$\mathrm{I}_2 = \lim_{\delta\to 0}\int_X f_2(x)\int_\delta^{1/\delta} t^2\mathscr{L}\mathscr{P}_{2t}g(x)\frac{dt}{t}d\mu(x) = \frac{1}{4}\int_X f_2(x)g(x)d\mu(x).$$

Combining the previous formulas for $\mathrm{I}_1$ and $\mathrm{I}_2$, we complete the proof. □

Recall that we set $u_s(\cdot) = u(\cdot,s)$ for any $s > 0$.

**Lemma 7.** *Suppose the complete Dirichlet metric measure space $(X,d,\mu,\mathscr{E})$ satisfies (2) with $Q > 1$ and admits (5). Let $0 \le V \in A_\infty(X)\cap RH_q(X)$ with $q > (Q+1)/2$. Assume that $u \in \mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}(X\times\mathbb{R}_+)$ with $\max\{-1/2,-1/2N\} < \alpha < 0$.*

*Then, there exists a positive constant $C$ such that, for every $s > 0$,*

$$|||\mu_{\nabla_t,u_s}|||_\alpha \le C\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}.$$

**Proof.** Let $B = B(x_B,r_B)$. It holds by Proposition 1 that

$$\frac{1}{[\mu(B)]^\alpha}\left(\int_0^{r_B}⨍_B|t\partial_t\mathscr{P}_tu_s|^2d\mu\frac{dt}{t}\right)^{1/2} = \frac{1}{[\mu(B)]^\alpha}\left(\int_0^{r_B}⨍_B|t\partial_tu(y,t+s)|^2d\mu(y)\frac{dt}{t}\right)^{1/2}.$$

If $r_B > s$, by the doubling property (2), we have that

$$\begin{aligned}&\frac{1}{[\mu(B)]^\alpha}\left(\int_0^{r_B}⨍_B|t\partial_t\mathscr{P}_tu_s|^2d\mu\frac{dt}{t}\right)^{1/2}\\ &\le \frac{1}{[\mu(B)]^\alpha}\left(\frac{1}{\mu(B)}\int_0^{r_B+s}\int_{B(x_B,r_B+s)}|t\partial_tu(y,t)|^2d\mu(y)\frac{dt}{t}\right)^{1/2}\\ &\le \frac{C}{[\mu(2B)]^\alpha}\left(\int_0^{2r_B}⨍_{2B}|t\partial_tu(y,t)|^2d\mu(y)\frac{dt}{t}\right)^{1/2}\\ &\le C\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}.\end{aligned}$$

Otherwise, $r_B \leq s$, Lemma 2 together with elementary integration implies that there exists a positive constant $C$ independent of $r_B$ and $s$ such that

$$\begin{aligned}
&\frac{1}{[\mu(B)]^{\alpha}}\left(\int_0^{r_B} \fint_B |t\partial_t \mathscr{P}_t u_s|^2 d\mu \frac{dt}{t}\right)^{1/2} \\
&\leq \frac{C}{[\mu(B)]^{\alpha}}\left(\int_0^{r_B} \fint_B \frac{t^2}{(t+s)^2}[\mu(B(y,t+s))]^{2\alpha}\|u\|^2_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} d\mu(y)\frac{dt}{t}\right)^{1/2} \\
&\leq \frac{C}{[\mu(B)]^{\alpha}}\left(\int_0^{r_B} \fint_B \left(\frac{t}{r_B}\right)^2 [\mu(B(y,r_B))]^{2\alpha}\|u\|^2_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} d\mu(y)\frac{dt}{t}\right)^{1/2} \\
&\leq C\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}},
\end{aligned}$$

which, together with the case $r_B > s$, means that

$$|||\mu_{\nabla_t, u_s}|||_{\alpha} \leq C\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}},$$

which thus finishes the proof. □

**Proposition 2.** *Suppose the complete Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (2) with $Q > 1$ and admits (5). Let $0 \leq V \in A_\infty(X) \cap RH_q(X)$ with $q > (Q+1)/2$. Assume that $u \in \mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}(X \times \mathbb{R}_+)$ with $\max\{-1/2, -1/2N\} < \alpha < 0$. Then, for any $s > 0$, we have $u_s \in L^{2,\alpha}(X)$ and there exists a constant $C > 0$, independent of $s$, such that*

$$\|u_s\|_{L^{2,\alpha}} \leq C\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}.$$

**Proof.** Since $u \in \mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}(X \times \mathbb{R}_+)$, it follows from Lemma 3 that $u_s \in \mathcal{M}_2$. Given a ball $B \subset X$, for any $L^2$ function $g$ supported on $B$, it follows from Lemmas 5, 6 and 7 that

$$\begin{aligned}
\left|\int_X u_s g d\mu\right| &= 4\left|\int_0^\infty \int_X t\partial_t \mathscr{P}_t u_s t\partial_t \mathscr{P}_t g d\mu \frac{dt}{t}\right| \\
&\leq C[\mu(B)]^{1/2+\alpha} |||\mu_{\nabla_t, u_s}|||_{\alpha} \|g\|_{L^2(B)} \\
&\leq C[\mu(B)]^{1/2+\alpha} \|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} \|g\|_{L^2(B)}.
\end{aligned}$$

This together with the $L^2$-duality argument shows that

$$\begin{aligned}
\frac{1}{[\mu(B)]^{\alpha}}\left(\fint_B |u_s|^2 d\mu\right)^{1/2} &= \frac{1}{[\mu(B)]^{1/2+\alpha}} \sup_{\|g\|_{L^2(B)}\leq 1} \left|\int_X u_s g d\mu\right| \\
&\leq C \sup_{\|g\|_{L^2(B)}\leq 1} \|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} \|g\|_{L^2(B)} \leq C\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}.
\end{aligned}$$

Then, by taking the supremum over all the ball $B$, it holds that

$$\|u_s\|_{L^{2,\alpha}} \leq C\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}},$$

which completes the proof. □

## 4. From Initial Value to Solution

In this section, we will show that every Morrey function $f$ induces a Carleson type measure $t|\nabla \mathscr{P}_t f|^2 d\mu dt$. In order to estimate the space derivation part $t|\nabla_x \mathscr{P}_t f|^2 d\mu dt$, we introduce a result of Jiang-Li [18] (Proposition 5.2), which establishes a Caccioppoli inequality for the Schrödinger Poisson semigroup in a tent domain $B(x_B, r_B) \times (0, r_B)$.

**Proposition 3.** *Suppose the complete Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (2) with $Q > 1$ and admits (5). Let $0 \le V \in A_\infty(X) \cap RH_q(X)$ with $q > \max\{1, Q/2\}$. Assume that $g$ satisfies for some $y \in X$ that*

$$\int_X \frac{|g(x)|}{(1 + d(x,y))\mu(B(y, 1 + d(x,y)))} d\mu(x) < \infty.$$

*Then, for any ball $B = B(x_B, r_B)$, it holds that*

$$\int_0^{r_B} \int_B |t\nabla_x \mathscr{P}_t g|^2 d\mu \frac{dt}{t} \le C \int_0^{2r_B} \int_{2B} \left( |t^2 \partial_t^2 \mathscr{P}_t g||\mathscr{P}_t g| + |\mathscr{P}_t g|^2 \right) d\mu \frac{dt}{t}.$$

**Theorem 2.** *Assume the complete Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (2) with $Q > 1$ and admits (5). Let $V \in A_\infty(X) \cap RH_q(X)$ with $q \ge \max\{1, Q/2\}$. If $f \in L^{2,\alpha}(X)$ with $-1/2 < \alpha < 0$, then $u(x,t) = \mathscr{P}_t f(x) \in \mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}(X \times \mathbb{R}_+)$. Moreover, there exists a constant $C > 0$ such that*

$$\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} \le C\|f\|_{L^{2,\alpha}}.$$

**Proof.** For any ball $B = B(x_B, r_B)$, it holds that

$$\left( \int_0^{r_B} \fint_B |t\nabla \mathscr{P}_t f|^2 d\mu \frac{dt}{t} \right)^{1/2} \le \sum_{k=1}^{\infty} \left( \int_0^{r_B} \fint_B |t\nabla \mathscr{P}_t f_k|^2 d\mu \frac{dt}{t} \right)^{1/2} =: \sum_{k=1}^{\infty} \mathrm{J}_k,$$

where $f_1 := f\chi_{4B}$ and $f_k := f\chi_{2^{k+1}B \setminus 2^k B}$ for $k \in \{2, 3, 4, \dots\}$.

For the term $\mathrm{J}_1$, we apply the $L^2$-boundedness of the Riesz operator $\nabla_x \mathscr{L}^{-1/2}$ to obtain that

$$\begin{aligned}
\left( \int_0^{r_B} \fint_B |t\nabla \mathscr{P}_t f_1|^2 d\mu \frac{dt}{t} \right)^{1/2} &\le \left( \frac{1}{\mu(B)} \int_0^{\infty} \int_X |t\nabla \mathscr{P}_t f_1|^2 d\mu \frac{dt}{t} \right)^{1/2} \\
&\le C \left( \frac{1}{\mu(B)} \int_0^{\infty} \int_X |t\sqrt{\mathscr{L}} \mathscr{P}_t f_1|^2 d\mu \frac{dt}{t} \right)^{1/2} \\
&\le C \left( \frac{1}{\mu(B)} \int_X |f_1|^2 d\mu \right)^{1/2} \\
&\le C[\mu(B)]^{\alpha} \|f\|_{L^{2,\alpha}}.
\end{aligned}$$

Since $f_k \in L^{2,\alpha}(X)$, it is easy to see $f_k \in \mathcal{M}_2$. Hence, $f_k$ satisfies the requirement in Proposition 3, which implies that, for any $k \in \{2, 3, 4, \dots\}$,

$$\mathrm{J}_k \le C \left( \int_0^{2r_B} \fint_{2B} \left( |t\partial_t \mathscr{P}_t f_k|^2 + |t^2 \partial_t^2 \mathscr{P}_t f_k||\mathscr{P}_t f_k| + |\mathscr{P}_t f_k|^2 \right) \frac{d\mu dt}{t} \right)^{1/2}.$$

Then, for any $x \in 2B$, we apply the Poisson upper bound to obtain

$$\begin{aligned}
&|\mathscr{P}_t f_k(x)| + |t\partial_t \mathscr{P}_t f_k(x)| + |t^2 \partial_t^2 \mathscr{P}_t f_k(x)| \\
&\le C \int_{2^{k+1}B \setminus 2^k B} \frac{t}{(t + d(x,y))} \frac{|f(y)|}{\mu(B(x, t + d(x,y)))} d\mu(y) \\
&\le C 2^{-k} \frac{t}{r_B} \fint_{2^{k+1}B} |f(y)| d\mu(y) \\
&\le C 2^{-k} \frac{t}{r_B} [\mu(2^{k+1}B)]^{\alpha} \|f\|_{L^{2,\alpha}} \\
&\le C 2^{-k} \frac{t}{r_B} [\mu(B)]^{\alpha} \|f\|_{L^{2,\alpha}},
\end{aligned}$$

which yields

$$\mathrm{J}_k \le C2^{-k}[\mu(B)]^{\alpha}\|f\|_{L^{2,\alpha}}.$$

Hence, it follows that

$$\|\mathscr{P}_t f\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} = \sup_{B\subset X} \frac{1}{[\mu(B)]^{\alpha}} \left( \int_0^{r_B} \fint_B |t\nabla \mathscr{P}_t f|^2 d\mu \frac{dt}{t} \right)^{1/2} \le \sum_{k=1}^{\infty} \mathrm{J}_k \le C\|f\|_{L^{2,\alpha}}.$$

This completes the proof. □

## 5. From Solution to Initial Value

In this section, we will show that, for every function $u \in \mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}(X\times\mathbb{R}_+)$, there is a function $f \in L^{2,\alpha}(X)$ such that $u(x,t) = \mathscr{P}_t f(x)$ with the desired norm control.

**Theorem 3.** *Suppose the complete Dirichlet metric measure space $(X,d,\mu,\mathscr{E})$ satisfies (2) with $Q > 1$ and admits (5). Assume $0 \le V \in A_\infty(X)\cap RH_q(X)$ with $q \ge (Q+1)/2$, and $\max\{-1/2,-1/2N\} < \alpha < 0$. If $u \in \mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}(X\times\mathbb{R}_+)$, then there exists a function $f \in L^{2,\alpha}(X)$ such that $u(x,t) = \mathscr{P}_t f(x)$. Moreover, there exists a constant $C > 0$, independent of $u$, such that*

$$\|f\|_{L^{2,\alpha}} \le C\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}.$$

**Proof.** Without loss of generality, we may assume $q > (Q+1)/2$ because of the self improvement of the $RH_q(X)$ class. Suppose $u \in \mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}(X\times\mathbb{R}_+)$. For any $0<\epsilon<1$, by Proposition 2, we have

$$\|u_\epsilon\|_{L^{2,\alpha}} \le C\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}. \tag{13}$$

Next, we will fix a point $x_0$ and look for a function $f \in L^{2,\alpha}(X)$ through $L^2(B(x_0,2^j))$-boundedness of $\{u_\epsilon\}$ for each $j\in\mathbb{N}$. Indeed, for every $j \in \mathbb{N}$, we use (13) to obtain

$$\int_{B(x_0,2^j)} |u_\epsilon(x)|^2 d\mu(x) \le C[\mu(B(x_0,2^j))]^{1+2\alpha}\|u\|^2_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}},$$

which implies that the family $\{u_\epsilon(\cdot)\}_{0<\epsilon<1}$ is uniformly bounded in $L^2(B(x_0,2^j))$. Then, the Eberlein–Šmulian theorem and the diagonal method imply that there exists a sequence $\epsilon_k \to 0$ $(k\to\infty)$ and a function $g_j \in L^2(B(x_0,2^j))$ such that $u_{\epsilon_k} \to g_j$ weakly in $L^2(B(x_0,2^j))$, for any $j\in\mathbb{N}$. Now, we define a function $f(x)$ by

$$f(x) = g_j(x),$$

if $x \in B(x_0,2^j)$, $j=1,2,\ldots$. It is easy to see that $f$ is well defined on $X = \bigcup_{j=1}^{\infty} B(x_0,2^j)$. We can check that, for any ball $B\subset X$,

$$\int_B |f(x)|^2 d\mu(x) \le C[\mu(B)]^{1+2\alpha}\|u\|^2_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}},$$

which implies that

$$\|f\|_{L^{2,\alpha}} \le C\|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}.$$

Finally, we will show that $u(x,t) = \mathscr{P}_t f(x)$. By Lemma 1, we know that $u(x,\cdot)$ is continuous on $\mathbb{R}_+$. This together with Proposition 1 yields that

$$u(x,t) = \lim_{k\to+\infty} u(x,t+\epsilon_k) = \lim_{k\to+\infty} \mathscr{P}_t u_{\epsilon_k}(x).$$

This reduces to verify that

$$\lim_{k\to+\infty} \mathscr{P}_t u_{\epsilon_k}(x) = \mathscr{P}_t f(x). \tag{14}$$

Indeed, we recall that $p_t^v(x,y)$ is the kernel of $\mathscr{P}_t$, and for any $\ell \in \mathbb{N}$, we write

$$\mathscr{P}_t u_{\epsilon_k}(x) = \int_{B(x,2^\ell t)} p_t^v(x,y) u_{\epsilon_k}(y) d\mu(y) + \int_{X\setminus B(x,2^\ell t)} p_t^v(x,y) u_{\epsilon_k}(y) d\mu(y).$$

Using the Poisson upper bound, the Hölder inequality and (13), we obtain

$$\begin{aligned} \left| \int_{X\setminus B(x,2^\ell t)} p_t^v(x,y) u_{\epsilon_k}(y) d\mu(y) \right| &\le C \sum_{i=\ell}^{\infty} 2^{-i} \fint_{B(x,2^{i+1}t)} |u_{\epsilon_k}(y)| d\mu(y) \\ &\le C \sum_{i=\ell}^{\infty} 2^{-i} [\mu(B(x,2^i t))]^\alpha \|u_{\epsilon_k}\|_{L^{2,\alpha}} \\ &\le C2^{-\ell} [\mu(B(x,t))]^\alpha \|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}}, \end{aligned}$$

where $C$ is a positive constant independent of $k$. One has

$$\begin{aligned} 0 &\le \lim_{\ell\to+\infty} \lim_{k\to+\infty} \left| \int_{X\setminus B\left(x,2^\ell t\right)} p_t^v(x,y) u_{\epsilon_k}(y) d\mu(y) \right| \\ &\le \lim_{\ell\to+\infty} C2^{-\ell} [\mu(B(x,t))]^\alpha \|u\|_{\mathrm{HL}^{2,\alpha}_{\sqrt{\mathscr{L}}}} = 0. \end{aligned}$$

Therefore, it holds that

$$\lim_{k\to+\infty} \mathscr{P}_t u_{\epsilon_k}(x) = \lim_{\ell\to+\infty} \lim_{k\to+\infty} \int_{B\left(x,2^\ell t\right)} p_t^v(x,y) u_{\epsilon_k}(y) d\mu(y) = \mathscr{P}_t f(x),$$

which yields (14) readily. Then, we show that

$$u(x,t) = \mathscr{P}_t f(x).$$

The proof of Theorem 3 is complete. □

## 6. Conclusions

In this article, we solved the Dirichelt problem for the Schrödinger equation on the metric measure space. We obtained that a Schrödinger harmonic function satisfies the Carleson type condition if and only if it is the Poisson extension of a Morrey function. This continues the line of research on the Dirichlet problem with boundary value in $L^p$ space and BMO space, extends the result in Song-Tian-Yan [8] from the Euclidean space to the metric measure space and improves the reverse Hölder index from $q \ge n$ to $q \ge (n+1)/2$.

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