mathematics

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Improved Hille Oscillation Criteria for Nonlinear Functional Dynamic Equations of Third-Order

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Abstract: This paper aims to improve Hille oscillation criteria for the third-order functional dynamic equation $\left\{p_2(\zeta)\phi_{\gamma_2}\left(\left[p_1(\zeta)\phi_{\gamma_1}(y^\Delta(\zeta))\right]^\Delta\right)\right\}^\Delta + a(\zeta)\phi_\gamma(y(g(\zeta))) = 0$, on an above-unbounded time scale $\mathbb{T}$. The obtained results improve related contributions reported in the literature without restrictive conditions on the time scales. To demonstrate the essential results, an example is presented.

Keywords: oscillation criteria; third order; dynamic equations; time scales

MSC: 34K11; 39A10; 39A99; 34N05

Citation: Hassan, T.S.; Ramadan, R.A.; Alsheekhhussain, Z.; Khedr, A.Y.; Menaem, A.A.; Odinaev, I. Improved Hille Oscillation Criteria for Nonlinear Functional Dynamic Equations of Third-Order. *Mathematics* **2022**, *10*, 1078. https://doi.org/10.3390/math10071078

Academic Editor: Alberto Cabada

Received: 20 February 2022
Accepted: 23 March 2022
Published: 28 March 2022

1. Introduction

Stefan Hilger presented the theory of dynamic equations on time scales in his Ph.D. thesis in 1988 in an attempt to unify continuous and discrete analysis, which has recently gained a lot of attention, see [1]. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the classical theories of differential and difference equations are represented by situations, where this time scale is equal to the reals or integers. There are a variety of different intriguing time scales that can be used in a variety of ways (see [2]). This novel theory of "dynamic equations" unites the related theories for differential equations and difference equations and extends these traditional cases to "in-between" circumstances. That is, when $\mathbb{T}=q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0 \text{ for } q > 1\}$ (which has major applications in quantum theory, see [3]), we may treat the so-called $q-$difference equations, which can be applied to different types of time scales such that $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{T}_n$, the set of the harmonic numbers. We assume that the reader is familiar with the fundamentals of time scales and time scale notation; see [2,4,5], for an excellent introduction to time scale calculus.

Oscillatory properties of solutions to dynamic equations on time scales are gaining popularity due to their applications in engineering and natural sciences. This work is on the asymptotic and oscillatory behavior of the third-order functional dynamic equation:

$$\left\{p_2(\zeta)\phi_{\gamma_2}\left(\left[p_1(\zeta)\phi_{\gamma_1}\left(y^\Delta(\zeta)\right)\right]^\Delta\right)\right\}^\Delta + a(\zeta)\phi_\gamma(y(g(\zeta))) = 0 \quad (1)$$

on an above-unbounded time scale $\mathbb{T}$, where $\phi_\beta(u) := |u|^{\beta-1}u$, $\beta > 0$; $\gamma_1, \gamma_2, \gamma := \gamma_1\gamma_2 > 0$; a is a positive $rd-$continuous function on $\mathbb{T}$; and $g : \mathbb{T} \to \mathbb{T}$ is a $rd-$continuous nondecreasing function , such that $\lim_{t\to\infty} g(t) = \infty$; and p_i, $i = 1,2$, are positive $rd-$continuous functions on $\mathbb{T}$ such that:

$$\int_{\zeta_0}^{\infty} \frac{\Delta t}{p_i^{1/\gamma_i}(t)} = \infty,\ i = 1,2. \tag{2}$$

Throughout this paper, we let:

$$H_i(\zeta,\tau) := \phi_{\gamma_{i-1}}\left(\int_{\tau}^{\zeta} \phi_{\gamma_{i-1}}^{-1}\left(\frac{H_{i-1}(t,\tau)}{p_{i-1}(t)}\right)\Delta t\right),\ i = 1,2,3,$$

with:

$$H_0(t,\tau) := \frac{1}{p_2^{1/\gamma_2}(t)},\ p_0 = \gamma_0 = 1$$

and:

$$y^{[i]}(\zeta) := p_i(\zeta)\phi_{\gamma_i}([y^{[i-1]}(\zeta)]^{\Delta}),\ i = 1,2, \quad \text{with } y^{[0]}(\zeta) = y. \tag{3}$$

By a solution of Equation (1) we mean a nontrivial real–valued function $y \in C^1_{\mathrm{rd}}[T_y,\infty)_{\mathbb{T}}$ for some $T_y \geq \zeta_0$ for a positive constant $\zeta_0 \in \mathbb{T}$ such that $y^{[1]}(\zeta)$, $y^{[2]}(\zeta) \in C^1_{\mathrm{rd}}[T_y,\infty)_{\mathbb{T}}$ and $y(\zeta)$ satisfies Equation (1) on $[T_y,\infty)_{\mathbb{T}}$, where C_{rd} is the space of right-dense continuous functions. Solutions that vanish in the neighborhood of infinity will be excluded from consideration. If a solution y of (1) is neither eventually positive nor eventually negative, it is said to be oscillatory; otherwise, it is nonoscillatory. For nonoscillatory solutions of (1), we assume that:

$$\mathcal{N}_0 := \left\{y(\zeta) :\ y^{[i-1]}(\zeta)\, y^{[i]}(\zeta) > 0,\ i = 1,2,\ \text{eventually}\right\}$$

and:

$$\mathcal{N}_1 := \left\{y(\zeta) :\ y^{[i-1]}(\zeta)\, y^{[i]}(\zeta) < 0,\ i = 1,2,\ \text{eventually}\right\}.$$

In this paper, we establish some Hille oscillation criteria known on second-order differential equations (see [6]) for the third-order functional dynamic equation. Our criteria improve related contributions reported in the literature without restrictive conditions on the time scales, contrary to some previous works, see Section 2.

This paper is organized as follows: after this introduction, we state some previous results for third-order dynamic equations on time scales in Section 2. The main results are given in Section 3 after several technical lemmas are derived. Some examples are introduced at the end of Section 3. Discussions and Conclusions are listed in Section 4.

2. Preliminaries

In this section, we present some oscillation criteria for dynamic equations connected to our main findings that will be related to our main results for Equation (1) and explain the important contributions of this work.

Erbe et al. [7] established Hille oscillation criteria for the third-order dynamic equation:

$$y^{\Delta\Delta\Delta}(\zeta) + a(\zeta)y(\zeta) = 0. \tag{4}$$

The following are the main findings of [7]:

Theorem 1 ([7]). *Every solution of Equation* (4) *is either oscillatory or tends to zero eventually provided that:*

$$\int_{\zeta_0}^{\infty}\int_{\omega}^{\infty}\int_{\tau}^{\infty} a(t)\Delta t\ \Delta\tau\ \Delta\omega = \infty \tag{5}$$

and:

$$\liminf_{\zeta\to\infty} \zeta \int_{\zeta}^{\infty} \frac{h_2(t,\zeta_0)}{\sigma(t)} a(t)\Delta t > \frac{1}{4}, \tag{6}$$

where $h_2(t, \zeta_0)$ is the Taylor monomial of degree 2*, see ([2] [Section 1.6]).*

Saker [8] considered the dynamic equation as:

$$\left\{p_2(\zeta)\left[y^{\Delta\Delta}(\zeta)\right]^{\gamma_2}\right\}^{\Delta} + a(\zeta)y^{\gamma_2}(g(\zeta)) = 0, \tag{7}$$

where $g(\zeta) \leq \zeta$, γ_2 is a quotient of odd positive integers, and p_2 is a nondecreasing functions on $\mathbb{T}$. Hille oscillation criteria for (7) have been established, one of which we give below.

Theorem 2 ([8] Theorem 3.4)**.** *Every solution of Equation* (7) *is either oscillatory or tends to zero eventually provided that:*

$$\int_{\zeta_0}^{\infty} \frac{\Delta t}{p_2^{1/\gamma_2}(t)} = \infty, \tag{8}$$

$$\int_{\zeta_0}^{\infty} \int_{\omega}^{\infty} \left[\frac{1}{p_2(\tau)} \int_{\tau}^{\infty} a(t)\Delta t\right]^{1/\gamma_2} \Delta\tau\Delta\omega = \infty, \tag{9}$$

and:

$$\liminf_{\zeta\to\infty} \frac{\zeta^{\gamma_2}}{p_2(\zeta)} \int_{\sigma(\zeta)}^{\infty} \left(\frac{h_2(g(t), \zeta_0)}{\sigma(t)}\right)^{\gamma_2} a(t)\Delta t > \frac{\gamma_2^{\gamma_2}}{l^{\gamma_2^2}(1+\gamma_2)^{1+\gamma_2}}, \tag{10}$$

where $l := \liminf_{\zeta\to\infty} \dfrac{\zeta}{\sigma(\zeta)}$.

Theorem 3 ([8] Corollary 3.5)**.** *Assume that* (5) *holds with $p_2(\zeta) = 1$ and $\gamma_2 = 1$. If:*

$$\liminf_{\zeta\to\infty} \zeta \int_{\sigma(\zeta)}^{\infty} \frac{h_2(g(t), \zeta_0)}{\sigma(t)} a(t)\Delta t > \frac{1}{4l}. \tag{11}$$

Every solution of the equation:

$$y^{\Delta\Delta\Delta}(\zeta) + a(\zeta)y(g(\zeta)) = 0 \tag{12}$$

is either oscillatory or tends to zero eventually.

When $g(\zeta) = \zeta$, condition (11) becomes:

$$\liminf_{\zeta\to\infty} \zeta \int_{\sigma(\zeta)}^{\infty} \frac{h_2(t, \zeta_0)}{\sigma(t)} a(t)\Delta t > \frac{1}{4l}. \tag{13}$$

When comparing (6) and (13), it is clear that [7] improves [8] for Equation (4) since:

$$\frac{1}{4l} \geq \frac{1}{4} \quad \text{and} \quad \zeta \int_{\sigma(\zeta)}^{\infty} \frac{h_2(t, \zeta_0)}{\sigma(t)} a(t)\Delta t \leq \zeta \int_{\zeta}^{\infty} \frac{h_2(t, \zeta_0)}{\sigma(t)} a(t)\Delta t.$$

Wang and Xu in [9] considered the third order dynamic equation:

$$\left(p_2(\zeta)\left[(p_1(\zeta)y^{\Delta}(\zeta))^{\Delta}\right]^{\gamma_2}\right)^{\Delta} + a(\zeta)y(\zeta) = 0,$$

under certain restrictive conditions on the time scales. Agarwal et al. [10] suggested some Hille type oscillation criteria to the third-order delay dynamic equation as follows:

$$\left(p_2(\zeta)(p_1(\zeta)y^{\Delta}(\zeta))^{\Delta}\right)^{\Delta} + a(\zeta)y(g(\zeta)) = 0, \tag{14}$$

where $g(\zeta) \leq \zeta$ on $[\zeta_0, \infty)_{\mathbb{T}}$ and under the canonical type assumptions:

$$\int_{\zeta_0}^{\infty} \frac{\Delta t}{p_i(t)} = \infty, \ i = 1, 2, \tag{15}$$

and:

$$\int_{\zeta_0}^{\infty} \frac{1}{p_1(\omega)} \int_{\omega}^{\infty} \frac{1}{p_2(\tau)} \int_{\tau}^{\infty} a(t)\Delta t \, \Delta\tau \, \Delta\omega = \infty. \tag{16}$$

One of these results in [10] reads as follows.

Theorem 4 ([10]). *Every solution of Equation (14) is either oscillatory or tends to zero eventually if (15) and (16) hold, and:*

$$\liminf_{\zeta \to \infty} H_1(\zeta, \zeta_0) \int_{\zeta}^{\infty} \frac{H_2(g(t), \zeta_0)}{H_1(\sigma(t), \zeta_0)} a(t)\Delta t > \frac{1}{4}. \tag{17}$$

The results in [10] included the results that were established in [7]. We note that the results obtained in [8,10] are proved only when $g(\zeta) \leq \zeta$ and cannot be applied when $g(\zeta) \geq \zeta$. Agarwal et al. [11] examined a generalized third-order delay dynamic Equation (1) and gave some new oscillation criteria under the canonical type conditions.

$$\int_{\zeta_0}^{\infty} \frac{\Delta t}{p_i^{1/\gamma_i}(t)} = \infty, \ i = 1, 2, \tag{18}$$

and:

$$\int_{\zeta_0}^{\infty} \left(\frac{1}{p_1(\omega)} \int_{\omega}^{\infty} \left(\frac{1}{p_2(\tau)} \int_{\tau}^{\infty} a(t)\Delta t \right)^{1/\gamma_2} \Delta\tau \right)^{1/\gamma_1} \Delta\omega = \infty. \tag{19}$$

We quote below one of the most interesting ones for Eq. due to Hille.

Theorem 5 ([11]). *Every solution of Equation (1) is either oscillatory or tends to zero eventually if (18) and: (19) hold, and:*

$$\liminf_{\zeta \to \infty} H_1^{\gamma_2}(\zeta, \zeta_0) \int_{\sigma(\zeta)}^{\infty} \left(\frac{H_2(\varphi(t), \zeta_0)}{H_1(t, \zeta_0)} \right)^{\gamma_2} a(t)\Delta t > \frac{\gamma_2^{\gamma_2}}{l^{\gamma_2}(1+\gamma_2)^{1+\gamma_2}}, \tag{20}$$

where $l := \liminf_{\zeta \to \infty} \frac{H_1(\zeta, \zeta_0)}{H_1(\sigma(\zeta), \zeta_0)}$

$$\varphi(\zeta) := \begin{cases} \zeta, & g(\zeta) \geq \zeta, \\ g(\zeta), & g(\zeta) \leq \zeta. \end{cases} \tag{21}$$

We note that the critical constant in (17) is $\frac{1}{4}$ and in (20) is $\frac{\gamma_2^{\gamma_2}}{l^{\gamma_2^2}(1+\gamma_2)^{1+\gamma_2}}$, which is $\frac{1}{4l} \geq \frac{1}{4}$ if $\gamma_2 = 1$ and depends on a concrete time scale; so the critical constant in [10] is better than the one in [11].

Recently, Hassan et al. [12] improved the results of [7–11] for Equation (14). We include one of intriguing ones for Equation (14).

Theorem 6 ([12]). *Every solution of Equation (14) is either oscillatory or tends to zero eventually if (15) and: (16) hold, and*

$$\liminf_{\zeta \to \infty} H_1(\zeta, \zeta_0) \int_{\zeta}^{\infty} \frac{H_2(\varphi(t), \zeta_0)}{H_1(t, \zeta_0)} a(t)\Delta t > \frac{1}{4}, \tag{22}$$

where $\varphi(\zeta)$ is defined as in (21).

We noted that, when $g(\zeta) = \zeta$ and $p_1(\zeta) = p_2(\zeta) = 1$, condition (22) improves condition (6); when $g(\zeta) \leq \zeta$ and $p_1(\zeta) = 1$, condition (22) improves condition (10); and when $g(\zeta) \leq \zeta$, condition (22) improves condition (17). In addition, the critical constant in (22) does not depend on a concrete time scale. The reader is directed to papers [6,13–31] and the sources listed therein.

As a result of the above findings, this paper intends to improve Hille oscillation conditions (6), (11), (13), (17) and (20) for the generalized dynamic Equation (1). All of the functional inequalities reported in this paper are assumed to hold in the eventually, that is, for all sufficiently large ζ.

3. Main Results

We begin this section with the preliminary lemmas listed below, which will be crucial in the proof of the main results. We omit the details proving the first lemma that follows directly from the canonical form ((2) holds) of Equation (1).

Lemma 1. *If $y(\zeta)$ is a nonoscillatory solution of Equation (1), then $y \in \mathcal{N}_0 \cup \mathcal{N}_1$, eventually.*

Lemma 2. *If $y \in \mathcal{N}_0$, then $\frac{|y^{[1]}(\zeta)|}{H_1(\zeta,\zeta_0)}$ is strictly decreasing on $(\zeta_0,\infty)_{\mathbb{T}}$ and:*

$$\frac{\phi_{\gamma_1}(y(\zeta))}{y^{[1]}(\zeta)} \geq \frac{H_2(\zeta,\zeta_0)}{H_1(\zeta,\zeta_0)}. \tag{23}$$

Proof. Without loss of generality, assume that:

$$y^{[i]}(\zeta) > 0,\ i = 0,1,2 \text{ and } y(g(\zeta)) > 0 \text{ on } [\zeta_0,\infty)_{\mathbb{T}}.$$

By using the fact that $y^{[2]}(\zeta)$ is strictly decreasing on $[\zeta_0,\infty)_{\mathbb{T}}$. Then for $\zeta \in [\zeta_0,\infty)_{\mathbb{T}}$,

$$\begin{aligned} y^{[1]}(\zeta) &\geq \int_{\zeta_0}^{\zeta} \phi_{\gamma_2}^{-1}\left(y^{[2]}(t)\right) H_0(t,\zeta_0)\,\Delta t \\ &\geq \phi_{\gamma_2}^{-1}\left(y^{[2]}(\zeta)\right) \int_{\zeta_0}^{\zeta} H_0(t,\zeta_0)\,\Delta t \\ &= \phi_{\gamma_2}^{-1}\left(y^{[2]}(\zeta)\right) H_1(\zeta,\zeta_0). \end{aligned}$$

Hence, we conclude that, for $\zeta \in (\zeta_0,\infty)_{\mathbb{T}}$,

$$\left(\frac{y^{[1]}(\zeta)}{H_1(\zeta,\zeta_0)}\right)^{\Delta} = \frac{H_0(\zeta,\zeta_0)\left\{\phi_{\gamma_2}^{-1}\left(y^{[2]}(\zeta)\right)H_1(\zeta,\zeta_0) - y^{[1]}(\zeta)\right\}}{H_1(\zeta,\zeta_0)H_1(\sigma(\zeta),\zeta_0)} < 0.$$

Thus $\frac{y^{[1]}(\zeta)}{H_1(\zeta,\zeta_0)}$ is strictly decreasing on $(\zeta_0,\infty)_{\mathbb{T}}$. Therefore, for $\zeta \in (\zeta_0,\infty)_{\mathbb{T}}$,

$$\begin{aligned} y(\zeta) &\geq \int_{\zeta_0}^{\zeta} \phi_{\gamma_1}^{-1}\left(\frac{y^{[1]}(t)}{H_1(t,\zeta_0)}\right)\left(\frac{H_1(t,\zeta_0)}{p_1(t)}\right)^{\frac{1}{\gamma_1}} \Delta t \\ &\geq \phi_{\gamma_1}^{-1}\left(\frac{y^{[1]}(\zeta)}{H_1(\zeta,\zeta_0)}\right) \int_{\zeta_0}^{\zeta} \left(\frac{H_1(t,\zeta_0)}{p_1(t)}\right)^{\frac{1}{\gamma_1}} \Delta t \\ &= \phi_{\gamma_1}^{-1}\left(\frac{y^{[1]}(\zeta)}{H_1(\zeta,\zeta_0)}\right) H_2^{\frac{1}{\gamma_1}}(\zeta,\zeta_0). \end{aligned}$$

That is,

$$y^{\gamma_1}(\zeta) \geq \frac{y^{[1]}(\zeta)}{H_1(\zeta,\zeta_0)} H_2(\zeta,\zeta_0) \tag{24}$$

Thus (23) holds for $\zeta \in (\zeta_0, \infty)_{\mathbb{T}}$. This completes the proof. $\square$

Lemma 3. *If $y(\zeta) \in \mathcal{N}_1$, then $y(\zeta)$ tends to a finite limit eventually.*

Proof. The proof is straightforward and hence is omitted. $\square$

Lemma 4. *Let:*

(A) *either,*

$$\int_{\zeta_0}^{\infty} a(t)\,\Delta t = \infty;$$

$$\int_{\zeta_0}^{\infty} \left(\frac{1}{p_2(\tau)} \int_{\tau}^{\infty} a(t)\,\Delta t \right)^{1/\gamma_2} \Delta\tau = \infty;$$

or,

$$\int_{\zeta_0}^{\infty} \left[\frac{1}{p_1(\omega)} \int_{\omega}^{\infty} \left(\frac{1}{p_2(\tau)} \int_{\tau}^{\infty} a(t)\,\Delta t \right)^{1/\gamma_2} \Delta\tau \right]^{1/\gamma_1} \Delta\omega = \infty.$$

If $y(\zeta) \in \mathcal{N}_1$, then $y(\zeta)$ tends to zero eventually.

Proof. The proof is similar to that of ([32], Theorem 2.1) and is therefore omitted. $\square$

Lemma 5. *Let $0 < \gamma_2 \leq 1$. If $y \in \mathcal{N}_0$, then for all large ζ,*

$$x(\zeta) \geq \gamma_2 \int_{\zeta}^{\infty} H_0(t,\zeta_0) x(t) x^{1/\gamma_2}(\sigma(t)) \Delta t + \int_{\zeta}^{\infty} \left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(\sigma(t),\zeta_0)} \right)^{\gamma_2} a(t) \Delta t, \tag{25}$$

where:

$$x(\zeta) := \frac{y^{[2]}(\zeta)}{(y^{[1]}(\zeta))^{\gamma_2}}. \tag{26}$$

Proof. Without loss of generality, assume that:

$$y^{[i]}(\zeta) > 0,\ i = 0,1,2 \text{ and } y(g(\zeta)) > 0 \text{ on } [\zeta_0, \infty)_{\mathbb{T}}.$$

By the product rule and the quotient rule, we get:

$$\begin{aligned} x^{\Delta}(\zeta) &= \frac{(y^{[2]}(\zeta))^{\Delta}}{(y^{[1]}(\sigma(\zeta)))^{\gamma_2}} + \left(\frac{1}{(y^{[1]}(\zeta))^{\gamma_2}} \right)^{\Delta} y^{[2]}(\zeta) \\ &= \frac{(y^{[2]}(\zeta))^{\Delta}}{(y^{[1]}(\sigma(\zeta)))^{\gamma_2}} - \frac{\left(\left(y^{[1]}(\zeta) \right)^{\gamma_2} \right)^{\Delta}}{(y^{[1]}(\sigma(\zeta)))^{\gamma_2}} \frac{y^{[2]}(\zeta)}{(y^{[1]}(\zeta))^{\gamma_2}}. \end{aligned}$$

From (1) and the definition of $x(\zeta)$, we see that for $\zeta \geq \zeta_0$,

$$x^{\Delta}(\zeta) = -\left(\frac{y^{\gamma_1}(g(\zeta))}{y^{[1]}(\sigma(\zeta))} \right)^{\gamma_2} a(\zeta) - \frac{\left(\left(y^{[1]}(\zeta) \right)^{\gamma_2} \right)^{\Delta}}{(y^{[1]}(\sigma(\zeta)))^{\gamma_2}} x(\zeta). \tag{27}$$

First, consider the case when $g(\zeta) \leq \zeta$, for all large ζ. From (24) and using the fact that $\frac{y^{[1]}(\zeta)}{H_1(\zeta,\zeta_0)}$ is strictly decreasing, we obtain:

$$y^{\gamma_1}(g(\zeta)) \geq \frac{y^{[1]}(g(\zeta))}{H_1(g(\zeta),\zeta_0)} H_2(g(\zeta),\zeta_0) \geq \frac{y^{[1]}(\sigma(\zeta))}{H_1(\sigma(\zeta),\zeta_0)} H_2(g(\zeta),\zeta_0). \tag{28}$$

Next, consider the case when $g(\zeta) \geq \zeta$, for all large ζ. Using the fact that $y(\zeta)$ is strictly increasing and (24), and using the fact that $\frac{y^{[1]}(\zeta)}{H_1(\zeta,\zeta_0)}$ is strictly decreasing, we obtain:

$$y^{\gamma_1}(g(\zeta)) \geq y^{\gamma_1}(\zeta) \geq \frac{y^{[1]}(\zeta)}{H_1(\zeta,\zeta_0)} H_2(\zeta,\zeta_0) \geq \frac{y^{[1]}(\sigma(\zeta))}{H_1(\sigma(\zeta),\zeta_0)} H_2(\zeta,\zeta_0), \tag{29}$$

It follows from (28) and (29) that there exists a $\zeta_1 \in (\zeta_0,\infty)_{\mathbb{T}}$ such that:

$$\frac{y^{\gamma_1}(g(\zeta))}{y^{[1]}(\sigma(\zeta))} \geq \frac{H_2(\varphi(\zeta),\zeta_0)}{H_1(\sigma(\zeta),\zeta_0)} \qquad \text{for } \zeta \in [\zeta_1,\infty)_{\mathbb{T}}.$$

Hence, we conclude that, for $\zeta \in [\zeta_1,\infty)_{\mathbb{T}}$,

$$x^{\Delta}(\zeta) \leq -\left(\frac{H_2(\varphi(\zeta),\zeta_0)}{H_1(\sigma(\zeta),\zeta_0)}\right)^{\gamma_2} a(\zeta) - \frac{\left(\left(y^{[1]}(\zeta)\right)^{\gamma_2}\right)^{\Delta}}{\left(y^{[1]}(\sigma(\zeta))\right)^{\gamma_2}} x(\zeta).$$

By the Pötzsche chain rule,

$$\begin{aligned} \left((y^{[1]}(\zeta))^{\gamma_2}\right)^{\Delta} &= \gamma_2 \int_0^1 [(1-h)y^{[1]}(\zeta) + hy^{[1]}(\sigma(\zeta))]^{\gamma_2 - 1} \mathrm{d}h \left[y^{[1]}(\zeta)\right]^{\Delta} \\ &\geq \gamma_2 \left[y^{[1]}(\sigma(\zeta))\right]^{\gamma_2-1} \left[y^{[1]}(\zeta)\right]^{\Delta}. \end{aligned}$$

Hence, by the fact that $y^{[2]}(\zeta)$ is strictly decreasing and (26),

$$\begin{aligned} x^{\Delta}(\zeta) &\leq -\left(\frac{H_2(\varphi(\zeta),\zeta_0)}{H_1(\sigma(\zeta),\zeta_0)}\right)^{\gamma_2} a(\zeta) - \gamma_2 \frac{\left[y^{[1]}(\zeta)\right]^{\Delta}}{y^{[1]}(\sigma(\zeta))} x(\zeta) \\ &\leq -\left(\frac{H_2(\varphi(\zeta),\zeta_0)}{H_1(\sigma(\zeta),\zeta_0)}\right)^{\gamma_2} a(\zeta) - \gamma_2 H_0(\zeta,\zeta_0) \frac{\left[y^{[2]}(\sigma(\zeta))\right]^{\frac{1}{\gamma_2}}}{y^{[1]}(\sigma(\zeta))} x(\zeta) \\ &= -\left(\frac{H_2(\varphi(\zeta),\zeta_0)}{H_1(\sigma(\zeta),\zeta_0)}\right)^{\gamma_2} a(\zeta) \\ &\quad -\gamma_2 H_0(\zeta,\zeta_0) x(\zeta) x^{1/\gamma_2}(\sigma(\zeta)), \end{aligned} \tag{30}$$

which implies that $x^{\Delta} < 0$. Integrating (30) from ζ to v, we have:

$$x(v) - x(\zeta) \leq -\int_{\zeta}^{v} \left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(\sigma(t),\zeta_0)}\right)^{\gamma_2} a(t)\Delta t - \gamma_2 \int_{\zeta}^{v} H_0(t,\zeta_0) x(t) x^{1/\gamma_2}(\sigma(t))\Delta t.$$

Taking into account that $x > 0$ and passing to the limit as $v \to \infty$, we get:

$$-x(\zeta) \leq -\int_{\zeta}^{\infty} \left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(\sigma(t),\zeta_0)}\right)^{\gamma_2} a(t)\Delta t - \gamma_2 \int_{\zeta}^{\infty} H_0(t,\zeta_0) x(t) x^{1/\gamma_2}(\sigma(t))\Delta t.$$

Thus, (25) holds for all large ζ. This completes the proof. □

Lemma 6. *Let $\gamma_2 \geq 1$. If $y \in \mathcal{N}_0$, then for all large ζ,*

$$x(\zeta) \geq \gamma_2 \int_{\zeta}^{\infty} H_0(t,\zeta_0) x^{1/\gamma_2}(t) x(\sigma(t))\Delta t + \int_{\zeta}^{\infty} \left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(t,\zeta_0)}\right)^{\gamma_2} a(t)\Delta t, \tag{31}$$

where $\varphi(\zeta)$ and $x(\zeta)$ are defined as in (21) and (26), respectively.

Proof. Without loss of generality, assume that:

$$y^{[i]}(\zeta) > 0,\ i = 0,1,2 \text{ and } y(g(\zeta)) > 0 \text{ on } [\zeta_0, \infty)_{\mathbb{T}}.$$

By the product rule and the quotient rule, we get:

$$\begin{aligned} x^{\Delta}(\zeta) &= \frac{\left(y^{[2]}(\zeta)\right)^{\Delta}}{(y^{[1]}(\zeta))^{\gamma_2}} + \left(\frac{1}{(y^{[1]}(\zeta))^{\gamma_2}}\right)^{\Delta} y^{[2]}(\sigma(\zeta)) \\ &= \frac{\left(y^{[2]}(\zeta)\right)^{\Delta}}{(y^{[1]}(\zeta))^{\gamma_2}} - \frac{\left((y^{[1]}(\zeta))^{\gamma_2}\right)^{\Delta}}{(y^{[1]}(\zeta))^{\gamma_2}} \frac{y^{[2]}(\sigma(\zeta))}{(y^{[1]}(\sigma(\zeta)))^{\gamma_2}}. \end{aligned}$$

From (1) and the definition of $x(\zeta)$, we see that for $\zeta \geq \zeta_0$,

$$x^{\Delta}(\zeta) = -\left(\frac{y^{\gamma_1}(g(\zeta))}{y^{[1]}(\zeta)}\right)^{\gamma_2} a(\zeta) - \frac{\left((y^{[1]}(\zeta))^{\gamma_2}\right)^{\Delta}}{(y^{[1]}(\zeta))^{\gamma_2}} x(\sigma(\zeta)). \tag{32}$$

First, consider the case when $g(\zeta) \leq \zeta$, for all large ζ. From (24) and using the fact that $\frac{y^{[1]}(\zeta)}{H_1(\zeta,\zeta_0)}$ is strictly decreasing, we obtain:

$$\begin{aligned} y^{\gamma_1}(g(\zeta)) &\geq \frac{y^{[1]}(g(\zeta))}{H_1(g(\zeta),\zeta_0)} H_2(g(\zeta),\zeta_0) \\ &\geq \frac{y^{[1]}(\zeta)}{H_1(\zeta,\zeta_0)} H_2(g(\zeta),\zeta_0). \end{aligned} \tag{33}$$

Next, consider the case when $g(\zeta) \geq \zeta$, for all large ζ. Using the fact that y is strictly increasing and (24), we have that:

$$y^{\gamma_1}(g(\zeta)) \geq y^{\gamma_1}(\zeta) \geq \frac{y^{[1]}(\zeta)}{H_1(\zeta,\zeta_0)} H_2(\zeta,\zeta_0). \tag{34}$$

It follows from (33) and (34) that there exists a $\zeta_1 \in (\zeta_0, \infty)_{\mathbb{T}}$ such that:

$$\frac{y^{\gamma_1}(g(\zeta))}{y^{[1]}(\zeta)} \geq \frac{H_2(\varphi(\zeta),\zeta_0)}{H_1(\zeta,\zeta_0)} \qquad \text{for } \zeta \in [\zeta_1, \infty)_{\mathbb{T}}.$$

Hence, we conclude that, for $\zeta \in [t_1, \infty)_{\mathbb{T}}$,

$$x^{\Delta}(\zeta) \leq -\left(\frac{H_2(\varphi(\zeta),\zeta_0)}{H_1(\zeta,\zeta_0)}\right)^{\gamma_2} a(\zeta) - \frac{\left((y^{[1]}(\zeta))^{\gamma_2}\right)^{\Delta}}{(y^{[1]}(\zeta))^{\gamma_2}} x(\sigma(\zeta)).$$

By the Pötzsche chain rule,

$$\left((y^{[1]}(\zeta))^{\gamma_2}\right)^{\Delta} \geq \gamma_2 \left[y^{[1]}(\zeta)\right]^{\gamma_2 - 1} \left[y^{[1]}(\zeta)\right]^{\Delta}.$$

Hence, by (26),

$$\begin{aligned} x^{\Delta}(\zeta) &\leq -\left(\frac{H_2(\varphi(\zeta),\zeta_0)}{H_1(\zeta,\zeta_0)}\right)^{\gamma_2} a(\zeta) - \gamma_2 \frac{\left[y^{[1]}(\zeta)\right]^{\Delta}}{y^{[1]}(\zeta)} x(\sigma(\zeta)) \\ &= -\left(\frac{H_2(\varphi(\zeta),\zeta_0)}{H_1(\zeta,\zeta_0)}\right)^{\gamma_2} a(\zeta) - \gamma_2 H_0(\zeta,\zeta_0) x^{1/\gamma_2}(\zeta) x(\sigma(\zeta)). \end{aligned} \tag{35}$$

Integrating (35) from ζ to v, we have:

$$x(v)-x(\zeta)\le-\int_{\zeta}^{v}\left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(t,\zeta_0)}\right)^{\gamma_2}a(t)\Delta t-\gamma_2\int_{\zeta}^{v}H_0(t,\zeta_0)x^{1/\gamma_2}(t)x(\sigma(t))\Delta t.$$

Taking into account that $x>0$ and passing to the limit as $v\to\infty$, we get:

$$-x(\zeta)\le-\int_{\zeta}^{\infty}\left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(t,\zeta_0)}\right)^{\gamma_2}a(t)\Delta t-\gamma_2\int_{\zeta}^{\infty}H_0(t,\zeta_0)x^{1/\gamma_2}(t)x(\sigma(t))\Delta t.$$

Thus, (31) holds for all large ζ. This completes the proof. □

The classification of the possible nonoscillatory solutions of Equation (1) will now be presented.

Theorem 7. *Let* $0<\gamma_2\le 1$. *If:*

$$\int_{\zeta_0}^{\infty}\left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(\sigma(t),\zeta_0)}\right)^{\gamma_2}a(t)\Delta t=\infty, \tag{36}$$

where $\varphi(\zeta)$ *is defined as in* (21), *then* $\mathcal{N}_0=\varnothing$.

Proof. Assume Equation (1) has a nonoscillatory solution $y(\zeta)\in\mathcal{N}_0$ such that $y(\zeta)>0$ and $y(g(\zeta))>0$ for $\zeta\in[\zeta_0,\infty)_{\mathbb{T}}$. Then:

$$y^{[i]}(\zeta)>0,\ i=1,2 \text{ and } y^{[3]}(\zeta)<0 \text{ on } [\zeta_0,\infty)_{\mathbb{T}}.$$

From (25), we have for $\zeta\in[\zeta_1,\infty)_{\mathbb{T}}$ and $\zeta_1\in(\zeta_0,\infty)_{\mathbb{T}}$,

$$-x^{\Delta}(\zeta)\ge\left(\frac{H_2(\varphi(\zeta),\zeta_0)}{H_1(\sigma(\zeta),\zeta_0)}\right)^{\gamma_2}a(\zeta).$$

Integrating the last inequality from ζ to v, we obtain:

$$x(\zeta)-x(v)\ge\int_{\zeta}^{v}\left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(\sigma(t),\zeta_0)}\right)^{\gamma_2}a(t)\Delta t,$$

and hence:

$$x(\zeta)\ge\int_{\zeta}^{\infty}\left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(\sigma(t),\zeta_0)}\right)^{\gamma_2}a(t)\Delta t$$

This is in contradiction with (36). The proof is now complete. □

Theorem 8. *Let* $0<\gamma_2\le 1$. *If:*

$$\liminf_{\zeta\to\infty}H_1^{\gamma_2}(\zeta,\zeta_0)\int_{\zeta}^{\infty}\left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(\sigma(t),\zeta_0)}\right)^{\gamma_2}a(t)\Delta t>\frac{\gamma_2^{\gamma_2}}{l^{\gamma_2(1-\gamma_2)}(1+\gamma_2)^{1+\gamma_2}}, \tag{37}$$

where:

$$l:=\liminf_{\zeta\to\infty}\frac{H_1(\zeta,\zeta_0)}{H_1(\sigma(\zeta),\zeta_0)} \tag{38}$$

and $\varphi(\zeta)$ *is defined as in* (21), *then* $\mathcal{N}_0=\varnothing$.

Proof. Assume Equation (1) has a nonoscillatory solution $y(\zeta)\in\mathcal{N}_0$ such that $y(\zeta)>0$ and $y(g(\zeta))>0$ for $\zeta\in[\zeta_0,\infty)_{\mathbb{T}}$. Then:

$$y^{[i]}(\zeta)>0,\ i=1,2 \text{ and } y^{[3]}(\zeta)<0 \text{ on } [\zeta_0,\infty)_{\mathbb{T}}.$$

As a result, (25) holds on $[\zeta_1,\infty)_{\mathbb{T}}$, for sufficiently large $\zeta_1\in[\zeta_0,\infty)_{\mathbb{T}}$. Now, for any $\varepsilon>0$, there exists a $\zeta_2\in[\zeta_1,\infty)_{\mathbb{T}}$ such that for $\zeta\in[\zeta_2,\infty)_{\mathbb{T}}$,

$$\frac{H_1(\zeta,\zeta_0)}{H_1(\sigma(\zeta),\zeta_0)}\geq l-\varepsilon\quad\text{and}\quad H_1^{\gamma_2}(\zeta,\zeta_0)x(\zeta)\geq H-\varepsilon, \tag{39}$$

where:

$$H:=\liminf_{\zeta\to\infty}H_1^{\gamma_2}(\zeta,\zeta_0)x(\zeta),\quad 0\leq H\leq 1$$

Multiplying both sides of (25) by $H_1(\zeta,\zeta_0)$, we obtain for $\zeta\in[\zeta_2,\infty)_{\mathbb{T}}$,

$$\begin{aligned}
& H_1^{\gamma_2}(\zeta,\zeta_0)\int_\zeta^\infty\left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(\sigma(t),\zeta_0)}\right)^{\gamma_2}a(t)\Delta t\\
\leq\;& H_1^{\gamma_2}(\zeta,\zeta_0)x(\zeta)\\
& -\gamma_2H_1^{\gamma_2}(\zeta,\zeta_0)\int_\zeta^\infty H_0(t,\zeta_0)x(t)x^{1/\gamma_2}(\sigma(t))\Delta t\\
\leq\;& H_1^{\gamma_2}(\zeta,\zeta_0)x(\zeta)\\
& -H_1^{\gamma_2}(\zeta,\zeta_0)(H-\varepsilon)^{1+1/\gamma_2}(l-\varepsilon)^{1-\gamma_2}\int_\zeta^\infty\frac{\gamma_2\,H_0(t,\zeta_0)}{H_1(t,\zeta_0)H_1^{\gamma_2}(\sigma(t),\zeta_0)}\Delta t\\
\leq\;& H_1^{\gamma_2}(\zeta,\zeta_0)x(\zeta)\\
& -H_1^{\gamma_2}(\zeta,\zeta_0)(H-\varepsilon)^{1+1/\gamma_2}(l-\varepsilon)^{1-\gamma_2}\int_\zeta^\infty\left(\frac{-1}{H_1^{\gamma_2}(t,\zeta_0)}\right)^\Delta\Delta t\\
=\;& H_1^{\gamma_2}(\zeta,\zeta_0)x(\zeta)-(H-\varepsilon)^{1+1/\gamma_2}(l-\varepsilon)^{1-\gamma_2},
\end{aligned}\tag{40}$$

since:

$$\begin{aligned}
\left(\frac{-1}{H_1^{\gamma_2}(t,\zeta_0)}\right)^\Delta &= \frac{\gamma_2\int_0^1[(1-h)H_1(t,\zeta_0)+hH_1(\sigma(t),\zeta_0)]^{\gamma_2-1}\mathrm{d}h\;H_0(t,\zeta_0)}{H_1^{\gamma_2}(t,\zeta_0)H_1^{\gamma_2}(\sigma(t),\zeta_0)}\\
&\leq \frac{\gamma_2H_0(t,\zeta_0)}{H_1(t,\zeta_0)H_1^{\gamma_2}(\sigma(t),\zeta_0)}.
\end{aligned}$$

Taking the lim inf of both sides of the inequality (40) as $\zeta\to\infty$, we get:

$$\liminf_{\zeta\to\infty}H_1^{\gamma_2}(\zeta,\zeta_0)\int_\zeta^\infty\left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(\sigma(t),\zeta_0)}\right)^{\gamma_2}a(t)\Delta t\leq H-(l-\varepsilon)^{1-\gamma_2}(H-\varepsilon)^{1+1/\gamma_2}.$$

By virtue of the fact that $\varepsilon>0$ are arbitrary, we conclude that:

$$\liminf_{\zeta\to\infty}H_1^{\gamma_2}(\zeta,\zeta_0)\int_\zeta^\infty\left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(\sigma(t),\zeta_0)}\right)^{\gamma_2}a(t)\Delta t\leq H-l^{1-\gamma_2}H^{1+1/\gamma_2}.$$

Letting $A=1$, $B=l^{1-\gamma_2}$, and $V=H$, and using inequality:

$$AV-BV^{1+1/\gamma_2}\leq\frac{\gamma_2^{\gamma_2}}{(1+\gamma_2)^{1+\gamma_2}}\frac{A^{1+\gamma_2}}{B^{\gamma_2}},\quad A\geq 0,\ B>0, \tag{41}$$

we achieve the following:

$$\liminf_{\zeta\to\infty}H_1^{\gamma_2}(\zeta,\zeta_0)\int_\zeta^\infty\left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(\sigma(t),\zeta_0)}\right)^{\gamma_2}a(t)\Delta t\leq\frac{\gamma_2^{\gamma_2}}{l^{\gamma_2(1-\gamma_2)}(1+\gamma_2)^{1+\gamma_2}},$$

which is a contradiction with (37). The proof is complete. □

The last theorem is based on the following assumption:

$$\int_{\zeta_0}^{\infty}\left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(\sigma(t),\zeta_0)}\right)^{\gamma_2}a(t)\Delta t<\infty.$$

Otherwise, (36) holds, implying that $\mathcal{N}_0=\varnothing$ according to Theorem 7.

Theorem 9. *Let* $\gamma_2\geq 1$. *If:*

$$\int_{\zeta_0}^{\infty}\left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(t,\zeta_0)}\right)^{\gamma_2}a(t)\Delta t=\infty, \tag{42}$$

where $\varphi(\zeta)$ *is defined as in* (21)*, then* $\mathcal{N}_0=\varnothing$.

Proof. The proof is similar to that of Theorem 7 and is therefore omitted. □

Theorem 10. *Let* $\gamma_2\geq 1$. *If:*

$$\liminf_{\zeta\to\infty}H_1^{\gamma_2}(\zeta,\zeta_0)\int_{\zeta}^{\infty}\left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(t,\zeta_0)}\right)^{\gamma_2}a(t)\Delta t>\frac{\gamma_2^{\gamma_2}}{l^{\gamma_2(\gamma_2-1)}(1+\gamma_2)^{1+\gamma_2}}, \tag{43}$$

where $\varphi(\zeta)$ *and* l *are defined as in* (21) *and* (38)*, respectively, then* $\mathcal{N}_0=\varnothing$.

Proof. Assume Equation (1) has a nonoscillatory solution $y(\zeta)\in\mathcal{N}_0$ such that $y(\zeta)>0$ and $y(g(\zeta))>0$ for $\zeta\in[\zeta_0,\infty)_{\mathbb{T}}$. Then:

$$y^{[i]}(\zeta)>0,\ i=1,2 \text{ and } y^{[3]}(\zeta)<0 \text{ on } [\zeta_0,\infty)_{\mathbb{T}}.$$

As a result, (31) holds on $[\zeta_1,\infty)_{\mathbb{T}}$, for sufficiently large $\zeta_1\in[\zeta_0,\infty)_{\mathbb{T}}$. Now, for any $\varepsilon>0$, there exists a $\zeta_2\in[\zeta_1,\infty)_{\mathbb{T}}$ such that (39) for $\zeta\in[\zeta_2,\infty)_{\mathbb{T}}$. Multiplying both sides of (31) by $H_1^{\gamma_2}(\zeta,\zeta_0)$ and using (39), we obtain for $\zeta\in[t_2,\infty)_{\mathbb{T}}$,

$$\begin{aligned}
&H_1^{\gamma_2}(\zeta,\zeta_0)\int_{\zeta}^{\infty}\left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(t,\zeta_0)}\right)^{\gamma_2}a(t)\Delta t\\
\leq\ & H_1^{\gamma_2}(\zeta,\zeta_0)x(\zeta)-\gamma_2H_1^{\gamma_2}(\zeta,\zeta_0)\int_{\zeta}^{\infty}H_0(t,\zeta_0)x^{1/\gamma_2}(t)x(\sigma(t))\Delta t\\
\leq\ & H_1^{\gamma_2}(\zeta,\zeta_0)x(\zeta)-H_1^{\gamma_2}(\zeta,\zeta_0)(H-\varepsilon)^{1+1/\gamma_2}(l-\varepsilon)^{\gamma_2-1}\int_{\zeta}^{\infty}\frac{\gamma_2H_0(t,\zeta_0)}{H_1^{\gamma_2}(t,\zeta_0)H_1(\sigma(t),\zeta_0)}\Delta t\\
\leq\ & H_1^{\gamma_2}(\zeta,\zeta_0)x(\zeta)-H_1^{\gamma_2}(\zeta,\zeta_0)(H-\varepsilon)^{1+1/\gamma_2}(l-\varepsilon)^{\gamma_2-1}\int_{\zeta}^{\infty}\left(\frac{-1}{H_1^{\gamma_2}(t,\zeta_0)}\right)^{\Delta}\Delta t\\
=\ & H_1^{\gamma_2}(\zeta,\zeta_0)x(\zeta)-(H-\varepsilon)^{1+1/\gamma_2}(l-\varepsilon)^{\gamma_2-1},
\end{aligned} \tag{44}$$

since:

$$\begin{aligned}
\left(\frac{-1}{H_1^{\gamma_2}(t,\zeta_0)}\right)^{\Delta} &= \frac{\gamma_2\int_0^1[(1-h)H_1(t,\zeta_0)+hH_1(\sigma(t),\zeta_0)]^{\gamma_2-1}\mathrm{d}h\ H_0(t,\zeta_0)}{H_1^{\gamma_2}(t,\zeta_0)H_1^{\gamma_2}(\sigma(t),\zeta_0)}\\
&\leq \frac{\gamma_2H_0(t,\zeta_0)}{H_1^{\gamma_2}(t,\zeta_0)H_1(\sigma(t),\zeta_0)}.
\end{aligned}$$

Taking the lim inf of both sides of the inequality (44) as $\zeta\to\infty$, we conclude that:

$$\liminf_{\zeta\to\infty}H_1^{\gamma_2}(\zeta,\zeta_0)\int_{\zeta}^{\infty}\left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(t,\zeta_0)}\right)^{\gamma_2}a(t)\Delta t\leq H-(l-\varepsilon)^{\gamma_2-1}(H-\varepsilon)^{1+1/\gamma_2}.$$

Since ε is arbitrary, we arrive at:

$$\liminf_{\zeta\to\infty} H_1^{\gamma_2}(\zeta,\zeta_0)\int_{\zeta}^{\infty}\left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(t,\zeta_0)}\right)^{\gamma_2} a(t)\Delta t \le H - l^{\gamma_2-1}\, H^{1+1/\gamma_2}.$$

Let:

$$A = 1, \quad B = l^{\gamma_2-1}, \quad \text{and} \quad V = H.$$

Using inequality (41) we have:

$$\liminf_{\zeta\to\infty} H_1^{\gamma_2}(\zeta,\zeta_0)\int_{\zeta}^{\infty}\left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(t,\zeta_0)}\right)^{\gamma_2} a(t)\Delta t \le \frac{\gamma_2^{\gamma_2}}{l^{\gamma_2(\gamma_2-1)}(1+\gamma_2)^{1+\gamma_2}},$$

which is a contradiction with (43). This completes the proof. □

Furthermore, Theorem 10 is based on the following assumption:

$$\int_{\zeta_0}^{\infty}\left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(t,\zeta_0)}\right)^{\gamma_2} a(t)\Delta t < \infty.$$

Otherwise, (42) holds, implying that $\mathcal{N}_0 = \varnothing$ according to Theorem 9.

By combining the conclusions of Theorems 7–10 with Lemma 3, we may set convergence of nonoscillatory solutions of the investigated Equation (1).

Theorem 11. *Let $0 < \gamma_2 \le 1$. If (36) or (37) holds, then every solution of Equation (1) is either oscillatory or tends to a finite limit eventually.*

Theorem 12. *Let $\gamma_2 \ge 1$. If (42) or (43) holds, then every solution of Equation (1) is either oscillatory or tends to a finite limit eventually.*

Moreover, by combining the conclusions of Theorems 7–10 with Lemma 4, we may set convergence (of zero) of nonoscillatory solutions of the investigated Equation (1).

Theorem 13. *Let $0 < \gamma_2 \le 1$. If (A) and either (36) or (37) hold, then every solution of Equation (1) is either oscillatory or tends to zero eventually.*

Theorem 14. *Let $\gamma_2 \ge 1$. If (A) and either (42) or (43) holds, then every solution of Equation (1) is either oscillatory or tends to zero eventually.*

Example 1. *Consider the third order dynamic equation:*

$$\left\{\zeta^{\gamma_2}\phi_{\gamma_2}\left(\left[\zeta^{1-\gamma_1}\phi_{\gamma_1}\left(y^{\Delta}(\zeta)\right)\right]^{\Delta}\right)\right\}^{\Delta} + \frac{\beta}{\zeta\alpha(\zeta,\zeta_0)}\phi_{\gamma}(y(g(\zeta))) = 0, \tag{45}$$

where $\beta > 0$, $0 < \gamma_2 \le 1$, and $\alpha(\zeta,\zeta_0) = H_1(\zeta,\zeta_0)H_2^{\gamma_2}(\varphi(\zeta),\zeta_0)$. It is easy to see that (2) is satisfied since:

$$\int_{\zeta_0}^{\infty}\frac{\Delta t}{p_1^{1/\gamma_1}(t)} = \int_{\zeta_0}^{\infty}\frac{\Delta t}{t^{1-\frac{1}{\gamma_1}}} = \infty$$

and:

$$\int_{\zeta_0}^{\infty}\frac{\Delta t}{p_2^{1/\gamma_2}(t)} = \int_{\zeta_0}^{\infty}\frac{\Delta t}{t} = \infty,$$

by ([5], Example 5.60). Additionally:

$$\begin{aligned}
&\liminf_{\zeta\to\infty} H_1^{\gamma_2}(\zeta,\zeta_0)\int_\zeta^\infty \left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(\sigma(t),\zeta_0)}\right)^{\gamma_2} a(t)\Delta t \\
= \;& \beta \liminf_{\zeta\to\infty} H_1^{\gamma_2}(\zeta,\zeta_0)\int_\zeta^\infty \frac{1/t}{H_1(t,\zeta_0)H_1^{\gamma_2}(\sigma(t),\zeta_0)}\Delta t \\
\geq \;& \frac{\beta}{\gamma_2}\liminf_{\zeta\to\infty} H_1^{\gamma_2}(\zeta,\zeta_0)\int_\zeta^\infty \left(\frac{-1}{H_1^{\gamma_2}(t,\zeta_0)}\right)^{\Delta}\Delta t = \frac{\beta}{\gamma_2}.
\end{aligned}$$

As a result of Theorem 13, every solution of (45) is either oscillatory or tends to zero eventually if:

$$0<\gamma_2\leq 1 \qquad \textit{and} \qquad \beta > \frac{1}{l^{\gamma_2(1-\gamma_2)}}\left(\frac{\gamma_2}{1+\gamma_2}\right)^{1+\gamma_2}.$$

Example 2. *Consider the third-order delay dynamic equation:*

$$\left\{\frac{1}{4\zeta^2}\left(\left[\frac{1}{9\zeta^2}\left(y^{\Delta}(\zeta)\right)^2\right]^{\Delta}\right)^2\right\}^{\Delta} + \frac{\delta}{\zeta^{13}}\phi_\gamma\left(y(\sqrt[4]{\frac{2}{3}}\zeta)\right) = 0, \quad \zeta\in[1,\infty), \tag{46}$$

in which $\delta>0$ are constants. It is obvious that condition (2) is fulfilled. Now:

$$\begin{aligned}
&\liminf_{\zeta\to\infty} H_1^{\gamma_2}(\zeta,\zeta_0)\int_\zeta^\infty \left(\frac{H_2(\varphi(t),\zeta_0)}{H_1(t,\zeta_0)}\right)^{\gamma_2} a(t)\Delta t \\
= \;& \delta \liminf_{\zeta\to\infty}\left(\zeta^2-1\right)^2\int_\zeta^\infty \left(\frac{\left(\sqrt{\frac{2}{3}}t^2-1\right)^3}{t^2-1}\right)^2 \frac{1}{t^{13}}\,\mathrm{d}t = \frac{2}{27}\delta
\end{aligned}$$

and:

$$\begin{aligned}
&\int_{\zeta_0}^\infty \left[\frac{1}{p_1(\omega)}\int_\omega^\infty \left(\frac{1}{p_2(\tau)}\int_\tau^\infty a(t)\,\Delta t\right)^{1/\gamma_2}\Delta\tau\right]^{1/\gamma_1}\Delta\omega \\
= \;& \sqrt[4]{27}\delta\int_{\zeta_0}^\infty\left[\omega^2\int_\omega^\infty \frac{1}{\tau^5}\,\Delta\tau\right]^{1/2}\Delta\omega = \infty.
\end{aligned}$$

Therefore, the conditions (A) and (43) are satisfied if $\delta>2$. Then, when $\delta>2$, every solution of Equation (46) is either oscillatory or tends to zero eventually, according to Theorem 14.

4. Discussions and Conclusions

(1) If $p_1(\zeta)=p_2(\zeta)=\gamma_1=\gamma_2=1$ and $g(\zeta)=\zeta$, it is clear that condition (43) becomes:

$$\liminf_{\zeta\to\infty}\zeta\int_\zeta^\infty \frac{h_2(t,\zeta_0)}{t}a(t)\Delta t > \frac{1}{4}$$

Due to:

$$\int_\zeta^\infty \frac{h_2(t,\zeta_0)}{t}a(t)\Delta t \geq \int_\zeta^\infty \frac{h_2(t,\zeta_0)}{\sigma(t)}a(t)\Delta t$$

Theorem 14 improves Theorem 1 for Equation (4).

(2) If $p_1(\zeta)=\gamma_1=1$, $g(\zeta)\leq\zeta$, and p_2 is a nondecreasing functions on $\mathbb{T}$, it is clear that conditions (37) and (43) become:

$$\liminf_{\zeta\to\infty}\frac{\zeta^{\gamma_2}}{p_2(\zeta)}\int_\zeta^\infty\left(\frac{h_2(g(t),\zeta_0)}{\sigma(t)}\right)^{\gamma_2}a(t)\Delta t > \frac{\gamma_2^{\gamma_2}}{l^{\gamma_2(1-\gamma_2)}(1+\gamma_2)^{1+\gamma_2}}$$

and:

$$\liminf_{\zeta\to\infty} \frac{\zeta^{\gamma_2}}{p_2(\zeta)} \int_{\zeta}^{\infty} \left(\frac{h_2(g(t),\zeta_0)}{t}\right)^{\gamma_2} a(t)\Delta t > \frac{\gamma_2^{\gamma_2}}{l^{\gamma_2(\gamma_2-1)}(1+\gamma_2)^{1+\gamma_2}},$$

respectively. Due to:

$$\int_{\zeta}^{\infty} \left(\frac{h_2(g(t),\zeta_0)}{t}\right)^{\gamma_2} a(t)\Delta t \geq \int_{\zeta}^{\infty} \left(\frac{h_2(g(t),\zeta_0)}{\sigma(t)}\right)^{\gamma_2} a(t)\Delta t \geq \int_{\sigma(\zeta)}^{\infty} \left(\frac{h_2(g(t),\zeta_0)}{\sigma(t)}\right)^{\gamma_2} a(t)\Delta t,$$

$$\frac{\gamma_2^{\gamma_2}}{l^{\gamma_2(1-\gamma_2)}(1+\gamma_2)^{1+\gamma_2}} \leq \frac{\gamma_2^{\gamma_2}}{l^{\gamma_2^2}(1+\gamma_2)^{1+\gamma_2}} \qquad \text{for } \frac{1}{2} \leq \gamma_2 \leq 1,$$

and

$$\frac{\gamma_2^{\gamma_2}}{l^{\gamma_2(\gamma_2-1)}(1+\gamma_2)^{1+\gamma_2}} \leq \frac{\gamma_2^{\gamma_2}}{l^{\gamma_2^2}(1+\gamma_2)^{1+\gamma_2}} \qquad \text{for } \gamma_2 \geq 1.$$

Theorem 13 improves Theorem 2 for Equation (10) when $\frac{1}{2} \leq \gamma_2 \leq 1$ and Theorem 14 improves Theorem 2 for Equation (10) when $\gamma_2 \geq 1$.
(3) If $\gamma_1 = \gamma_2 = 1$ and $g(\zeta) \leq \zeta$, then condition (43) becomes:

$$\liminf_{\zeta\to\infty} H_1(\zeta,\zeta_0) \int_{\zeta}^{\infty} \frac{H_2(g(t),\zeta_0)}{H_1(t,\zeta_0)} a(t)\Delta t > \frac{1}{4}$$

Due to:

$$\int_{\zeta}^{\infty} \frac{H_2(g(t),\zeta_0)}{H_1(t,\zeta_0)} a(t)\Delta t \geq \int_{\zeta}^{\infty} \frac{H_2(g(t),\zeta_0)}{H_1(\sigma(t),\zeta_0)} a(t)\Delta t$$

Theorem 14 improves Theorem 4 for the Equation (14).
(4) If $\gamma_1 = \gamma_2 = 1$, Theorem 14 will be reduced to Theorem 6 for the Equation (14).
(5) Following the preceding discussion, the results in this paper improve the results of [7,8,10–12].
(6) It would be interesting to establish Hille oscillation criteria to third-order dynamic Equation (1) supposing

$$\int_{\zeta_0}^{\infty} \frac{\Delta t}{p_i^{1/\gamma_i}(t)} < \infty,\ i = 1,2.$$

Author Contributions: Funding acquisition, R.A.R., Z.A. and A.Y.K.; Investigation, T.S.H., R.A.R. and Z.A.; Methodology, T.S.H., R.A.R., Z.A., A.Y.K., A.A.M. and I.O.; Writing—original draft, T.S.H.; Writing—review & editing, R.A.R., Z.A., A.A.M. and I.O. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Scientific Research Deanship at the University of Ha'il—Saudi Arabia through project number RG-21 101.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that they have no competing interest. There are not any non-financial competing interests (political, personal, religious, ideological, academic, intellectual, commercial, or any other) to declare in relation to this manuscript.

References

1. Hilger, S. Analysis on measure chains—A unified approach to continuous and discrete calculus. *Results Math.* **1990**, *18*, 18–56. [CrossRef]
2. Bohner, M.; Peterson, A. *Dynamic Equations on Time Scales: An Introduction with Applications*; Birkhäuser: Boston, MA, USA, 2001.
3. Kac, V.; Chueng, P. *Quantum Calculus*; Universitext; Springer: Berlin, Germany, 2002.
4. Agarwal, R.P.; Bohner, M.; O'Regan, D.; Peterson, A. Dynamic equations on time scales: A survey. *J. Comput. Appl. Math.* **2002**, *141*, 1–26. [CrossRef]
5. Bohner, M.; Peterson, A. *Advances in Dynamic Equations on Time Scales*; Birkhäuser: Boston, MA USA, 2003.

6. Hille, E. Non-oscillation theorems. *Trans. Am. Math. Soc.* **1948**, *64*, 234–252. [CrossRef]
7. Erbe, L.; Peterson, A.; Saker, S.H. Hille and Nehari type criteria for third-order dynamic equations. *J. Math. Anal. Appl.* **2007**, *329*, 112–131. [CrossRef]
8. Saker, S.H. Oscillation of third-order functional dynamic equations on time scales. *Sci. China Math.* **2011**, *54*, 2597–2614. [CrossRef]
9. Wang, Y.; Xu, Z. Asymptotic properties of solutions of certain third-order dynamic equations. *J. Comput. Appl. Math.* **2012**, *236*, 2354–2366. [CrossRef]
10. Agarwal, R.P.; Bohner, M.; Li, T.; Zhang, C. Hille and Nehari type criteria for third order delay dynamic equations. *J. Differ. Equ. Appl.* **2013**, *19*, 1563–1579. [CrossRef]
11. Agarwal, R.P.; Hassan, T.S.; Mohammed, W. Oscillation criteria for third-order functional half-linear dynamic equations. *Adv. Differ. Equ.* **2017**, *2017*, 111.
12. Hassan, T.S.; Almatroud, A.O.; Al-Sawalha, M.M.; Odinaev, I. Asymptotics and Hille-type results for dynamic equations of third order with deviating arguments. *Symmetry* **2021**, *13*, 2007 [CrossRef]
13. Baculikova, B. Oscillation of second-order nonlinear noncanonical differential equations with deviating argument. *Appl. Math. Lett.* **2019**, *91*, 68–75. [CrossRef]
14. Bohner, M.; Hassan, T.S.; Li, T. Fite-Hille-Wintner-type oscillation criteria for second-order half-linear dynamic equations with deviating arguments. *Indag. Math.* **2018**, *29*, 548–560. [CrossRef]
15. Chatzarakis, G.E.; Džurina, J.; Jadlovská, I. Oscillatory and asymptotic properties of third-order quasilinear delay differential equations. *J. Inequalities Appl.* **2019**, *2019*, 23. [CrossRef]
16. Erbe, L. Existence of oscillatory solutions and asymptotic behavior for a class of third order linear differential equations. *Pacific J. Math.* **1976**, *64*, 369–385. [CrossRef]
17. Erbe, L.; Hassan, T.S.; Peterson, A. Oscillation of third-order nonlinear functional dynamic equations on time scales. *Differ. Eq. Dynam. Syst.* **2010**, *18*, 199–227. [CrossRef]
18. Moaaz, O.; El-Nabulsi, R.A.; Muhsin, W.; Bazighifan, O. Improved Oscillation Criteria for 2nd-Order Neutral Differential Equations with Distributed Deviating Arguments. *Mathematics* **2020**, *8*, 849. [CrossRef]
19. Fite, W.B. Concerning the zeros of the solutions of certain differential equations. *Trans. Am. Math. Soc.* **1918**, *19*, 341–352. [CrossRef]
20. Hassan, T.S. Oscillation of third-order nonlinear delay dynamic equations on time scales. *Math. Comput. Model.* **2009**, *49* , 1573–1586. [CrossRef]
21. Hassan, T.S.; Sun, Y.; Abdel Menaem, A. Improved oscillation results for functional nonlinear dynamic equations of second order. *Mathematics* **2020**, *8*, 1897. [CrossRef]
22. Han, Z.; Li, T.; Sun, S.; Zhang, M. Oscillation behavior of solutions of third-order nonlinear delay dynamic equations on time scales. *Commun. Korean Math. Soc.* **2011**, *26*, 499–513. [CrossRef]
23. Li, T.; Han, Z.; Sun, S.; Zhao, Y. Oscillation results for third-order nonlinear delay dynamic equations on time scales. *Bull. Malays. Math. Sci. Soc.* **2011**, *34*, 639–648.
24. Li, T.; Han, Z.; Sun, Y.; Zhao, Y. Asymptotic behavior of solutions for third-order half-linear delay dynamic equations on time scales. *J. Appl. Math. Comput.* **2011**, *36*, 333–346. [CrossRef]
25. Hovhannisy, G. On oscillations of solutions of third-order dynamic equation. *Abstr. Appl. Anal.* **2012**, *2012*, 715981.
26. Wintner, A. A criterion of oscillatory stability. *Quart. Appl. Math.* **1949**, *7*, 115–117. [CrossRef]
27. Wintner, A. On the nonexistence of conjugate points. *Am. J. Math.* **1951**, *73*, 368–380. [CrossRef]
28. Senel, M.T. Behavior of solutions of a third-order dynamic equation on time scales. *Senel. J. Inequal. Appl.* **2013**, *2013*, 47. [CrossRef]
29. Sun, Y.; Han, Z.; Sun, Y.; Pan, Y. Oscillation theorems for certain third-order nonlinear delay dynamic equations on time scales. *Electron. J. Qual. Theory Diff. Equ.* **2011**, *75*, 1–14. [CrossRef]
30. Yu, Z.; Wang, Q. Asymptotic behavior of solutions of third-order dynamic equations on time scales. *J. Comput. Appl. Math.* **2009**, *255*, 531–540. [CrossRef]
31. Hassan, T.S.; El-Matary, B.M. Oscillation criteria for third order nonlinear neutral differential equation. *PLOMS Math.* **2021**, *1*, 12.
32. Hassan, T.S.; Kong, Q. Asymptotic behavior of third order functional dynamic equations with γ-Laplacian and nonlinearities given by Riemann-Stieltjes integrals. *Electron. J. Qual. Theory Differ. Equ.* **2014**, *40*, 21. [CrossRef]